Fundamentals of Conduction and Radiation
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Lecture - 14
Fins with Non-Uniform Cross-Section Area

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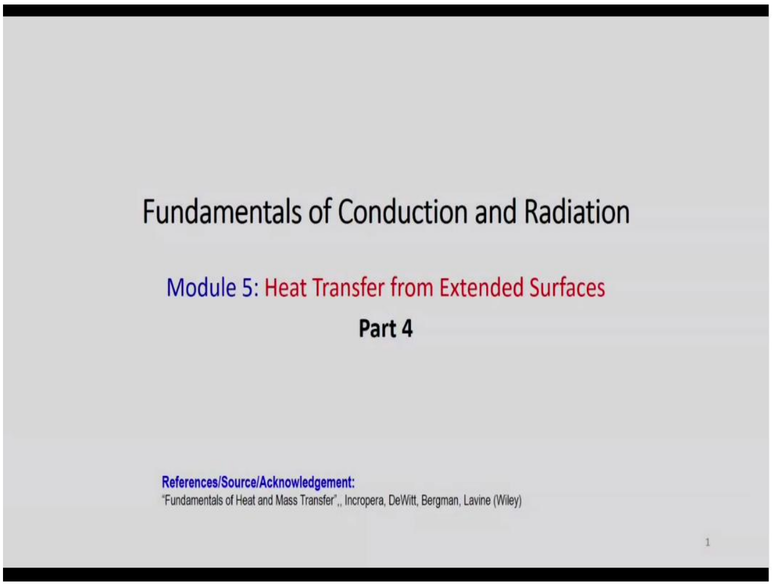

Hello everyone. So, in last few classes, we have done the analysis of a single fin with uniform cross-section. So, we have found the temperature distribution as well as the heat transfer rate. Today, first will see the analysis of an array of fins and then will see the analysis of fin with non-uniform cross-section and at last will solve some problems on fin with uniform cross-section. So, today the same module you are continuing heat transfer from extended surfaces.

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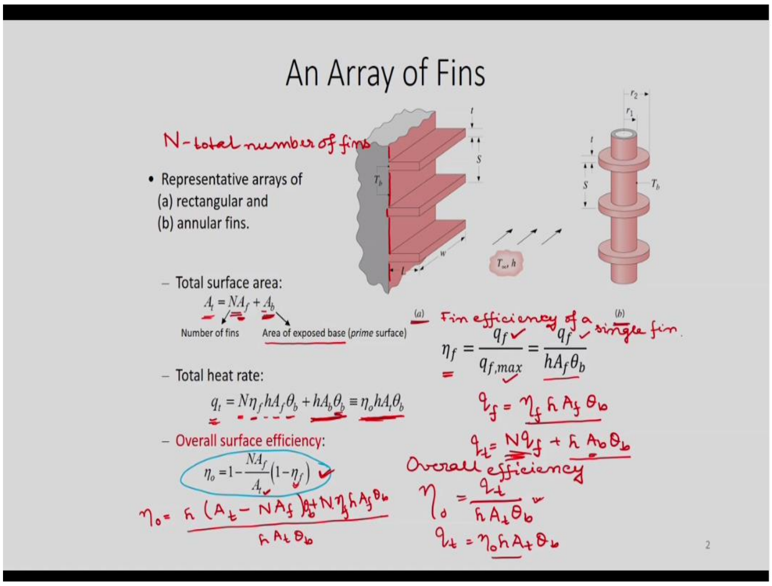

And you can see this is array of fins, so this is your array of rectangular fins and this figure is your array of annular fins. So, there are many fins are there. So, these fins are connected with the base surface and here let us say that if you have N number of fins, then what will be their total surface area?

So, if we say that for a single fin, your fin area is A_f, then if number of fins is N, then you have total $N \times A_f$. But at the same time, you see here, here you have some base surface also. So, this is known as exposed base okay which is your prime surface on which your fins are attached.

So, from this surface also there will be heat loss due to convection. So, total surface area will be your fin surface area of total N fins and what will be the base area. So, in this case, we are telling the total base area A_b or the exposed base okay, not considering this area, only this and this, so total whatever you have this we consider as total A_b which is area of exposed base.

So, total heat transfer area will be

$$A_t = N \times A_f + A_b$$

Now, η_f the efficiency already we have defined as heat transfer from a fin and maximum possible heat transfer. So, maximum possible heat transfer means if that entire fin is maintained at the base temperature T_b , then what will be your heat transfer? It will be

$$= hA_f\theta_b$$

So,

$$\eta_f = \frac{q_f}{hA_f\theta_b}$$

This is the fin efficiency of a single fin. So, from here for a particular fin what will be your q_f.

$$q_f = \eta_f hA_f\theta_b$$

So, now if you have a N number of fins then total heat transfer from the fin will be

$$q = N\eta_f hA_f\theta_b$$

And what will be the heat transfer from the prime or exposed base,

$$q_b = hA_b\theta_b$$

Because A_b is the total exposed area, h is the heat transfer coefficient and temperature difference is θ_b. So, total heat transfer from the array of fins will be

$$q_t = N\eta_f hA_f\theta_b + hA_b\theta_b$$

Now, if we define the overall efficiency as η_o then we can write it

$$\eta_o = \frac{q_t}{hA_t\theta_b}$$

Where, total surface area is A_t. So, we are defining the overall efficiency of this array of fins as the ratio of total heat transfer from the array of fins plus heat transfer from the exposed base to the maximum possible heat transfer from the arrays of fins. Here A_t is the total area right including the fin area as well as the exposed base area. So,

$$q_t = \eta_o hA_t\theta_b$$

So, that we have written here okay. So, now if you do the simplification, then you will get η_o as

$$\eta_o = 1 - \frac{NA_f}{A_t}\left(1-\eta_f\right)$$

So, you remember in this case here N is the total number of fins okay, A_f is the fin area of a single fin, A_t is the total heat transfer area including the fin area as well as the exposed base area and η_f is the efficiency of a single fin. So, this is the overall efficiency of an array of fins okay. So, you will be able to calculate that overall efficiency of an array of fins.

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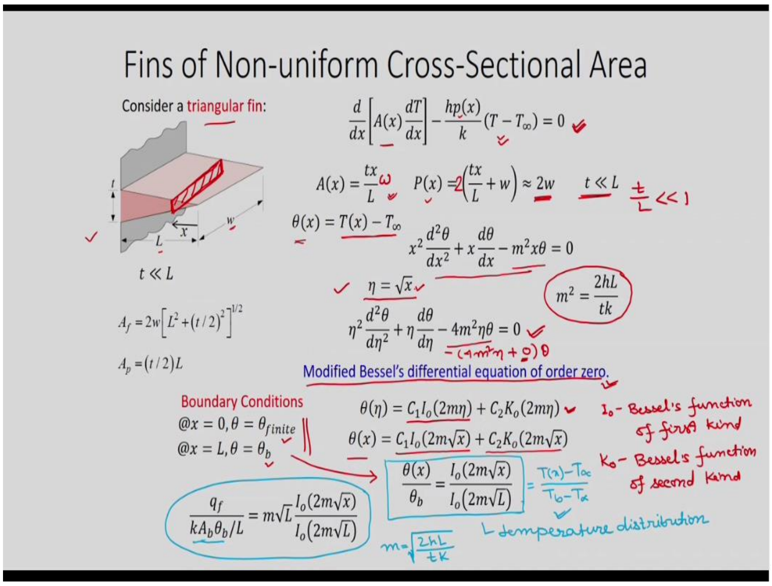

So, next now we will consider one fin with non-uniform cross-section okay. So, as these analysis is very cumbersome so will not go into details, just I will tell you the steps and how to find it and what is the final expression of the temperature distribution as well as the heat transfer rate. So, let us consider first one fin with non-uniform cross section and will consider a simplified fin that is our triangular fin.

So, you consider this triangular fin okay, with base length is t and length of the fin is L and width of the fin is w and x is measured from from the fin tip. So at x =0 some finite temperature will be there. So,

$$at\ x = 0;\ \theta = \theta_{finite}$$

And x = L you have base, so base temperature T_b and theta is equal to θ_b.

$$at\ x = L;\ \theta = \theta_b$$

So, here you can see that your cross-sectional area is varying along x. Now this was our governing equation of a non-uniform fin, what we derived in the first class.

$$\Rightarrow \frac{d}{dx}\left(A(x)\frac{dT}{dx}\right) - \frac{hp(x)}{K}(T - T_\infty) = 0$$

So, you can see but later we assume that A(x) is constant, so we take it outside but now in this case your cross-sectional area is varying with x, so you have to keep it inside the derivative.

Where, p is the perimeter. Now what will be the cross-sectional area at a distance x from the tip? It will be

$$A(x) = \frac{txw}{L}$$

Now, what is the perimeter? Perimeter will be

$$P(x) = 2\left(\frac{tx}{L} + w\right)$$

But lets assume that the fin base length t is very much smaller than the length of the fin or t<<L. So we can write

$$P(x) = 2w; for\ t \ll L$$

So, with this assumption and if we define the excess temperature

$$\theta(x) = T(x) - T_\infty$$

Then this governing equation you can convert it into

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - m^2 x\theta = 0$$

Where,

$$m^2 = \frac{2hL}{tk}$$

So, now we have to solve it. So, first we will transform it with a transformation $\eta = \sqrt{x}$ okay, to bring it down to a differential equation such that we have a solution for that differential equation. For that reason, we are taking this transformation. So, if you put it the equation becomes

$$\eta^2 \frac{d^2\theta}{d\eta^2} + \eta \frac{d\theta}{d\eta} - 4m^2\eta\theta = 0$$

Can you remember what this equation is? So, this equation resembles with the modified Bessel's differential equation of order zero. Because the last we can write as

$$-(4m^2\eta + 0)\theta$$

Where, the 0 defines the order. So, now you know the solution of this modified Bessel's differential equation. So, the solution is

$$\theta(\eta) = C_1 I_0(2m\eta) + C_2 K_0(2m\eta)$$

Or,

$$\theta(x) = C_1 I_0\left(2m\sqrt{x}\right) + C_2 K_0\left(2m\sqrt{x}\right)$$

I_0 is known as the Bessel function of first kind, and K_0 is known as Bessel's function of second kind. Now, we have two boundary conditions

$$at\ x = 0;\ \theta = \theta_{finite}$$

$$at\ x = L;\ \theta = \theta_b$$

So, you put these boundary conditions and find the C_1 and C_2. The final form for temperature distribution will be

$$\frac{\theta(x)}{\theta_b} = \frac{I_0\left(2m\sqrt{x}\right)}{K_0\left(2m\sqrt{L}\right)} = \frac{T(x) - T_\infty}{T_b - T_\infty}$$

So, this is the temperature distribution for this triangular fin. Now, if you want to calculate what is the heat transfer rate? From Fourier's law we can calculate the heat transfer rate taking

$$q_f = KA \frac{d\theta}{dx}\bigg|_{x=L}$$

As x=L signifies the base. The final form will be

$$\frac{q_f}{KA_b\theta_b/L} = m\sqrt{L} \frac{I_0\left(2m\sqrt{x}\right)}{K_0\left(2m\sqrt{L}\right)}$$

Where

$$m^2 = \frac{2hL}{tk}$$

So, this is the heat transfer rate for the non-uniform fin, for the triangular fin with non-uniform cross-section okay. So, you can actually do this analysis whatever way the steps I have discussed and you can refer some books to find what is the modified Bessel equation and also what is the Bessel function of first kind and second kind.

So, that you can find and what is its derivative because you need to find the derivative of this functions okay. So, those you can find from some text books. So, now we have finished this analysis of fins okay and now we will solve 3 problems and you will understand how to solve a problem for a fin with uniform cross-section.

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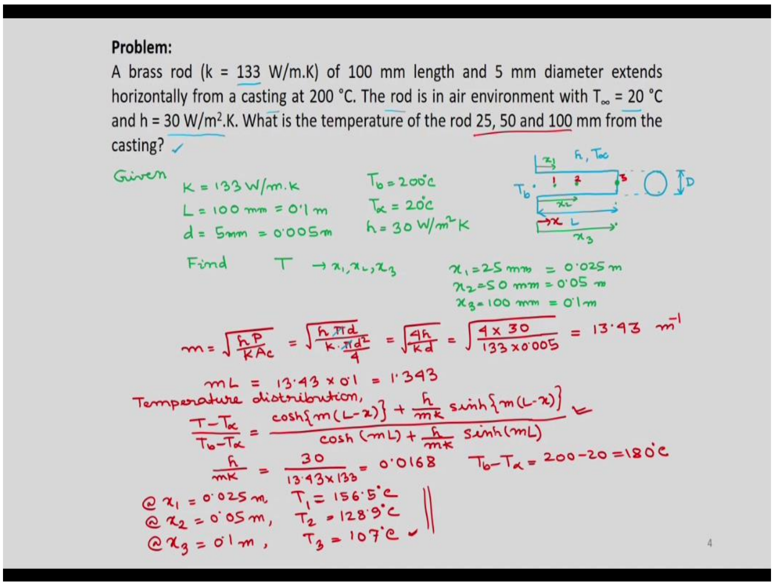

So, first problem is just first read. A brass rod with thermal conductivity 133 W/mK of 100 mm length and 5 mm diameter extends horizontally from a casting at 200 ºC and the rod is in an air environment with T_∞ is 20 ºC and heat transfer coefficient 30 W/m^{2}K. What is the temperature of the rod 25, 50 and 100 mm from the casting okay?

So, what is the problem, problem is that you have given a brass rod. Its thermal conductivity is given, its length is given and its diameter is given. So, it is a pin fin kind of thing where you have a diameter of 5 mm. So, now here it is written that it extends horizontally from a casting at 200 ºC. So, that means it is actually attached to a casting surface where base temperature is 200 ºC.

And ambient temperature it is 20 ºC and heat transfer coefficient of the ambient air is 30 W/m^{2}K. So, you have to find the temperature at 3 different locations, one is 25 mm from the casting that means from the base, 50 mm and 100 mm, 100 mm means at the tip. So this is a uniform cross-sectional fin with see its cross-section circular because it is a pin fin kind of thing. So, this is your diameter and this is your length and this is your base. So you have a T_b here and ambient is h, T_∞ and you have to find the temperature at 3 different locations.

So what are given?

$$K = 133 \frac{W}{mK}$$

$$L = 100mm = 0.1\, m$$

$$d = 5\, mm = 0.005\, m$$

$$T_b = 200℃$$

$$T_\infty = 20℃$$

$$h = 30 \frac{W}{m^2 K}$$

Locations to find temperatures are

$$x_1 = 25\, mm = 0.025\, m$$

$$x_2 = 50\, mm = 0.05\, m$$

$$x_3 = 100\, mm = 0.1\, m$$

So, we have understood the problem right. So, you have a pin fin whose base temperature is given and ambient heat transfer coefficient and the temperature ambient temperature is given. So, as nothing is specified so we will assume that from the tip of the fin also you have a convection right.

So, it will be a convective tip boundary condition, so you have to use the temperature distribution for a case where fin with convective heat transfer condition okay. So, that analysis whatever we have done earlier so that temperature distribution we will consider. So, let us solve it. So, first you solve the value of m because m is required right. What is m? So, m is

$$= \sqrt{\frac{hP}{KA_c}} = \sqrt{\frac{h\pi d}{\frac{K\pi d^2}{4}}} = \sqrt{\frac{4h}{Kd}} = \sqrt{\frac{4 \times 30}{133 \times 0.0005}} = 13.43\, m^{-1}$$

As it is circular fin perimeter and cross-section is taken for the same.

Now mL will be

$$mL = 13.43 \times 0.1 = 1.343$$

So, now what is the temperature distribution? Temperature distribution in this case, as I told before you will take for the case with convective heat transfer tip okay. So, you will get

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh\{m(L-x)\} + \frac{h}{Km}\sinh\{m(L-x)\}}{\cosh(mL) + \frac{h}{Km}\sinh(mL)}$$

So, I think already you have derived this in last few classes. So, you remember this expression. So, now

$$\frac{h}{Km} = \frac{30}{13.43 \times 133} = 0.0168$$

$$T_b - T_\infty = 200 - 20 = 180\ ℃$$

So all terms are known, T_∞, T_b, m is known at location x is known, so you will be able to find the temperature at 3 different locations at x1, x2 and x3. Then, you will get

$$T_1 = 156.5℃$$

$$T_2 = 128.9℃$$

$$T_3 = 107℃$$

So, now we could find the temperature at 3 different locations. What is the temperature at the base? Base temperature is 200 ºC and what is the temperature at the tip, it is 107 ºC. So, you can see that your temperature gradually decreases along the fin length okay and x obviously is measured from the fin base okay. So, you can see from here. So, these are the required answers. So, you will be able to solve this kind of problem okay. So, you solve some problems from the text book and also we will give some assignments. So, those problems also you solve. Now, let us take the second problem.

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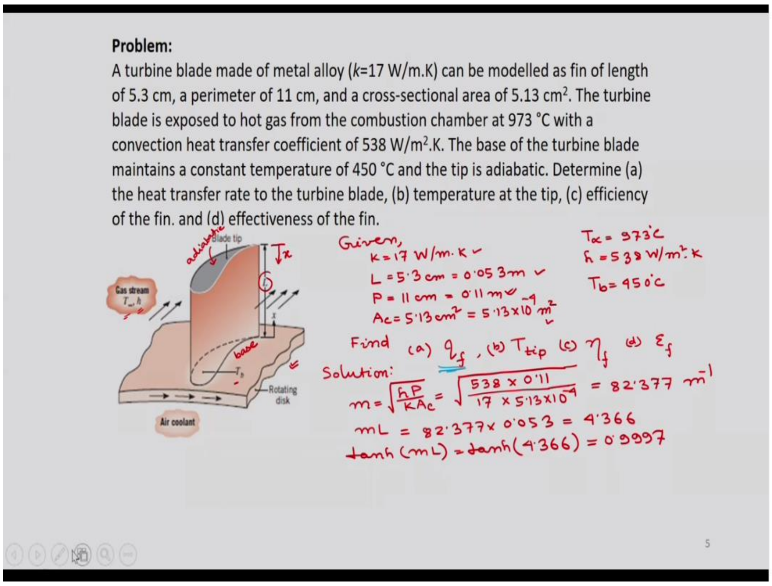

So, first I will read the question. A turbine blade made of metal alloy where k is 17 W/mK can be modeled as fin of length of 5.3 cm, a perimeter of 11 cm and a cross-sectional area of 5.13 cm^2. The turbine blade is exposed to hot gas from the combustion chamber at 973 ºC with a convection heat transfer coefficient of 538 W/m^2K.

The base of the turbine blade maintains a constant temperature of 450 ºC and the tip is adiabatic. Determine the heat transfer rate to the turbine blade, temperature at the tip, efficiency of the fin and effectiveness of the fin. So, I think first try to understand the problem. So, this is your fin which is your turbine blade okay.

So, here you can see that it is your blade tip and this is your fin base and obviously temperature will be T_b and whatever steam is coming, whatever temperature that is your T_∞ and here your perimeter and cross-sectional area given okay. So, already given so you first the given data you write.

$$K = 17 \frac{W}{mK}$$

$$L - 5.3\ cm = 0.053\ m$$

$$P = 11\ cm = 0.11\ m$$

$$A_C = 5.13\ cm^2 = 5.13\ \times 10^{-4} m^2$$

$$T_\infty = 973\ ℃$$

$$h = 538 \frac{W}{m^2 K}$$

$$T_b = 450℃$$

Here it is given that the tip is adiabatic tip. So, there is no heat loss from the fin tip. So, you have to consider when you are doing the analysis. So, temperature distribution or heat transfer rate you have to consider for the uniform cross-sectional fin with adiabatic tip condition.

Because ambient temperature is higher than your fin base, so obviously heat transfer will takes place from the steam to the turbine blade. Here we can take x=0 signifies the fin tip. So, you know given data, now let us do the solution. So, obviously we have to find m, so what is m?

$$= \sqrt{\frac{hP}{KA_c}} = \sqrt{\frac{538 \times 0.11}{17 \times 5.13 \times 10^{-4}}} = 82.377\ m^{-1}$$

$$mL = 82.377 \times 0.053 = 4.366$$

Now

$$\tanh mL = \tanh 4.366 = 0.9997$$

So, now first we have to calculate this q_f right. So, let us do in the next slide.

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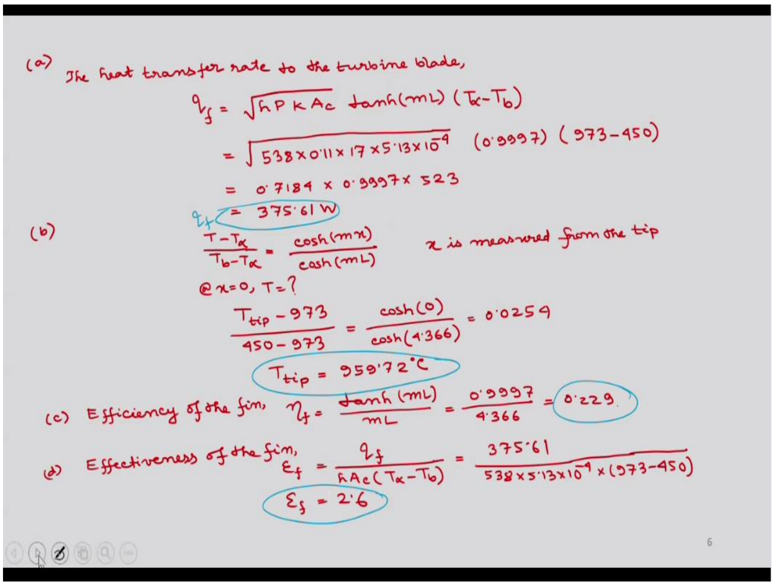

So, the heat transfer rate to the turbine blades because your ambient temperature is higher than the base temperature will be

$$q_f = \sqrt{hPKA_c}\theta_b \tanh mL$$

Now here θ_b will be $T_\infty - T_b$ as the heat transfer is taking place from the ambient to the fin. So, you know all the values, you put here.

$$= \sqrt{538 \times 0.11 \times 17 \times 5.13 \times 10^{-4}}(973 - 450)(0.9997)$$

$$= 0.7184 \times 0.9997 \times 523$$

$$= 375.61\ W$$

So, we found the heat transfer rate to the turbine blade. So, this is your part a okay. Now, what is part b? Part b is you have to calculate the temperature at the tip.

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh(mx)}{\sinh(mL)}$$

Here we are taking x from the tip, so at x equal to 0 what is the temperature we have to calculate. So the temperature distribution when x is measured from the tip for fin with adiabatic tip is

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh(mx)}{\sinh(mL)}$$

So, now you put the values,

$$\frac{T(tip) - 973}{450 - 973} = \frac{\cosh(0)}{\sinh(4.366)} = 0.0254$$

$$T(tip) = 959.72℃$$

Obviously, you can see, it is lower than the ambient temperature okay that means steam temperature. So, steam temperature is 973 but at the tip you have 959 almost, so it is lower than that.

Now, next you have to calculate the efficiency okay, so what is efficiency formula for this,

$$\eta_f = \frac{\tanh mL}{mL} = \frac{0.9997}{4.366} = 0.229$$

So, next you have to calculate what is the fin effectiveness? So, fin effectiveness is

$$\varepsilon_f = \frac{q_f}{hA_c(T_b - T_\infty)} = \frac{375.61}{538 \times 5.13 \times 10^{-4} \times (973 - 450)} = 2.6$$

So, what is the effectiveness we got, 2.6 and when we discuss about the fin performance factor, we have already discussed that if effectiveness is greater than 2 then it is convenient to use the fin okay. So, for this problem all the answers we have found. Now let us consider the last problem okay.

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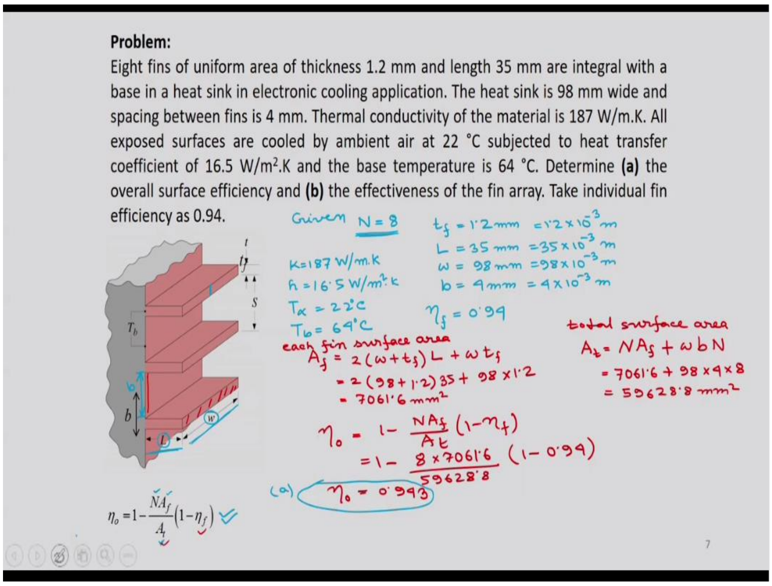

So, in the last problem let me first read out. Eight fins of uniform area of thickness 1.2 mm and length 35 mm are integral with a base in a heat sink in electronic cooling applications. So, you can see that it is the case of an array of fins because here you have total eight fins okay. The heat sink is 98 mm wide and spacing between fins is 4 mm. Thermal conductivity of the material is 187 W/mK.

All exposed surfaces are cooled by ambient air at 22 ºC subjective to heat transfer coefficient of 16.5 W/m^{2}K and the base temperature is 64 ºC. Determine the overall surface efficiency and the effectiveness of fin array. Take individual fin efficiency as 0.94. So, it is a problem of an array of fins, so total number of fins in this case, what are the given things.

$$number\ of\ fins, N = 8$$

$$fin\ thickness\ of\ each\ fin, t_f = 1.2\ mm = 1.2 \times 10^{-3}\ m$$

$$length\ of\ fin, L = 35\ mm = 0.035\ m$$

$$width\ of\ fin, w = 0.098\ m$$

$$fin\ spacing\ (distance\ between\ fins), b = .004\ m$$

$$K = 87 \frac{W}{mK}$$

$$h = 16.5 \frac{W}{m^2 K}$$

$$T_\infty = 2.2°\text{C}$$

$$T_b = 64°\text{C}$$

$$efficiency\ of\ single\ fin, \eta_f = 0.94$$

Now what will be the fin area of each fin? It will be for a rectangular fin

$$A_f = 2(w + t_f)L + wt_f$$

Where, $2(w + t_f)$ is the perimeter and L is length and wt_f corresponds to the fin tip area. Putting all the values it will be

$$= 7061.6\ mm^2$$

Now total surface area will be summation of total fin surface area plus the exposed surface are except the fins

$$A_t = A_f N + wbN$$

$$= (7061.6 + 98 \times 4) \times 8$$

$$= 59628.8\ mm^2$$

Now

$$\eta_o = 1 - \frac{NA_f}{A_t}(1 - \eta_f)$$

Here A_f is the fin area of a single fin and A_t is the total heat transfer area includes the fin area as well as the exposed base area and η_f is the efficiency of a single fin. We have calculated A_f and A_t and we know η_f. So,

$$\eta_o = 1 - \frac{8 \times 7061.6}{59628.8}(1 - 0.94) = 0.943$$

So this is your overall efficiency. Now what will be the fin effectiveness? So, fin effectiveness is your total heat transfer from the fin divided by the heat transfer from the base if there is no fin okay. So in the next slide let us calculate these.

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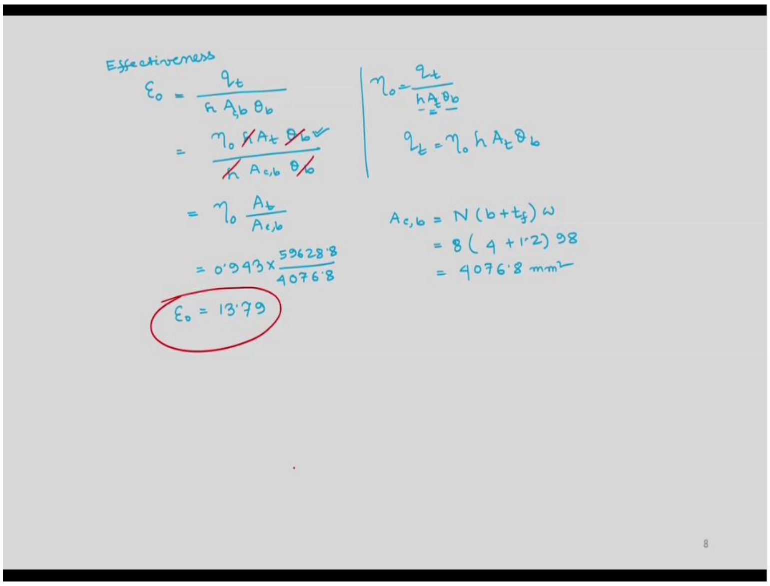

So effectiveness as we know will be

$$\varepsilon_f = \frac{q_t}{hA_{c.b}\theta_b}$$

Where, $A_{c.b}$ is the area of the surface without the fins. That is fin base area plus the exposed surface are. Now from the efficiency relation we know

$$q_t = \eta_o h A_t \theta_b$$

Putting it above and doing simplification

$$\varepsilon_f = \eta_o \frac{A_t}{A_{c.b}}$$

Now what will be $A_{c.b}$? It will be fin base area plus the exposed area for all the fins.

$$A_{c.b} = (t_f w + bw)N$$

$$= Nw(t_f + b)$$

$$= 8 \times 98(1.2 + 4)$$

$$= 4076.8\ mm^2$$

Hence effectiveness becomes

$$\varepsilon_f = 0.943 \times \frac{59628.8}{4086.8} = 13.79$$

So, this is the effectiveness of the array of fins. So we have calculated the overall efficiency of the array of fins as well as the effectiveness of the array of fins. So, whatever problem today we have solved okay, three problems, so different problems we have considered. First problem with convective tip, the second one is adiabatic tip and the third one is arrays of fins. So, you solve some problems from the textbook as well as the assignment problems.