Fundamentals of Conduction and Radiation
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Lecture - 13
Fins with Uniform Cross-Section Area II

Hello everyone. So in last two classes we have studied the temperature distribution in fins with uniform cross-section. So we considered infinite fin and finite fin with adiabatic tip as well as in last class we have done the temperature distribution for a fin with uniform cross section with convective boundary condition. So let us recapitulate what we have done.

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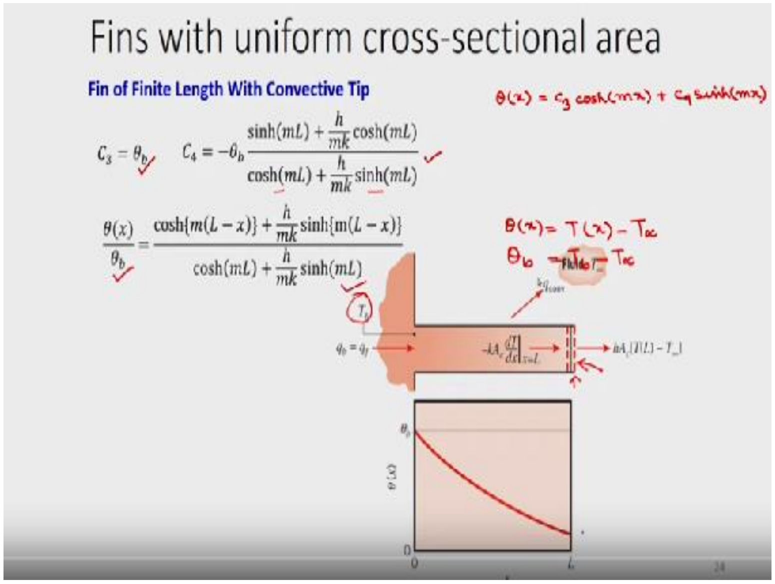

So for a fin of finite length with convective tip we have the equation as

$$\theta(x) = C_3 \cosh(mx) + C_4 \sinh(mx)$$

So we have started with that then we have applied to different boundary conditions; you have specified base temperature as well as at the tip you have a convective boundary condition. With that we found the constant C_3 and C_4. So

$$C_3 = \theta_b$$

$$C_4 = -\theta_b \frac{\left\{\sinh(mL) + \dfrac{h}{Km}\cosh(mL)\right\}}{\left\{\cosh(mL) + \dfrac{h}{Km}\sinh(mL)\right\}}$$

Then we put these values to the solution of theta x and we found the final temperature distribution as

$$\frac{\theta(x)}{\theta_b} = \frac{\cosh\{m(L-x)\} + \frac{h}{Km}\sinh\{m(L-x)\}}{\cosh(mL) + \frac{h}{Km}\sinh(mL)}$$

So this is the temperature distribution where

$$\theta(x) = T(x) - T_\infty$$

$$\theta_b = T_b - T_\infty$$

So let us see what how the temperature varies here. So if you see in this case, so this is the fin okay, so you see here so this is the base okay. So this is the base temperature and here convective boundary condition is applied.

So if you see the temperature variations or high temperature at the base and low temperature at the fin tip. So you can see that $\theta(x)$ decays from θ_b along length and it reaches θ_L at x=L. So this is the temperature distribution for the fin with convective boundary conditions.

So now we are interested to know what is the heat transfer rate for this fin; so for this condition for the fin with the convective boundary condition. So let us find the heat transfer rate for this fin.

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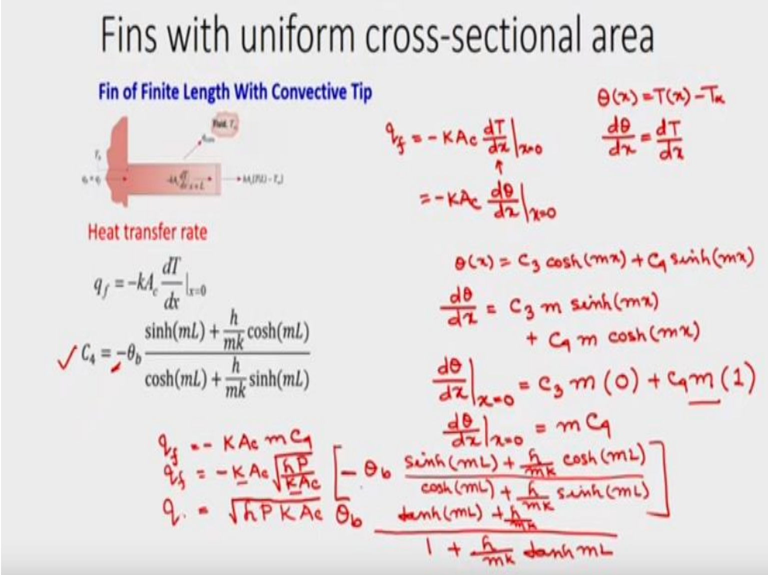

So we know that q_f, at x = 0, or at fin base is actually heat conducted to the fin and is equal to heat convected through the periphery of the fin surfaces and at the end of the fin surface because at the fin tip also there will be convection right. So let us calculate now heat transfer rate. So we know

$$q_f = -KA_c \frac{dT}{dx}\bigg|_{x=0}$$

And we know that

$$\theta(x) = T(x) - T_\infty$$

So you can find the derivative

$$\frac{d\theta}{dx} = \frac{dT}{dx}$$

We can substitute with this

$$q_f = -KA_c \frac{d\theta}{dx}\bigg|_{x=0}$$

So we know the temperature distribution you can directly take the derivative and put it here. But let us take the initial solution what we have written

$$\theta(x) = C_3 \cosh(mx) + C_4 \sinh(mx)$$

So if you take the derivative of this with respect to x so let us write what is this?

$$\frac{d\theta}{dx} = m[C_3 \sinh mx + C_4 \cosh mx]$$

So this is the expression. Now let us find what is the temperature gradient at x = 0? That means at the fin base. So let us write

$$\frac{d\theta}{dx}\bigg|_{x=0} = m[C_3 \times 0 + C_4 \times 1]$$

So that means first term is 0. So

$$\frac{d\theta}{dx}\bigg|_{x=0} = mC_4$$

So now you put it in q_f

$$q_f = -KA_c m C_4$$

$$= -KA_c \sqrt{\frac{hP}{KA_c}} \left\{ -\theta_b \frac{\left\{\sinh(mL) + \frac{h}{Km}\cosh(mL)\right\}}{\left\{\cosh(mL) + \frac{h}{Km}\sinh(mL)\right\}} \right\}$$

So this is the expression of q_f. Now let us do some simple algebra just to write it in a simplified form. So what we can do?

$$= \sqrt{hPKA_c}\left\{\theta_b \frac{\left\{\sinh(mL) + \frac{h}{Km}\cosh(mL)\right\}}{\left\{\cosh(mL) + \frac{h}{Km}\sinh(mL)\right\}}\right\}$$

so now let us divide the numerator and denominator with $\cosh(mL)$ so that we can write it

$$q_f = \sqrt{hPKA_c}\theta_b \frac{\left\{\tanh(mL) + \frac{h}{Km}\right\}}{\left\{1 + \frac{h}{Km}\tanh(mL)\right\}}$$

So this is the expression for the heat transfer rate for the fin with convective boundary conditions. The same way now we can find the heat lost due to convection from the periphery of the fin as well as the fin tip. So I am not going to derive it but I will write the expression and you can do as homework. So you can see here

$$q_f = \int_{A_f} h(T - T_\infty)\, dA_s + hA_c(T - T_\infty)|_{x=L}$$

The first term corresponds to the periphery of the fin and the second term to the fin tip. So this heat transfer rate will be equal to whatever we have earlier from the fin base. Now there will be some heat loss from the from the fin tip. So let us calculate that in next slide.

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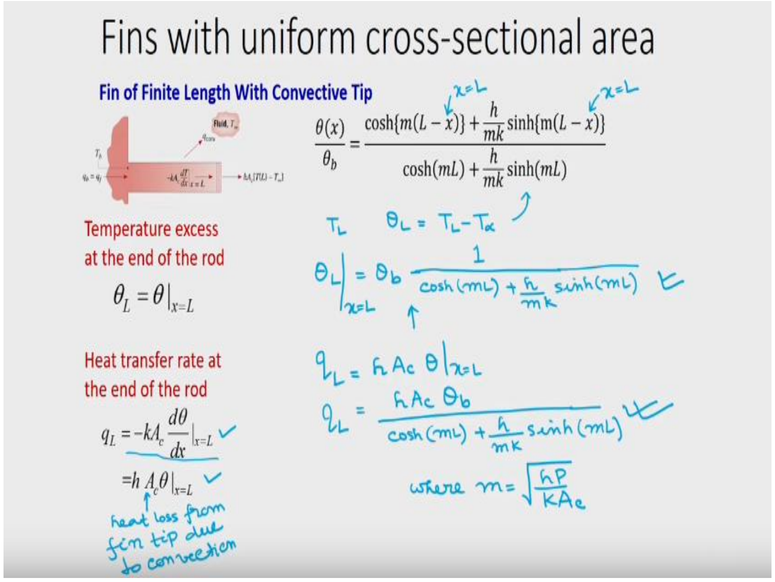

Let us derive it. So now you see temperature excess at the end of the rod. So we can define that as

$$\theta_L = T_L - T_\infty$$

So T_L or θ_L you can find from this expression right. If we put x = L in the temperature distribution we will get the temperature at the fin tip.

So let us write θ_L

$$\left.\frac{\theta(x)}{\theta_b}\right|_L = \frac{\cosh\{m(L-L)\} + \frac{h}{Km}\sinh\{m(L-L)\}}{\cosh(mL) + \frac{h}{Km}\sinh(mL)}$$

$$= \frac{1+0}{\cosh(mL) + \frac{h}{Km}\sinh(mL)}$$

$$\Rightarrow \theta_L = \theta_b \frac{1}{\cosh(mL) + \frac{h}{Km}\sinh(mL)}$$

So this is the excess temperature at the fin tip. Now let us find the heat loss from the fin tip only. So we can write

$$q_L = -KA_c \left.\frac{d\theta}{dx}\right|_{x=L} = hA_c\theta|_{x=L}$$

So it is heat loss from fin tip due to convection. So we know the θ_L. So it will be easy to find what will be heat loss from the fin tip.

$$q_L = hA_c\theta|_{x=L} = \frac{hA_c\theta_b}{\cosh(mL) + \frac{h}{Km}\sinh(mL)}$$

So this is the heat transfer from the fin tip, only fin tip okay, due to convection so that will be equal to also we can find out from this expression $-KA_c \left.\frac{d\theta}{dx}\right|_{x=L}$, but it is simple to use, so we found what is the heat transfer rate from the fin tip. So now we have completed the analysis of heat transfer for a fin with convective boundary condition. Now we will go to the next fin with specified temperature at the tip okay.

So now we will do the heat transfer analysis. First we will find the temperature distribution then we will find the heat transfer rate for the fin.

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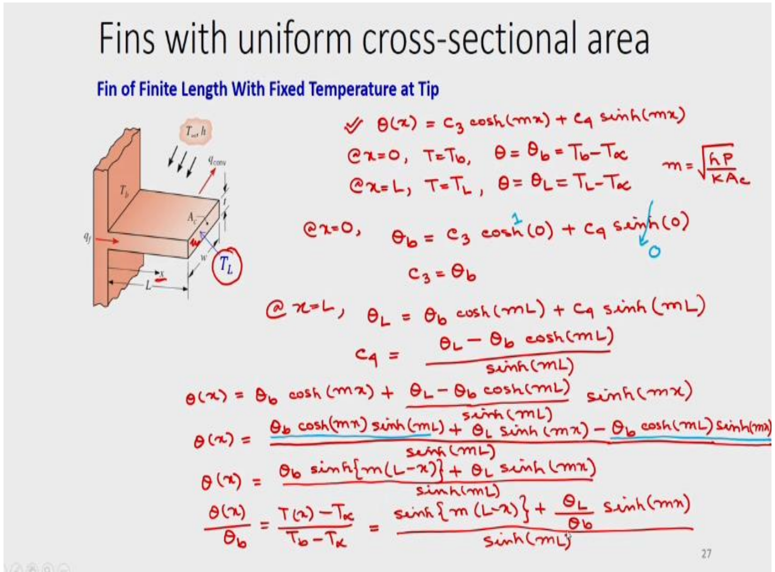

So now consider this fin of finite length and x we are measuring from the fin base and at the fin tip we have a specified temperature T_L okay, it may be connected to some liquid where it is maintaining the constant temperature. So that constant temperature is T_L we may consider.

So now we have two different boundary conditions at fin base we have T_b temperature is constant as well as fin tip we have the temperature T_L which is constant. So the general solution let us write

$$\theta(x) = C_3 \cosh(mx) + C_4 \sinh(mx)$$

We need 2 boundary conditions so

$$at\ x = 0; T = T_b;\ \theta_b = T_b - T_\infty$$

$$at\ x = L; T = T_L;\ \theta_L = T_L - T_\infty$$

Now we apply these 2 boundary conditions and find the constants C_3 and C_4. So first you put at x = 0

$$\theta_b = C_3 \cosh(0) + C_4 \sinh(0)$$

In this case $\sinh(0)$ is 0 and $\cosh(0)$ is 1, hence C_3 will be θ_b.

$$C_3 = \theta_b$$

Okay so 1 constant we have found, now we apply the second boundary conditions at x = L

$$\theta_L = \theta_b \cosh(mL) + C_4 \sinh(mL)$$

So C_4 you can write,

$$C_4 = \frac{\theta_L - \theta_b \cosh(mL)}{\sinh(mL)}$$

So now we know the constants C_3 and C_4 you put the value in the $\theta(x)$ expression then you can find the temperature distribution for the case fixed temperature at the fin tip.

$$\theta(x) = \theta_b \cosh(mx) + \frac{\theta_L - \theta_b \cosh(mL)}{\sinh(mL)} \sinh(mx)$$

So this is the temperature distribution but always we write in a simplified form so let us do the simple algebra here to write it in a simple form.

$$\Rightarrow \theta(x) = \frac{\theta_b \cosh(mx)\sinh(mL) + \theta_L \sinh(mx) - \theta_b \cosh(mL)\sinh(mx)}{\sinh(mL)}$$

Now you will see whether further you can simplify. So in the numerator we can write the 1st and 3rd term together in terms of $\sinh(A - B)$.

$$\Rightarrow \theta(x) = \frac{\theta_b \sinh\big(m(L-x)\big) + \theta_L \sinh(mx)}{\sinh(mL)}$$

So in nondimensional form

$$\frac{\theta(x)}{\theta_b} = \frac{\sinh\big(m(L-x)\big) + \frac{\theta_L}{\theta_b}\sinh(mx)}{\sinh(mL)}$$

So this is the temperature distribution for the fin with fixed tip temperature okay. So now we are interested to know the heat transfer rate okay so that we will find now.

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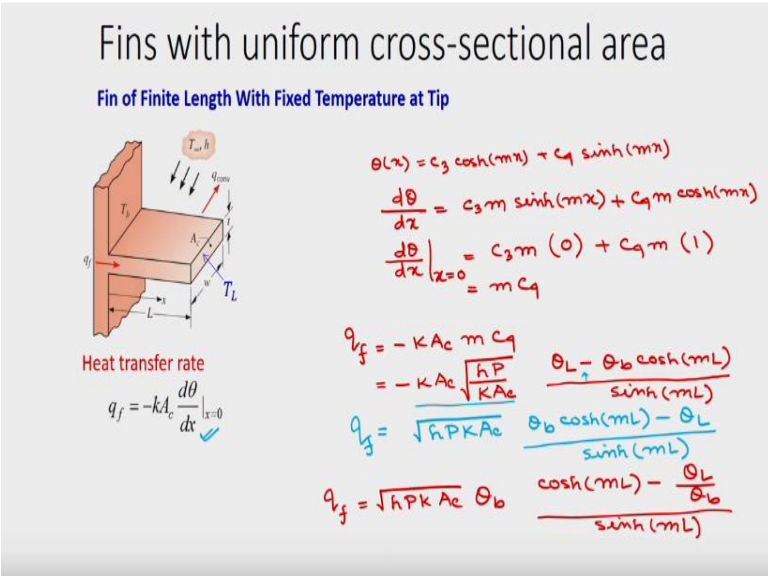

So heat transfer rate is

$$q_f = -KA_c \left.\frac{d\theta}{dx}\right|_{x=0}$$

But

$$\theta(x) = C_3 \cosh(mx) + C_4 \sinh(mx)$$

Hence

$$\frac{d\theta}{dx} = m[C_3 \sinh mx + C_4 \cosh mx]$$

at x = 0

$$\left.\frac{d\theta}{dx}\right|_{x=0} = mC_4$$

Hence

$$q_f = -KA_c m C_4$$

Putting value of C_4

$$= -KA_c \sqrt{\frac{hP}{KA_c}} \left\{\frac{\theta_L - \theta_b \cosh(mL)}{\sinh(mL)}\right\}$$

$$= \sqrt{hPKA_c} \left\{\frac{\theta_b \cosh(mL) - \theta_L}{\sinh(mL)}\right\}$$

$$= \sqrt{hPKA_c}\,\theta_b \left\{\frac{\cosh(mL) - \frac{\theta_L}{\theta_b}}{\sinh(mL)}\right\}$$

So this is the heat transfer from the fin with fixed temperature at the tip. Here we have written it in a simplified form. Similarly you can find the heat loss from the fin from the surrounding surface as well as from the fin tip and it will be equal to whatever we have derived here. So both way you can do okay; heat loss from the surfaces fin surfaces as well as you can also find the heat transfer rate from the fin base okay.

So that is the heat conducted and it is simple energy balance. So let us summarize whatever we have done in the next slide.

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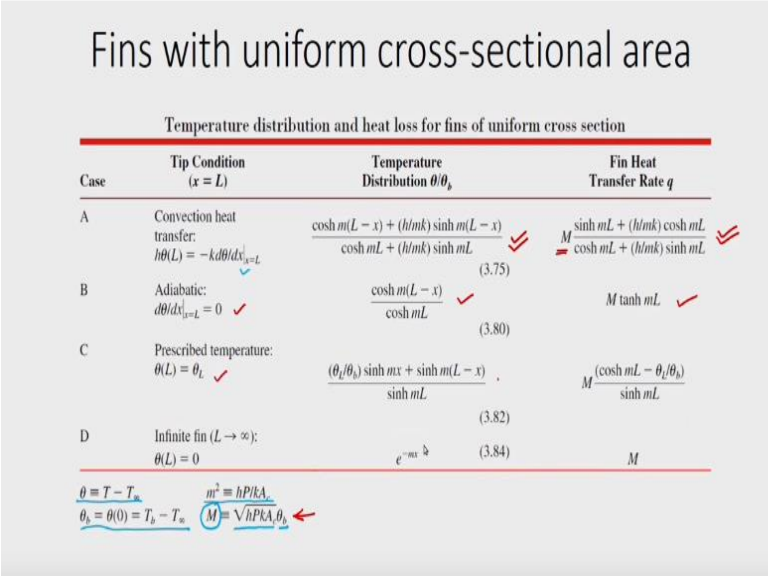

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate q
A	Convection heat transfer: $h\theta(L) = -kd\theta/dx\|_{x=L}$	$\frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ (3.75)	$M\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$
B	Adiabatic: $d\theta/dx\|_{x=L} = 0$	$\frac{\cosh m(L-x)}{\cosh mL}$ (3.80)	$M \tanh mL$
C	Prescribed temperature: $\theta(L) = \theta_L$	$\frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ (3.82)	$M\frac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx} (3.84)	M

$\theta \equiv T - T_\infty$ $\quad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\quad M \equiv \sqrt{hPkA_c}\,\theta_b$

So we have considered 4 different types of fins one is infinite fin, second is finite fin with adiabatic tip then finite fin with convective tip then finite fin with fixed temperature at the tip. We have found the temperature distribution in each cases as well as the heat transfer rate in each cases. So let us summarize in this slide. So before going to that you see that

$$\theta = T - T_\infty$$

$$\theta_b = T_b - T_\infty$$

$$m^2 = \frac{hP}{KA_c}$$

$$M = \sqrt{hPKA_c}\theta_b$$

The table lists all the formula for temperature distribution and heat transfer rate of the 4 types of fins we have discussed. You can see that for infinite fin it is the simplest where temperature distribution is e^{-mx} and heat transfer rate q is M. Here just I want to mention that in the adiabatic tip whatever expression we have derived it is somewhat different because in this case we have taken the x from the fin base okay x is measured from there. But in our case we considered from the fin tip.

So with that now let us see that when you are using fin or whether it is effective or not how you will determine okay so for a particular application whether you need fin or not how you will judge? So there are two parameters so in the next slide let us go.

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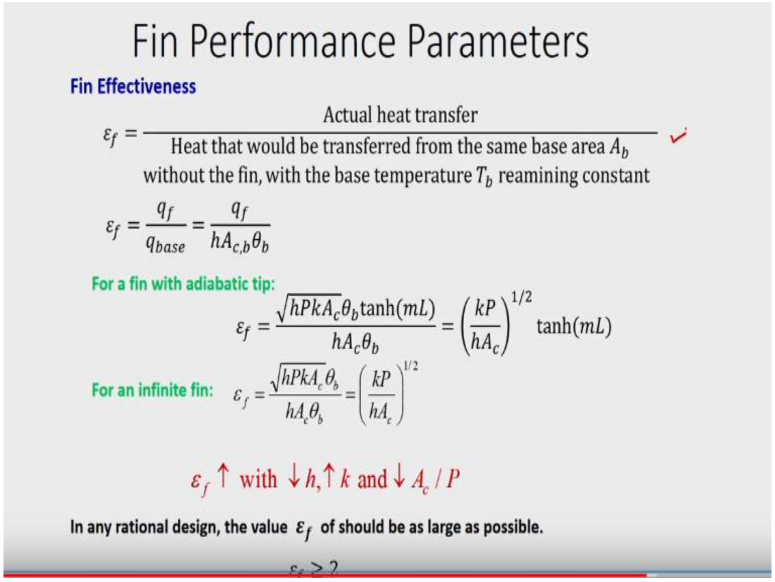

So there are two different parameters are there those are known as fin performance parameters one is effectiveness, fin effectiveness another is fin efficiency. Let us first define the fin effectiveness. So fin effectiveness is defined as

$$\varepsilon_f = \frac{actual\ heat\ transfer}{\begin{array}{c} heat\ that\ would\ be\ transferred\ from\ the\ same\ base\ area\ A_b without\ the\ fin, \\ with\ the\ base\ temperature\ T_b\ remaining\ constant \end{array}}$$

So, that means say when you are using the fin for a particular application so obviously your aim is to increase the heat transfer right or increasing the heat transfer area. So that is your aim but at the same time when you are attaching fin with the system you are actually increasing the conductive resistance distance because in the fin when the heat transfer will take place there will be a conductive resistance.

So you are increasing the conductive resistance. For that reason we are defining the fin effectiveness as the ratio actual heat transfer and if you assume that there is no fin but from that fin base with the same area A_b what is the heat transfers. So ratio of these two is defined as fin effectiveness. So if q_f is actual heat transfer and q_{base} is heat that would be transferred from the same base area A_b without the fin with the base temperature T_b remaining constant. Then

$$\varepsilon_f = \frac{q_f}{q_{base}} = \frac{q_f}{hA_{c,b}\,\theta_b}$$

So this q_f we have already found for different cases. And what is the heat transfer from the fin base without the fin? If the cross sectional area is A_c and heat transfer coefficient is h then q_{base} is nothing but $hA_{c,b}\,(T_b - T_\infty)$. Because T_b is the constant temperature at the fin base and T_∞ is the ambient temperature. Essentially it is θ_b . So already we have found the different cases so first you consider fin with adiabatic tip. So we know the heat transfer in this case which we have already calculated. Let us put the value in the numerator

$$\varepsilon_f = \frac{\sqrt{hPKA_c}\theta_b \tanh mL}{hA_c\,\theta_b} = \left(\frac{KP}{hA_c}\right)^{\frac{1}{2}} \tanh mL = \frac{\tanh mL}{m}$$

So this is the expression and for adiabatic fin. For infinite fin

$$\varepsilon_f = \left(\frac{KP}{hA_c}\right)^{\frac{1}{2}}$$

Now you can see how you can increase the effectiveness because fin effectiveness in general this should be very high okay. Then only you can use the fin okay and in general if effectiveness of a fin is greater than or equal to 2 then you can use this fin okay.

So you can see effectiveness is greater than equal to 2 so now how you can increase the effectiveness so you can see from these expressions so you can see in this that if h you can decrease then obviously your epsilon f will increase or K thermal conductivity of the fin material if you increase then obviously your effectiveness will increase.

At the same time A_c/P if you decrease so cross sectional area divided by the perimeter if you decrease then your effectiveness of the fin will increase okay. Now let us consider another parameter that is known as fin efficiency okay.

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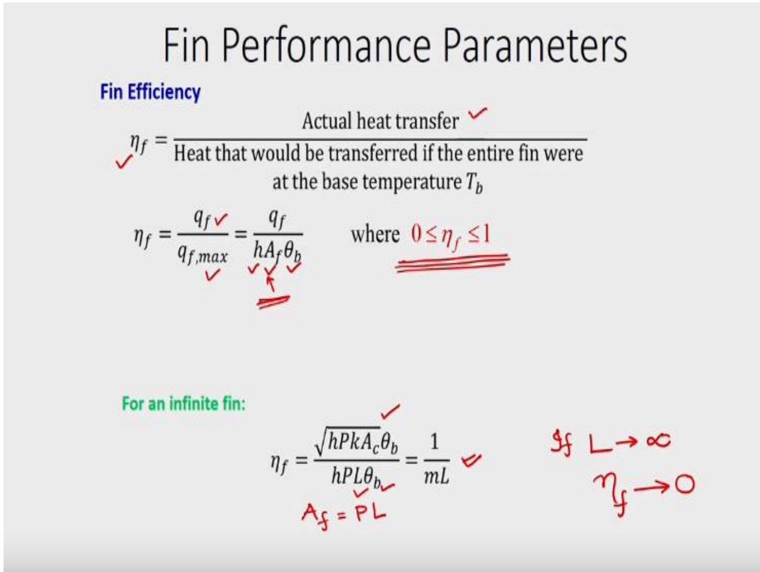

So this fin efficiency that is defined as

$$\eta_f = \frac{actual\ heat\ transfer}{heat\ that\ would\ be\ transferred\ if\ the\ entire\ fin\ were\ at\ the\ base\ temperature, T_b}$$

So now you see, you have a fin but you assume that the entire fin is at base temperature T_b. So what will be the heat transfer because that is the maximum possible heat transfer from the fin right. So how we can determine that, so it will be

$$\eta_f = \frac{q_f}{q_{f,max}} = \frac{q_f}{hA_f\theta_b}$$

We have already found what is the heat transfer from the fin for different cases and what is the maximum possible heat transfer from the fin if the fin is maintained at temperature T_b which is the base temperature so that will product of h, fin surface area and $(T_b - T_\infty)$ which is nothing but θ_b. A_f is the total fin heat transfer area right. So you can see that it is the maximum possible heat transfer. So you can see that η_f minimum value can be 0 and maximum value can be. So η_f will be between 0 and 1. So now you consider for an infinite fin.

$$\eta_f = \frac{\sqrt{hPKA_c}\theta_b}{hPL\theta_b} = \frac{1}{mL}$$

So infinite fin you know the q_f is $\sqrt{hPKA_c}\theta_b$ and what is the maximum possible heat transfer in this case? So A_f if in this case is PL okay because it is an infinite fin so at that fin tip there will be no heat loss because that is maintained at T_∞.

So there will be no heat loss there in this case so the fin area will be essentially your PL. So finally efficiency turns out to be 1/mL. So now if L tends to infinity what will be the value of η_f? As for the infinite fin, length is considered infinite. So if L tends to infinite then your fin efficiency will tends to 0. So what does it mean?

So if you use infinite fin your L is infinite then your fin efficiency will decrease so you can see it tends to 0; so its minimum value is 0. So now let us consider adiabatic tip so next slide let us go.

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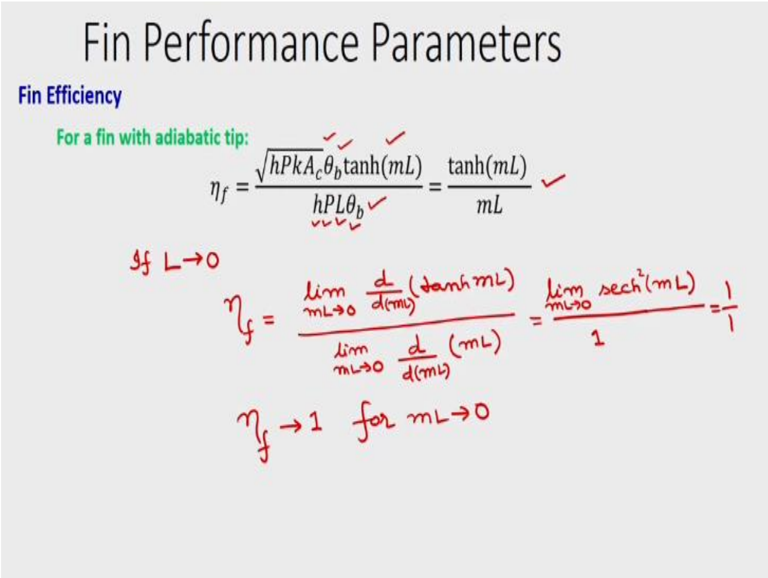

So now in fin efficiency for an adiabatic tip now you see here

$$\eta_f = \frac{\sqrt{hPKA_c}\theta_b \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL}$$

So in the adiabatic tip obviously from the fin tip there is no heat loss because it is insulated okay. So there would be no heat loss so the only heat loss it will take place from the periphery of the surface and what is the area of that? If P is the perimeter and L is the length then PL is the heat transferred area right of the fin. So this is the expression for the efficiency of adiabatic tip fin.

Now you consider if L tends to 0 what will happen in this case for an adiabatic tip fin. Here $\tanh 0$ is 0 and mL is 0. So in this case L hopital's rule we will use. So

$$\eta_f = \frac{\lim\limits_{mL\to 0} \frac{d}{d(mL)}(\tanh mL)}{\lim\limits_{mL\to 0} \frac{d}{d(mL)}(mL)} = \frac{\lim\limits_{mL\to 0} \operatorname{sech}^2 mL}{1} = \frac{1}{1} = 1$$

This is the maximum value of the efficiency. Actually mL tends to 0 means there is no fin. In that case you will get η_f tends to 1. So when you are attaching a fin in a system then obviously your η_f will decrease. So now you have to see that how maximum you can η_f you can get okay with the minimum length because if you increase the length of the fin your material cost also will increase right.

It will be heavy as well as it will be costlier. So with low cost and with low weight you have to design the fin such a way that you can maximize the heat transfer. So in this case we have seen that for an infinite fin you will get the minimum efficiency that is 0 and for adiabatic tip when your length or mL tends to 0 then you will get maximum fin efficiency as 1. So today we will stop here so in the next class we will consider the non-uniform temperature distribution in non-uniform fin. Thank you.