Fundamentals of Conduction and Radiation

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Lecture - 16
Graphical Approach

Hello everyone. So, today we will have the second lecture on two-dimensional steady state heat conduction. So, in last class, we have used analytical approach to solve two-dimensional heat conduction and we have found the temperature distribution. So, let us see the expression.

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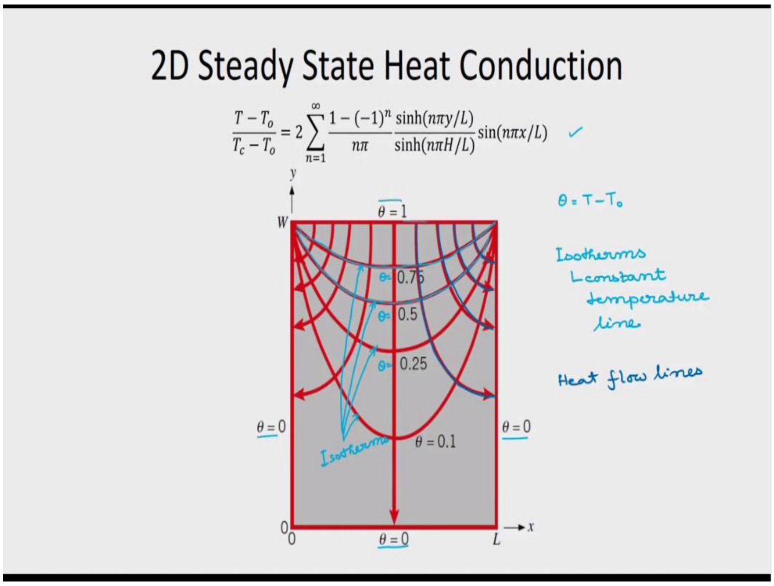

So, we can see that this is the analytical expression okay

$$\frac{T-T_0}{T_c-T_0} = 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}\sin\left(\frac{n\pi x}{L}\right)$$

And in last class we have solved one problem and discussed that first two terms if you take, then it converges fast. So first 3 or 4 terms if you use, then you will get a reasonable accuracy. So, now if we plot the temperature, how it will look like in two-dimensional domain. So, let us see this picture.

So, this is the θ we have already discussed that it is T - T_0 okay, so obviously on the side and bottom wall θ=0 and top wall we are considering θ =1. Three walls are having zero temperature and top wall is having temperature 1. So, if you solve this equation, this analytical approach whatever we have shown, so every x, y location you will get the temperature.

And if you plot a constant temperature line, then it will look like this. So, obviously θ = 1 itself it is a constant temperature line okay. So, this is one temperature line but this is a boundary, then another say θ = 0.75, if you draw so it will look like this, like we have drawn here okay. So, it is a constant temperature line.

And constant temperature lines are known as isotherms. So, θ = 0.5 if you plot, then it will look like this. And value of isotherms will decrease as we go down because left, right and bottom walls are having temperature zero okay. So, here it is θ = 0.25 and here θ = 0.1. So these all are isotherms. You can draw other lines also say θ = 0.26 okay, so you can plot because you have θ (x, y) at every location. So, from that analytical expression, you can calculate and you can join these lines. So, we can use some post processing software like Tecplot or Matlab and you can draw these isotherms. Now, perpendicular to this if we draw some lines, so those are known as heat flow lines. So, here all these perpendicular lines if you draw, so let us say these lines okay, so these lines are just orthogonal to the isotherms.

So, these lines are known as heat flow lines. So, what are heat flow lines? Heat flow lines are those lines through which actually your heat flows okay. So, these are always perpendicular to the isotherms. So, let us see in the next slide.

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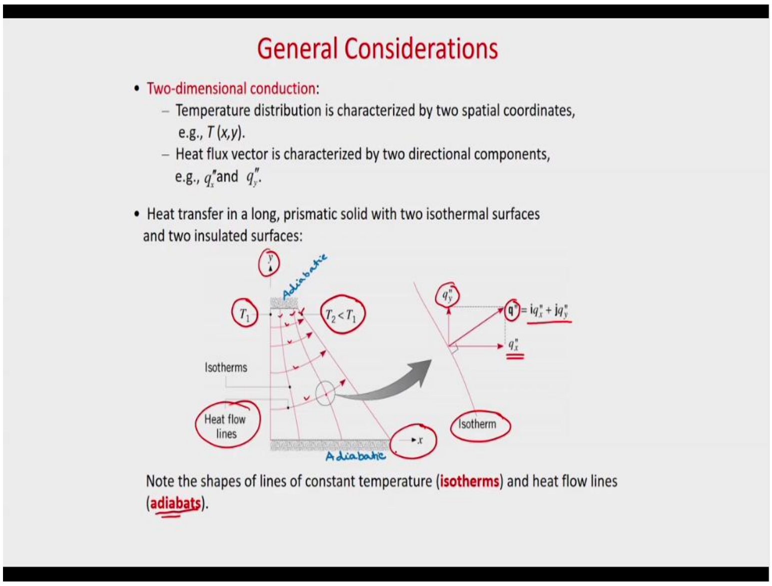

So, let's consider this figure where T is function of (x, y) and top and bottom walls are adiabatic. Adiabatic wall means no heat loss okay. So, through these walls, there will be no heat loss okay. So, now left wall is maintained at temperature T_1 okay and this slant right wall is maintained at T_2 which is less than T_1 okay. So, two boundary conditions are adiabatic boundary condition, top and bottom and left wall is isothermal surface maintained at temperature T_1 and right wall is a slant wall but it is maintained at constant temperature T_2 okay. So, isotherms actually will look like this. So, these are all isotherms lines but heat will flow just perpendicular to these lines.

So, these are heat flow lines we can see, okay. And at any point if you see perpendicular to this line you will have the heat flux vector q'' okay. As heat flux is a vector quantity, so at any point if you draw the perpendicular line on the isotherm, then it will give the direction of your heat flux vector okay. So, that is

$$q = i\, q_x'' + j q_y''$$

I and j corresponds to flux in x and y directions respectively. So, you can see that this is your x direction, so you can have q_x here and this is your y direction, so q_y is acting in this direction. But perpendicular to this isotherm you have q'' . These heat flow lines also are known as adiabats okay. So, constant temperature lines are known as isotherms and heat flow lines are known as adiabats we should remember okay. So, now in last class, we have used analytical approach where we have used separation of variables approach to find the temperature distribution.

But today another method we will learn that is graphical method. Earlier it was used but nowadays we have numerical techniques to solve this type of problem. So, generally this graphical method is not used but in industrial application if you need some quantitative value then you can use this graphical approach with some approximation but it will not give good accuracy in the solution.

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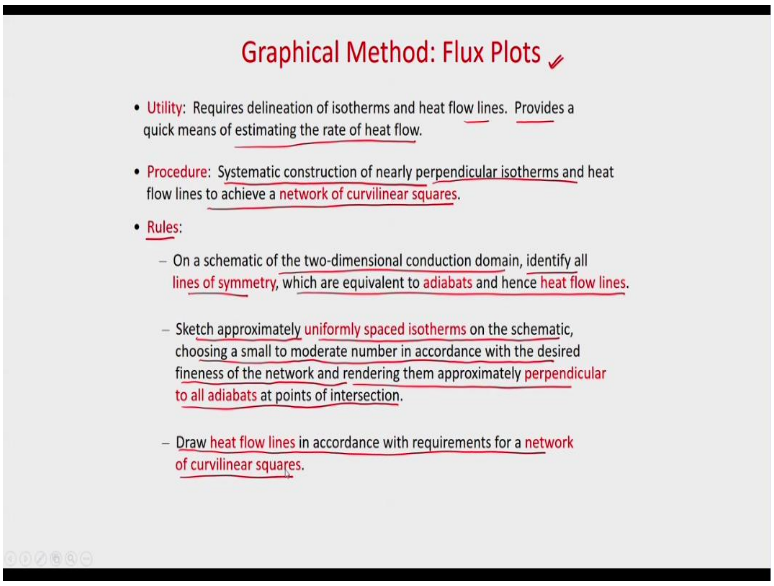

So, first I will read them and later I will explain how to use graphical methods to find the heat flux or heat transfer rate for a particular geometry. So, what is the utility? So, delineation of isotherms and heat flow lines provides a quick means of estimating the rate of heat flow.

Because already there are some results are tabulated for a particular geometry, so you can use it or you can draw these isotherms and the adiabats and you can calculate the heat transfer rate. The procedure is you can see here systematic construction of nearly perpendicular isotherms and heat flow lines to achieve a network of curvilinear squares. So, you have to draw the curvilinear squares that mean $\Delta x \cong \Delta y$. Then, perpendicular to these isotherms we have to draw the heat flow lines. Then, that way you can construct some curvilinear squares.

The rules are on a schematic of the two-dimensional conduction domain, identify all lines of symmetry, which are equivalent to adiabats and hence heat flow lines. So, generally if there are some symmetry lines, then there will be no heat flow across this line because these lines are adiabats okay. So, there will be heat flow lines, across which there will be no heat flow.

Sketch approximately uniformly spaced isotherms on the schematic, choosing a small to moderate number in accordance with the desired fineness of the network and rendering them approximately perpendicular to all adiabats at points of interaction.

So, when these isotherms and adiabats cross each other, you just draw it perpendicular and try to maintain a curvilinear square where$\Delta x = \Delta y$. Then, draw heat flow lines in accordance with requirements of a network of curvilinear squares. So, next slide now I will explain.

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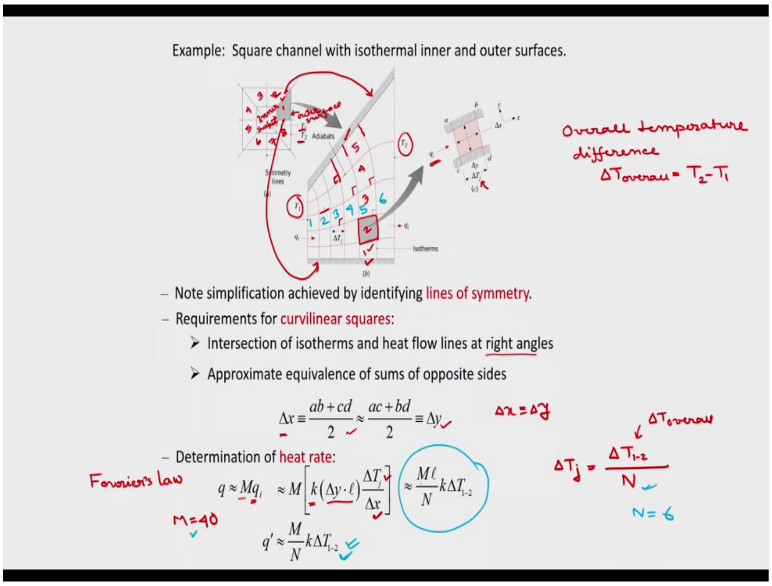

So, you can see this is a square channel with isothermal inner and outer surface. So, you can see this is your inner surface and this is outer surface okay. So, we are considering a two-dimensional square channel and inner surface is maintained at temperature T_1.

You can see, this is your temperature T_1 and outer surface is maintained at a constant temperature T_2. So, these are all isotherms. So, now you try to find where the symmetry lines are. So, you can see from this figure that it is almost symmetry of this line and this quarter will be symmetry of this quarter. So, you can choose in this prism this and this lines as symmetry lines okay.

So, these two are symmetry lines because these symmetry lines will be heat flow lines and those are adiabats, so there will be no heat flow across these lines. So, you can see now it is a zoomed view we have shown here. So, this surface is shown here okay and this surface is shown here. So, we can see this is now one adiabats heat flow lines and this bottom also it is an adiabat and left wall now it is inner surface, so temperature T_1 and outer surface is T_2 okay. So, the temperature difference, overall temperature difference is T_2 -T_1.

So, let's say this is overall temperature difference, as we know the T_2 is greater than T_1.

$$\Delta T_{overall} = T_2 - T_1$$

So, now let's draw the isotherms and the adiabats okay. So these two lines are already adiabats because these are symmetry lines okay. So, following these two lines, draw some curvilinear lines and perpendicular to this adiabats. So, these are isotherms okay, because this T_2 line and T_1 line are also isotherms. So these are isotherms and these are heat flow lines. So, now you draw this perpendicular such a way that you will get $\Delta x = \Delta y$, or you will get curvilinear square. So, you can see the curvilinear square. So, now you take one such domain, so this zoomed view we have shown here.

So, you can see these a, b, c, d are corners and this is Δy and if you tell Δx along the heat flow lines, then the distance here it is Δx and this is your Δy and the temperature drop from here to here is ΔT_j okay. So, now in this curvilinear square, you can use Fourier's law of heat conduction.

So, what will be the Fourier's law of heat conduction? q will be product of K and A and the temperature difference divided by the length okay. So, now after drawing this curvilinear squares okay, so you maintain the intersection of isotherms and heat flow lines at right angles okay. And approximate equivalence of sum of opposite side. So, Δx would be

$$\Delta x = \frac{ab + cd}{2} \approx \frac{ac + bd}{2} = \Delta y$$

So, that means $\Delta x = \Delta y$ you try to maintain while plotting these adiabats and isotherms okay. So, now you can write the Fourier's law of heat conduction, for M number of curvilinear squares in the domain each having heat transfer rate of q_i. So, in this case, you see 1, 2, 3, 4, 5 okay. So, there are 5 here, so you will get M into qi. So, it will be $5q_i$ for this particular domain but now it is we have taken symmetry, so if you try to get from the full domain, then there is 1, 2, 3, 4, 5, 6, 7, 8. So, again you have to multiply 8 okay. So, 8 multiplied by 5 means 40, so M will be 40 in this case okay. So this q_i is the heat flow rate for a particular square domain.

So, in this case k is the thermal conductivity of the material. And what is the area? Area is $\Delta y . l$, l is length in the z direction and the temperature difference for this particular domain is ΔT_j. And the distance is Δx. So, q becomes

$$q \approx M q_i \approx M \left[k (\Delta y . l) \frac{\Delta T_j}{\Delta x} \right] \approx \frac{Ml}{N} k \Delta T_{1-2}$$

Where

$$\Delta T_{1-2} = N\Delta T_j = T_2 - T_1$$

And N is the number of squares along the adiabat. So how many are there? You can see in this direction, so if you take different color, it will be easy to find, so let us say this is 1, 2, 3, 4, 5, 6. So, along this there are 6 squares. So in this case N=6 and M = 40. So, now heat transfer per unit length is

$$q' = \frac{q}{l} \approx \frac{M}{N} k\Delta T_{1-2}$$

So, now which way we have calculated this heat transfer rate? Just we have used some graphical method. We have drawn isotherms and adiabats just perpendicular to each other and we have tried to maintain the curvilinear squares where $\Delta x = \Delta y$. So, in that way using the Fourier's law for a particular domain, we could find heat conduction q_i and there are total M number of squares.

So, it will be Mq_i and from there the overall temperature ΔT_{1-2} divided by number of squares along the heat flow lines we have considered as N, then $\Delta T_{1-2} = N\Delta T_j$. So, using this graphical technique, we have calculated this heat transfer rate. So, it depends on the person who is drawing, how good he is to draw this isotherms and adiabats just perpendicular and maintaining $\Delta x = \Delta y$.

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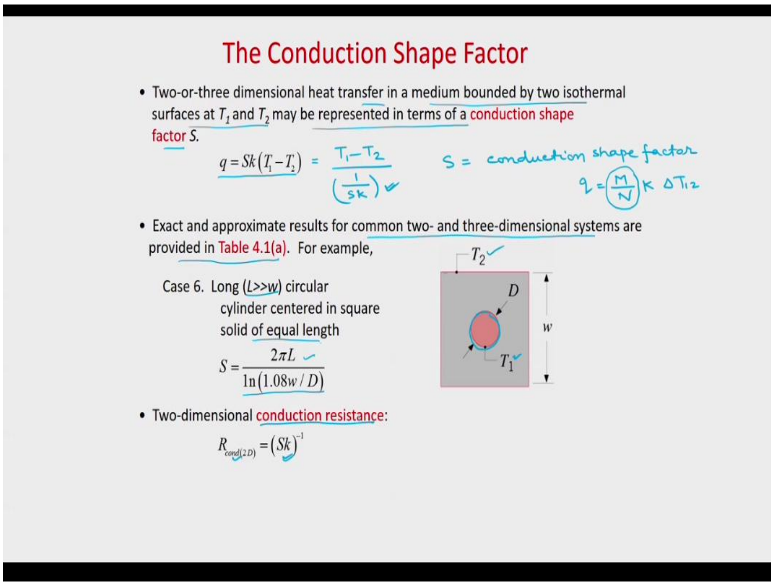

So, now we will introduce using this approach the conduction shape factor. So, what is conduction shape factor? Generally two and three-dimensional heat transfer, in a medium bounded by two isothermal surfaces T_1 and T_2, maybe represented in terms of conduction shape factor S.

$$q = Sk(T_1 - T_2) = \frac{T_1 - T_2}{\frac{1}{Sk}}$$

If you equate it to the equation in earlier slide

$$Sk(T_1 - T_2) = \frac{M}{N} k\Delta T_{1-2}$$

$$\Rightarrow S = \frac{M}{N}$$

And this 1/Sk you have already learnt is the thermal resistance right. So, electrical resistance we have discussed in earlier classes. So, it is equivalent to the thermal resistance and that is known as conduction resistance okay,. You can write

$$R_{cond(2D)} = \frac{1}{Sk}$$

Where S is the conduction shape factor and k is the thermal conductivity of the material.

So, now this exact and approximate results for common two and three-dimensional systems are provided in table 4.1 of the book fundamentals of heat and mass transfer by Incropera and DeWitt.

So, here you can see for this case where one circle or cylinder is subscribed by a square. Cylinder temperature is maintained at T_1 where diameter of the cylinder is D and outside wall temperature is T_2 and the width is w. So, it is a square domain and one cylinder is kept inside whose temperature is T_1. So, for this case actually it is given that if it is a long cylinder okay, L >> w okay, where L is the length of the cylinder, then,

$$S = \frac{2\pi L}{\ln\left(\frac{1.08w}{D}\right)}$$

So, this approximation results are tabulated in different books you can see, here I will show some of these relations in next slides.

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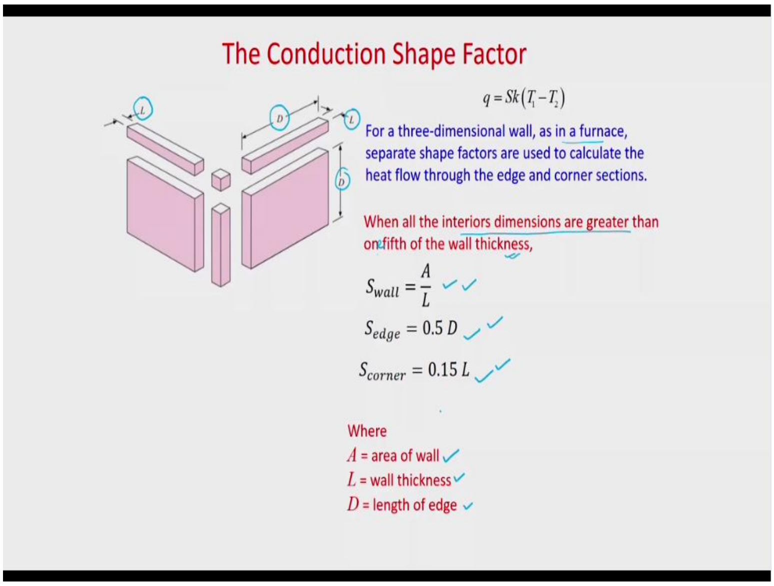

Now, let's consider the three-dimensional case okay and consider the shape factor. So, you can see for three-dimensional wall as in a furnace, separate shape factors are used to calculate the heat flow through the edge and corner section. So, you have sides, you have edges, you have corners okay. So, for that if you get a finite thickness wall if you consider in a three-dimensional situation, then shape factor will be different for different walls, edges and corners. So, when all the interiors dimensions are greater than one fifth of the wall thickness okay, then you can write

$$S_{wall} = \frac{A}{L}$$

$$S_{edge} = 0.5D$$

$$S_{corner} = 0.15L$$

Where A is the area of wall and L is the wall thickness and D is length of edge.

So, in these cases, we are calculating separate shape factor for different regions and if you have N number of such walls, corners and edges just you have to multiply by that. So, if you consider one cubical shape where you have thickness is L, so how many walls will be there in a cubical shape? So, it will be 6 right. How many edges will be there? You tell me how many edges will be there? So, it will be 12 and corners; corners will be 8 okay. So, in that way, you have to just multiply this shape factor with number of corners, number of edges and number of walls. So, we will solve some problems, that time you will understand better that how we will use these shape factors.

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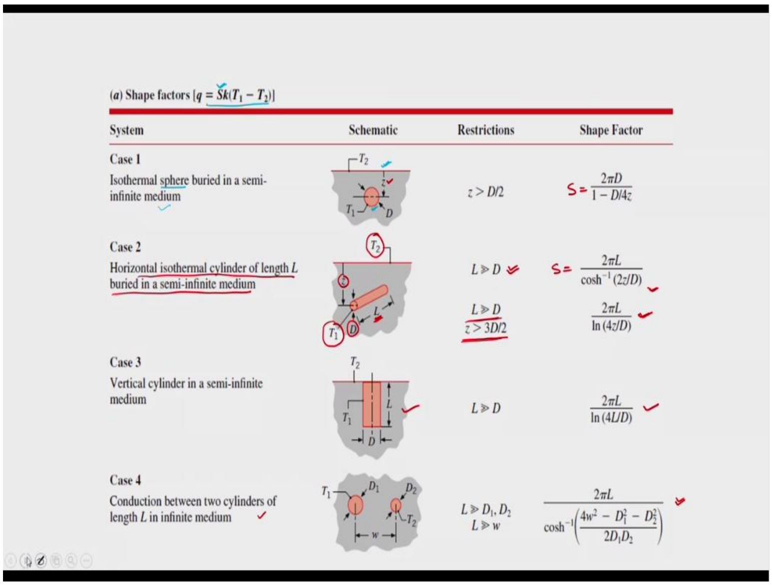

So, now in a tabular format for different conditions, shape factors are given that you can see. So, q obviously is given as $Sk(T_1 - T_2)$ where S is the shape factor. Now, there are different cases okay. Case 1 here you can see isothermal sphere buried in a semi-infinite medium. You know what is semi-infinite medium right. One side you have finite wall and other directions are infinite. So, in that case, you can see if T_2 is the surface temperature and T_1 is the temperature of sphere, so, you get the shape factor S,

$$S = \frac{2\pi D}{1 - \frac{D}{4z}}$$

Where, z is the distance of center of this sphere from the surface okay.

So, similarly you can see 2nd case. Here you can see horizontal isothermal cylinder of length L buried in a semi-infinite medium. D is the diameter of cylinder, z is the distance of center of this cylinder from the surface and surface is maintained at constant temperature T_2, and cylinder surface is maintained at T_1 okay. So the top surface can be assumed as an isothermal wall. So, there are two relations for two conditions, you can see

$$S = \frac{2\pi L}{\cosh^{-1}\frac{2z}{D}} \qquad for\ L \gg D$$

$$S = \frac{2\pi L}{\ln\left(\frac{4z}{D}\right)} \qquad for\ L \gg D\ and\ z > \frac{3D}{2}$$

In both the cases it is long cylinder as $L \gg D$.

So, similarly you will get many situations like vertical cylinder in a semi-infinite medium, you can see this is the relation okay. Conduction between two cylinders of length L in infinite medium. So you can see this, so these different cases you can see in some book okay.

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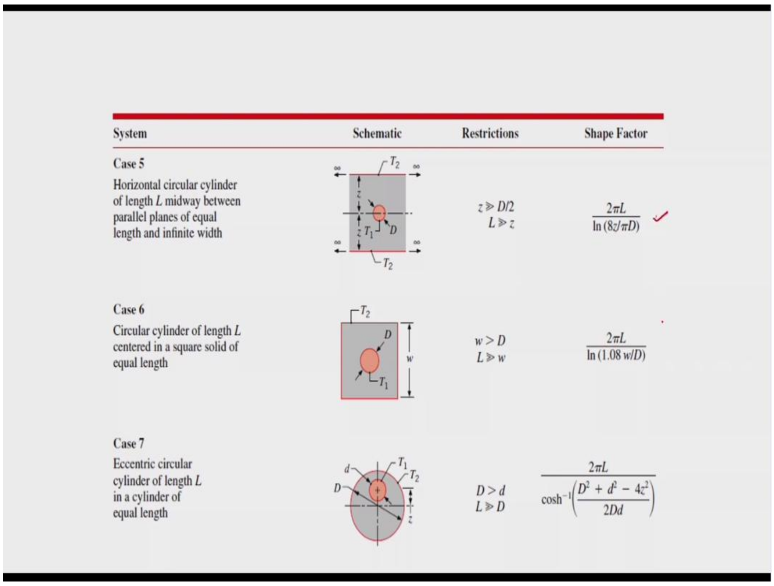

And you can find the shape factor here for other cases also. So, you can see from some book okay.

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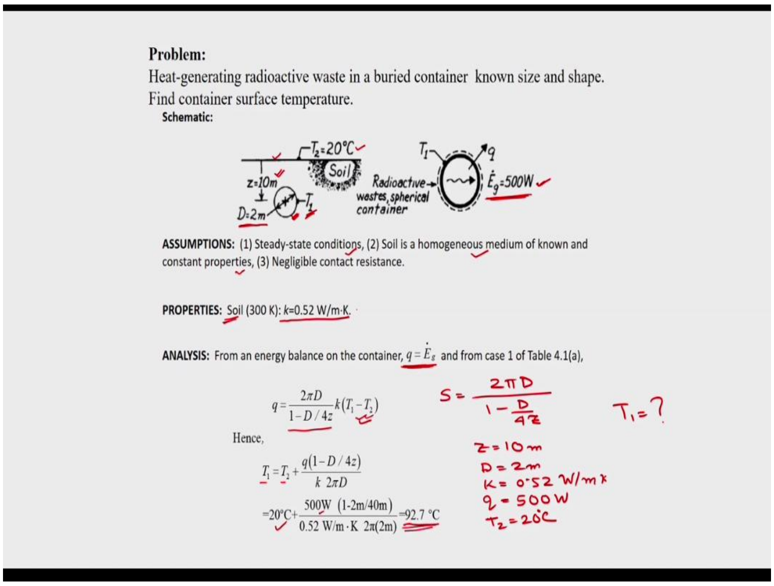

Now, we will solve some problems to understand this graphical method as well as we will use the shape factor here. So, one case you just see, one heat generating radioactive waste is there okay which is actually generating heat okay with $\dot{E}_g$ of 500 W. So, we have to find the container surface temperature T_1 where z from the surface it is kept at distance 10 m. This surface is maintained at temperature T_2 =20 ºC.

And T_1 you have find. And diameter of this radioactive waste material is D= 2 m okay. So, now let us make the assumption that steady state conditions, soil is a homogenous medium of known and constant properties and negligible contact resistance and thermal conductivity K is 0.52 W/mK for the soil okay.

So, with this now from the energy balance you can see whatever heat is generated inside this cylinder all will be actually going out, $\dot{E}_{out}$ because it is a steady state condition. And that $\dot{E}_{out}$ is nothing but the heat transfer rate because q right, so q will be $\dot{E}_g$. So, here you can see from the energy balance that

$$q = \dot{E}_g$$

So, now for this particular case, shape factor is

$$S = \frac{2\pi D}{1 - \frac{D}{4z}}$$

You can see from the shape factor table from any book for this, where D is the diameter and z is the distance from the surface okay. So

$$q = \frac{2\pi D}{1 - \frac{D}{4z}} k(T_1 - T_2)$$

Now we know the values of k, T_2, D, z and q. Now with this given conditions, we have to find the temperature T_1 okay. Hence

$$T_1 = T_2 + \frac{q\left(1 - \frac{D}{4z}\right)}{2\pi D} = 92.7℃$$

So, if you calculate from this you will get 92.7 ºC. So, using the conduction shape factor method actually we could find the temperature T_1 in this problem okay. So, let us take another problem.

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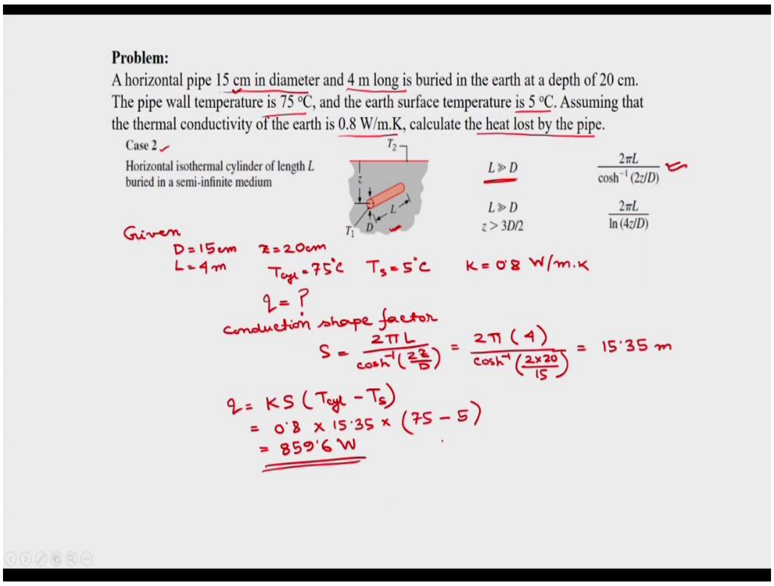

So, I am reading the problem first. A horizontal pipe 15 cm in diameter and 4 m long is buried in the earth at a depth of 20 cm. The pipe wall temperature is 75 ºC and the earth surface temperature is 5 ºC. Assuming that the thermal conductivity of earth is 0.8 W/mK, calculate the heat lost by the pipe. So, in this case T_1, T_2 are given, k is given. And for this particular case we have to find the conduction shape factor and the heat transfer rate q.

So you can see we can use case 2 here okay. Horizontal isothermal cylinder of length L buried in a semi-infinite medium. So, this already we have discussed. Given are

$$D = 15\ cm$$

$$L = 4m$$

$$z = 20\ cm$$

$$T_{cyl} = 75°\text{C}$$

$$T_s = 5°\text{C}$$

$$k = 0.8\ W/mK$$

Now, you have to find what is the heat transfer loss okay. So, now for this case L >>D so, we can use the expression

$$S = \frac{2\pi L}{\cosh^{-1}\frac{2z}{D}}$$

So, all values are known. Putting the values

$$S = \frac{2\pi 4}{\cosh^{-1}\frac{2 \times 20}{15}} = 15.35\ m$$

So, now conduction shape factor we have calculated and what will be the heat lost,

$$q = kS(T_{cyl} - T_s) = 0.8 \times 15.35 \times (75 - 5) = 859.6\ W$$

So, this is the answer. The heat lost for this case will be 859.6 W. So, let us solve another problem.

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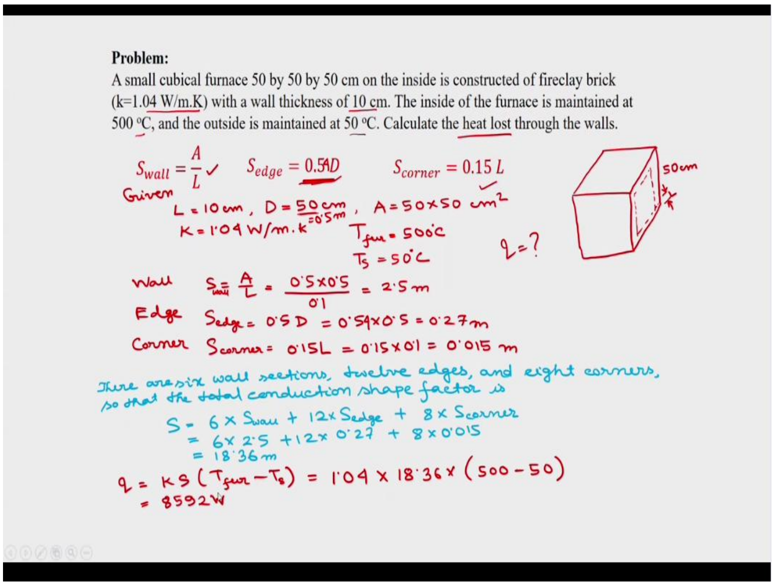

So, I am reading this problem. A small cubical furnace $50 \times 50 \times 50\ cm$ on the inside is constructed of fireclay brick of thermal conductivity 1.04 W/mK with a wall thickness of 10 cm. The inside of the furnace is maintained at 500 ºC and the outside is maintained at 50 ºC. Calculate the heat lost through the walls.

So, you can see that it is a cubical enclosure of finite thickness okay. So, it is a three-dimensional case, so we have learnt that shape factor will be different for the walls, edges as well as the corners. Which are,

$$S_{wall} = \frac{A}{L}$$

$$S_{edge} = 0.5D$$

$$S_{corner} = 0.15L$$

Where A is the area of wall and L is the wall thickness and D is length of edge. Let's write the given quantities

$$L = 10\ cm\ ; D = 50; A = 50 \times 50\ cm^2; k = 1.04 \frac{W}{mK}; T_{fur} = 500℃; T_s = 50℃$$

So, it is the same way we will calculate the shape factor but in this particular case, you have different walls, so there will be 6 walls because 4 plus 2; so 6 walls as it is a cubical furnace. How many edges will be there? So, edges will be 12 and how many corner points are there, 8 corner points. So, whatever the shape factor we have written, so it will be multiplied by number of walls, number of edges or number of corners okay.

So, first for one wall let us calculate the shape factor. So,

$$S_{wall} = \frac{A}{L} = \frac{0.5 \times 0.5}{0.1} = 2.5 \text{ m}$$

In SI unit we are writing. So, it is for one wall okay. Then, for one edge

$$S_{edge} = 0.54D = 0.54 \times 0.5 = 0.27\, m$$

Sometimes the relation is 0.54D okay. Now for corner

$$S_{corner} = 0.15 \times 0.1 = .015\, m$$

So, now total shape factor you calculate, total shape factor okay. So, there are 6 wall sections, 12 edges and 8 corners, so that the total conduction shape factor is

$$S = 6 \times S_{wall} + 12 \times S_{edge} + 8 \times S_{corner}$$

$$= 6 \times 2.5 + 12 \times 0.27 + 8 \times 0.015 = 18.36\, m$$

So, now you know the total shape factor and you know the thermal conductivity, you know the temperature difference, $T_{fur} - T_s$, so you will be able to calculate the heat loss.

$$q = kS(T_{fur} - T_s)$$

$$= 1.04 \times 18.36 \times (500 - 50) = 8592\, W$$

So, we have solved 3 problems okay and we have used the shape factors which are already tabulated for different conditions. And using the conduction shape factor method, we have calculated either temperature of the surface where the heat transfer rate is given or the heat loss we have calculated where the temperatures of the body are given.

So, we have in these two classes, we have discussed analytical approach as well as graphical approach but as I told you that graphical approach is hardly used nowadays. Next class we will study the numerical methods in heat conduction; and that is widely used nowadays for a complex geometries. Thank you.