Fundamentals of Conduction and Radiation
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Lecture - 12
Heat Transfer from Extended Surfaces (Part - 2)

So we have derived the temperature distribution for an infinite fin with uniform cross section.

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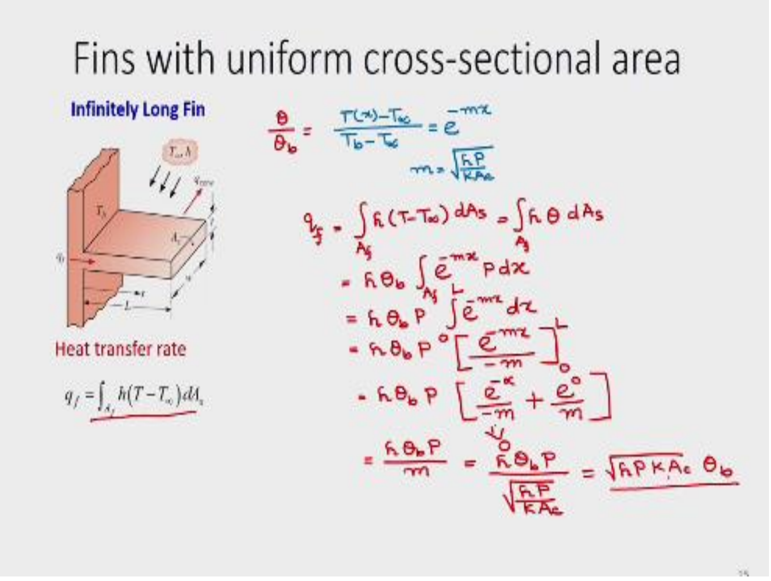

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx}$$

Where

$$m = \sqrt{\frac{hP}{KA_c}}$$

So already we have discussed that h is the heat transfer coefficient of the surrounding fluid and P is the perimeter A_c is the cross section of the fin and K is the thermal conductivity of the fin. Now let us find the heat transfer rate from this fin. For that let us take this heat transfer rate

$$q_f = \int_{A_f} h(T - T_\infty) dA_s = \int_{A_f} h\theta dA_s$$

So what we are doing. So we are doing the heat loss from this surrounding faces and as it is infinitely long so from the end of the fin means from the tip of the fin there is no heat loss. So we are actually integrating the surface, surface from where the heat transfer is taking place. So now we know $T - T_\infty$ is nothing but θ. dA_s is nothing but Pdx .Now putting the expression for θ and dA_s

$$q_f = h\theta_b P \int_0^L e^{-mx} dx$$

We can take h, θ_b and P outside the integral as they are constants. And x is varying from 0 to L. So now you can do this integral

$$= h\theta_b P \left[\frac{e^{-mx}}{-m}\right]_0^L$$

So L tends to infinity hence,

$$= h\theta_b P \left[\frac{e^{-\infty}}{-m} + \frac{e^0}{m}\right]$$

So $e^{-\infty}$ is 0. So first term it is 0 and e^0 is 1 so it will be 1/m. So you will get

$$= \frac{h\theta_b P}{m} = \frac{h\theta_b P}{\sqrt{\frac{hP}{KA_c}}}$$

$$= \sqrt{hPKA_c}\theta_b$$

So we have derived it for the infinitely long fin. So now you consider that from the base of the fin what is the heat transfer is taking place because that whatever heat transferred from the fin base actually that is heat loss due to the convection from the fin surface. It is a simple energy balance. The heat is transferred from the base to the fin that equal to heat loss from the fin to the surrounding surface.

So we have derived just heat loss to the surrounding fluids. Now we will consider the heat transfer from the base, because this should be equal. So let us see in the next slide.

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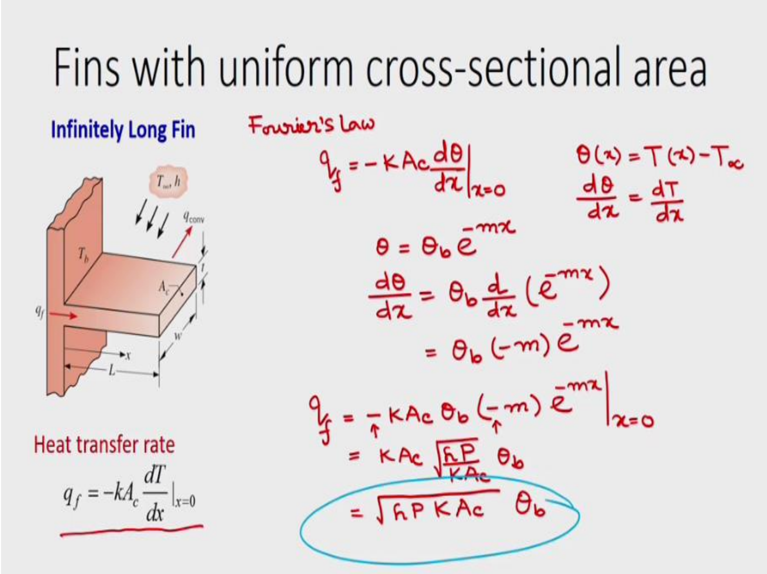

So now we can write Fourier's law at fin base

$$q_f = -KA_c \frac{dT}{dx}\bigg|_{x=0}$$

x = 0 is your fin base. So at x = 0 what is the heat transfer rate? So now as $\frac{dT}{dx} = \frac{d\theta}{dx}$ you write

$$q_f = -KA_c \frac{d\theta}{dx}\bigg|_{x=0}$$

So now x we are taking in the positive x direction and your heat transfer is also taking place in the positive x direction. And now θ is

$$\theta = \theta_b e^{-mx}$$

Which, we have already derived the temperature distribution. Now taking the derivatiave

$$\frac{d\theta}{dx} = \theta_b \frac{d}{dx}(e^{-mx}) = \theta_b(-m)e^{-mx}$$

you put it at q_f.

$$q_f = -KA_c\theta_b(-m)e^{-mx}|_{x=0}$$

Putting the expression for m

$$q_f = KA_c\left(\sqrt{\frac{hP}{KA_c}}\right)\theta_b = \sqrt{hPKA_c}\theta_b$$

So that is the heat transfer and from the base. So you can see that this is the heat transfer rate from the fin base and it should be equal to the heat loss due to convection towards the surrounding. So let us see in the earlier slide what we have derived so you can see that is also same. So you have an energy balance. The amount of heat conducted through the base actually that is convected to the surrounding fluid by convection.

Okay so now let us summarize what we have done now so we considered one infinitely long fin and first we have derived the temperature distribution and we have seen that temperature distribution actually exponentially decays along the x, and we have then found the heat transfer rate from the fin. So now we will consider adiabatic tip okay.

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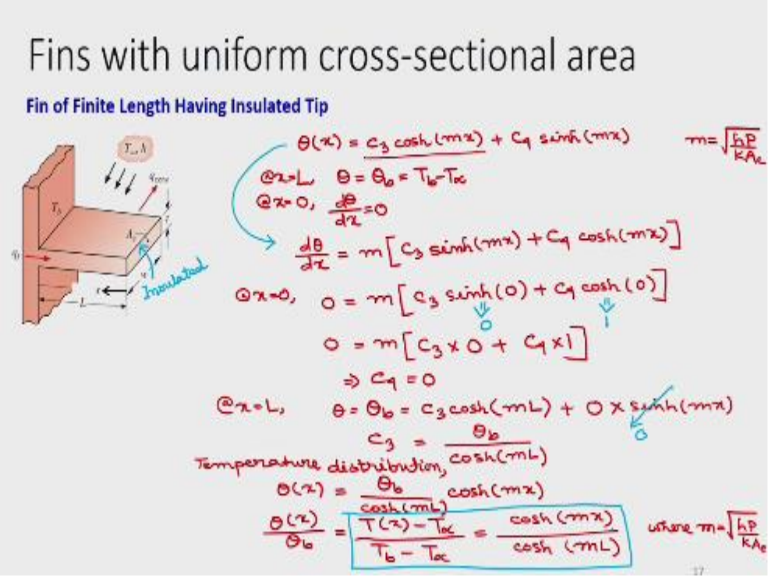

So, adiabatic tip means fin of finite length having insulated tip. So you can consider this fin of uniform cross section and so this is insulated. So if it is insulated, from the tip of the fin there will be heat loss that means we have told that q_f at tip will be 0. And if it is 0 then we have found already that the temperature gradient dT / dx at the fin tip also will be 0. So now let us consider the generalized solution of the energy equation for uniform cross section what we have derived.

In this case now which one we will consider exponential or hyperbolic function? As it is finite fin so we have to consider in hyperbolic function. So we have already derived.

$$\theta(x) = C_3 \cosh(mx) + C_4 \sinh(-mx)$$

So this is the solution for finite fin. So now we need two boundary conditions to find the 2 constants C_3 and C_4. In this case I forgot to tell that we will consider the x from the fin tip. So you can see that q_f the direction is opposite to the x direction. Now what are the boundary conditions? One already we know at base temperature is constant. So

$$at\ x = L, \theta(L) = T_b - T_\infty = \theta_b$$

And as already we have discussed that the fin tip is adiabatic so no heat loss so at x = 0. So,

$$at\ x = 0; \frac{d\theta}{dx} = 0$$

We are taking x from the fin tip just to get a simplified solution okay nothing else. So we can consider from here also but we will see that we will get a very simplified form of this solution that is why we are considering or we are measuring x from the fin tip. So what will be dθ/dx? So it will be

$$\frac{d\theta}{dx} = m[C_3 \sinh mx + C_4 \cosh mx]$$

$$\left.\frac{d\theta}{dx}\right|_{x=0} = m[C_3 \sinh 0 + C_4 \cosh 0] = 0$$

Putting the value of sinh and cosh functions

$$= m[C_3 \times 0 + C_4 \times 1] = 0$$

$$= mC_4 = 0$$

As m cannot be zero C_4 must be zero. Hence,

$$C_4 = 0$$

So now apply the other boundary condition at the fin base which is located at x = L.

$$\theta(L) = \theta_b = C_3 \cosh(mL) + C_4 \sinh(-mL)$$

So obviously second term is 0, so you will get,

$$C_3 = \frac{\theta_b}{\cosh(mL)}$$

So now you put the values to get the temperature distribution.

$$\theta(x) = \frac{\theta_b}{\cosh(mL)} \cosh(mx)$$

C_4 is 0 so that term we will not write; now you just divide θ_b on both sides.

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh(mx)}{\cosh(mL)}$$

So this is the temperature distribution for adiabatic tip. So now consider the heat transfer rate right, so how we will find? So we can find in both the ways. But now we will use only whatever heat transfer is taking place at base right. So at base what is the heat transfer is taking place.

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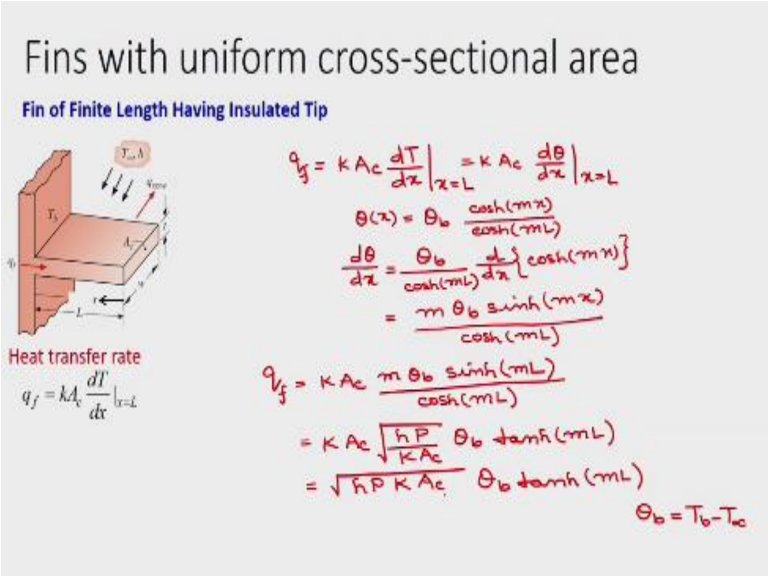

So if you consider that let us go to the next slide. Now you can see Fourier's Law we write $-KA_c\frac{dT}{dx}$ okay, but as in this case x we have measured from the fin tip towards the fin base so it is a negative direction to the way your heat flux is there. So q_b or q_f is acting in the negative x direction that is why this minus, minus will be plus and we will write

$$q_f = KA_c\frac{dT}{dx}\bigg|_{x=L}$$

So we know the, so we can write in terms of θ

$$q_f = KA_c\frac{d\theta}{dx}\bigg|_{x=L}$$

And what is your temperature distribution? It is

$$\theta(x) = \theta_b\frac{\cosh(mx)}{\cosh(mL)}$$

So now taking derivative

$$\frac{d\theta}{dx} = \frac{\theta_b}{\cosh(mL)}\frac{d}{dx}\{\cosh mx\}$$

θ_b and $\cosh(mL)$ are constants so you can take it outside of the derivative.

$$= \frac{m\theta_b}{\cosh(mL)}\sinh mx$$

Putting x = L

$$= \frac{m\theta_b}{\cosh(mL)}\sinh mL$$

So the heat transfer rate from the base putting the value of $\frac{d\theta}{dx}$ is

$$q_f = KA_c\frac{m\theta_b}{\cosh(mL)}\sinh mL$$

$$= KA_c \sqrt{\frac{hP}{KA_c}} \theta_b \tanh mL$$

$$= \sqrt{hPKA_c} \theta_b \tanh mL$$

So this is the heat transfer rate for a adiabatic tip, where

$$T_b - T_\infty = \theta_b$$

So we are not doing here heat loss due to convection. You can do as homework because it is the same analysis whatever we have done for earlier case where we consider the infinite fin. So just integrate over the heat transfer area and then you calculate the heat transfer rate. It will be same as this expression.

So now you just compare this two heat transfer rate for an infinite fin and a finite fin with adiabatic tip okay.

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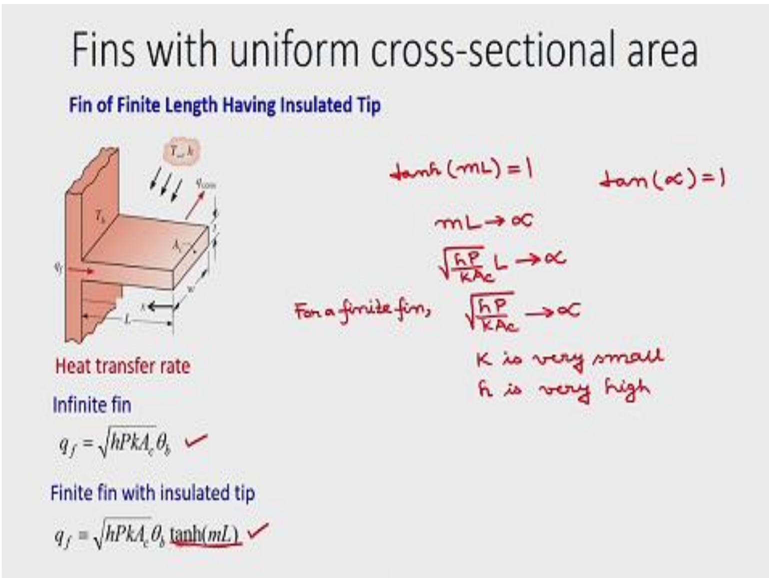

So infinite fin we have calculated you can see here,

$$q_f = \sqrt{hPKA_c} \theta_b$$

and for the case of finite fin with insulated tip

$$q_f = \sqrt{hPKA_c} \theta_b \tanh mL$$

So if you compare it you can see, if $\tanh mL$ becomes 1 then both heat transfer rate will be same. That means the heat transfer from the infinite fin will be equal to heat transfer from the finite fin with insulated tip. So when you will get this situation? So let us see. So that means you are telling that

$$\tanh mL = 1$$

But we know

$$\tanh \infty = 1$$

So that means

$$mL = \infty$$

$$\Rightarrow \sqrt{\frac{hP}{KA_c}} L = \infty$$

Then the heat transfer from the infinite fin will be equal to heat transfer from the finite fin with adiabatic tip. So in this case now you see when this will become your infinity. So L should be very large then obviously you will get ∞, but let us take that you have a finite length okay, so if you consider finite length then $\sqrt{\frac{hP}{KA_c}}$ for a finite fin, cannot be infinite.

So for a finite fin obviously $\sqrt{\frac{hP}{KA_c}}$ should be tends to infinity. So when you can get it? You can get when K is very small or h is very high. So in these two cases or scenario you will get heat transfer rate from the infinite fin will be equal to heat transfer rate from the finite fin with adiabatic tip.

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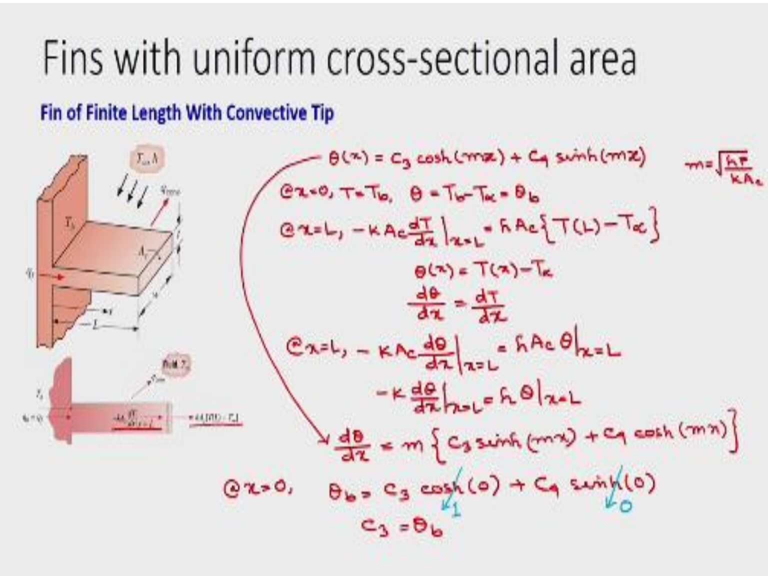

Now let us take the next case where we will consider fin with finite length with convective tip. So now we are seeing that from the tip also your heat loss is taking place due to convection. So in this case we have already discussed that the heat transfer or heat conduction through the fin will be equal to heat loss to the surrounding at the fin tip. So this is the case you can see here

$$-KA_c \frac{dT}{dx}\bigg|_{x=L} = hA(T_L - T_\infty)$$

This is one boundary condition. So in this case x we have considered from the fin base or heat transfer is taking place in the positive x direction. So now what is the solution we will consider? Now we will consider,

$$\theta(x) = C_3 \cosh(mx) + C_4 \sinh(mx)$$

This is the solution and convenient to use for a finite fin where

$$m = \sqrt{\frac{hP}{KA_c}}$$

So now we need two boundary conditions. So one boundary condition is

$$at\ x = 0, \theta(0) = T_b - T_\infty = \theta_b$$

And another boundary condition is convective boundary condition at the tip so that you can write

$$at\ x = L, -KA_c \frac{dT}{dx}\bigg|_{x=L} = hA_c(T_L - T_\infty)$$

Again

$$\frac{dT}{dx} = \frac{d\theta}{dx}$$

Hence, omitting the common term A_c from both sides and writing in terms of θ

$$at\ x = L, -K \frac{d\theta}{dx}\bigg|_{x=L} = h\theta|_{x=L}$$

So these are the two boundary conditions. So first take the derivative of this θ (x) so you will get,

$$\frac{d\theta}{dx} = m\{C_3 \sinh(mx) + C_4 \cosh(mx)\}$$

Now you apply the boundary conditions. So first boundary condition if you apply,

$$at\ x = 0, \theta_b = C_3 \cosh(0) + C_4 \sinh(0)$$

So you can see from here that sinh(0) is 0 and cosh(0) is 1. So your second term will become 0, hence

$$\theta_b = C_3$$

So one constant we have found so another constant now we apply the second boundary conditions so you will find C_4.

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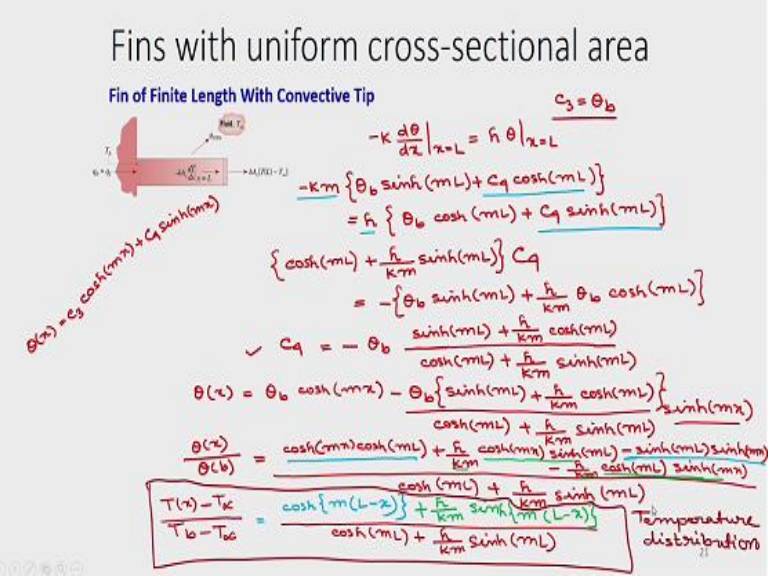

So now you can write the next boundary conditions as

$$-K\left.\frac{d\theta}{dx}\right|_{x=L} = h\theta|_{x=L}$$

So now you put the value. So it will be

$$-Km\{\theta_b \sinh(mL) + C_4 \cosh(mL)\} = h\{\theta_b \cosh(mL) + C_4 \sinh(mL)\}$$

Right hand side is just the temperature distribution θ at x = L and we have substituted the value of C_3 as θ_b. Now you do the algebra just to write it in a simplified form and find the find the value of C_4. So now rearranging the equation and taking the C_4 terms to one side

$$\left\{\cosh(mL) + \frac{h}{Km}\sinh(mL)\right\} C_4 = -\left\{\theta_b \sinh(mL) + \frac{h}{Km}\theta_b \cosh(mL)\right\}$$

Hence we can write C_4 as

$$C_4 = -\theta_b \frac{\left\{\sinh(mL) + \frac{h}{Km}\cosh(mL)\right\}}{\left\{\cosh(mL) + \frac{h}{Km}\sinh(mL)\right\}}$$

So this is the expression of C_4 and C_3 is θ_b. Now you put in the θ expression to get the temperature distribution for a finite fin with convective tip. So you can write

$$\theta(x) = \theta_b \cosh(mx) - \theta_b \frac{\left\{\sinh(mL) + \frac{h}{Km}\cosh(mL)\right\}}{\left\{\cosh(mL) + \frac{h}{Km}\sinh(mL)\right\}} \sinh(mx)$$

So we have got the temperature distribution but now let us write it in a simplified form.

$$\frac{\theta(x)}{\theta_b} = \frac{\cosh(mx)\cosh(mL) + \frac{h}{Km}\cosh(mx)\sinh(mL) - \sinh(mL)\sinh(mx) - \frac{h}{Km}\sinh(mx)\cosh(mL)}{\cosh(mL) + \frac{h}{Km}\sinh(mL)}$$

Now it is a very big expression. If we take the 1st and 3rd term from the numerator we can write them as $\cosh(A - B)$ format. Similarly we can write the 2nd and 4th term as $\sinh(A - B)$. So doing the changes

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh\{m(L-x)\} + \frac{h}{Km}\sinh\{m(L-x)\}}{\cosh(mL) + \frac{h}{Km}\sinh(mL)}$$

So this is the temperature distribution for the finite fin with convective tip. So, now next step is to calculate the heat transfer rate. So heat transfer rate how we will calculate? So you know the temperature distribution and heat transfer rate at the base you just calculate what is the heat conduction then that q_f will be your total heat transfer rate from the fin.

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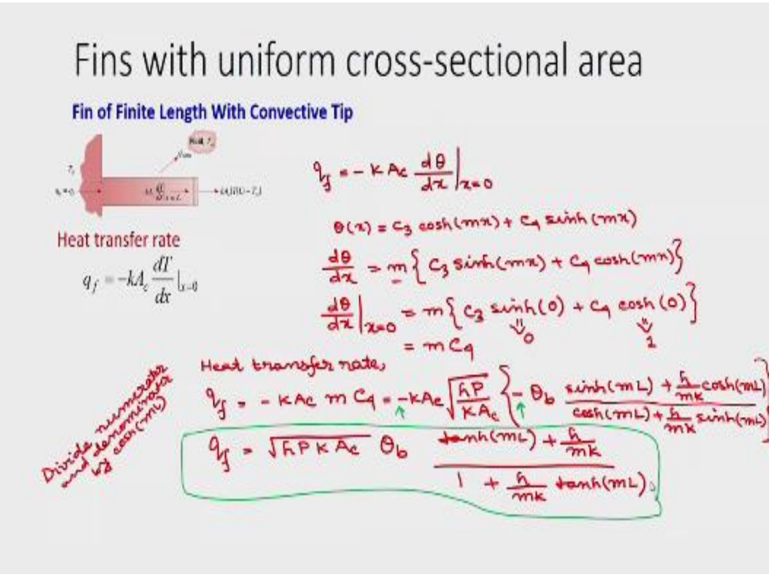

So in the next slide, so if you write

$$q_f = -KA_c \left.\frac{d\theta}{dx}\right|_{x=0}$$

Location is taken at x=0 as we are calculating at the fin base. So now θ you know expression, so from there you have to take the derivative and find what is dθ/dx. So let us take this at x equal to 0 so in a simplified form we will write okay first so that it will be easier to understand. So

$$\theta(x) = C_3 \cosh(mx) + C_4 \sinh(mx)$$

Values of C_3 and C_4 already we have found. So we can substitute later first find dθ/dx okay.

$$\frac{d\theta}{dx} = m\{C_3 \sinh(mx) + C_4 \cosh(mx)\}$$

Putting value of x

$$\frac{d\theta}{dx} = m\{C_3 \sinh(0) + C_4 \cosh(0)\}$$

$$\Rightarrow \frac{d\theta}{dx} = m\{C_3 \times 0 + C_4 \times 1\} = mC_4$$

Putting it in q_f expression

$$q_f = -KA_c mC_4$$

$$= -KA_c\sqrt{\frac{hP}{KA_c}}\left\{-\theta_b \frac{\left\{\sinh(mL) + \frac{h}{Km}\cosh(mL)\right\}}{\left\{\cosh(mL) + \frac{h}{Km}\sinh(mL)\right\}}\right\}$$

$$= \sqrt{hPKA_c}\left\{\theta_b \frac{\left\{\sinh(mL) + \frac{h}{Km}\cosh(mL)\right\}}{\left\{\cosh(mL) + \frac{h}{Km}\sinh(mL)\right\}}\right\}$$

Dividing both numerator and denominator by $\cosh(mL)$

$$= \sqrt{hPKA_c}\,\theta_b \frac{\left\{\tanh(mL) + \frac{h}{Km}\right\}}{\left\{1 + \frac{h}{Km}\tanh(mL)\right\}}$$

So this is the q_f so in this case now you can see that if you are considering a finite fin with convective tip then q_f will be this.

And same analysis you can do with the heat loss due to convection, but in this case as you are considering convective tip. So you have to the heat loss from the periphery of the fin as well as the heat loss from the fin tip. So both you have to add then you will get the same expression as now we have derived here. Thank you.