Fundamentals of Conduction and Radiation

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Lecture - 15
Method of Separation of Variables

Hello everyone, so in last few classes already you have studied one dimensional heat conduction and also steady state heat conduction. In those cases, the solution was very easy because with those assumptions whatever you have considered with that you got an ordinary differential equation. So you got exact solution of those. Now today we will consider two dimensional heat conduction where temperature is function of two spatial coordinates and we will consider steady state.

So in this case you will get partial differential equation, because T is function of x and y. So it is somewhat more complicated and we will see how we can solve these types of problems. There are different application of two dimensional heat conduction like if you see one long bar. So in x, y plane if you consider it so we will get a two dimensional heat conduction and if the third direction z if it is very long or infinite then you can neglect the temperature gradient in that direction.

So in that case you will consider it as a two-dimensional heat conduction. So let us see the governing equation in two-dimensional heat conduction. So you can see that we will study the 2D means two dimensional steady state heat conduction.

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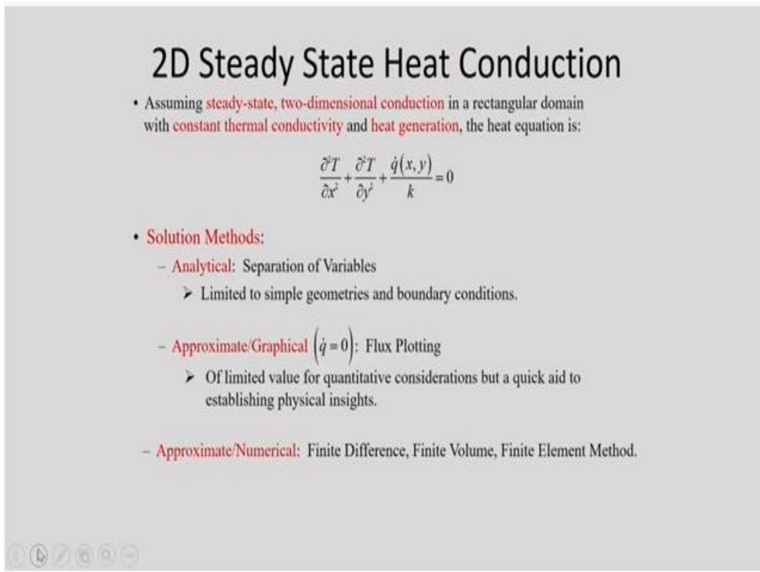

And this is the governing equation you can see with uniform heat generation.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\dot{q}(x,y)}{K}=0$$

There are different solution techniques we can use. One is analytical solution, so we can use separation of variables method where we can solve this. We can convert this partial differential equation to ordinary differential equation. Then we can solve this problem. So it is limited to simple geometries and simple boundary conditions where you have an orthogonal coordinates. Then only this method can be used.

Another approach is approximate or graphical okay. So here we do the flux plotting but it is also of limited value for quantitative considerations but a quick aid to establishing physical insights. And thirdly you can also use approximate or numerical solution. In numerical solution, the partial differential equation actually we use some discretization techniques and you discretize the partial differential equation and solve using some numerical techniques.

So there are different discretization techniques like finite difference method, finite volume method or finite element method. So in those discretization methods you can discretize the governing equation and you can solve. So today we will consider the analytical solution and we will solve using separation of variables method.

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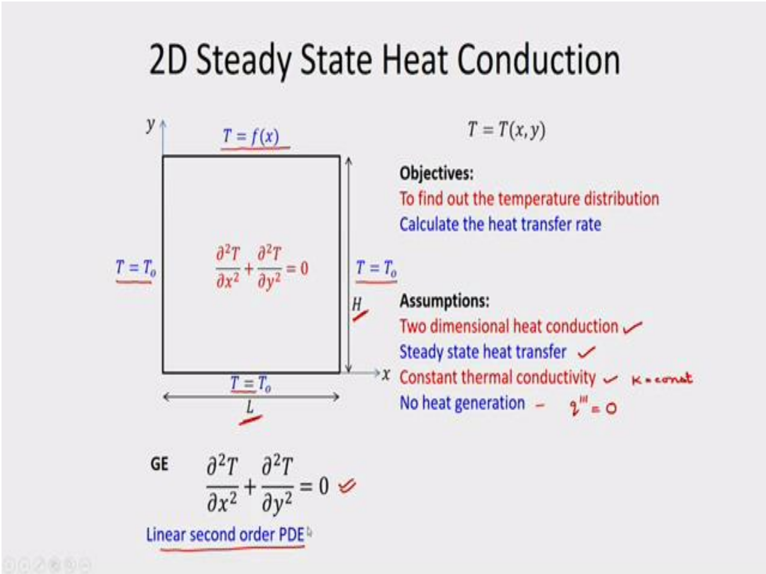

So let us consider two dimensional plate okay. So here you can see it is a two dimensional plate of length L and height H okay. So it is a rectangular plate and of length L and height H. And we need to find the temperature distribution inside this plate. So what are the objectives? Objectives are to find the temperature distribution and once we find the temperature distribution we need to find the heat transfer rate.

So for this problem, let us consider the boundary condition like this. So in the left wall and bottom wall and the right wall you have some fixed temperature T_0 . And the top boundary condition you can see that some temperature is given which is function of x okay.

So what are the assumptions we are taking here? Assumptions we are taking here that it is a two dimensional heat conduction okay. So T is function of x and y. Steady state heat transfer that means your temporal term is 0.

Then constant thermal conductivity that means K is constant okay and no heat generation. So q''' = 0 okay. So with these assumptions we can write the governing equation as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

So this is the equation and you can see that it is a linear and second order PDE and also it is a homogeneous equation.

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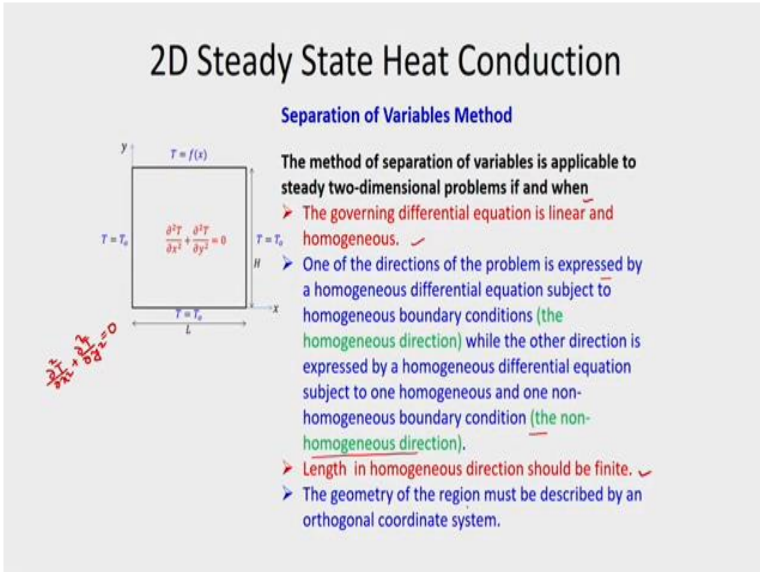

How to solve this problem? So we do not have the exact solution of this; but we can analytically solve this using separation of variables method. So it is one technique in which we can convert these partial differential equation to ordinary differential equation. So using the separation of variables method, we can convert it into ordinary differential equation.

But there are some limitations of using separation of variables method that let us see first. So you can see the method of separation of variables is applicable to steady two dimensional problems if and when the governing differential equation is linear and homogeneous. Obviously whatever heat conduction equation we have considered in our study that is linear and homogeneous.

So we can use separation of variables method. But if you have a uniform heat generation then your governing equation is non-homogeneous. In that case directly you cannot use separation of variables method. Other limitation is one of the direction of the problem is expressed by a homogeneous differential equation subject to a homogeneous boundary condition, which is known as the homogeneous direction.

And while the other direction is expressed by homogeneous differential equation subject to one homogeneous and one non-homogeneous boundary condition, which is known as the non-homogeneous direction okay. So here the boundary condition what we will be considering, in one direction say let us say x or y we should have both the boundary condition as homogeneous boundary conditions.

And if two boundary conditions in one directions are homogeneous boundary conditions, then it is known as homogeneous direction and in other direction we should have one homogeneous direction and one non-homogeneous boundary conditions. So in that case it is known as non-homogeneous direction. Homogeneous boundary condition we will discuss just in the next slide.

Another limitation is length in homogeneous direction should be finite. So when we will apply the separation of variables method, the length in homogeneous directions should be finite. If it is infinite, then we cannot consider or we cannot use this separation of variables method. The other limitation is the geometry of the region must be described by an orthogonal coordinate system. So it is important to note that this separation of variables method we can apply where you have an orthogonal coordinate system.

Like in Cartesian coordinate you can have rectangular okay or square geometry. If it is cylindrical or spherical, you can have orthogonal coordinate system. So if you use a sphere and radial heat conduction, if you use then obviously you can use this separation of variables method. So this is one of the other limitations.

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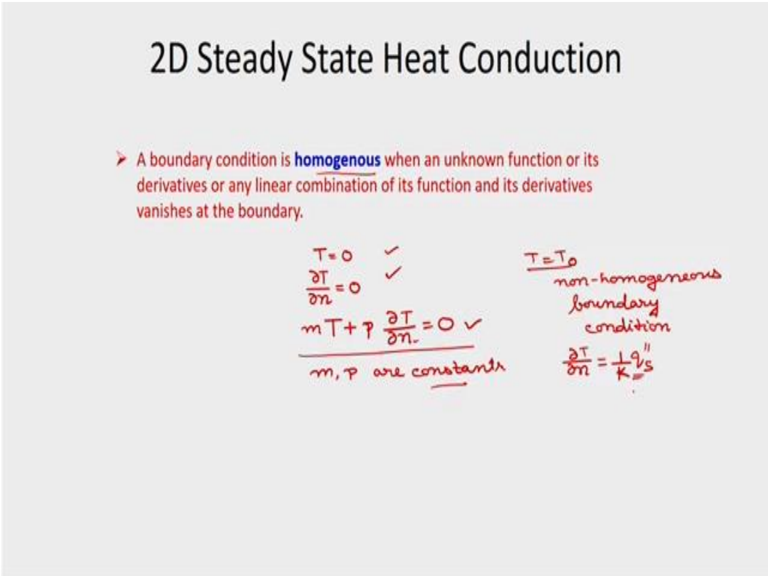

So what is homogeneous direction? So a boundary condition is homogeneous when an unknown function or its derivatives or any linear combination of its function and its derivatives vanishes at the boundary. So if you write say

$$T = 0$$

$$\frac{\partial T}{\partial n} = 0$$

Where, n is any direction. Or if we take combination of both

$$mT + p\frac{\partial T}{\partial n} = 0$$

Where, m and p are constants than these are considered to be homogeneous boundary conditions. So you can see here either the value of the temperature is 0. So that is Dirichlet type boundary condition or the gradient of temperature at the boundary is 0. That is known as homogeneous Neumann boundary condition. And the combination of these two is equal to 0 where m, P are constants and n is the coordinate x or y.

So in this form if you write the boundary condition then it is known as homogeneous boundary conditions. If say let us say T =T_0 , then obviously it is a non-homogeneous boundary condition. So even if you write the gradient say

$$\frac{\partial T}{\partial n} = \frac{1}{K} q_s''$$

Where, q_s is known. So in this way if you write the boundary condition then also it is a non-homogeneous boundary condition.

So now we understood what homogeneous boundary condition is. That is the value of the temperature or its gradients are 0 or any combination of this two with a multiplication of these constants is 0, then these are known as homogeneous boundary condition. In our case you can see the boundary conditions we have used. So these are non-homogeneous boundary condition. These four are non-homogeneous boundary conditions.

So we cannot directly use separation of variables method. So we have to use some transformation so that at least in one direction boundary condition becomes homogeneous, so that we can apply separation of variables methods. So you can see in this problem. So if we use some transformation θ

$$\theta(x, y) = T(x, y) - T_0$$

Then you can see that at the boundary this T_0 - T_0 will become 0 and your x direction will become homogeneous direction. But y direction one boundary condition will be homogeneous. And on the top boundary condition will be non-homogeneous. So it will be non-homogeneous direction and x will be homogeneous direction.

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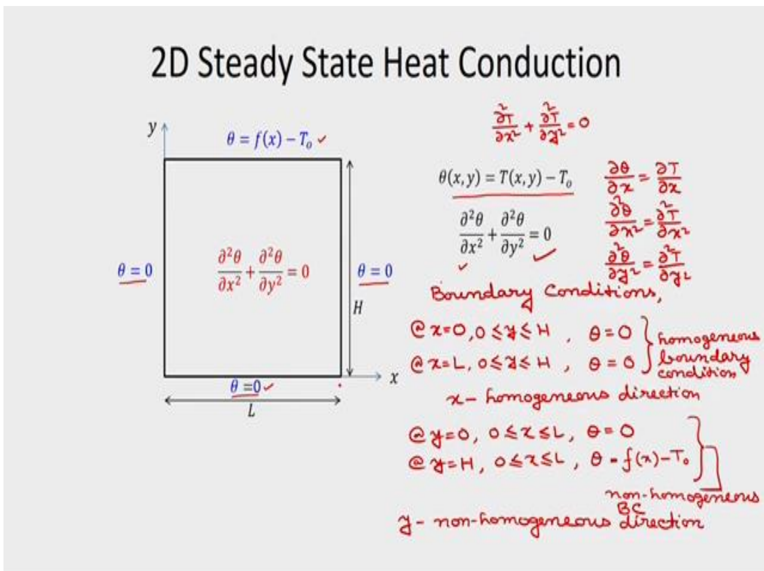

So you can see we have used this transformation here. So boundary conditions becomes

$$at\ left\ wall;\ x = 0, 0 \leq y \leq H, \theta = 0$$

$$at\ right\ wall, x = L, 0 \leq y \leq H, \theta = 0$$

$$at\ bottom\ wall;\ y = 0, 0 \leq x \leq L, \theta = 0$$

$$at\ top\ wall; y = H, 0 \leq x \leq L, \theta = f(x) - T_0$$

So you can see that x directions you have homogeneous direction as you have homogeneous boundary conditions. But here you can see in the bottom it is homogeneous but at top it is non-homogeneous. So it is non-homogeneous direction. Now converting T in terms θ in the governing equation

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

Because from the relation of T and θ we can write that

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 \theta}{\partial y^2}$$

Now this governing equation is also linear and homogeneous. So separation of variables method can be used. Your x direction is homogeneous direction and y direction is non-homogeneous direction, and also more importantly we have used a Cartesian system where x, y coordinates are orthogonal.

So x is becoming homogeneous direction and y is non-homogeneous direction and also we have orthogonal coordinate system. So we can use separation of variables method.

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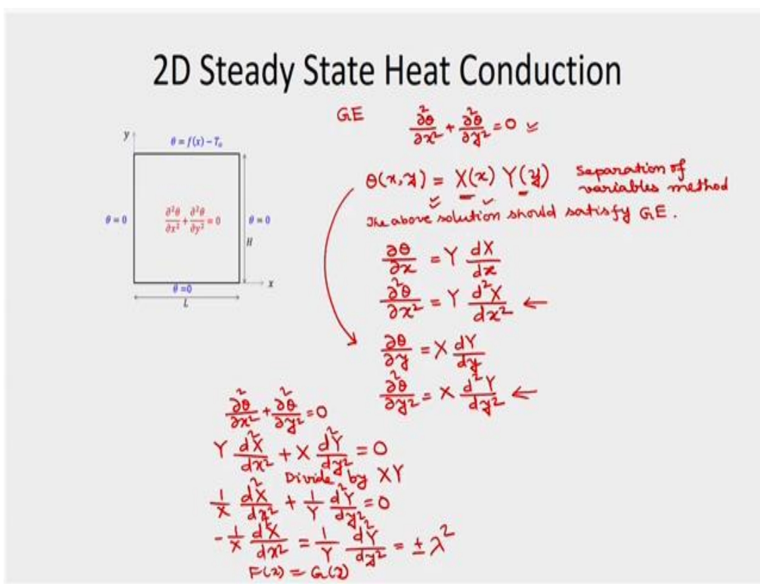

So now our governing equation is now

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

So what do you do in the separation of variables method? We seek a solution $\theta(x, y)$ such that it is product of two functions and each function is function of only one coordinate system. Let us see what I am telling.

$$\theta(x, y) = X(x)Y(y)$$

So if we are seeking the solution of $\theta(x, y)$ okay, its solution we can find as individual functions or solution which is function of one coordinate x and another solution which is function of y only. So you can see, these are the solutions independent direction x and y. So this is the separation of variables method.

Here I should tell that if you have more than two variables or independent direction, say if θ is function of x, y, z and t. Then you can write product of 4 functions X, Y, Z, and T. So that way you can use separation of variables method.

So now if this is the solution then it should satisfy this governing equation. So now let us put, so what we can write

$$\frac{\partial\theta}{\partial x} = Y\frac{dX}{dx}$$

$$\frac{\partial^2\theta}{\partial x^2} = Y\frac{d^2X}{dx^2}$$

Right side we are writing d instead of ∂ as Y is not a function of x and it comes out of the derivative. Similarly

$$\frac{\partial\theta}{\partial y} = X\frac{dY}{dy}$$

$$\frac{\partial^2\theta}{\partial y^2} = X\frac{d^2Y}{dy^2}$$

So now you put all these values in the governing equation because it should satisfy the governing equation. So if you write then will get

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

So now you can see that at least we could convert this partial differential equation to ordinary differential equation using separation of variables method. So now we will rearrange it and divide both side by XY

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2}$$

So you can see that left hand side is function of x only and right hand side it is functional of y only. So you can write that

$$F(x) = G(y)$$

So obviously there it should be equal to some constant. So

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda^2$$

So let's discuss this $\pm\lambda^2$ in the next slide.

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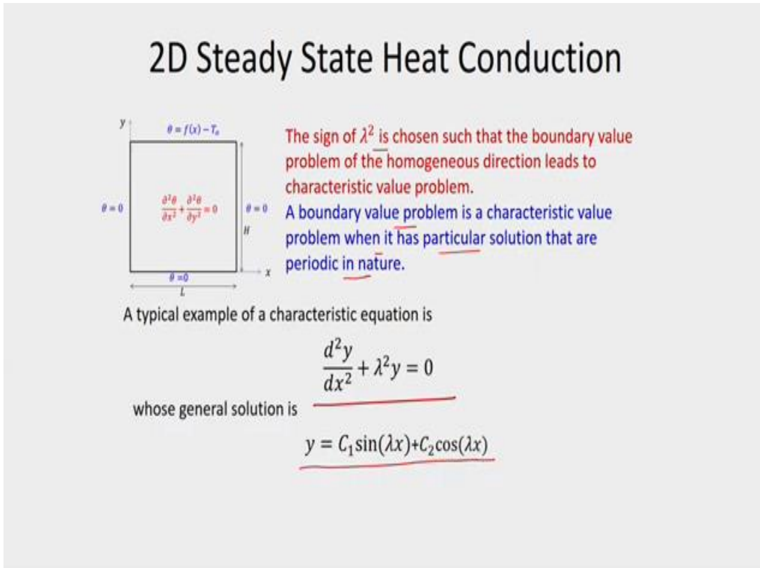

The sign of λ^2 is chosen such that the boundary value problem of the homogeneous direction leads to a characteristic value problem. So you have to choose the value of λ^2 such that in homogeneous direction you will get a characteristic value problem. What is characteristic value problem? A boundary value problem is a characteristic value problem when it has particular solutions that are periodic in nature.

What does it mean? That means when you will get the solution in homogeneous direction that solution itself will be periodic in nature. So let us take one equation. So it is a typical example of characteristic equation. So

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0$$

So if you write in this form what will be the solution of this? So you can see the solution, is

$$y = C_1 \sin(\lambda x) + C_2 \cos(\lambda x)$$

Where C_1 and C_2 are integration constants and you see that sin lambda x cos lambda x will give you this periodic solution y. So these type of solutions, this periodic solution you should get in homogenous direction. So what is homogeneous direction in our problem, x is the homogeneous direction. So in homogeneous direction you should get solution in this way so that you will get a periodic solution.

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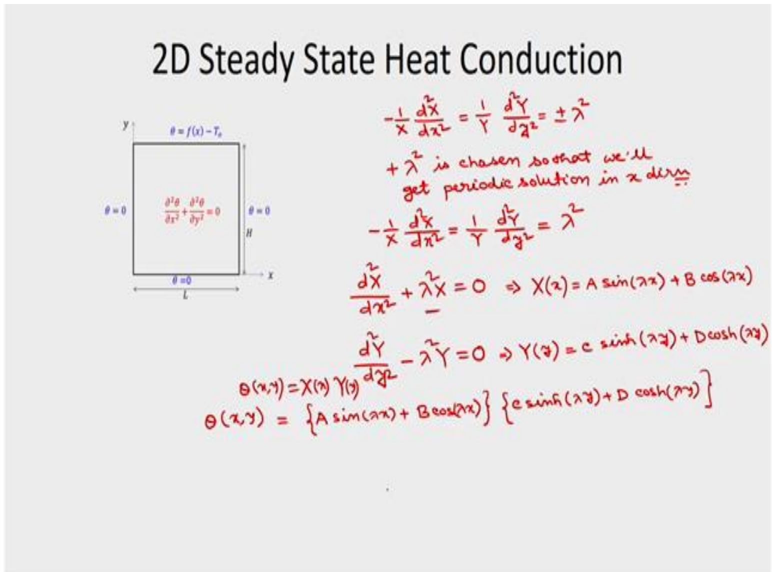

So now let us go the next slide. So we have written that

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda^2$$

So this as left hand side and right hand side are function of x and y respectively. λ^2 for convenience we have taken, but $\pm$ now we have to choose such way that in homogeneous direction we will get the periodic solution.

So you tell me what will be the sign of this λ^2 ? Let's choose $+\lambda^2$ so that we will get periodic solution in x direction. x direction is homogenous direction right. So now if you write

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2$$

So we can write

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

And

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0$$

So these are 2 homogeneous differential equations which you can actually integrate and you can apply the boundary conditions. So what will be the solution of this? What will be the solution you tell me? So this is your solution will be

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x)$$

And for the second equation, the solution you can have in two forms, in exponential form or hyperbolic form. So we have discussed when we were solving the problem of fins. So now we have finite directions. So which solution we should take? We will take the solution in hyperbolic function. So that it will be convenient for this approach. So we will be writing

$$Y(y) = C\sinh(\lambda y) + D\cosh(\lambda y)$$

So now we got the solutions okay. Now as you know the individual function X and Y, you can write what $\theta(x, y)$ as

$$\theta(x, y) = \{A\sin(\lambda x) + B\cos(\lambda x)\}\{C\sinh(\lambda y) + D\cosh(\lambda y)\}$$

So now we have boundary conditions. So we have to apply one by one and we have to find these constants. So now we got the product of two solutions right X and Y. And we got now we have four boundary conditions. Now one by one we will apply. So let us go in the next slide.

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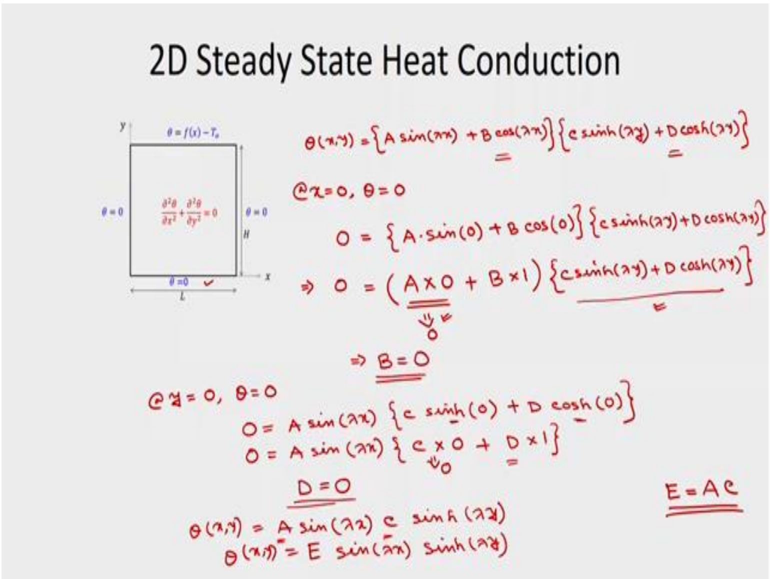

So what is the first boundary condition.

$$x = 0, \theta = 0$$

Putting this we will get

$$0 = \{A\sin(0) + B\cos(0)\}\{C\sinh(\lambda y) + D\cosh(\lambda y)\}$$

$$= \{A \times 0 + B \times 1\}\{C\sinh(\lambda y) + D\cosh(\lambda y)\}$$

The second term in the bracket cannot be zero, and to get zero in the left hand side B must be zero. So

$$B = 0$$

Another boundary condition let us apply on the bottom boundary here

$$y = 0;\ \theta = 0$$

So that means you will get

$$0 = \{A\sin(\lambda x)\}\{C\sinh(0) + D\cosh(0)\}$$

$$0 = \{A\sin(\lambda x)\}\{C \times 0 + D \times 1\}$$

So to have the solution you can see A cannot be 0. So in this case, only D must be 0. So you can write

$$D = 0$$

So now you can see, B =0 and D =0. So now we will write the solution

$$\theta(x, y) = A\sin(\lambda x)\, C\sinh(\lambda y)$$

So you can write E=AC, and write it

$$= E\sin(\lambda x)\sinh(\lambda y)$$

Another two boundary conditions we are left to apply. So at x = L and y = H. So those we will apply and we will find the other constants. So let us see.

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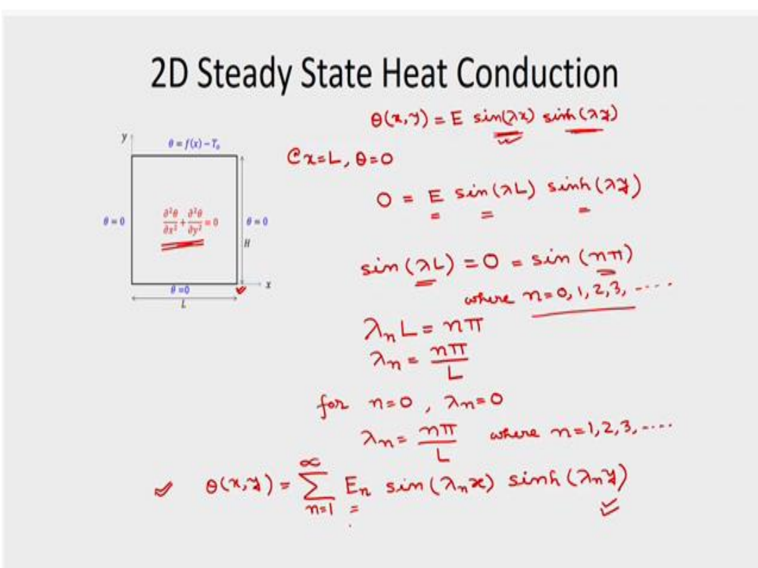

So another condition is

$$at\ x = L, \theta = 0$$

Now our temperature distribution is

$$\theta(x, y) = E \sin(\lambda x) \sinh(\lambda y)$$

Putting the values

$$0 = E \sin(\lambda L) \sinh(\lambda y)$$

So now you see L is constant right. So L is the length of the rectangular plate okay. So L is constant. Now, E cannot be 0 because if E is 0 then you will not get any solution. Because then your solution will be 0 only. And also $\sinh(\lambda y)$ cannot be 0 then also you will get the solution as 0. So only possible thing is that only $\sin(\lambda L)$ can be 0 okay.

$$\sin(\lambda L) = 0$$

Now for what values of sin you will get the 0. So it will be

$$\sin(\lambda L) = 0 = \sin(n\pi)$$

Where, n can be 0, 1, 2, 3, to ∞ right. So for different values of n, you will get the solution 0. So,

$$\lambda_n L = n\pi$$

We have replaced λ with suffix n, because we will get different values of λ for different values of n. if we put in the solution, you will get a different solution.

So

$$\lambda_n = \frac{n\pi}{L}$$

Now for n =0, $\sin(\lambda x) = 0$ and $\sinh(\lambda y) = 0$. So the hyperbolic function also will not contribute to the solution. So n cannot be zero. So n is equal to 1, 2, 3, to ∞ okay and it cannot be 0.

So now for different value of λ_n you will get a different solution. And as it is a linear governing equation we have considered which is the equation.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

It is a linear equation. Then the solution also will be summation of all the solutions right and it is applicable if you have a linear equation. So for different values of n, you will get different values of λ_n and for different values of λ_n you will get a different solution.

And as your governing equation is linear, you can sum up all the solutions. So in this case now we will write

$$\theta(x,y) = \sum_{n=1}^{\infty} E_n \sin(\lambda_n L) \sinh(\lambda_n y)$$

n will vary from 1 to ∞ as n=0 will not contribute to the solution. As you have a different values of λ_n you will get a different solution and as your governing equation is linear you can sum up all the solutions. You see, n = 1 you will get a solution plus n = 2, n = 3, so you just sum it up to infinity and all the summation of results you will get the solution of $\theta(x,y)$.

Now we have another boundary condition is left, right. Using which we have to find the constant E_n. Constant En is still unknown and for different values of n you will get different values of constant E_n okay. So now let us apply the top boundary condition which is non-homogeneous boundary condition. So let us go to the next slide.

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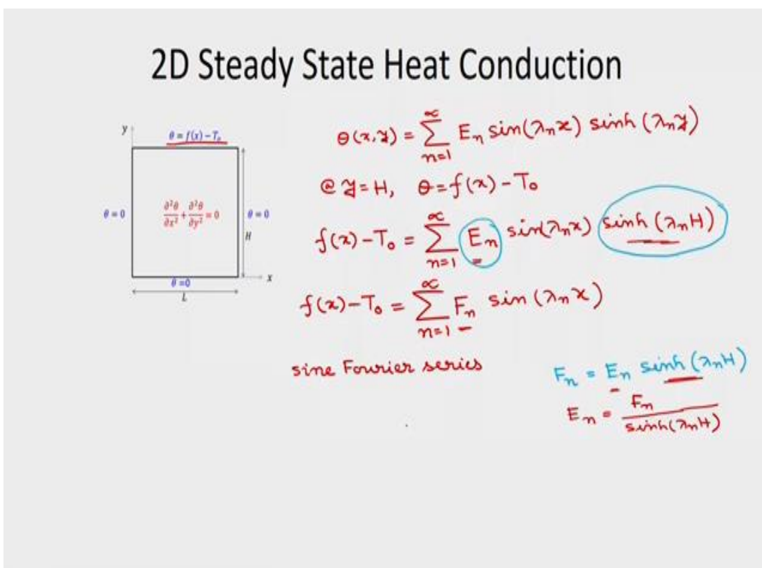

So the solution we got

$$\theta(x,y) = \sum_{n=1}^{\infty} E_n \sin(\lambda_n x) \sinh(\lambda_n y)$$

So for different value of λ_n you are summing up all the solutions. Now your top boundary condition is

$$y = H, \theta = f(x) - T_0$$

So this is your top boundary condition. So now you write it

$$f(x) - T_0 = \sum_{n=1}^{\infty} E_n \sin(\lambda_n x) \sinh(\lambda_n H)$$

So now you can see that E_n and $\sinh(\lambda_n H)$ is also constant. So taking them together if you write a new constant

$$F_n = E_n \sinh(\lambda_n H)$$

So if you write this then you will get

$$f(x) - T_0 = \sum_{n=1}^{\infty} F_n \sin(\lambda_n x)$$

So now can you tell me what this equation is? It is a sine Fourier series right. Now if you know the sine Fourier series, from here you can find the value F_n which is a constant for different value of n. Once you get F_n then you can find E_n.

$$E_n = \frac{F_n}{\sinh(\lambda_n H)}$$

Solution for F_n we will do in the next slide; how to find this constant from this sine Fourier series or in any general Fourier series how we will find the constants.

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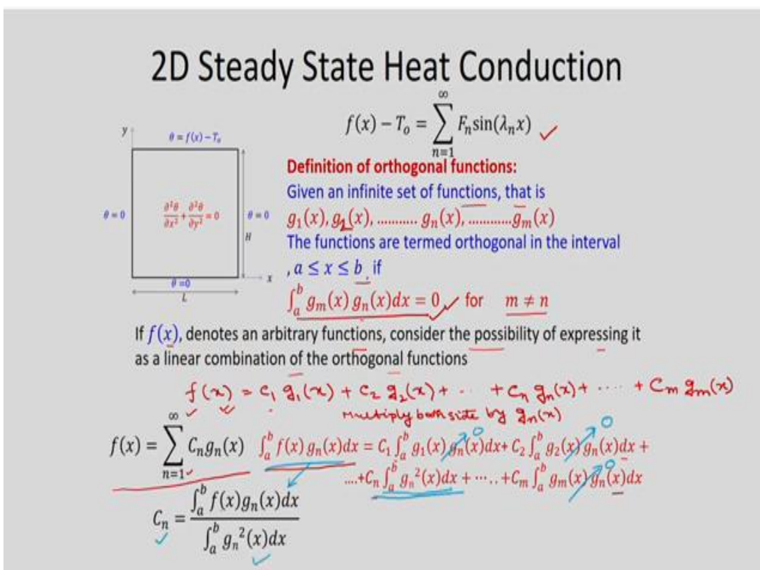

So before calculating this constant, let us see what is orthogonal function. So given a infinite set of functions, that is let us say we have $g_1(x), g_2(x), g_3(x), g_n(x)$ and $g_m(x)$. This function are termed orthogonal in the interval $a \le x \le b$, if

$$\int_a^b g_m(x) g_n(x) dx = 0 \qquad for\ m \neq n$$

So for m = n it is non-zero, but if m = n then this integral will be 0. Then you can say that $g_m(x)$ is orthogonal set. Now if we define an arbitrary function expressing it as a linear combination of the orthogonal function.

$$f(x) = C_1 g_1(x) + C_2 g_2(x) + C_3 g_3(x) + \cdots + C_n g_n(x) + \cdots + C_m g_m(x)$$

Where, C_1, C_2, C_3, C_n, C_m are constants. So you can express this $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} C_n g_n(x)$$

Now let's multiply both sides of the equation with $g_n(x)$ and take integral from a to b. Then we have

$$\int_a^b f(x) C_n g_n(x) dx$$
$$= C_1 \int_a^b g_1(x) g_n(x) dx + C_2 \int_a^b g_2(x) g_n(x) dx + \cdots + C_n \int_a^b g_n^2(x) dx + \cdots$$
$$+ C_m \int_a^b g_m(x) g_n(x) dx$$

C_1, C_2, C_3, C_n, C_m are constants, so you can take them out from the integral. So if it is a orthogonal function already we have discussed that

$$\int_a^b g_m(x) g_n(x) dx = 0$$

So if it is so then the all the other terms you can see will become 0 okay. So only one term left is

$$C_n \int_a^b g_n^2(x) dx$$

So now from here this constant C_n easily you can find out. So C_n you can write it like

$$C_n = \frac{\int_a^b f(x) C_n g_n(x) dx}{\int_a^b g_n^2(x) dx}$$

So you can find the constant C_n. So the same principle now we will use to find the constant F_n in this equation. Only we have to ensure that sin $(\lambda_n x)$ is an orthogonal set.

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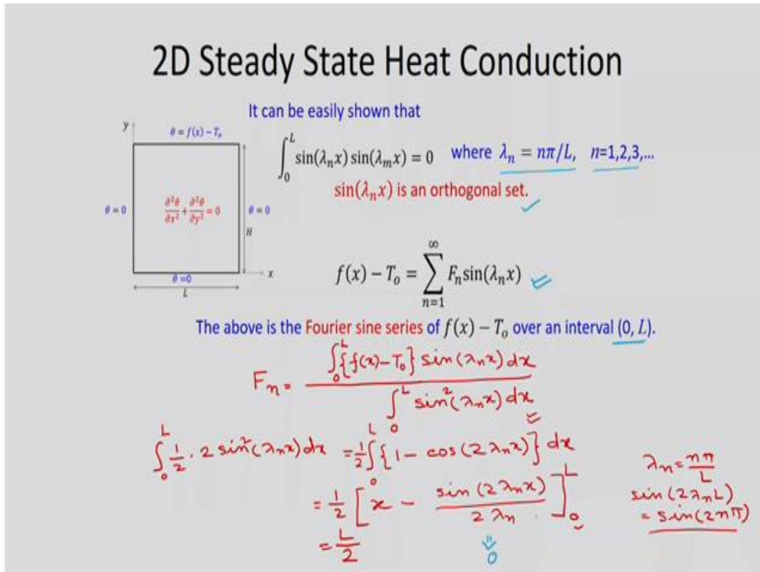

So here you can see that if we write here

$$\int_0^L \sin(\lambda_n x)\sin(\lambda_m x) = 0$$

For

$$\lambda_n = \frac{n\pi}{L} \; and \; n = 1,2,3, \ldots \ldots$$

So you just do it you will find that for $m \neq n$ all this sets will become 0. So you can prove that that your sin $\lambda_n x$ is an orthogonal set. So if it is orthogonal set, then this equation whatever we have derived in last slide that you can use and find the constant F_n. We know

$$f(x) - T_0 = \sum_{n=1}^{\infty} F_n \sin(\lambda_n x)$$

So F_n will be

$$F_n = \frac{\int_0^L \{f(x) - T_0\}\sin(\lambda_n x)\,dx}{\int_0^L \sin^2(\lambda_n x)\,dx}$$

So now applying the last boundary condition at y = H, We will find the constant F_n from the sine Fourier series. Now the denominator we can write

$$\int_0^L \sin^2(\lambda_n x)\,dx = \frac{1}{2}\int_0^L 2\sin^2(\lambda_n x)\,dx = \frac{1}{2}\int_0^L \{1 - \cos(2\lambda_n x)\}dx$$

Putting the limits

$$= \frac{1}{2}\left[x - \frac{\sin 2\lambda_n x}{2\lambda_n}\right]_0^L$$

$$= \frac{1}{2}\left[L - \frac{\sin 2\lambda_n L}{2\lambda_n}\right] - \frac{1}{2}\left[0 - \frac{\sin(2\lambda_n \times 0)}{2\lambda_n}\right]$$

Now

$$\lambda_n = \frac{n\pi}{L}$$

So

$$\lambda_n L = n\pi$$

Hence

$$\sin 2\lambda_n L = \sin 2n\pi = 0 \ \ for\ n = 1,2,3,4 \ldots ..$$

So the integral will be

$$\int_0^L \sin^2(\lambda_n x)\, dx = \frac{L}{2}$$

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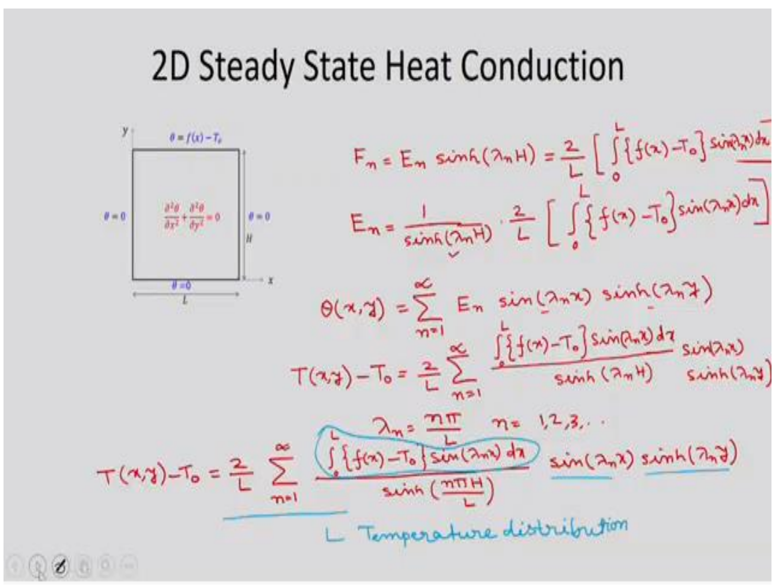

Now F_n becomes

$$F_n = E_n \sinh(\lambda_n H) = \frac{2}{L}\left[\int_0^L \{f(x) - T_0\} \sin(\lambda_n x)\, dx\right]$$

So E_n will be

$$E_n = \frac{1}{\sinh(\lambda_n H)} \frac{2}{L}\left[\int_0^L \{f(x) - T_0\} \sin(\lambda_n x)\, dx\right]$$

So now you put the value of E_n in the solution. So we get

$$\theta(x,y) = T(x,y) - T_0 = \frac{2}{L}\sum_{n=1}^{\infty}\frac{1}{\sinh(\lambda_n H)}\left[\int_0^L \{f(x) - T_0\}\sin(\lambda_n x)\,dx\right]\sin(\lambda_n x)\sinh(\lambda_n y)$$

Where,

$$\lambda_n = \frac{n\pi}{L}\ for\ n = 1,2,3, \dots \dots$$

So now you put the values of λ_n in the above equation and finally you will get the temperature distribution.

If you know the $f(x)$ then easily you can find the temperature distribution. So for this particular case whatever we have considered, now let us write that temperature distribution putting the value of λ_n

$$T(x,y) - T_0 = \frac{2}{L}\sum_{n=1}^{\infty}\frac{1}{\sinh\left(\frac{n\pi H}{L}\right)}\left[\int_0^L \{f(x) - T_0\}\sin(\lambda_n x)\,dx\right]\sin(\lambda_n x)\sinh(\lambda_n y)$$

So this is the temperature distribution. And the integral inside this you can find once the function $f(x)$ is known okay. So now let us consider that upper boundary is having a constant temperature T_c okay. Then you can evaluate this integral right. So in the next slide let us evaluate this integral.

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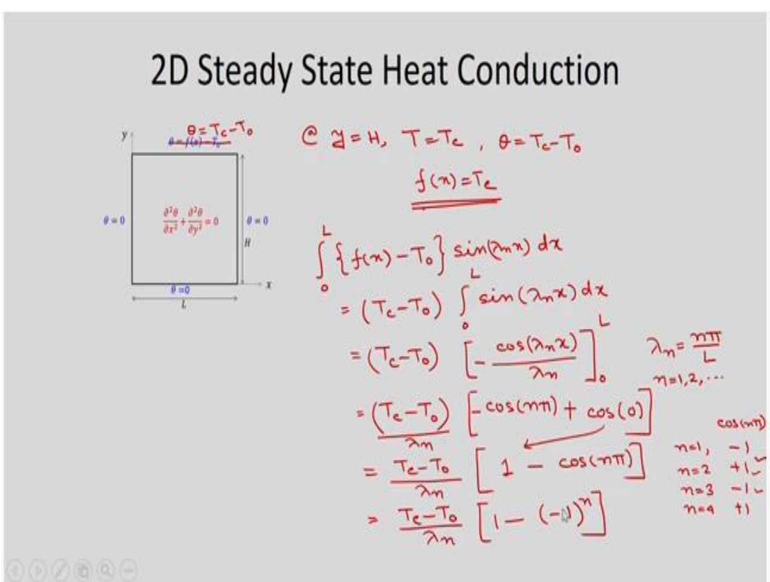

So

$$at\ y = H, T = T_c, \theta = T_c - T_0$$

So we can write

$$f(x) = T_c$$

So let's evaluate the integral first

$$\int_0^L \{f(x) - T_0\} \sin(\lambda_n x)\, dx$$

We can put T_c instead of $f(x)$ and take them outside the integral as they are constants. So it becomes

$$= (T_c - T_0) \int_0^L \sin(\lambda_n x)\, dx$$

Integrating

$$= -(T_c - T_0) \left[\frac{\cos \lambda_n x}{\lambda_n}\right]_0^L$$

So now you put the limits of this integral and find the value

$$= -\frac{(T_c - T_0)}{\lambda_n} (\cos n\pi - \cos 0)$$

As $\lambda_n = \frac{n\pi}{L}$

Here $\cos 0$ is 1. Let's check the value of $\cos n\pi$. Now for n=1 it is -1; for n=2 it is 1; for n=3 it is again -1; for n=4 it becomes 1. So we can write it as $(-1)^n$. So the equation becomes

$$= \frac{(T_c - T_0)}{\lambda_n} [1 - (-1)^n]$$

So now we found this integral. Now you can write for this case the temperature distribution.

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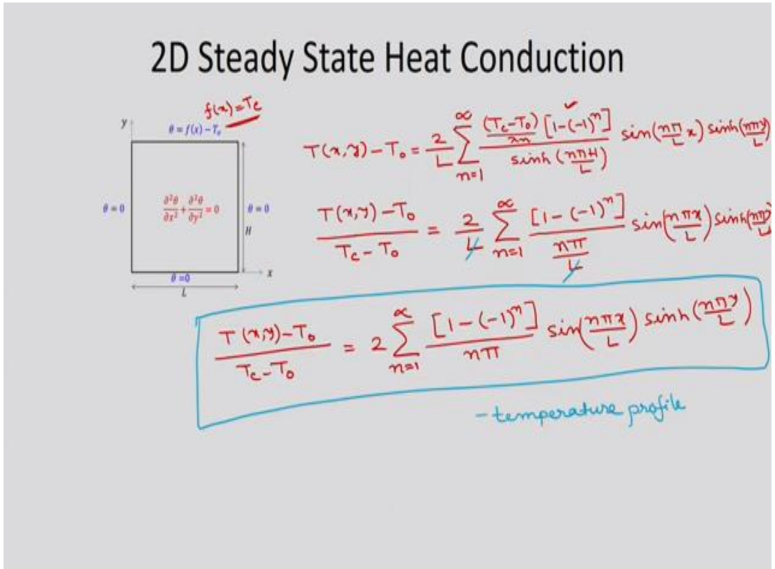

So we can write temperature distribution is like this.

$$T(x,y) - T_0 = \frac{2}{L}\sum_{n=1}^{\infty} \frac{1}{\sinh\left(\frac{n\pi H}{L}\right)} \left[\frac{(T_c - T_0)}{\lambda_n}\ [1-(-1)^n]\right] \sin(\lambda_n x)\sinh(\lambda_n y)$$

Let us rearrange it. So T_c - T_0 is constant. So you can take it and divide in the left hand side. We can also put the value of $\lambda_n = \frac{n\pi}{L}$. So it will come as

$$\frac{T(x,y) - T_0}{T_c - T_0} = \frac{2}{L}\sum_{n=1}^{\infty} \frac{1}{\sinh\left(\frac{n\pi H}{L}\right)} \left[\frac{L}{n\pi}\ [1-(-1)^n]\right] \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

$$= 2\sum_{n=1}^{\infty} \left[\frac{[1-(-1)^n]}{n\pi}\right] \frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)} \sin\left(\frac{n\pi x}{L}\right)$$

Here, T_c - T_0 will become some kind of non-dimensional temperature. So this is the final temperature distribution for the case where we have considered the upper boundary temperature is constant. So now you can see that we have found the temperature distribution in a two dimensional steady state condition using separation of variables method where θ is function of x and y. And we found the temperature distribution in terms of summation because for different value of n you got the different value of λ and as it is a linear partial differential equation so obviously you can write the solution as a summation of all solutions.

So using that method now we found the temperature profile. So here you have to know how many terms you will consider or how many n you will consider. So generally it is seen that n within 100 it will converge fast. So you need not to take up to infinity even in first four or five values it will converge. So let us take one example

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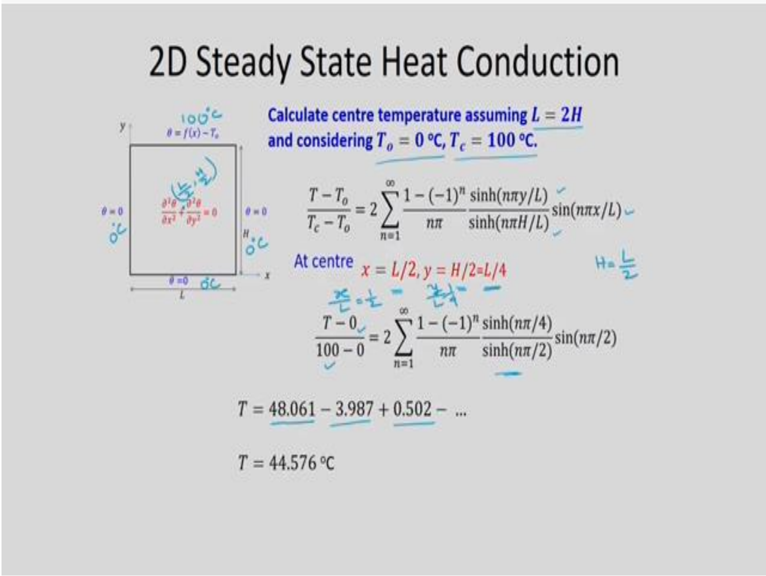

Then we will find that within three values it will converge. So if you see this is the problem where calculate centre temperature assuming L = 2H okay and considering T_0 as 0 ºC and T_c is 100 ºC okay. So here it is 100 ºC okay on the top. And this is 0 ºC. This is also 0 ºC and this is also 0 ºC okay. So with this condition and L = 2H.

So your length is twice of the height. Now you have to calculate the temperature at the centre. So at the centre means somewhere here where it is L/2 and H/2 okay. At this location you have to find what will be the temperature. So you know the temperature distribution. You see, it is

$$\frac{T-T_0}{T_c-T_0}=2\sum_{n=1}^{\infty}\left[\frac{[1-(-1)^n]}{n\pi}\right]\frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}\sin\left(\frac{n\pi x}{L}\right)$$

So now you can see that at centre what will be your x,

$$x=\frac{L}{2}$$

$$y=\frac{H}{2}=\frac{L}{4}\quad as\ L=2H$$

You have to find at centre. So putting the values of x and y and the temperatures it will be

$$\frac{T-0}{100-0} = 2\sum_{n=1}^{\infty}\left[\frac{[1-(-1)^n]}{n\pi}\right]\frac{\sinh\left(\frac{n\pi}{4}\right)}{\sinh\left(\frac{n\pi}{2}\right)}\sin\left(\frac{n\pi}{2}\right)$$

With this now use your calculator and find the values for n = 1 first, then you will get 48.061. Then now you put n = 2, if you put n = 2 and find this value you will get -3.987. You see the second term has become so much less than the first term. Now you put n = 3, then if you calculate this you will get 0.502. You see this value is much smaller than the second value and way much smaller than the first value. So that means it is converging faster. If you consider only first three terms, you are going to get the temperature value at least up to one decimal place correct.

So other terms I am neglecting because I do not need to consider because it will be very small value okay. So T will be finally

$$T = 48.061 - 3.987 + 0.502 - \cdots$$

$$T = 44.576\ ℃$$

So you can see that from this example, that if you consider only first few terms in the series you will see that this series will converge and you will get at least few decimal places correct.

So I will stop here today. Thank you.