Fundamentals of Conduction and Radiation
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Lecture 11
Heat Transfer from Extended Surfaces

Hello everyone. So today, we will study the heat transfer from extended surfaces. Already, in earlier few classes, you have studied heat transfer in one dimension, in Cartesian, cylindrical, and spherical coordinates. Today, we will see this application of one dimensional heat transfer in Cartesian coordinate.

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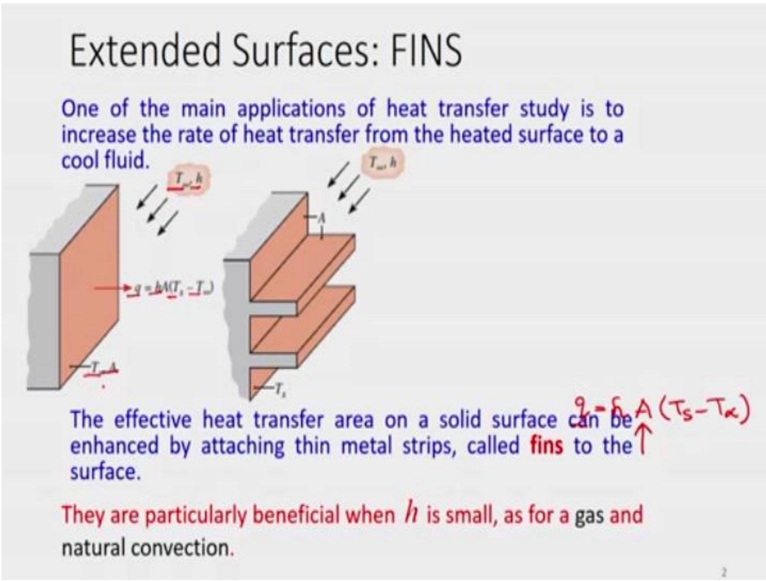

Before going to that, let us see that there are many applications of heat transfer study, where our main aim is to enhance the heat transfer from the hot surface. So you consider this hot surface, where temperature of the surface is T_s and area of the surface is A, and it is cooled by the surrounding fluid where temperature of surrounding fluid is T_∞ and the heat transfer coefficient is h. So this is your T_∞ and this is your h.

So if you want to know what the heat transfer rate from the surface is, then you can see that you can write heat transfer rate

$$q = hA(T_s - T_\infty)$$

Where h is the heat transfer coefficient of this surrounding fluid and A is the area and this is the Ts, is the surface temperature and T_∞ is the surrounding fluid temperature.

If you want to enhance the heat transfer q, how you can do it? So if we assume that T_s the surface temperature is constant, so there are two ways to enhance the heat transfer. You can increase the h or you can increase the area. In most of the cases, the surrounding fluid is same, then heat transfer coefficient will be constant and you cannot increase it more to get more heat transfer q.

So only possible way here, you can see that if you can write

$$q = hA(T_s - T_\infty)$$

So only possible way to achieve this is to enhance the heat transfer area. So in the next slide, let us see that if you attach some small metal strip to this heat transfer area, then you can increase the overall heat transfer area A. So in this case, you can actually enhance the heat transfer rate q.

So the effective heat transfer area on a solid surface can be enhanced by attaching thin metal strips. These are called fins to the surface and mostly this type of fins are used when heat transfer coefficient of the surrounding fluid is small in case of like, if it is a gas medium or in natural convection, mostly the heat transfer coefficient is very low. So in these cases, if you use these metal strips which is known as fin or extended surface, then you can enhance the heat transfer rate. Before going to the thermal analysis, let us see a few applications of this problem.

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So you have seen the motor bike, right. So in the motor bike, just below the oil tank, you will find the engine of the motor bike and if you notice carefully, then you will find that in the engine there are some metal strips attached to that and these are the metal strips, where these metal strips are attached with the engine, which is cylinder. So combustion takes place inside the cylinder.

And this cylinder wall will be very hard and naturally from these extended surfaces just by enhancing the heat transfer area, you can increase the heat transfer rate. Another application you can see in the car radiator, which you will find in front of the car, so there this hot fluid is cooled by the radiator and there you can see that these are the metal strips. So you can see these are some applications of the fins.

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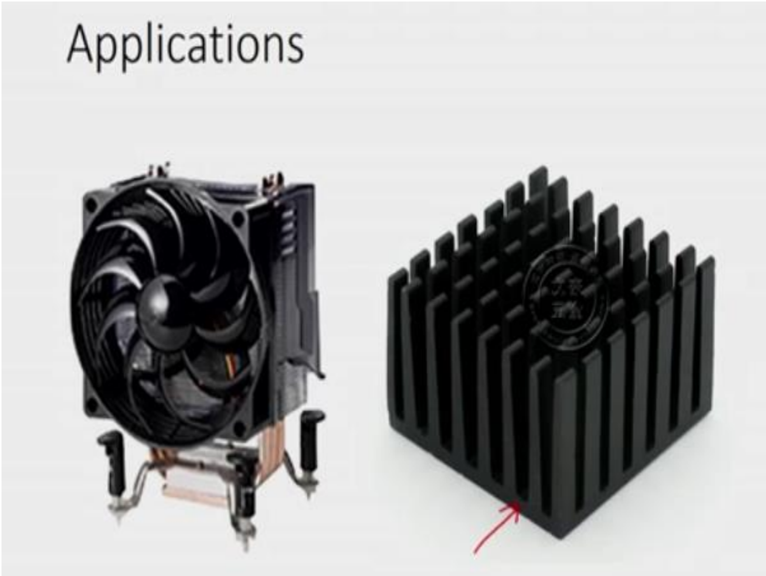

Have you seen inside your desktop where the processor is attached? You can see this type of fins is attached over the processor and over that some fans are attached. Here you can have the forced convection and through these fins you can enhance the heat transfer rate from the processor as well.

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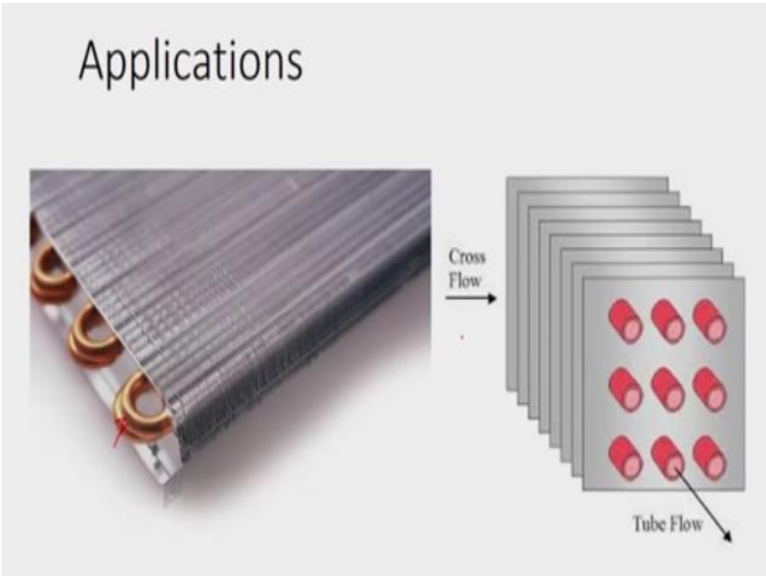

So here are some application in the heat exchanger areas. So these are copper tube through which your hot fluid flows, and over it you can see some metal strips are added. These are known as fins. So these are kind of cross flow heat exchanger. Here you can see this is the tube, so through this tube your hot fluid comes out and cold fluid goes over it, so it is kind of cross fluid exchanger. So this is one type of application.

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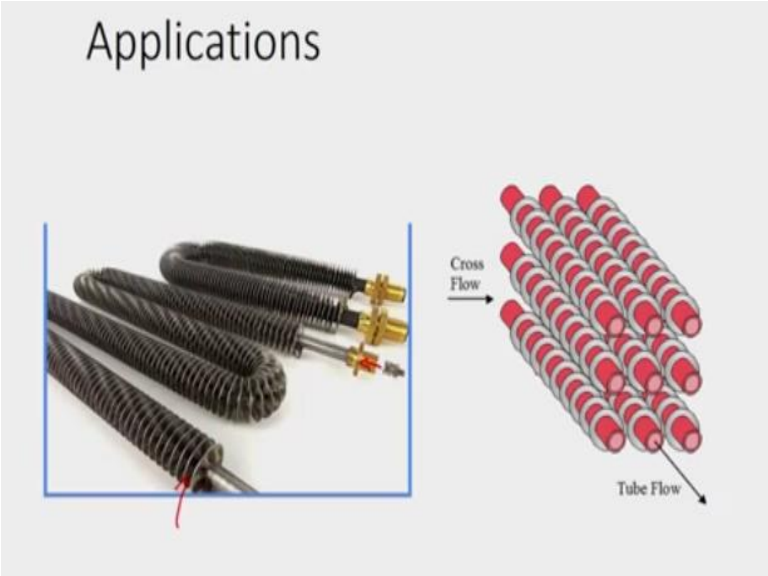

Another type of application you can see as annular fin, where the fin are circular in shape and attached to this pipe. So you can see this is some fins, circular fins. These are known as annular fins and through these pipes hot fluid flows. So you can see this is a cross flow kind of heat exchanger. So this is tube flow and the air or some other liquid flows over it. So you have extended surfaces, so you can enhance the heat transfer in these ways.

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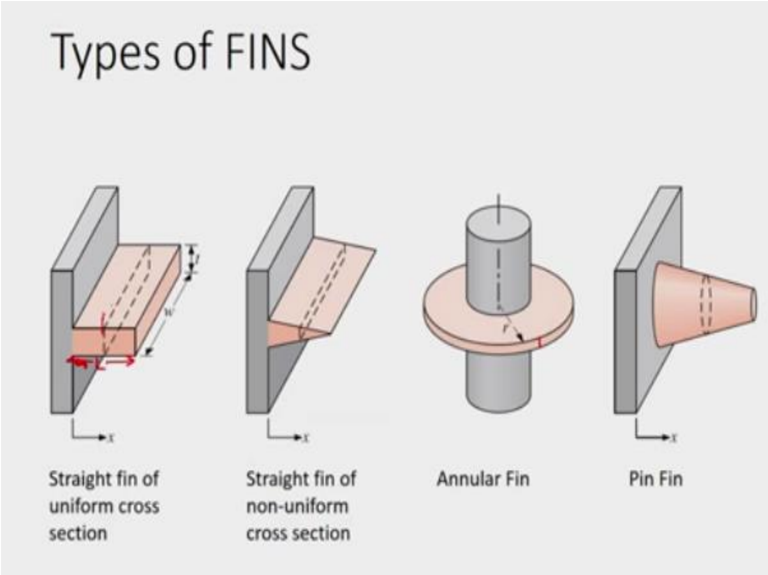

There are different types of fins. Now let us see that you have either uniform cross section fins or non-uniform cross section fins. So first one you can see, so this is a straight fin with uniform cross section. So W is the width of the fin. Here L is the length of the fin, and T is the thickness of the fin. So in this case, it is a uniform cross section. If you see this cross section, here cross section it is constant. So it is uniform cross section. But in this case you can see it is a triangular fin, and your area is decreasing along x. So it is a non-uniform cross section. In this case, you can see annular fin, so already we have seen some examples of annular fin. So in this case, these circular metal strips are attached to the surface of the cylinder and these are annular fins, so you can see here, it is uniform cross section. But in this case, these are some pin fins and in these pin fins actually your cross section is varying along the x. So there are different types of fins we can see. So now why do we need to study these heat transfer analysis in fins? Because we need to find what is the temperature distribution inside the fin. So for that, let us derive the energy equation, which is applicable in this fin.

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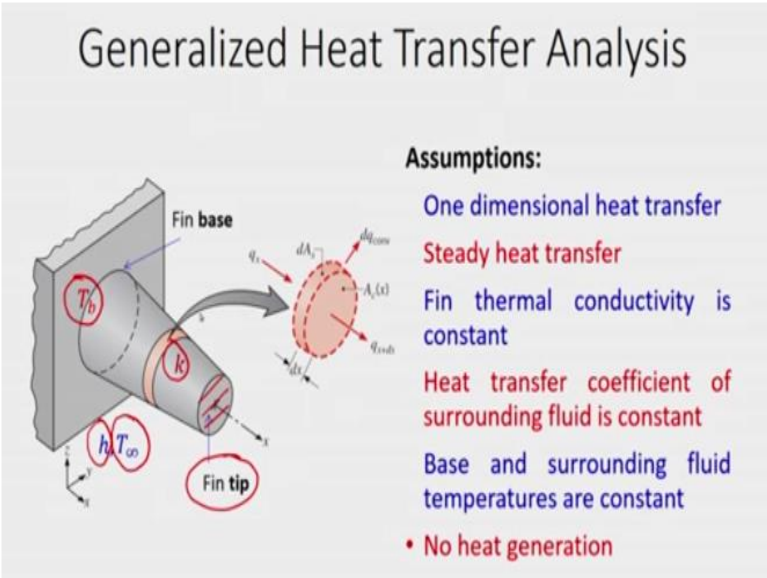

We will first make the generalized heat transfer analysis where we will consider a fin with varying cross section. So you can see that this is kind pin fin, we have considered where your area is decreasing along x. So before doing the analysis, let us take some assumptions, so that we can simplify the analysis. So first assumption, we will consider that it is one dimensional heat transfer. When can you consider one dimensional heat transfer?

So when the thickness and the width of the fin of the fin are very small compared to the length of the fin, then you have the assumptions of one dimensional heat transfer. So next we will consider that it is a steady state heat transfer. So your any temperature at any point will not vary with time. So it has reached a steady state, then after that we are doing the heat transfer analysis. Next we will consider that fin thermal conductivity constant.

So if you see this is the fin thermal conductivity K and this K is constant for the fin material. So that means it is a homogenous material. Next we will consider that heat transfer coefficient surrounding fluid is constant. So in this case, you can see you have surrounding fluid heat transfer coefficient h, and we will consider that h is constant. Next we will consider that base and surrounding fluid temperatures are constant. So what is fin base?

So when the fin is attached to the heat transfer area, so that is known as fin base and this fin base, we will consider temperature T_b, which is the fin base temperature and that we will assume as constant. Then, surrounding fluid temperature, which is nothing but T_∞ in this case, in this case, this T_∞ we will assume as constant. And the other end of the fin is known as fin tip, so this is the other end of the fin. So this is known as, this known as fin tip.

And the last assumptions we will consider that there is no heat generation. So in the fin, we will consider that q''', which is the heat generation per unit volume that we will consider as 0. So now we will do the generalized heat transfer analysis. First, we will consider one elemental volume at a distance x from the base of dx distance, we will consider this elemental volume and we will do the energy balance.

So in this case, you can see your dA_s, is the heat transfer area and heat is lost due to convection that is your dq_{conv} through this surface dA_s. And your heat conduction will take place at a distance x, that is your q_x in the x direction, and after $x + \Delta x$ distance, you will get that $q_{x+\Delta x}$ that is the heat conduction rate.

So in this case, you can see that your A_c, which is nothing by the cross sectional area at a distance x, so that is function of x as well as your dA_s, which is the heat transfer area that is also function of x, because we have considered a generalized fin where your fin is of non-uniform cross section. So now we will do the study.

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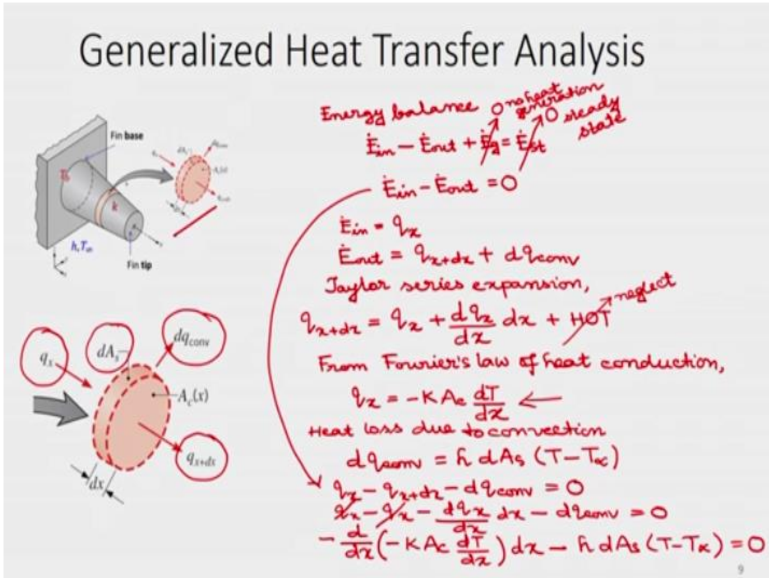

So let us consider the energy balance in this elemental area. So if you do the energy balance, what is the energy balance equation? That is your

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st}$$

So in this case, you can see $\dot{E}_{in}$ is energy coming in the elemental volume, $\dot{E}_{out}$ is the energy going out of the elemental volume. $\dot{E}_{gen}$ is the energy generation inside the volume and $\dot{E}_{st}$ is the rate of change of thermal energy with respect to time. But in this case we have assumed steady state, so this $\dot{E}_{st}$ is 0. Also you have considered that there is no heat generation. So this $\dot{E}_{gen}$ is 0, because there is no heat generation. So you can write

$$\dot{E}_{in} - \dot{E}_{out} = 0$$

Okay, so this is the energy balance equation. So what are the values of these quantities? So energy in is only q_x you can see. But energy going out through this conduction that is your $q_{x+\Delta x}$ and also you can see that your heat is going out due to convection. Because outside fluid temperature is T_∞ and also you have the heat transfer coefficient h. So through the periphery of the surface, your heat is lost due to convection. So you can write

$$\dot{E}_{in} = q_x$$

$$\dot{E}_{out} = q_{x+\Delta x} + dq_{conv}$$

So now we will do the Taylor series expansion of this $q_{x+\Delta x}$. So, it will be

$$q_{x+\Delta x} = q_x + \frac{dq_x}{dx}dx + HOT$$

We will neglect the higher order terms (HOT), only we will consider the first two terms. Here we will not write Δ; we will write d because we are considering one dimensional heat conduction. So from Fourier's law of heat conduction, you can write

$$q_x = -KA_c\frac{dT}{dx}$$

And what is your dq_{conv}, heat loss due to convection? It is

$$dq_{conv} = hdA_s(T - T_\infty)$$

Here, dA_s is the heat transfer area and through which your convection is taking place and temperature difference is $T - T_\infty$, where T_∞ is the surrounding fluid temperature. So now you got all these heat transfer.

Now you put it in the energy equation.

$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$\Rightarrow q_x - (q_{x+dx} + dq_{conv}) = 0$$

Putting the expression from Taylor's series

$$\Rightarrow q_x - \left(q_x + \frac{dq_x}{dx}dx + dq_{conv}\right) = 0$$

Putting the expressions for all the terms

$$\Rightarrow -\frac{d}{dx}\left(-KA_c\frac{dT}{dx}\right) - hdA_s(T - T_\infty) = 0$$

So let us further simplify it. So we have already assumed that thermal conductivity of the fin material is constant. So K is constant. So you can take it outside the derivative and you divide it. So in the next slide, let us do it.

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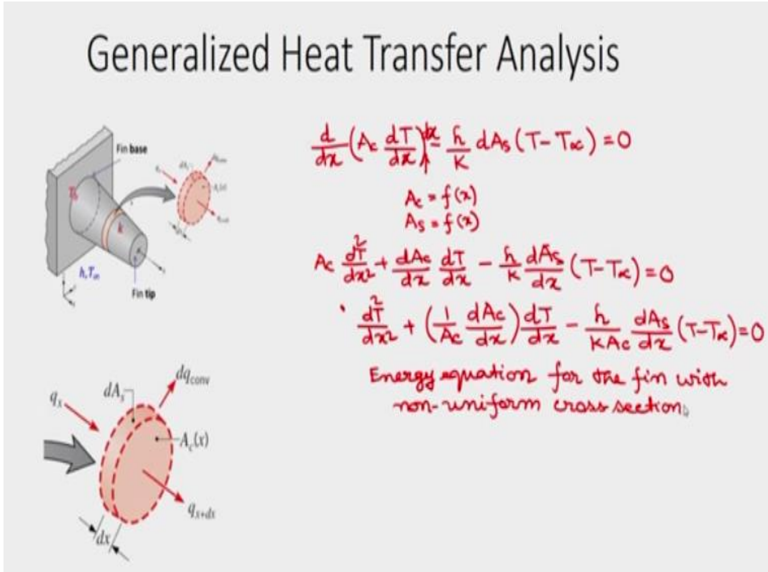

$$\Rightarrow \frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{K} dA_s (T - T_\infty) = 0$$

So this is the equation. Here

$$A_c = f(x)$$
$$A_s = f(x)$$

So if you take the derivative, so you can write

$$A_c \frac{d^2T}{dx^2} + \frac{dA_c}{dx}\frac{dT}{dx} - \frac{h}{K}\frac{dA_s}{dx}(T - T_\infty) = 0$$

So now let us divide both sides with A_c. So you can write

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \frac{h}{KA_c}\frac{dA_s}{dx}(T - T_\infty) = 0$$

So this is the generalized energy equation for the fin where you have non-uniform cross section and A_c, which is the cross sectional area, is function of x as well as your heat transfer area dA_s or A_s is function of x and we have assumed that K is thermal conductivity is constant. So this is the energy equation for the fin with non-uniform cross section.

So I think you have understood the derivation. Now we will make some special assumptions and we will simplify it further.

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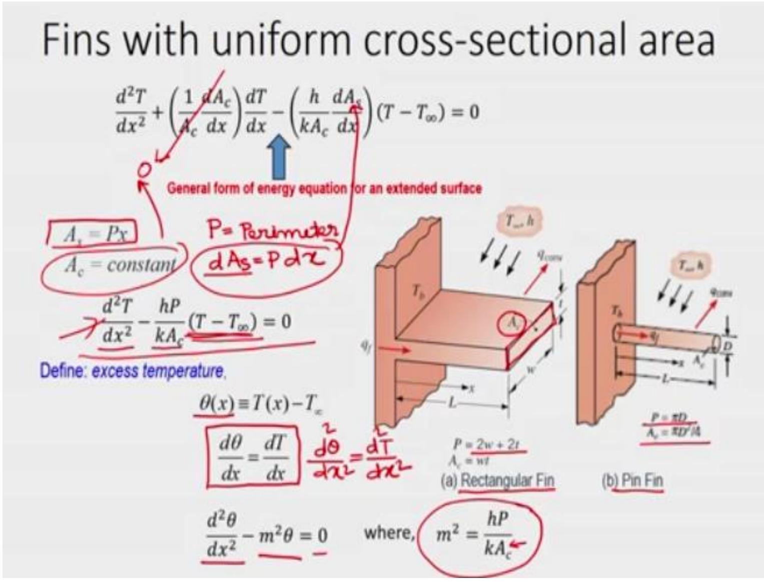

So this is the general form of energy equation, already we have written. So now consider uniform cross section fin. So in this case, we have considered a rectangular fin and pin fin. So in the rectangular fin, you can see this is having a uniform cross section. That means, A_c is not a function of x, it is constant in this case as well as your surface area A_s, we can write

$$A_s = Px$$

Where, P is the perimeter.

$$P = 2w + 2t$$

If you consider pin fin, in the pin fin also you can see that your perimeter will be

$$P = \pi d$$

And cross sectional area will be

$$A_c = \frac{\pi d^2}{4}$$

And as d is constant, in this case, you can consider that cross sectional area A_c is constant.

So let us use this A_c is constant and A_s = Px, then you can write

$$\frac{dA_s}{dx} = P$$

Putting this expression and as A_c is constant its derivative will be zero. Hence, the generalized equation reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{KA_c}(T - T_\infty) = 0$$

So this is the equation for fins with uniform cross-sectional area in a simplified form. Now we will define one excess temperature. So excess temperature is the temperature difference of the surface minus the surrounding fluid temperature. So if you write that, so we will define excess temperature as θ, which is

$$\theta = T(x) - T_\infty$$

So now if you take the derivative with respect to x,

$$\frac{d\theta}{dx} = \frac{dT}{dx}$$

And

$$\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2}$$

So if you substitute this now, in energy equation here, you can write

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

Where,

$$m^2 = \frac{hP}{KA_c}$$

Where, h is the heat transfer coefficient, P is the perimeter, which is constant in this case. K is the thermal conductivity and A_c is the cross sectional area, which is also constant. So you can see that this m^2, whatever we have considered that is constant. You can see that this equation is linear, it is homogeneous and it is second order differential equation. So what will be the solution of it?

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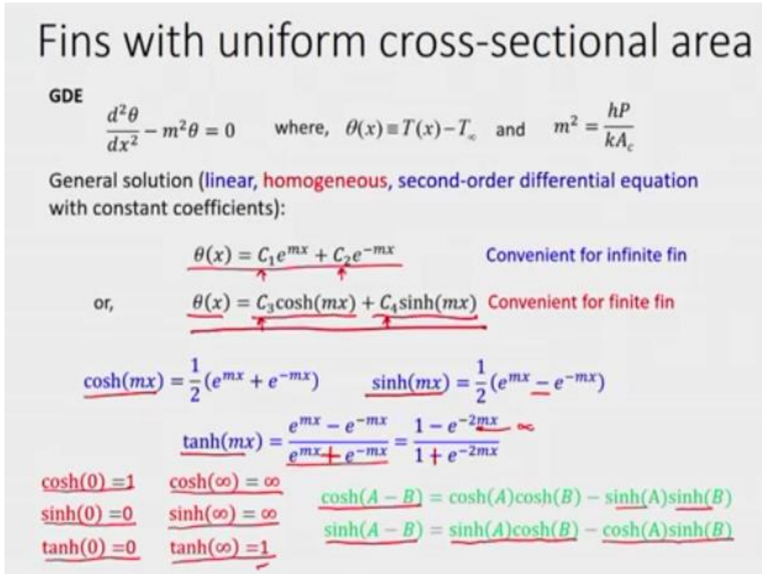

So now what will be the solution of it? So it is a constant coefficient because m is constant, so you can have

$$\theta(x) = C_1 \exp mx + C_2 \exp -mx$$

So exponential form we have written the solution where C1 and C2 are the constant, that we need to find applying the boundary condition. Later we will see those things and in another form, also you can write this solution of this second order differential equation in trigonometric form.

$$\theta(x) = C_3 \cosh mx + C_4 \sinh -mx$$

So here C3 and C4 are the constants we need to find applying the boundary condition, just we are writing C3, C4 because we have used C1 and C2 in the first equation. So these are constant anyway. So in the two forms we can write the solution, but you please understand very carefully which solution, where we will use.

The first solution what we have written in exponential form, it is convenient to use when you fin is infinitely long okay. So if you have infinite fin, then you can use this exponential form. In other solution also, if you find that one direction is infinite, then you try to write the solution of the differential equation in exponential form. Then you will get the solution very easily. Otherwise, it will be very complicated. And the other form we have written in hyperbolic function form. So in hyperbolic function form, generally it is convenient when you have a finite length fin okay. Please remember.

So one is exponential form, which is convenient to use in infinite fin or infinitely long fin and the other form in a hyperbolic function, we have written that is convenient to use in finite length fin okay. So when we have used this hyperbolic function, let us see what is the values of this functions and what is the relation between this hyperbolic function and the exponential function because it will be convenient when we will solve this energy equation for different problems.

So you can see here

$$\cosh mx = \frac{1}{2}(\exp mx + \exp -mx)$$

$$\sinh mx = \frac{1}{2}(\exp mx - \exp -mx)$$

$$\tanh mx = \frac{\exp mx - \exp -mx}{\exp mx + \exp -mx} = \frac{1 - e^{-2mx}}{1 + e^{-2mx}}$$

So now what is the value of this hyperbolic function

$$\cosh 0 = 1$$

$$\sinh 0 = 0$$

$$\tanh 0 = 0$$

$$\cosh \infty = \infty$$

$$\sinh \infty = \infty$$

$$\tanh \infty = 1$$

And

$$\cosh(A - B) = \cosh A \cosh B - \sinh A \sinh B$$

$$\sinh(A - B) = \sinh A \cosh B - \cosh A \sinh B$$

We will use all these when we will do the solution of these energy equation for different problems. So you please remember these things. Now we will go to the next slide.

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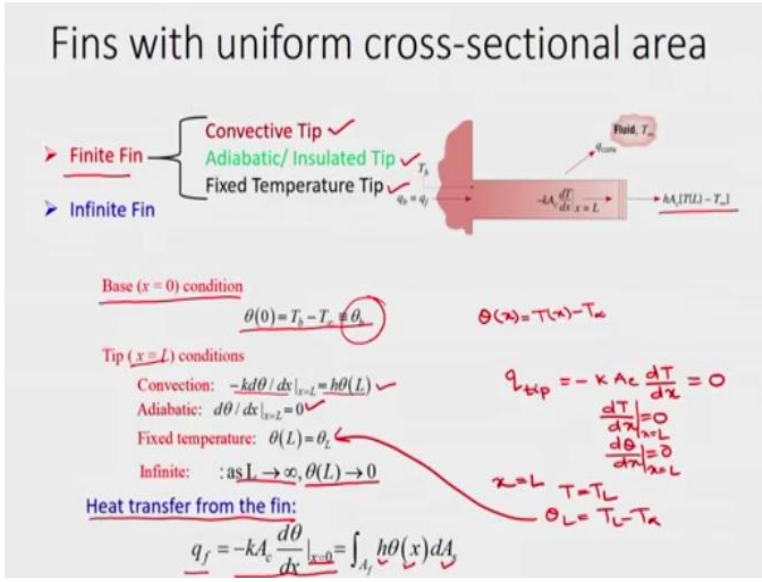

We will consider four different types of fins, one is infinite fin, where your length of the fin is very long, so that the temperature from the base it will take long time to go to the fin tip. So fin tip temperature will be almost same as the surrounding temperature. So if T_∞ is the surrounding temperature, then fin tips temperature will be T_∞. And in finite fin, we will consider three different types of fin.

One is convective tip where convection will take place from the fin tip and then we will consider adiabatic or insulated tip where we will consider that no heat loss from the fin tip. And then we will consider fixed temperature tip where the fin tip is attached to some other fluid, then your temperature may be constant at that case, so we will consider the fin tip temperature T_L is constant.

So you can see, now how many boundary condition do you need to solve these equations. So it is a second order differential equation, so we need two boundary conditions okay. So one boundary condition you know very easily, because you know the base temperature we had assumed that it is constant. So your T_b, which is the base temperature, it is constant. So

$$at\ x = 0, \theta(0) = T_b - T_\infty = \theta_b$$

Now at the other end, we have considered different types of fin, so depending on the different types, you can have different boundary conditions. So first you can see that in convection, so in convection in this case in the figure, you just look in, so whatever heat is conducted through this fin from the tip, it will be convected, right. So this is a simple energy balance. So

$$-K\frac{d\theta}{dx}\bigg|_{x=L} = h\theta|_L$$

So there we are neglecting that there is any heat loss due to radiation, so it will be just simply conduction equal to convection. So this is the energy balance at the fin tip.

Now if you consider adiabatic boundary condition, so in adiabatic boundary condition, we are telling that from the fin tip there will be no heat loss that means your heat transfer rate from the fil will be 0. So from that we can write

$$\frac{d\theta}{dx}\bigg|_{x=L} = 0$$

Another type of boundary condition we have considered that is fixed temperature. So at the fin tip we have specified the temperature as T_L. So that means

$$\theta|_L = T_L - T_\infty = \theta(L)$$

Now in the case of infinite fin, I told that your temperature at the fin tip will be equal to the surrounding temperature T_∞ so essentially it is

$$as\ L \to \infty;\ \theta(L) \to 0$$

So all these cases, now we will consider and we will find what is the temperature distribution inside the fin as well as what is the heat transfer rate for the fin okay. So we will derive that. How we will find the heat transfer from the fin?

So you can see, from the fin base whatever heat is conducted, that will be convected through the fin surface. Heat transfer area whatever you have considered from there your heat loss will take place. So this should be equal. So now we are writing

$$q_f = -KA_c\frac{d\theta}{dx}\bigg|_{x=0} = \int_{A_f} h\theta(x)dA_s$$

So that is the conduction heat transfer at x=0 which is equal to the convection heat transfer over the whole fin surface area and the temperature difference that is in this case θ(x) and dAs is the elemental heat transfer area.

So now if you integrate it, you will get the heat transfer from the fin. So either of this two form you can use, either heat conduction at the base or the heat transfer from the surrounding surface. So you should be careful when you are using convective tip, so one will be surrounding surface as well as from the fin tip there will be heat loss, so that you have to take care, okay. In case of adiabatic and in case of infinite fin, you will see that your heat transfer from that fin tip will be 0.

So you do not need to consider, but for the convective tip boundary condition, you need to consider. So you please keep in mind while finding the heat transfer rate for a fin okay.

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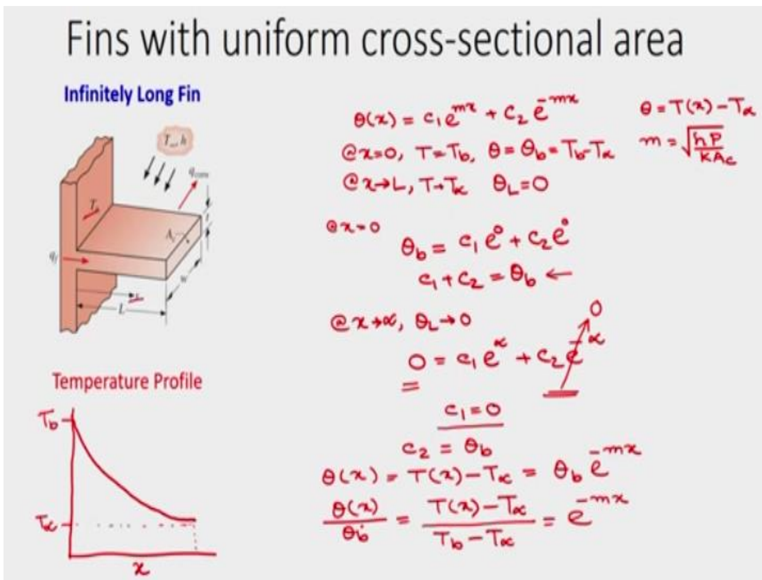

So next, now we will go to infinitely long fin. So now we are considering infinite fin where we need to find the temperature distribution inside the fin, and the heat transfer rate in this particular case. So you can see this we have considered, so here x we have measured from the fin base, T_b is the base temperature, surrounding temperature is T_∞ and heat transfer coefficient is h and A_c, where it is cross sectional area is constant.

So in this case, I have told from, what is your solution. So it is an infinite fin. So what solution you will consider? We will consider the exponential form of the solution, so already we have solved, right. So that is

$$\theta(x) = C_1 \exp mx + C_2 \exp -mx$$

Where

$$\theta = T(x) - T_\infty$$

$$m = \sqrt{\frac{hP}{KA_c}}$$

So we need two boundary conditions in this case. Those are

$$at\ x = 0; T = T_b\ or\ \theta = \theta_b$$

And another boundary condition we told that it is a long fin, so at the fin tip your temperature will be same of as surrounding temperature, so that will be T_∞.

$$at\ x \to \infty; T \to T_\infty\ or\ \theta_L = 0$$

So these are the two boundary conditions. Now you apply and find the constant C_1 and C_2 and find the temperature distribution. So first apply at x = 0, $\theta = \theta_b$, so you can write

$$\theta_b = C_1 \exp 0 + C_2 \exp -0$$

$$\Rightarrow \theta_b = C_1 + C_2$$

Now you put at $x = \infty$,

$$0 = C_1 \exp \infty + C_2 \exp -\infty$$

So now tell me, so left hand side is 0, right hand side $\exp -\infty$ is 0, because it is $1/\exp \infty$. So this term is 0, so second term in the right hand side is 0, left hand side is 0, so C_1 must be 0, otherwise $\exp \infty$ is infinite right, so C_1 must be 0. So what will be C_2 then from this equation? So C_2 is

$$C_2 = \theta_b$$

So it is very simple. So now you can write the temperature distribution

$$\theta(x) = T(x) - T_\infty = \theta_b \exp -mx$$

Or,

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx}$$

Now we have found the temperature distribution for the infinite fin. So you can see that temperature varies exponentially, right. It decays exponentially along the longitudinal direction x. Obviously your base temperature T_b is higher than the T_∞. So if you plot it, the temperature profile, it will decay exponentially from T_b to T_∞. So today I will stop here. So in the next class, we will find the heat transfer rate for this infinite fin.