Fundamentals of Conduction and Radiation
Prof. Amaresh Dalal and Dipankar N. Basu
Department of Mechanical Engineering
Indian Institute of Technology - Guwahati

Lecture - 19
Lumped Capacitance Approach - II

Hello everyone. So in last class, we have started the transient heat conduction where we have considered first the lumped system analysis. We can apply lumped system analysis when temperature does not vary spatially, that means temperature is function of time only. So in those cases we have derived just in general considering uniform heat generation, the temperature distribution with time.

Then finally just putting the heat generation equal to 0, we have derived the temperature distribution like this.

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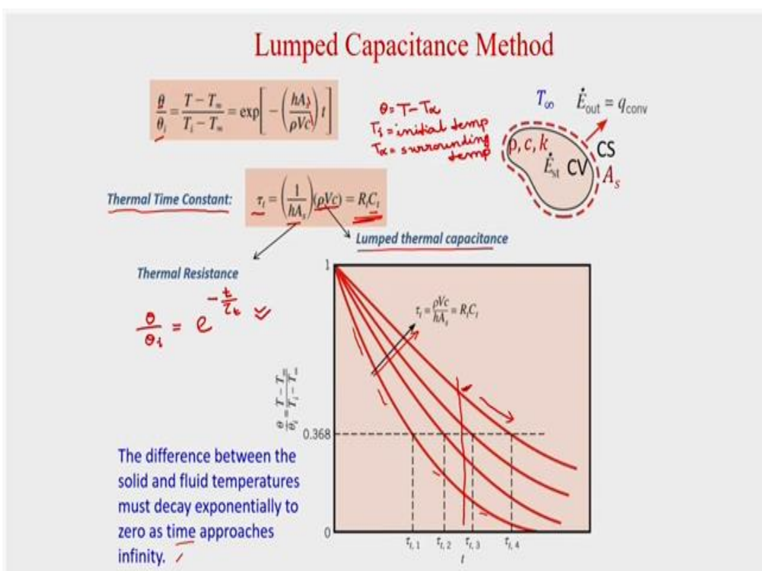

So you can see that we have derived

$$\frac{\theta}{\theta_i} = \frac{(T - T_\infty)}{(T_i - T_\infty)} = e^{-\left(\frac{hA_s}{\rho cV}\right)t}$$

Where, T_i and T_∞ are the initial, and surrounding temperatures, h is the heat transfer coefficient, c is specific heat, V is volume A_s is the heat transfer area and ρ is the density of the solid. And also we defined Biot number which is

$$Bi = \frac{hL_c}{K}$$

If Biot number is less than 0.1 then this lumped system analysis is valid. So we can now define one thermal time constant τ_t.

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho Vc) = R_t C_t$$

Where, R_t is the thermal resistance due to convection that is $\left(\frac{1}{hA_s}\right)$ and $C_t$is the lumped thermal capacitance which is ρVc. So now if you put it there then you will get

$$\frac{\theta}{\theta_i} = \frac{(T - T_\infty)}{(T_i - T_\infty)} = e^{-\frac{t}{\tau_t}}$$

So this is the temperature distribution. So now let us plot the temperature distribution with time. So in the Y axis we will plot $\frac{\theta}{\theta_i}$ which is the non-dimensional temperature and in X direction we will put time t okay. For a particular τ_t , you can see that it exponentially decays. And it will go to 0 when T = T_∞. So in this case if you increase the τ_t, then you can see that so for a particular time, temperature decrease will be less for a higher τ_t okay. So if your thermal time constant is higher, then at a particular time you will find that your temperature drop will be less okay.

So from this you can see for a different τ_t how your temperature actually decays exponentially with time. So the difference between the solid and fluid temperature must decay exponentially to 0 at time approaches to infinity okay. So now we have found the temporary distribution. Now what is the next goal? Next goal is to find the heat transfer rate. So in this case we are considering transient heat conduction.

So at the particular time we will have some heat transfer rate and from time 0 to t if you integrate this heat transfer rate then we will get what is that total heat transfer from the system. So let us calculate that.

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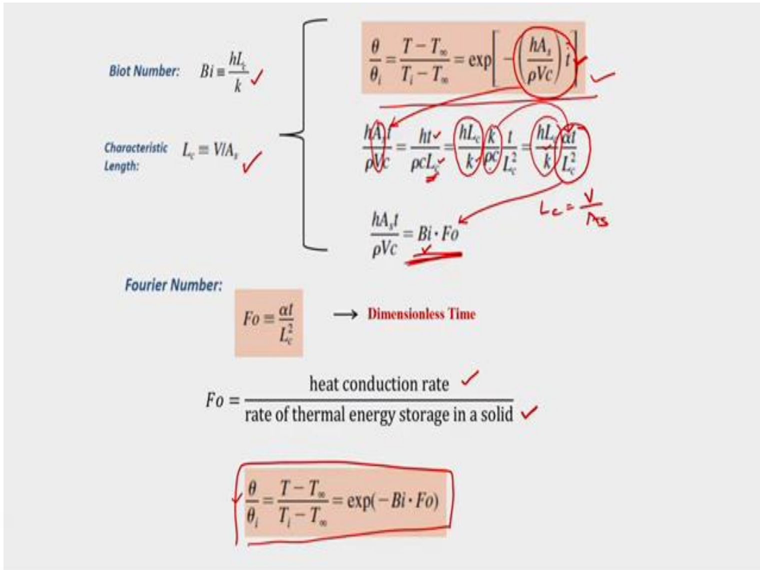

Okay before going to that, let us define the Fourier number okay. So Biot number already we have defined as

$$Bi = \frac{hL_c}{K}$$

Now let's rearrange the term $\left(\frac{hA_s}{\rho cV}\right) t$

$$\left(\frac{hA_s}{\rho cV}\right) t = \frac{ht}{\rho c}\left(\frac{A_s}{V}\right) = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2} = Bi.Fo$$

Here, $\frac{k}{\rho c}$ as we have learned earlier is your thermal diffusivity α right. And $\frac{\alpha t}{L_c^2}$ is known as Fourier number, which is representation of dimensionless time. It can be defined as

$$Fo = \frac{heat\ conduction\ rate}{rate\ of\ thermal\ energy\ storage\ in\ a\ solid}$$

So the temperature distribution now we can write in terms of Biot number and Fourier number.

$$\frac{\theta}{\theta_i} = \frac{(T - T_\infty)}{(T_i - T_\infty)} = e^{-Bi.Fo}$$

Fourier number is a one new non-dimensional number you have learnt which is nothing but the dimensionless time and it is defined as

$$Fo = \frac{\alpha t}{L_c^2}$$

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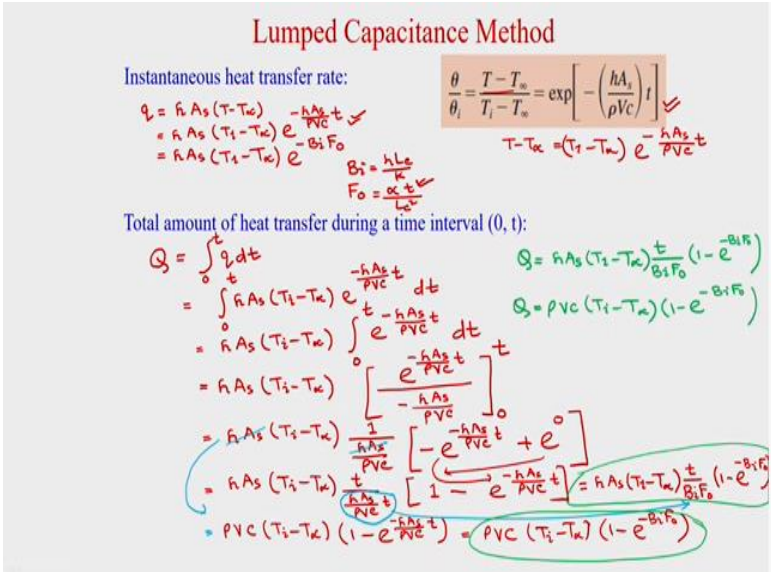

Now, we need to calculate the heat transfer rate at a particular time. Then we will find the total heat transfer in a time range 0 to t. As it is transient heat conduction, total time we have to integrate is from time 0 to t. So what is your instantaneous transfer rate? Now,

$$\frac{\theta}{\theta_i} = \frac{(T - T_\infty)}{(T_i - T_\infty)} = e^{-\left(\frac{hA_s}{\rho cV}\right)t}$$

And instantaneous heat transfer rate is

$$q = hA_s(T - T_\infty)$$

Where, T is the instantaneous temperature. Putting the value of $T - T_\infty$

$$q = hA_s(T_i - T_\infty)e^{-\left(\frac{hA_s}{\rho cV}\right)t}$$

So this is the instantaneous heat transfer rate and now this we can write in terms of Biot number and Fourier number.

$$q = hA_s(T_i - T_\infty)e^{-Bi.Fo}$$

So you can see that h is constant in this case and A_s is also constant and temperature difference between initial and surrounding Ti - T_∞ also constant. Biot number is also constant for a particular system.

So at a particular time you can calculate the Fourier number and you will get the instantaneous heat transfer rate. Now you want to calculate the total heat transfer rate from the system okay. So then you have to integrate this q from time 0 to t.

So total amount of heat transfer

$$Q = \int_0^t q dt$$

$$= \int_0^t hA_s(T_i - T_\infty) e^{-\left(\frac{hA_s}{\rho cV}\right)t} dt$$

So now in this case I already told that your heat transfer coefficient, heat transfer area and the temperature difference Ti - T_∞ are constant. So we can take it out from the integral. So we can write

$$= hA_s(T_i - T_\infty) \int_0^t e^{-\left(\frac{hA_s}{\rho cV}\right)t} dt$$

So now you will be able to integrate it right. It is exponential functions. So

$$= hA_s(T_i - T_\infty) \left[\frac{e^{-\left(\frac{hA_s}{\rho cV}\right)t}}{-\left(\frac{hA_s}{\rho cV}\right)} \right]_0^t$$

So it is simple. So now will write

$$= hA_s(T_i - T_\infty) \left(\frac{1}{\left(\frac{hA_s}{\rho cV}\right)} \right) \left[-e^{-\left(\frac{hA_s}{\rho cV}\right)t} + e^0 \right]$$

Multiply t in the numerator and denominator

$$= hA_s(T_i - T_\infty) \left(\frac{t}{\left(\frac{hA_s}{\rho cV}\right) t} \right) \left[1 - e^{-\left(\frac{hA_s}{\rho cV}\right)t} \right]$$

$$= hA_s(T_i - T_\infty) \left(\frac{t}{Bi.Fo} \right) \left[1 - e^{-Bi.Fo} \right]$$

So this is one expression and another way we can write by cancelling hA_s from the previous expression

$$= \rho cV(T_i - T_\infty) \left[1 - e^{-\left(\frac{hA_s}{\rho cV}\right)t} \right]$$

$$= \rho cV(T_i - T_\infty) \left[1 - e^{-Bi.Fo} \right]$$

So you please remember that in the Fourier number, you have the time t okay. So at a particular time you need to calculate the instantaneous heat transfer rate. So you will find the Fourier number at that particular time okay. So you have to remember that okay. When we will solve some numerical problem; you calculate the Fourier number at that time t okay.

And when you are calculating the total heat transfer, then you have to integrate from 0 to t. So now we have written two expressions. So you can use whichever is convenient. The expressions are

$$Q = hA_s(T_i - T_\infty)\left(\frac{t}{Bi.Fo}\right)\left[1 - e^{-Bi.Fo}\right]$$

$$Q = \rho c V(T_i - T_\infty)\left[1 - e^{-Bi.Fo}\right]$$

So let us first solve two problems, so that you will be able to understand how this heat transfer is taking place and you will appreciate whatever we have studied in this last class and todays class okay.

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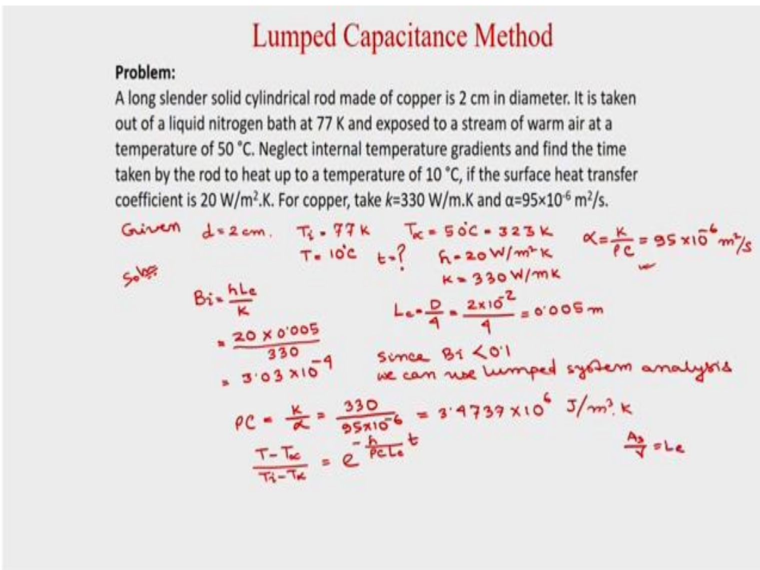

So this is the problem, first I am reading this problem and try to understand what it says. A long slender solid cylindrical rod made of copper is 2 cm in diameter. It is taken out of a liquid nitrogen bath at 77 K and exposed to a stream of warm air at a temperature of 50 ºC. Neglect internal temperature gradients and find the time taken by the rod to heat up to a temperature of 10 ºC, If the surface heat transfer coefficient is 20 W/m^2K. For copper take k is equal to 330 W/mK and thermal diffusivity α is equal to $95X10^{-6}$ m^2/s.

So now you have a case where you have a long slender cylindrical rod okay. So it is a long so when you calculate L_c the characteristic length. What will be the characteristic length in this case? We have studied in the last class. That will be R/2 right.

So let us write what are given okay, then it will be easy to solve this problem.

$$diameter = 2\ cm$$

$$intial\ temperature\ T_i = 77K$$

$$T_\infty = 50°\text{C} = 323\ K$$

$$final\ temp\ T = 10°\text{C}$$

$$h = 20\frac{W}{m^2K}$$

$$K = 330\frac{W}{mK}$$

$$\alpha = \frac{K}{\rho C} = 95 \times 10^{-6}\frac{m^2}{s}$$

Your T_i is 77 K which is the liquid nitrogen temperature from which the rod was taken out okay. T_∞ will be temperature of the air stream which is used to heat it.

So what first we will do? So first we have to find whether lumped system analysis is valid or not okay. For that what we need to do? We need to calculate the Biot number. So L_c you know which is R/2 or D/4 from the last class and h is given, k is given. So you can calculate the Biot number. So,

$$L_c = \frac{D}{4} = \frac{2 \times 10^{-4}}{4} = 0.005\ m$$

$$Bi = \frac{hL_c}{K} = \frac{20 \times 0.005}{330} = 3.03 \times 10^{-4}$$

So this is very less right.

If Biot number is less than 0.1, we can use lumped system analysis. So Biot number less than 0.1 okay. Since Biot number is less than 0.1 we can use lumped system analysis okay. So now,

$$\rho C = \frac{K}{\alpha} = \frac{330}{95 \times 10^{-6}} = 3.4737 \times 10^6\frac{J}{m^3.K}$$

Now the temperature distribution will be

$$\frac{(T - T_\infty)}{(T_i - T_\infty)} = e^{-\left(\frac{h}{\rho c L_c}\right)t}$$

As $A_s/V = L_c$ we are writing it directly. Now here everything is known you see, T is known okay it is your 10 ºC, T_∞ is known T_i is known h, ρ, L_c; everything is known. So now you can find the time t from this expression. Now let us go to the next slide and let us put.

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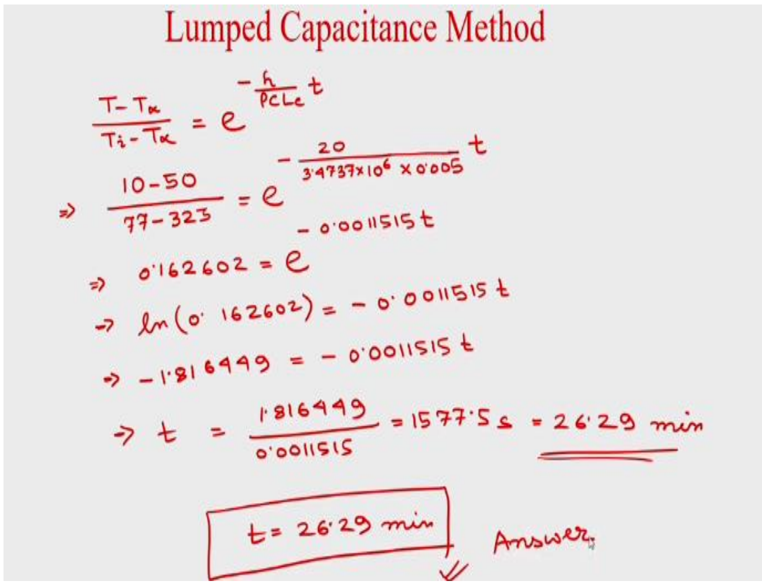

So we can write

$$\frac{(T - T_\infty)}{(T_i - T_\infty)} = e^{-\left(\frac{h}{\rho c L_c}\right)t}$$

$$\Rightarrow \frac{(10 - 50)}{(77 - 323)} = e^{-\left(\frac{20}{3.4737 \times 10^6 \times 0.005}\right)t}$$

In the left hand side numerator temperatures are in ºC and denominator temperatures are in K. As we are taking difference this does not cause any discrepancy.

$$\Rightarrow 0.162602 = e^{-0.0011515t}$$

$$\Rightarrow \ln 0.162602 = -0.0011515t$$

$$\Rightarrow t = \frac{1.816449}{0.0011515} = 1577.5\ s = 26.29\ min$$

In the problem you are asked to find the time so that it will reach that temperature 10 ºC. So now you can see that if the initial temperature is 77 K so to reach 10 ºC it will take 26.23 minutes okay so this is the answer.

So t = 26.29 min if you are asked in seconds you write in second 1577.5 s. So you understood right how to calculate. So the most important thing here is that you have to calculate Biot number first okay. You do not know whether you can use the lumped system analysis as Biot number is much greater than the one or order of 1 then you have to use distributed system because spatial variation maybe there. That is the why first you calculate the L_c. You know how to calculate L_c for different shape of the body. Then you calculate the Biot number, and then you go ahead if the lumped system analysis is valid. Okay, so let us take another problem.

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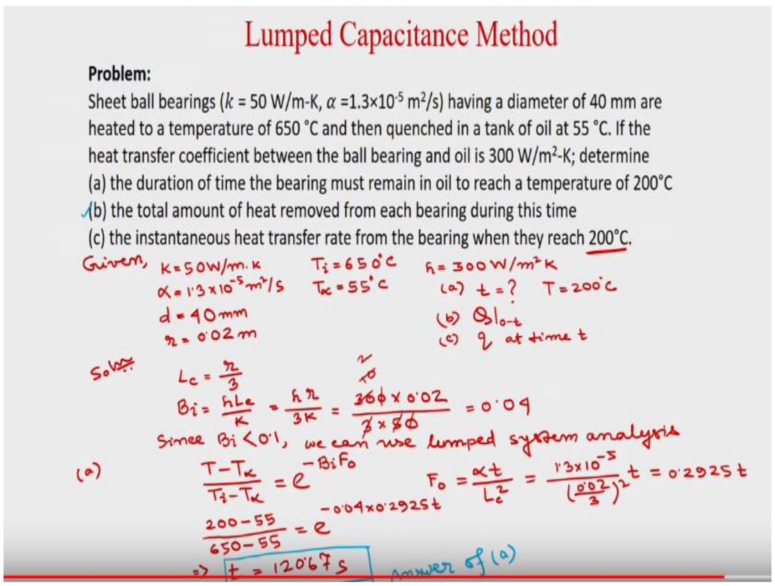

So I am reading first this problem. Sheet ball bearings thermal conductivity is 50 W/mK and thermal diffusivity is $1.3 \times 10^{-5}\frac{m^2}{s}$, having a diameter of 40 mm okay, are heated to a temperature of 650 ºC and then quenched in a tank of oil at 55 ºC. If the heat transfer coefficient between the ball bearing and oil is 300 W/m^2K. Determine (a) the duration of time the bearing must remain in oil to reach a temperature of 200 ºC (b) the total amount of heat removed from each bearing during this time and (c) the instantaneous heat transfer rate from the bearing when they reach 200 ºC. So what is this problem? So one sheet ball bearing is there okay its properties are given. Now diameter is also given 40 mm.

So now it is heated to a temperature 650 ºC. So now this is your T_i right? Then it is kept in a tank of oil so that is T_∞, 55 ºC. So these are given then your heat transfer coefficient of the oil is given

300 W/m^2K. Now first pass have to calculate that how much time it will take to reach the temperature 200 ºC from 650 ºC, which was initial temperature okay.

So we will use the similar analysis what we have done earlier. Then you have to calculate the total amount of heat removed from each bearing during this time. So whatever time you will calculate from time t=0 to that time the total amount of heat removed we have to calculate. So that we have already calculated and we have explained in two different ways. And thirdly you have to calculate the instantaneous heat transfer rate from the bearing when they reach 200 ºC.

So let us calculate. So first we will write what are the data given

$$K = 50\frac{W}{mK}$$

$$\alpha = 1.3 \times 10^{-5}\frac{m^2}{s}$$

$$T_i = 650℃$$

$$T_\infty = 55℃$$

$$h = 300\frac{W}{m^2K}$$

$$diameter = 40\,mm \;\Rightarrow r = 0.02\,m$$

So you see this is the ball bearing and how it looks? Ball bearing is spherical in shape. So for a spherical shape what is the L_c? It is r/3 okay. So

$$L_c = \frac{r}{3}$$

Now,

$$Bi = \frac{hL_c}{K} = \frac{hr}{3K} = \frac{300 \times 0.02}{3 \times 50} = .04$$

So Biot number is 0.04 which is less than 0.1, so we can use that lumped system analysis. So now first solve the first problem (a),

$$\frac{(T - T_\infty)}{(T_i - T_\infty)} = e^{-Bi.Fo}$$

What is Fourier number in this case? Fourier number is

$$Fo = \frac{\alpha t}{L_c^2}$$

Putting the values

$$= 1.3\frac{\times 10^{-5}}{\left(\frac{0.02}{3}\right)^2}t = 0.2925t$$

Now, putting this value in the temperature distribution

$$\Rightarrow \frac{(200 - 55)}{(650 - 55)} = e^{-0.04\times 0.2925t}$$

$$\Rightarrow t = 120.67\ s$$

So this is answer of first part okay that is (a). Now in problem (b) you can see here, just read it; the total amount of heat removed from each bearing during this time; so from the time 0 to 120.67 second the that total amount of heat removed that you have to calculate. So we have already calculated that right, in two expressions. The easier expression will take.

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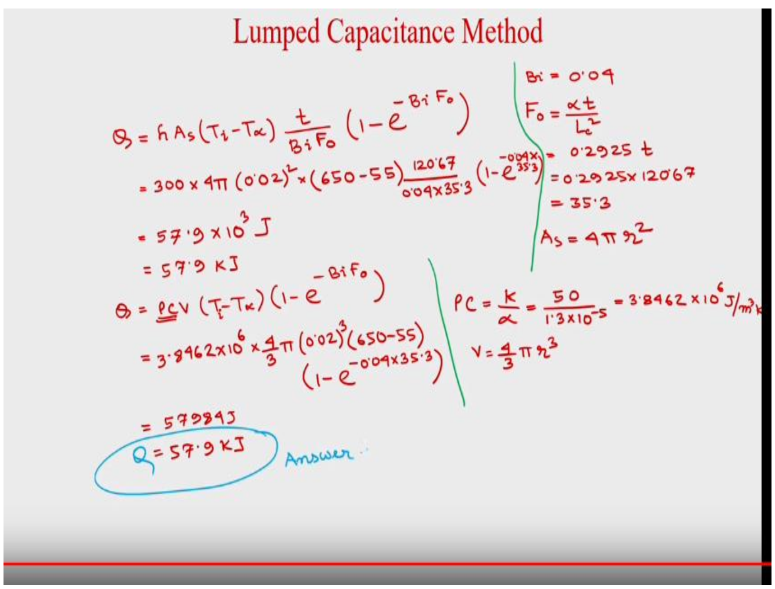

So we have written your

$$Q = hA_s(T_i - T_\infty)\left(\frac{t}{Bi.Fo}\right)\left[1 - e^{-Bi.Fo}\right]$$

So you know the time t, so Fourier number is known and all other terms are known. So it will be easier to use this expression. Another expression also you can write but you have to calculate the ρC from the expression of thermal diffusivity.

First you calculate the Fourier number okay. Fourier number is

$$Fo = \frac{\alpha t}{L_c^2} = 0.2925t = 0.2925 \times 120.67 = 35.3$$

So all the other terms are you know; so now you put the values

$$Q = 300 \times 4\pi(.02)^2(650 - 55)\left(\frac{120.67}{0.04 \times 35.3}\right)[1 - e^{-0.04 \times 35.3}]$$

$$57.9 \times 10^3 J = 57.9\ kJ$$

Here, as it is a spherical shape the heat transfer area will be $4\pi r^2$. Let's see whether we can get this expression from the other formula

$$Q = \rho c V(T_i - T_\infty)\left[1 - e^{-Bi.Fo}\right]$$

Here,

$$\rho c = \frac{K}{\alpha} = \frac{50}{1.3 \times 10^{-5}} = 3.8462 \times 10^6 \frac{J}{m^3 K}$$

$$V = \frac{4}{3}\pi r^3$$

Putting this

$$Q = 3.8462 \times 10^6 \times \frac{4}{3}\pi(.02)^3(650 - 55)\,[1 - e^{-0.04 \times 35.3}]$$

$$= 57984\ J = 57.9\ kJ$$

Okay so this is the answer for the b part okay. We have calculated the total heat removed. Now for the third part you have to find the instantaneous heat transfer rate at time t = 120.67 s okay. So that let us calculate. So you know the expression of q right.

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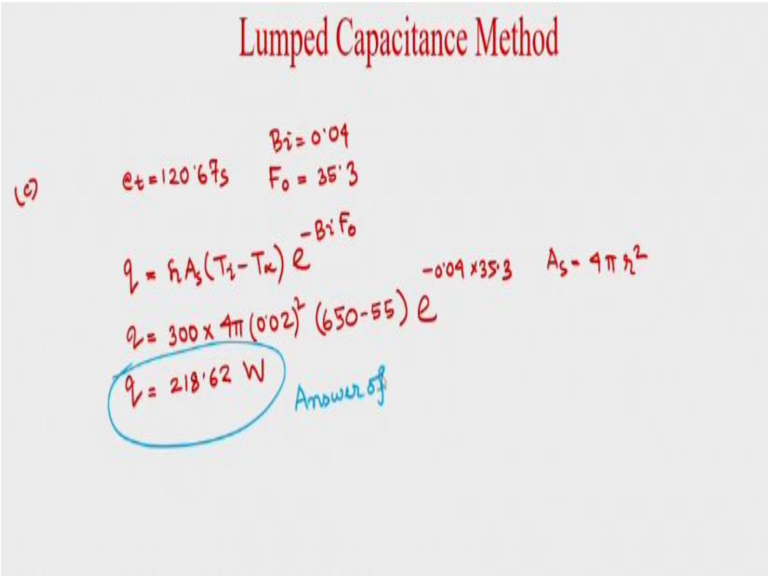

Now Fourier at that time is

$$Fo = 35.3$$

Hence instantaneous heat removal rate will be,

$$q = hA_s(T_i - T_\infty)e^{-Bi.Fo}$$
$$= 300\ \times 4\pi(0.02)^2 \times (650 - 55) \times e^{-0.04\times 35.3}$$
$$= 218.62\ W$$

This is your answer of (c) okay.

So it is instantaneous heat transfer rate at that time okay, at time t = 120.67 s. And total heat transfer rate or total heat transfer from the body from time 0 to 120.67 seconds that we have already shown in the earlier in b part okay so that will be 57.9 kJ okay. So now in these two examples you have learned how to solve this transient problem where T is a function of t that means you can use lumped system analysis.

So most important thing you have to remember is, first you have to calculate the Biot number which is hL_c/k; and L_c depending on the shape of the body you calculate okay. So for a cylinder we have written r/2 for a sphere we have written r/3 but if it is a cube you know what we have calculated and for a plane wall also we have calculated. So you have to find this Biot number and check whether it is less than 0.1 or not.

If less than 0.1 then your lumped system analysis and valid and you can use the temperature distribution that we have derived.

$$q = hA_s(T_i - T_\infty)e^{-\left(\frac{hA_s}{\rho cV}\right)t}$$

Also in terms of Biot number and Fourier number.

$$q = hA_s(T_i - T_\infty)e^{-Bi.Fo}$$

So what is Fourier number? Fourier number is your dimension less time and it is

$$Fo = \frac{\alpha t}{L_c^2}$$

So for a given time if you want to calculate the Fourier number then you put that value of t or if you need to find the t then other things you put in the expression and find the time t. So I think from these two examples problems you have learnt and will appreciate from these two problems the applications of transient heat conduction. Okay so thank you.