**Fundamentals of Conduction and Radiation**
**Amaresh Dalal and Dipankar N. Basu**
**Department of Mechanical Engineering**
**Indian Institute of Technology, Guwahati**

**Lecture - 17**
**Method of Superposition**

Hello everyone. So, today we will continue with the two-dimensional steady state heat conduction. So, in earlier classes, we have seen that if there are one homogeneous direction and another direction is non-homogeneous, then we can solve the problem using separation of variables method. But now if two directions are non-homogeneous then how will you solve that problem, so that we will discuss today.

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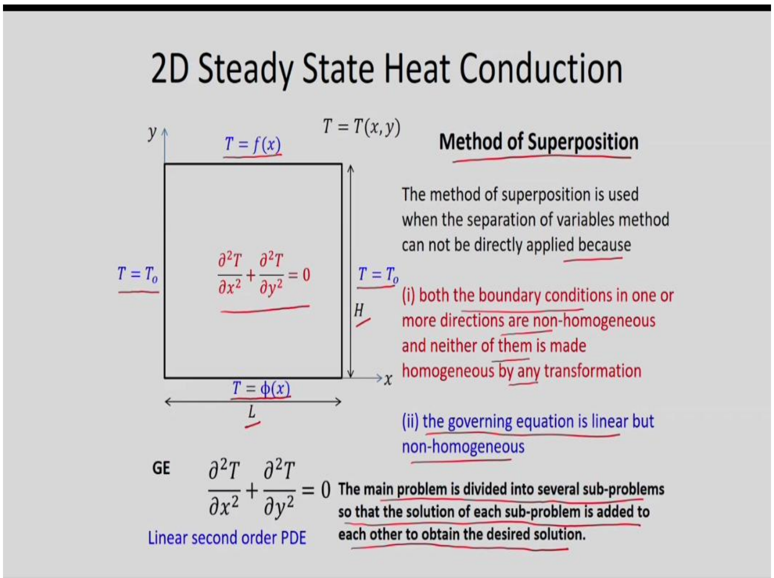


We will learn today method of superposition okay. So, this will apply for steady state two-dimensional heat conduction. So, now you see this problem okay. So, we have a rectangular bar okay of length L and height H. We need to find the temperature distribution inside this solid. So, obviously governing equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Obviously, T is function of x and y. So with the assumption that it is a steady state and there is no heat generation, this is the governing equation. Now, you see the boundary conditions, so in left hand side this wall along x = 0 along y, you can see $T = T_0$ okay and right side wall where at x = L along y, $T = T_0$, top wall at y = H T =f(x). So, it is varying temperature and also at y = 0 along x, T =f(x).

So, you can see that in x and y, both directions are non-homogeneous. Even if you use some transformation technique still you will not be able to convert it to homogeneous in one direction. Unless you do that you will not be able to solve this problem using separation of variables method. So, another method we will use that is known as method of superposition and it is applicable if the governing equation is linear.

So in this problem, you see the method of superposition is used when the separation of variables method cannot be directly applied because both the boundary conditions in one or more directions are non-homogeneous and neither of them is made homogeneous by any transformation. And another one is the governing equation is linear but non-homogeneous. Say if you consider a two-dimensional case with uniform heat generation, then governing equation will become non-homogeneous.

So, in that case you cannot use directly separation of variables method but we can use method of superposition. So, the main problem actually is divided into several sub-problems, so that the solution of each sub-problem is added to each other to obtain the desired solution and that you can do if the governing equation is linear. So, let us see which way we can transform this problem and which way we can use the method of superposition. In this case, we have not considered heat generation, only it is two-dimensional heat conduction and boundary conditions in both directions it is non-homogeneous.

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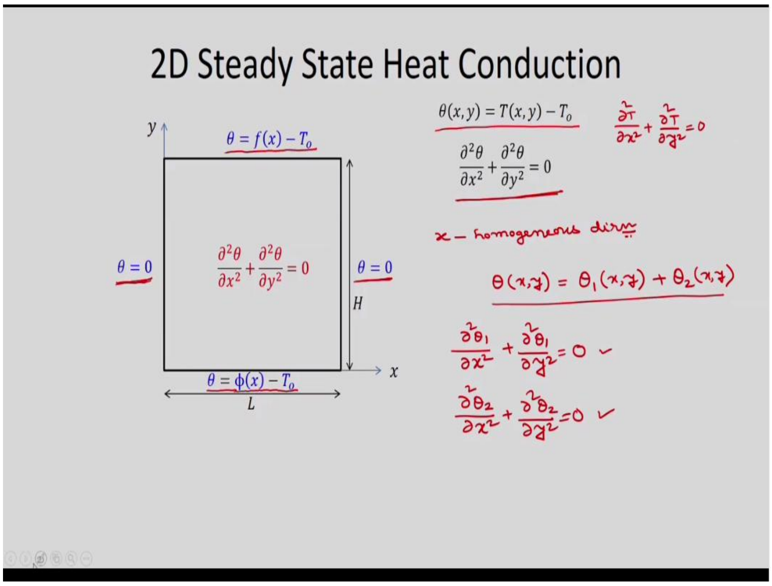


So you see here, so let us take one transformation

$$\theta(x, y) = T(x, y) - T_0$$

So, if you use this transformation what will be the governing equation in this case? So, it will be with conversion of T to θ

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$$

What about the boundary conditions? So, earlier it was $T = T_0$ in the left wall, so obviously if you put $T = T_0$, so it will become $\theta = 0$. Similarly, at $x = L$ also it will become $\theta = 0$ because $T = T_0$ here too. But on the top wall you can see that at $y = H$, it is becoming $\theta = f(x) - T_0$ and at $y = 0$ bottom wall, you can see $\theta = \varphi(x) - T_0$. So, here you can see that your x direction is homogeneous direction because both the boundary conditions you can see it is 0, but in the y direction both boundary conditions are non-homogeneous. So, you cannot directly use separation of variables method because to use separation of variables method in non-homogeneous direction, one should be homogeneous boundary condition and another should be non-homogenous boundary condition.

So, you can see that here x is your homogenous direction. y is non-homogenous direction and both the boundary conditions are non-homogeneous. So, now we will use superposition technique so that we will find the solution of θ which is

$$\theta(x,y)=\theta_1(x,y)+\theta_2(x,y)$$

Where, $\theta_1$ and $\theta_2$ are the solutions of two individual problems where we can use separation of variables method but this superposition $\theta(x,y)=\theta_1(x,y)+\theta_2(x,y)$ we can do, if the governing equation is linear.

And in this case our governing equation is linear and also homogeneous, so we can write

$$\theta(x,y)=\theta_1(x,y)+\theta_2(x,y)$$

So, if you put it in the governing equation, so you will get one equation as

$$\frac{\partial^2\theta_1}{\partial x^2}+\frac{\partial^2\theta_1}{\partial y^2}=0$$

Another governing equation will be

$$\frac{\partial^2\theta_2}{\partial x^2}+\frac{\partial^2\theta_2}{\partial y^2}=0$$

So, if you put these here, so you are going to get these two individual governing equations but now we will find the boundary condition such that we can use separation of variables method. So, we are splitting this problem into two sub-problems where we can use separation of variables method. So, let us see how we can split the boundary conditions.

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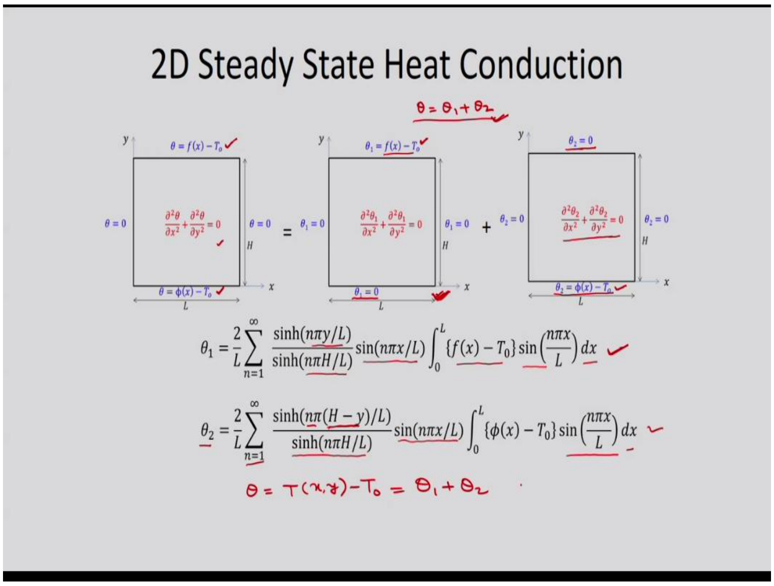


So, here you can see, you look in this slide, this is our main problem where both are non-homogeneous boundary conditions but now we are writing

$$\theta = \theta_1 + \theta_2$$

$$\theta_1 = 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}\sin\left(\frac{n\pi x}{L}\right)\int_0^L \{f(x)-T_0\}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\theta_2 = 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\frac{\sinh\left(\frac{n\pi(H-y)}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}\sin\left(\frac{n\pi x}{L}\right)\int_0^L \{\varphi(x)-T_0\}\sin\left(\frac{n\pi x}{L}\right)dx$$

So this is the solution of $\theta_1$ with the boundary condition $\theta_1 = 0$ and $\theta_1 = f(x) - T_0$ at y = H. So, you can see that in this case, although y direction is non-homogenous direction but one boundary condition is homogeneous and one boundary condition is non-homogeneous, so you can use separation of variables method.

And another problem we are splitting where your governing equation is

$$\frac{\partial^2\theta_2}{\partial x^2} + \frac{\partial^2\theta_2}{\partial y^2} = 0$$

And on the top wall we are putting $\theta_2 = 0$ and bottom wall $\theta_2 = \varphi(x) - T_0$ . So, you can see in this case, your y = 0, it is non-homogeneous boundary condition but y = H, it is it is homogeneous boundary condition. So, x direction in both the cases is homogeneous direction.

Now, y direction is the non-homogeneous direction with one homogeneous boundary condition and one non-homogenous boundary condition. So, both these problems we can solve using separation of variables method but we have used method of superposition just to split the original problem into two. So, now you can see the boundary conditions. So, if you sum it up $\theta_1 + \theta_2$, it should be $\theta$; so you see $\theta_1$ is f(x) - $T_0$ and $\theta_2$=0; so $\theta$ is equal to $\theta_1 + \theta_2$.

Similarly, this one $\theta_1 = 0$, $\theta_2 = \varphi(x) - T_0$. If you add it $\theta_1 + \theta_2$, you are going to get this one and the governing equation is linear, so this you can write this. So, now you can solve individually these two problems. Already the first problem we have already solved it and this is the solution you can see.

$$\theta_1 = 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}\sin\left(\frac{n\pi x}{L}\right)\int_0^L \{f(x) - T_0\}\sin\left(\frac{n\pi x}{L}\right)dx$$

So, this is the solution already we have carried out, so directly we are writing. Now, similar way $\theta_2$ solution you can do using separation of variables method and you will get this solution.

$$\theta_2 = 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\frac{\sinh\left(\frac{n\pi(H-y)}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}\sin\left(\frac{n\pi x}{L}\right)\int_0^L \{\varphi(x) - T_0\}\sin\left(\frac{n\pi x}{L}\right)dx$$

We are writing H-y in the second solution because non-homogeneous condition is in the bottom plane. So, now you got the solution of $\theta_1$ and $\theta_2$. So, the final solution is

$$\theta(x,y) = T(x,y) - T_0 = \theta_1 + \theta_2$$

So, in this case, if you know the function f(x) and $\varphi$(x), then you can integrate this case and you can write the temperature distribution. So, now we have learnt if the problem is having both the boundary conditions non-homogeneous in non-homogeneous direction, then how to use superposition technique to solve the problem. So, now in the next slide what we will do?

Say if you have uniform heat generation, then the governing equation itself it is a non-homogeneous governing equation. So, obviously you cannot directly use separation of variables method. So, again we will use superposition technique in the governing equation so that we will split into two problems and we can solve the problem.

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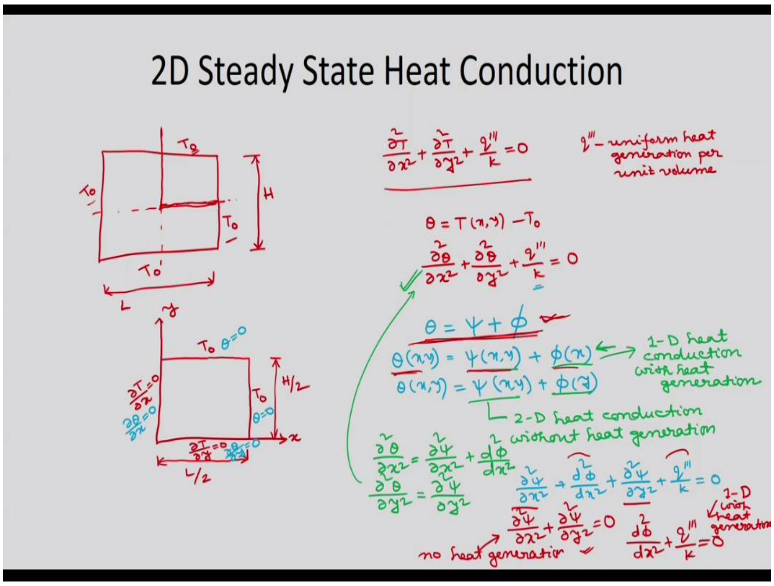


So, let us say we have a rectangular block with symmetry okay. So it is maintained at $T_0$, it is $T_0$, this is your $T_0$ and this is your $T_0$ okay. So, all the boundaries are maintained at temperature $T_0$ and it is a rectangular block of length let us say L and height is H okay. So, in this case, let us have the governing equation as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q'''}{K} = 0$$

So, $q'''$ is the uniform heat generation per unit volume okay. So, in this case, you can see this is your governing equation but you cannot directly use the separation of variables method because the governing equation is non-homogeneous. So, we will split this problem into two. Before that let us see that whether we can divide this problem into simpler problems because you see all boundaries are maintained at temperature $T_0$.

So, obviously this and this will be line of symmetry okay. So, one quadrant you can solve, this one that is one fourth of the problem you can solve okay. So this will be your H/2, and this will be your L/2 okay with this temperature is $T_0$, this temperature is $T_0$ but this is symmetry line, so symmetry line means there will be heat transfer =0. So, here you can write

$$\frac{\partial T}{\partial y} = 0$$

And in this case

$$\frac{\partial T}{\partial x} = 0$$

So, now one fourth of the problem we are solving because these are the symmetry lines and new boundary condition you know. So, now we will transform this equation as

$$\theta(x,y) = T(x,y) - T_0$$

So, your governing equation you can write as

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{q'''}{K} = 0$$

Now what will be your boundary condition? It will be when $T = T_0$; $\theta = 0$ and when $\frac{\partial T}{\partial x} = 0$; $\frac{\partial \theta}{\partial x} = 0$ and when $\frac{\partial T}{\partial y} = 0$; $\frac{\partial \theta}{\partial y} = 0$.

Now let's write

$$\theta(x,y) = \psi(x,y) + \varphi(x)$$

So, what is this? So, it will be $\psi$ is function of x, y and $\varphi$ is function of x only and θ is function of x, y. Another solution also you can write

$$\theta(x,y) = \psi(x,y) + \varphi(y)$$

So, now you are splitting the original problem into two. So the first term in both the solution will be solution of two-dimensional heat conduction without heat generation. The second terms corresponds to 1D heat conduction with heat generation.

So, now we are splitting it in terms of $\psi(x,y)$ which is the solution of two-dimensional heat conduction without heat generation, so the governing equation will become linear. And second problem which is function of one coordinate x or y but with heat generation, so it is 1D heat conduction with heat generation and you can solve it easily okay. So, now if you put the 1st solution in the governing equation we will get

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2}$$

And

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial^2\psi}{\partial y^2}$$

As $\varphi$ is a function of x only and its derivative with y will become 0. So, now you put these two expressions in the governing equation. So, what you are going to get? So, if you put it here in this governing equation, you are going to get

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{q'''}{K} = 0$$

So, now what we will do? Now, we will write 1st and 3rd term together and 2nd and 4th term together. So, you are going to get

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{q'''}{K}=0$$

So, now you can see that 1st problem is with no heat generation and 2nd problem is 1D and with heat generation. So, now it is easy to solve and now we can also transform the boundary conditions accordingly. After that you can solve using separation of variables method this two-dimensional problem and the 1D problem anyway directly you can integrate and find the φ(x) okay. Then, your final solution you can write

$$\theta(x,y)=\psi(x,y)+\varphi(x)$$

So, this just I have given some idea that if you have a non-homogeneous boundary condition how you can solve okay. So, just you can try this but it will not be in your syllabus. Now, another way you can solve that is imaging technique. So, how we will solve this problem, this type of problem let us see.

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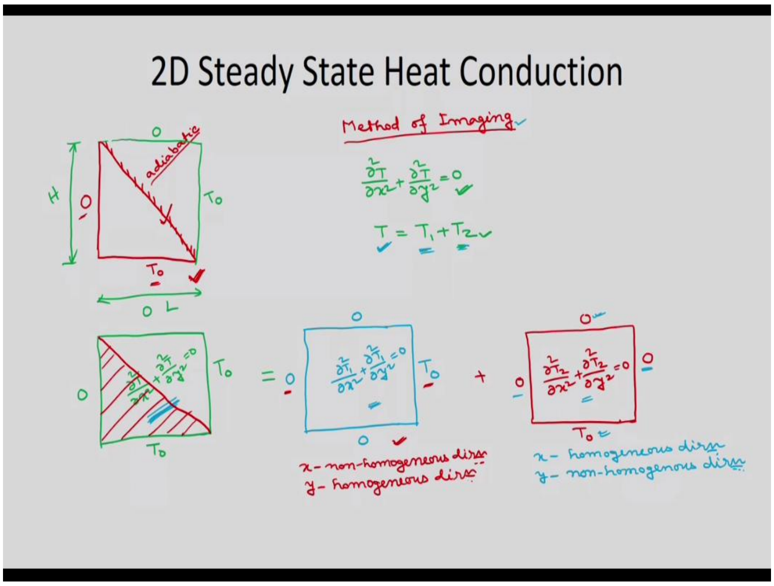


So, let us consider one right angle triangle like this okay. Third direction is infinite, so we are considering a two-dimensional case. So it is a bar of triangular cross-section okay right angular cross-section. Now, the hypotenuse is maintained at adiabatic condition. Adiabatic means there will be no heat transfer across the surface, okay. And let us say base is maintained at $T_0$ okay, and the height is maintained at 0 okay. So, method of imaging we will use to solve this problem.

So, now first question is can you solve this using separation of variables method? So, according to the rules we have considered earlier, in this case we cannot solve using separation of variables method because one of the boundary (hypotenuse) does not lie in the x or y direction okay. So, obviously we cannot solve using separation of variables method because that was one of the condition okay.

So, what we will use, as it is adiabatic, so we can use that as a symmetry line okay. And we will transform this problem into a rectangular problem like this okay. We can convert it into a rectangle by using a mirror image of the domain over the hypotenuse taking hypotenuse as the symmetry line. So the boundary conditions of the height and base will reflect on the newly formed sides. So now it is a rectangular geometry with some length L okay and height H. So, obviously you can use separation of variables method because the boundaries are aligned with the either the x coordinate or y coordinate.

Now, we have to see about the boundary conditions okay. Obviously their boundary conditions, in both the direction it is non-homogeneous. So, we can split into two problems such that one direction it will become homogeneous and one direction it will become one boundary condition homogeneous with another non-homogeneous. So, if the governing equation here

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

So, this problem actually triangular problem, you cannot solve using separation of variable method considering it alone. But what we have done, we have considered a symmetry line, this adiabatic boundary condition line and we are converted this problem into a rectangular bar problem. So now we can solve using separation of variables method with appropriate boundary conditions.

Now, as we cannot directly use, we will split into two problems so that your non-homogeneous direction becomes homogeneous. So, let us say

$$T = T_1 + T_2$$

$T_1$ is the solution of temperature with one problem and $T_2$ is the solution of temperature of second problem. So, that we can use separation of variables method as this governing equation is linear, we can write this $T = T_1 + T_2$. So, this problem now what we can do, we can split.

In one problem we will use

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = 0$$

But boundary condition will write 0, 0, 0 and $T_0$ okay (as given in the figure). And another problem will write

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0$$

So, the boundary conditions will be, here this will be 0, this will be 0 okay, this will be 0, this will be $T_0$ (see figure). Now you see these two pictures, so this is the solution of $T_1$ okay. With the boundary condition, x direction is non-homogeneous direction with one homogenous boundary condition and one non-homogeneous boundary condition. And y direction both 0, so y direction is homogeneous direction okay. So, now for this problem you can solve using separation of variables method okay. So, you can solve for $T_1$.

Now, for the 2nd problem you can see in our x direction, you have both 0, so x direction is homogeneous direction and y is this is 0, this is $T_0$, so it is non-homogeneous direction with one homogeneous and one non-homogeneous boundary condition. So, obviously you can use separation of variables method and find the temperature $T_2$. So, here x is homogenous direction and y is non-homogenous direction with one homogeneous boundary condition and one non-homogenous boundary conditions okay.

So, easily you can solve using separation of variables method and you find from here $T_1$ and from here $T_2$. So, $T_1$, $T_2$ you will get and in the final solution obviously T is function of x, y; $T_1$ is function of x, y; $T_2$ is function of x, y. So, you can solve and find the temperature T using the method of imaging for this triangular bar okay.

So you can actually plot the temperature distribution of this half because this will be a symmetric line, so obviously this triangular bar you will get the temperature distribution same as whatever you have solved for the rectangular domain. So, in this way just you can solve some problem okay using method of superposition or method of imaging okay.

So, you split into convenient sub-problems such that you can apply separation of variables method or if these governing equations are non-homogenous, you divide it into sub problems such that you can solve those problems. So, I have already shown that one for this non-homogenous governing equation, one we have splitted into linear and homogeneous governing equation and one is 1D heat conduction with heat generation.

So, obviously you will be able to solve these two problems. So, with appropriate method, you can use and solve some problems using these methods okay. So, here we will stop this two-dimensional heat conduction. Thank you.