Fundamentals of Conduction and Radiation

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Lecture - 10
Special 1-D Heat Conduction Situations – Part 2

Good morning everyone, welcome to the second lecture of our module 4, where we are talking about some special situations of the application of 1-D steady state heat conduction. While in the previous week, we have developed the 1-D steady-state version of the generalized heat diffusion equation and introduced the concept of thermal resistance.

In this week, we are talking about situations where the concept of thermal resistance truly speaking is not applicable and we have to go for the solution of corresponding ordinary differential equation. In the previous lecture, we have talked about the situation involving the heat generation in Cartesian and spherical geometry.

So using the corresponding differential equation, we have solved that using suitable boundary condition. In both the cases, we have taken boundary conditions; particularly for the Cartesian one we have taken the boundary condition to be temperature specified one that is the Dirichlet condition on both sides and when both sides are having the same temperature then we have got the concept of the plane of symmetry.

And using that concept of plane of symmetry I have analysed the cylindrical coordinate system. Now today we shall be seeing a few more cases or rather a different approach of solving the equation associated with the heat generation or conduction involving heat generation. In addition, couple of other special cases related to the steady state 1-D heat conduction will be introduced briefly.

So what we want to do first is to go for a dimensionless approach. The equations that we have solved in the previous lecture that is 1-D steady state heat conduction involving heat generation, the same kind of equations will be attempted to solve, but following a dimensionless approach. Before I proceed, I must answer that why we should going for a dimensionless approach.

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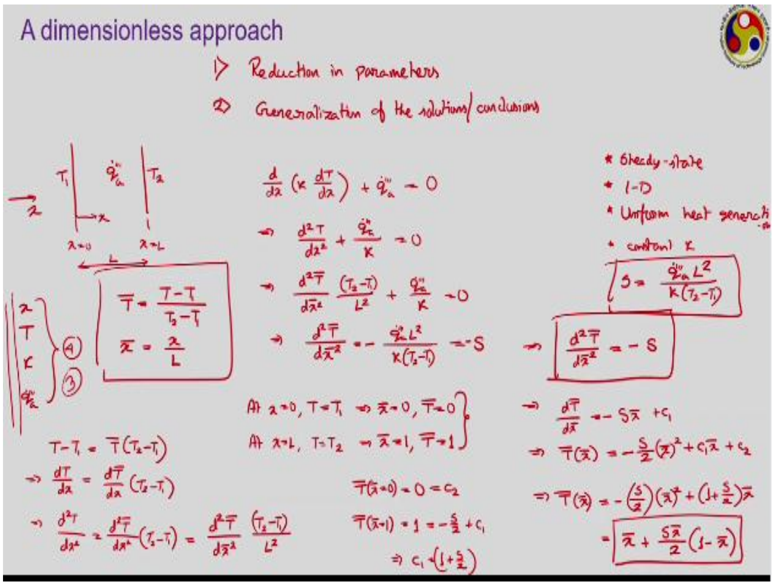

There may be quite a few different reasons that we can identify. Number one is the reduction in number of parameters. We can say from here that, in a standard diffusion equation, whatever number of parameters that we have; when we convert that equation with suitable non-dimensional numbers or dimensionless groups and finally get a corresponding dimensionless version of the same equation, then the number of parameters is expected to be much lower.

Those who have done the course on fluid mechanics may have got the idea or may have been introduced to the concept of Buckingham Pi theorem where again the same way you try to convert an equation to a dimensionless form and identify the corresponding important dimensionless number. We are not of course using the Buckingham Pi theorem, but idea is quite similar.

Another very important outcome of going for this dimensionless approach is the generalization of the solutions. Or we can say the conclusions like suppose you are trying to solve the heat conduction equation in 2 different systems, both are cylindrical geometries, but one is something like our domestic pipe which is having a diameter of say 15 mm and another is a microchannel which is having a diameter of 50 µm.

Now both are facing or both are being subjected to the flow of same kind of fluid and accordingly some kind of or rather let us forget about the fluid flow situation let us talk about the conduction heat transfer. So you are having 2 solid cylindrical bars one is having a diameter of 15 mm other is having a diameter of 50 µm.

Now we can easily solve the corresponding conservation equation or corresponding governing equation to get that solution or temperature profile; however, how can you compare these 2 geometries, you definitely cannot compare them because of such wide disparity in their dimensions, but if we allow ourselves to go for the dimensionless version of the same equations.

Instead of getting the temperature profile, if we get some non-dimensional version of the temperature profile in terms of some non-dimensional coordinate framework then whatever solution we are getting that should be equally applicable to both the systems. As long as the magnitude of the dimensionless groups remains the same in both the systems, their behaviour also should be very similar.

That is why the solutions can be generalized only when we go for a dimensionless approach. So this is the primary reason that we go for the non-dimensionalization of the equations. Therefore, let us try to see how we can develop a dimensionless version of the equation. Our geometry remains the same. We are going for the Cartesian coordinate system. This is our primary direction of heat transfer, that is x.

Let us say this is the origin of the coordinate system, so this is x = 0, this is x = L and this surface is maintained at some temperature T_1, this surface is maintained at temperature T_2 and in between we are having this much of volumetric energy generation which is uniform over the entire system.

Now what is the corresponding governing equation? We have already used yesterday that we know that the generalized equation can be written as

$$\frac{d}{dx}\left(K\frac{dT}{dx}\right) + \dot{q_G}''' = 0$$

So what are the conditions we are imposing here? We are imposing

*steady state

*1D heat transfer

*uniform heat generation

*constant properties (K)

Property means I am talking about the thermo physical properties; the thermal conductivity. As temperature is also a property just to avoid confusion let us say the constant K, we are talking about constant thermal conductivity. We now know how to solve this equation using this boundary condition.

But here our objective is not to go for a solution of this equation rather try to convert this equation to a dimensionless form. To convert this to a dimensionless form we have to identify what are the most important variables. You can see there are 4 variables. There is x, which is independent variable, there is T, which is the dependent variable and also 2 additional parameters of thermal conductivity and the rate of volumetric energy generation.

So there are 4 parameters that is involved into this. So we have to identify some non-dimensional group. Now should we go for all the 4 non dimensional group? Probably that is not required. Let us just try to investigate. Let us define $\bar{T}$ as some dimensionless temperature

$$\bar{T} = \frac{T - T_1}{T_2 - T_1}$$

Now as you can see the dimension of both the numerator and denominator is of temperature, so this is a dimensionless group. Similarly, we are defining $\bar{x}$ as the dimensionless length as

$$\bar{x} = \frac{x}{L}$$

Where L can be the length scale of a system like in this case L refers to this particular distance from one end to the other end. So these 2 are our dimensionless groups.

So let us try to modify the previous equation. We have taken K as constant, so we could have written this one as

$$\frac{d^2T}{dx^2} + \frac{\dot{q}_G{}'''}{K} = 0$$

Now,

$$T - T_1 = \bar{T}(T_2 - T_1)$$

Here T_1 and T_2 are constants because these are the boundary temperatures. So if you differentiate this equation with respect to x once then we have

$$\frac{dT}{dx} = \frac{d\bar{T}}{dx}(T_2 - T_1)$$

Differentiating once more

$$\frac{d^2T}{dx^2} = \frac{d^2\bar{T}}{dx^2}(T_2 - T_1)$$

And now we have to convert this x in terms of this $\bar{x}$. How we can do this? You know the same way we can get a relation between dx and $d\bar{x}$. So we can replace this x as $\bar{x}L$; accordingly it becomes

$$\frac{d^2T}{dx^2} = \frac{d^2\bar{T}}{d\bar{x}^2}\frac{(T_2 - T_1)}{L^2}$$

Now, take it back here

$$\frac{d^2\bar{T}}{d\bar{x}^2}\frac{(T_2 - T_1)}{L^2} + \frac{\dot{q_G}'''}{K} = 0$$

$$\frac{d^2\bar{T}}{d\bar{x}^2} = -\frac{\dot{q_G}'''L^2}{K(T_2 - T_1)} = -S$$

Let us call this parameter to be equal to S.

Because you can see this entire term is a constant as per the imposed condition, thermal conductivity and this volumetric energy generation rate they are constants, L is the dimension on the system. T_1, T_2 are the boundary temperature, so everything is constant. So S is also a constant. Therefore, you can easily find the solution for this. So our final dimensionless equation becomes

$$\frac{d^2\bar{T}}{d\bar{x}^2} = -S$$

This is the dimensionless form of the corresponding equation. What will be your boundary conditions to find a solution for this? Your boundary conditions are given as

$$at\ x = 0, T = T_1 \Rightarrow at\ \bar{x} = 0, \bar{T} = 0$$

$$at\ x = L, T = T_2 \Rightarrow at\ \bar{x} = 1, \bar{T} = 1$$

These are your modified boundary conditions. Now if you attempt to solve the newly developed dimensionless equation what we are going to get? By differentiating it once we get

$$\frac{d\bar{T}}{d\bar{x}} = -S\bar{x} + C_1$$

C_1 is the constant of integration. Integrating for second time, it becomes

$$\bar{T}(\bar{x}) = -\frac{S\bar{x}^2}{2} + C_1\bar{x} + C_2$$

Now you know how to find a solution. You can easily put the boundary conditions. So it will be

$$\bar{T}(\bar{x}=0)=0 \Rightarrow C_2=0$$

$$\bar{T}(\bar{x}=1)=1=-\frac{S}{2}+C_1 \Rightarrow C_1=\left(1+\frac{S}{2}\right)$$

Accordingly final dimensionless temperature profile becomes something like this

$$\bar{T}(\bar{x})=-\frac{S\bar{x}^2}{2}+\left(1+\frac{S}{2}\right)\bar{x}$$

$$=\bar{x}+\frac{S\bar{x}}{2}(1-\bar{x})$$

So this becomes the dimensionless temperature profile. Now look at this equation first, this equation that I am talking about. Initially we had 4 variables, 4 parameters to deal with as I have reported here, but in this equation how many parameters we have? We have 3. We have $\bar{T}$, we have $\bar{x}$. $\bar{T}$ is the dependent variable, $\bar{x}$ is the independent variable. And that K and $\dot{q_G}'''$, the 2 other parameters which are associated with the system definition they are now being clubbed into this parameter S along with the dimension of the system that is L and also the 2 boundary temperature T_1 and T_2, everything is incorporated within this S. So S is also very important parameter.

Change in any one of this, that is either the length of the system L, or the rate of volumetric energy generation $\dot{q_G}'''$ or any of the boundary temperatures T_1 and T_2, S is going to change. S has been defined as

$$S=\frac{\dot{q_G}'''L^2}{K(T_2-T_1)}$$

We can clearly see there is a reduction in at least one parameter. And secondly, as the solution is a dimensionless one so it is applicable to any kind of system of any dimension and having any boundary temperature values as long as it is having an uniform volumetric energy generation within this and the boundary temperatures are specified. You can easily find the value of the S using the system dimensions and the boundary temperatures and the value of $\dot{q_G}'''$ and you can impose this temperature profile.

Absolute magnitude of temperature of course will vary from one system to another but their temperature profile in dimensionless sense will be exactly the same. So this dimensionless approach is very useful particularly when this end temperatures T_1 and T_2 are known to us.

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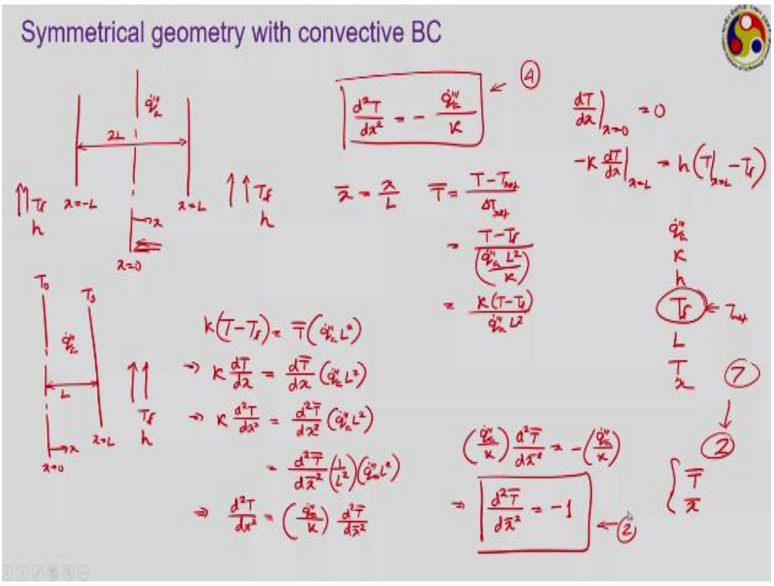

Let us see another situation of applying this boundary condition. Now our geometry, we are going to go for a symmetrical geometry in this case. Let us take Cartesian geometry only but a symmetrical one. That is we have a block, this is the centre line of the block, the coordinate starts from here. So this is identified at x = 0. Let us say the block is having a thickness of 2L.

Thickness of 2L such that one end is marked at x = L, other end is marked at x =- L. We can easily solve this if the boundary temperatures are known because that is what exactly we have done instead of just having the length scale as L we are having 2L. But here it is said that both sides of this particular block is being subjected to flow of a fluid stream.

Fluid is flowing with the temperature T_∞ on both sides and corresponding characteristic convective heat transfer coefficients are h, the same. So now we can see it is a very symmetric geometry and on both sides, you are having same boundary conditions and also the volumetric energy generation is uniform over this, plus constant properties, in that case, it is a symmetrical geometry.

Therefore, instead of going for the solution of this complete geometry, we can just stick to half of this, because then this particular plane will act like a plane of symmetry. So let us take half of this particularly domain for the further processing. So this is our geometry, this is x coordinate direction x = 0, x = L, this distance is L. We are having volumetric energy generation inside this and of course external fluid stream is flowing with temperature T_∞ and convective heat transfer coefficient is h.

Let us mark the centreline temperature to be T_0 but we actually do not know the value of T_0, and let us mark the surface temperature to be T_s, again this is something that also we do not know. T_0 and T_s are not known, we have to identify them by using the other information. The only thing that we know is only temperature is this T_∞.

The external fluid stream temperature is known, dimension of the system that is this L, volumetric energy generation rate, convective heat transfer coefficient h, thermal conductivity K, these are all known, but this internal temperature T_0 and T_s these are not known. You have to identify them using the dimensionless approach and the boundary condition that is given.

So we know that as we are assuming constant thermal conductivity then in this particular case again our governing equation will be

$$\frac{d^2T}{dx^2} + \frac{\dot{q_G}'''}{K} = 0$$

What are your boundary conditions? What can be your boundary condition in this case? We do not know any of this temperature T_0 or T_s, but we know that this dotted line where this T_0 prevails is a plane of symmetry.

We know there will be no heat transfer across this because the centre line is behaving like a mirror. So that whatever happens on one side can exactly be converted to its mirror image to get the idea on the other side, so rate of heat transfer across this will be 0, that means your 1 boundary condition is

$$\left.\frac{dT}{dx}\right|_{x=0} = 0$$

What is happening at x = L? at x = L, we are having a convective boundary condition. So whatever amount of energy that is receiving by conduction to this x = L surface is getting carried away by convection by the external stream. So we can write again the boundary condition of the third kind there. So you can say boundary condition of the second kind the Neumann boundary condition is provided at x = 0.

Whereas we are having at x = L, the boundary condition of the third kind. So it is

$$-K\frac{dT}{dx}\bigg|_{x=L} = h(T|_{x=L} - T_\infty)$$

So these are your boundary conditions in dimensional form. Now you have to convert this to a dimensionless form. So

$$\bar{x} = \frac{x}{L}$$

Problem is that T_0 and T_s none of them are known to us. And therefore, we cannot go for a definition like before; then what we have to do? See the standard form of non-dimensionalizing temperature is

$$\bar{T} = \frac{T - T_{ref}}{\Delta T_{ref}}$$

The ΔT_{ref} is some reference temperature difference. So in the numerator we need to have a reference temperature and in the denominator we need to have a reference temperature difference. Now your T_{ref} from the given information, if I list what are the information is given to us. Given informations are $\dot{q_G}'''$, thermal conductivity, heat transfer coefficient, external fluid temperature and length of this particular domain L.

Then maybe T_∞ is the only temperature that is known to us and we can use this one as our T_{ref}. Now what about ΔT_{ref}, of course ΔT_{ref} is not there we have just only one temperature like in the previous case we had both T_1 and T_2 known to us. So accordingly we can easily define a ΔT_{ref} as T_1 -T_2 or T_2 -T_1, but here we know only one temperature that is T_∞.

Then how can you define a temperature difference or a reference temperature difference, just check it out what can be the solution that you can identify? Remember you have to use the specified variables. If you look carefully if you form a group of this form

$$\frac{\dot{q_G}'''L^2}{K}$$

What is the dimension of this quantity? Just check it out. $\dot{q_G}'''$ is energy generation rate per unit volume. So it is W/m^3, L^2 is m^2, K is W/mK,. So we can see that the dimension leaves to be only K or rather it has a dimension of temperature and this is the one that we are going to use as the ΔT_{ref}. So the definition of $\bar{T}$ now becomes

$$\bar{T} = \frac{T - T_\infty}{\frac{\dot{q_G}'''L^2}{K}}$$

See here though we have only 1 temperature specified still we can define a suitable reference temperature difference by logical combination of the parameters. So now we have to dimensionalize this equation,

$$\bar{T} = \frac{T - T_\infty}{\frac{\dot{q_G}'''L^2}{K}} = \frac{K(T - T_\infty)}{\dot{q_G}'''L^2}$$

Or,

$$K(T - T_\infty) = \bar{T}\,(\dot{q_G}'''L^2)$$

Differentiating with respect to x,

$$\Rightarrow K\frac{dT}{dx} = \frac{d\bar{T}}{dx}(\dot{q_G}'''L^2)$$

Differentiating once more,

$$\Rightarrow K\frac{d^2T}{dx^2} = \frac{d^2\bar{T}}{dx^2}(\dot{q_G}'''L^2)$$

Remember here K and $\dot{q_G}'''$ and L all are constants. We have to now convert this x to $\bar{x}$. So to convert x to $\bar{x}$ we have to replace this one with $L\bar{x}$ and accordingly this becomes

$$\Rightarrow K\frac{d^2T}{dx^2} = \frac{d^2\bar{T}}{d\bar{x}^2}(\dot{q_G}''')$$

$$\Rightarrow \frac{d^2T}{dx^2} = \left(\frac{\dot{q_G}'''}{K}\right)\frac{d^2\bar{T}}{d\bar{x}^2}$$

So if we put this expression back into the original equation that is here. Now just see what we are getting. So we now can replace this second derivative as

$$\left(\frac{\dot{q_G}'''}{K}\right)\frac{d^2\bar{T}}{d\bar{x}^2} = -\frac{\dot{q_G}'''}{K}$$

That gives our final dimensionless equation as,

$$\frac{d^2\bar{T}}{d\bar{x}^2} = -1$$

So such a simple form of equation we have got. Like in the previous case when we had 2 boundary temperatures specified, we actually had 3 variables available or rather 3 parameters available in the final dimensionless equation.

So from 4, the number of parameters was reduced to 3. Look at this case; here we have more parameters involved. Like we have already made a list, see we have the parameters involved as temperature. We have x as the independent variable, then we have the length scale L and all this that I have listed which is 7 parameters. But now it has converted only to 2, may not be 2 actually because we are yet to treat the boundary condition, but as of now we have got just 2 variables that is $\bar{T}$ and $\bar{x}$. At least you can look at the governing equation. In the governing equation itself there were 4 variables and here you have only 2. So we have successfully reduced the total number of variables.

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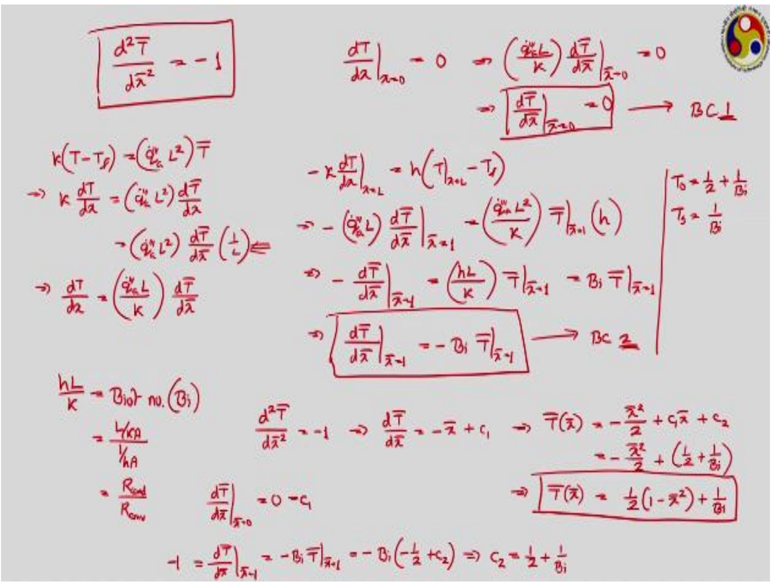

Now let us treat the boundary conditions. So now our equation is

$$\frac{d^2\bar{T}}{d\bar{x}^2} = -1$$

This is the converted form of the ordinary differential equation, the governing equation. Now our boundary condition was

$$\left.\frac{dT}{dx}\right|_{x=0} = 0$$

Now let me go back to the previous slide see here we had something a form of the dT/dx or we can differentiate this also. So from the temperature if I just check out the definition we already had

$$K(T - T_\infty) = \bar{T}\,(\dot{q_G}'''L^2)$$

This was from the definition of $\bar{T}$, and now if we differentiate it once,

$$\Rightarrow K\frac{dT}{dx} = \frac{d\bar{T}}{dx}(\dot{q_G}'''L^2)$$

Exactly this we have done earlier. Now if we replace this x in terms of $\bar{x}$ and then it becomes

$$\Rightarrow K\frac{dT}{dx} = \frac{d\bar{T}}{d\bar{x}}(\dot{q_G}'''L^2)\left(\frac{1}{L}\right)$$

Accordingly

$$\Rightarrow \frac{dT}{dx} = \left(\frac{\dot{q_G}'''L}{K}\right)\frac{d\bar{T}}{d\bar{x}}$$

Now,

$$\left.\frac{dT}{dx}\right|_{x=0} = 0 = \left(\frac{\dot{q_G}'''L}{K}\right)\frac{d\bar{T}}{d\bar{x}}$$

Now the term in the bracket that these are all constant $\dot{q_G}'''$ L and K, so you can neglect that, therefore your first boundary condition now becomes

$$\frac{d\bar{T}}{d\bar{x}} = 0$$

So this is the first boundary condition, boundary condition number 1. Now what was the second boundary condition, the convective boundary condition, that is,

$$-K\left.\frac{dT}{dx}\right|_{x=L} = h(T|_{x=L} - T_\infty)$$

So putting the expressions

$$-(\dot{q_G}'''L)\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = h\left(\frac{\dot{q_G}'''L^2}{K}\right)\bar{T}\Big|_{\bar{x}=1}$$

So simplifying

$$-\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = \left(\frac{hL}{K}\right)\bar{T}|_{\bar{x}=1}$$

Now the term in the bracket hL/k what is that, what is the dimension of this quantity? That has to be dimensionless and this dimensionless number is generally called as biot number. Bi is the symbol for this. What is the physical significance of this? If you just inspect that term hL/k, here if we just reorient the term as

$$Bi = \frac{hL}{K} = \frac{\left(\frac{L}{KA}\right)}{\left(\frac{1}{KA}\right)} = \frac{R_{cond}}{R_{conv}}$$

Now what is L/KA, when you are going for such 1D concept of the or rather 1D formulation using the thermal resistance concept, the numerator is the conduction resistance, the denominator is the convection resistance. So accordingly

$$-\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = \left(\frac{hL}{K}\right)\bar{T}|_{\bar{x}=1} = Bi\,\bar{T}|_{\bar{x}=1}$$

So finally, the second boundary condition now becomes

$$\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = -Bi\,\bar{T}|_{\bar{x}=1}$$

So this is the boundary condition number 2. So look at one important change. The total number of variables now as I have mentioned in the equation we had two variables $\bar{T}$ and $\bar{x}$ and after non-dimensionlising the boundary condition we have a new one that is involved that is the Biot number. So instead of 2 we are having actually 3 variables in this dimensionless formulation.

So we have reduced, successfully reduced the total number of parameters from 7 to 3 and we have converted the equation to a dimensionless form. Corresponding boundary conditions to dimensionless forms as well and we have a generalized situation when we are talking about a such a symmetrical geometry subjected to convective boundary condition and very easily you can solve this set of equation and you can get the solution for this.

Like if we are looking for the solution then what we are going to have? What solution you can expect from this? So if we just go back to the equation

$$\frac{d^2\bar{T}}{d\bar{x}^2} = -1$$

Integrating it

$$\frac{d\bar{T}}{d\bar{x}} = -\bar{x} + C_1$$

$$\bar{T}(x) = -\frac{\bar{x}^2}{2} + C_1\bar{x} + C_2$$

So if we put the boundary conditions, now the first boundary condition

$$\frac{d\bar{T}}{d\bar{x}} = 0$$

Which gives C_1 equal to 0. Second boundary condition

$$\left.\frac{d\bar{T}}{d\bar{x}}\right|_{\bar{x}=1} = -Bi\,\bar{T}|_{\bar{x}=1} = -Bi\left(-\frac{1}{2} + C_2\right)$$

$d\bar{T}$ /d$\bar{x}$ at $\bar{x}$ = 1 is -1, so now we have C_2 to be equal to

$$C_2 = \left(\frac{1}{2} + \frac{1}{Bi}\right)$$

So correspondingly the final solution is,

$$\bar{T}(x) = -\frac{\bar{x}^2}{2} + \frac{1}{2} + \frac{1}{Bi}$$

$$= \frac{1}{2}(1 - \bar{x}^2) + \frac{1}{Bi}$$

So this is the final temperature profile that we are getting for this case. Again as long as we are dealing with such a geometry we do not need to look for different solution for different configurations.

Once the geometries are similar that is a symmetric Cartesian geometry with convective boundary condition specified on both sides then we can just calculate the value of corresponding biot number and we can go for this solution or rather this final equation that we have developed to create the corresponding temperature profiles. And now putting $\bar{x}$ =0 you can get the value of T_0. Putting $\bar{x}$ = 1 you can get the value of T_s as well. So the values are,

$$T_0 = \frac{1}{2} + \frac{1}{Bi}$$

$$T_s = \frac{1}{Bi}$$

So we have the extreme temperatures also as outcome of the solution, but we could have also calculated the temperature from a different ways, following a different pattern. See here we have solved the dimensionless equation to get the corresponding temperature profile and then putting different values of $\bar{x}$ we are getting the corresponding values of temperature.

Like putting $\bar{x}$ = 1 we have just got the value of T_s, but there is something else also you could have tried. Just think about over the half of the geometry how much is the total amount of energy that got generated? What is the magnitude of this total energy that got generated?

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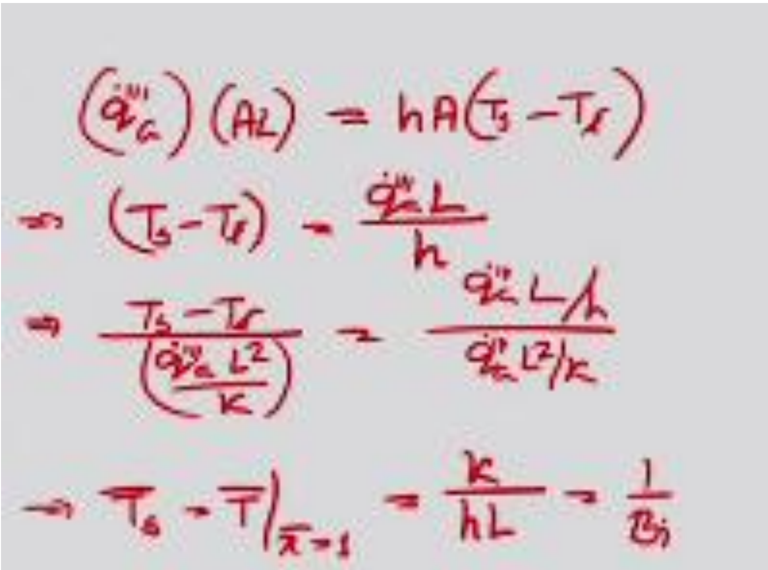

$\dot{q_G}'''$ is the amount of energy generation per unit volume and over half geometry how much is the volume, its cross section area multiplied by length L. Now in order to maintain the steady state exactly the same amount of energy must be convected away by the convective stream. So from the convective stream we know that this should be is equal to

$$\dot{q_G}'''(AL) = hA(T_s - T_\infty)$$

$$\Rightarrow (T_s - T_\infty) = \frac{\dot{q_G}'''L}{h}$$

Now just think about what are the dimensionless definition of temperature, if we divide this T_s -T_∞ by $\frac{\dot{q_G}'''L^2}{K}$, it becomes the dimensionless value. So dividing both side

$$\Rightarrow \frac{(T_s - T_\infty)}{\frac{\dot{q_G}'''L^2}{K}} = \frac{\frac{\dot{q_G}'''L}{h}}{\frac{\dot{q_G}'''L^2}{K}}$$

Then what is the left hand side? As per our definition this is

$$\bar{T}_s = \bar{T}|_{\bar{x}=1}$$

Hence simplifying the right side

$$\bar{T}_s = \bar{T}|_{\bar{x}=1} = \frac{K}{hL} = \frac{1}{Bi}$$

We got the same solution there also. So the surface temperature you could have calculated just by performing an energy balance as well. However, to know the value of T_0 of course we need to go for the final solution. So this is a situation of uniform energy generation.

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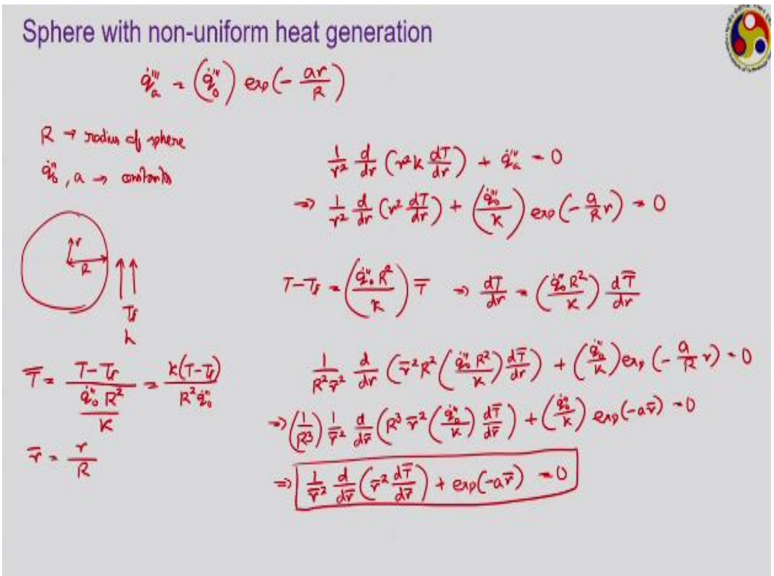

Let us quickly take care of one situation with non-uniform heat generation and for that, we are choosing a spherical geometry. We have not dealt spherical geometry in this week, that is why I have gone for this problem, the spherical geometry with non-uniform energy generation. Let us just assume one big sphere made of some nuclear material and because of the decay heat generation it is having energy generation within it, but in a non-uniform way.

So the entire energy generation is happening in the radial direction of the sphere and the energy generation can be represented as some

$$\dot{q_G}''' = \dot{q_0}'''exp\left(-\frac{ar}{R}\right)$$

$\dot{q_0}'''$is a constant number. Where R is the radius of the sphere that we are concerned about and this $\dot{q_0}'''$and a are constants; r of course is the radial coordinate direction.

So what will be your conservation equation or governing equation in this case? Just think about the spherical version of the generalized heat diffusion equation that you have developed earlier, here we are talking about the 1D steady state version of that. So we have

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2K\frac{dT}{dr}\right) + \dot{q_G}''' = 0$$

Let us assume K to be constant, so we can take this K out as well. In that case it becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{\dot{q_G}'''}{K} = 0$$

This is the governing equation here. This $\dot{q_G}'''$ is not constant rather that is a function of r. So to get that into picture let us represent the form of this $\dot{q_G}'''$, so we have

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{\dot{q_0}'''exp\left(-\frac{ar}{R}\right)}{K} = 0$$

Now we have to dimensionalize this equation. So we go for again our boundary condition. Let us say we are talking about a sphere. I am extremely poor in these drawings but this is the centre of this, this is the radius R, so this is our radial coordinate direction r and it is being subjected to convective boundary condition a stream of temperature T_∞ and convective heat transfer coefficient of h.

So the only known temperature here is T_∞ and therefore we have to go for some definition of that ΔT_{ref} using the $\dot{q_0}'''$. So use something very similar to what we have done in the previous case. So we define $\bar{T}$ as

$$\bar{T} = \frac{T - T_\infty}{\frac{\dot{q_0}'''R^2}{K}} = \frac{K(T - T_\infty)}{\dot{q_0}'''R^2}$$

And $\bar{r}$ has been non dimensionalized. This is quite straightforward we can easily define this

$$\bar{r} = \frac{r}{R}$$

So we have to convert this equation to its non-dimensional form. So how we can do this? We have

$$T - T_\infty = \left(\frac{\dot{q_0}'''R^2}{K}\right)\bar{T}$$

Accordingly

$$\frac{dT}{dr} = \left(\frac{\dot{q_0}'''R^2}{K}\right)\frac{d\bar{T}}{dr}$$

So if we are going back to the corresponding governing equation we now have r being replaced by $\bar{r}R$, so

$$\frac{1}{R^2\bar{r}^2}\frac{d}{dr}\left(\bar{r}^2R^2\left(\frac{\dot{q_0}'''R^2}{K}\right)\frac{d\bar{T}}{dr}\right) + \frac{\dot{q_0}'''}{K}exp\left(-\frac{ar}{R}\right) = 0$$

Now let's convert remaining r into the $\bar{r}$ notation. We have now putting $\bar{r} = \frac{r}{R}$

$$\frac{1}{R^3\bar{r}^2}\frac{d}{d\bar{r}}\left(\bar{r}^2R^3\left(\frac{\dot{q_0}'''}{K}\right)\frac{d\bar{T}}{d\bar{r}}\right) + \frac{\dot{q_0}'''}{K}exp(-a\bar{r}) = 0$$

$$\Rightarrow \frac{1}{\bar{r}^2}\frac{d}{d\bar{r}}\left(\bar{r}^2\frac{d\bar{T}}{d\bar{r}}\right) + exp(-a\bar{r}) = 0$$

So this is the converted form the equation where we are taking care of the spatial variation in the rate of heat generation as well. And what about the boundary condition in this case?

In this case your boundary conditions will be; at the centre line again there should not be any kind of the heat flux, or it has to be 0 to maintain a symmetry to have a finite value of the temperature.

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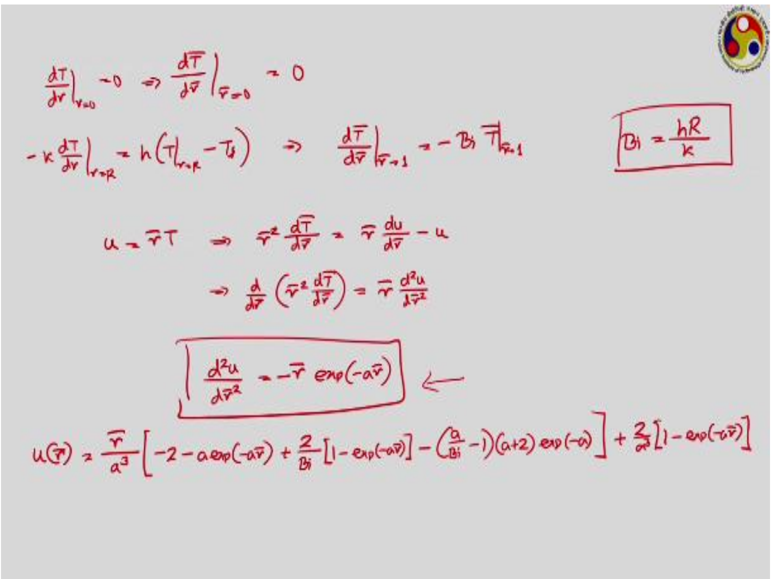

So you should have

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \Rightarrow \left.\frac{d\bar{T}}{d\bar{r}}\right|_{\bar{r}=0} = 0$$

Similarly,

$$-K\left.\frac{dT}{dr}\right|_{r=0} = h(T|_{r=R} - T_{\infty}) \Rightarrow \left.\frac{d\bar{T}}{d\bar{r}}\right|_{\bar{r}=1} = -Bi\,\bar{T}|_{\bar{r}=1}$$

Where the definition of biot number is the same as in the previous case,

$$Bi = \frac{hR}{K}$$

So we now have the governing equation and we have solved the 2 boundary conditions and from this now we can easily solve this, actually I am not going for the complete solution, but generally one transformation we do to get the solution done just seeing this equation you have this $\bar{r}$ and $\bar{T}$ both are involved into this.

Therefore, we generally define a new variable u as

$$u = \bar{r}T$$

Once you use this then in the governing equation we can change this term

$$\left(\bar{r}^2\frac{d\bar{T}}{d\bar{r}}\right) = \bar{r}\frac{du}{d\bar{r}} - u$$

And differentiating once more with respect to d/d$\bar{r}$

$$\frac{d}{d\bar{r}}\left(\bar{r}^2\frac{d\bar{T}}{d\bar{r}}\right) = \bar{r}\frac{d^2u}{d\bar{r}^2}$$

So your final equation now becomes

$$\frac{d^2u}{d\bar{r}^2} = -\bar{r}exp(-a\bar{r})$$

Even simpler form then what we had in the previous slide. Both the boundary condition can be converted in terms of u also. It is not required, but we can easily perform the integration to be the solution in terms of u; I am just going to give you the final form in terms of u that is a quite complicated form that why I am not solving it.

I am just going to give you the final solution which I have written

$$u(r) = \frac{\bar{r}}{a^3}\left[-2 - a\exp(-a\bar{r}) + \frac{2}{Bi}[1 - \exp(-\bar{a})] - \left(\frac{a}{Bi} - 1\right)(a+2)\exp(-a)\right] + \frac{2}{a^3}[1 - \exp(-a\bar{r})]$$

So a really complicated form of equation that we are getting from the solution because of this spatial variation in the volumetric energy generation. So this problem I have chosen just to give you an idea about what may happen or how to approach when the energy generation is not constant rather it is variable. So we have discussed the situation with 1D steady state heat conduction with volumetric energy generation mostly uniform volumetric energy generation, and also special case with non-uniform volumetric energy generation. I would like to close this lecture by discussing quickly 2 special cases. One case is conduction with variable thermal conductivity. So far we have always assumed that K to be constant, but if K is a variable then what will happen.

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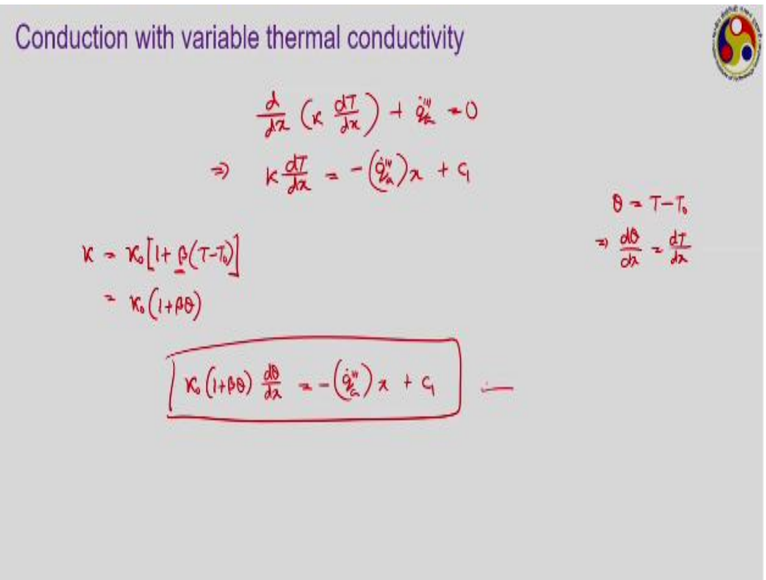

Just sticking to the Cartesian coordinate system let us say we have our governing equation as

$$\frac{d}{dx}\left(K\frac{dT}{dx}\right) + \dot{q_G}''' = 0$$

Let us assume that the volumetric energy generation is constant here in this case. Then by integrating it once, we have

$$K\frac{dT}{dx} = -(\dot{q_G}''')x + C_1$$

Now the problem is that K may have some spatial dependency.

For most of the common heat transfer situations that we deal with K may be assumed to be constant, but if you are talking about a very large temperature change over the system that you are dealing with the K can also vary. K generally follows a linear profile of temperature, but if you go to higher temperature then it can follow quadratic or even cubic behaviour as well.

Just to demonstrate one simple case let us assume K to have a form something like

$$K = K_0[1 + \beta(T - T_0)]$$

Where T_0 is some reference temperature, K_0 is a value of thermal conductivity at the temperature and this β is referred as the thermal coefficient of thermal conductivity. Just to remove that T - T_0 part sometimes we will represent this one as

$$K = K_0[1 + \beta\theta]$$

Where θ is nothing but T -T_0, accordingly

$$\frac{d\theta}{dx} = \frac{dT}{dx}$$

So putting this into situation we now can write the equation as

$$K_0[1 + \beta\theta]\frac{dT}{dx} = -(\dot{q_G}''')x + C_1$$

Now the situation is definitely more complicated than we had in a previous case.

But this still can be solved once the mathematical variation of K with temperature is known we can go for mathematical solution, but the equations are definitely more complicated like we can see in this case. Another special case of 1D conduction is where we have to go for a conjugate approach and they take the convection into picture with much further detail.

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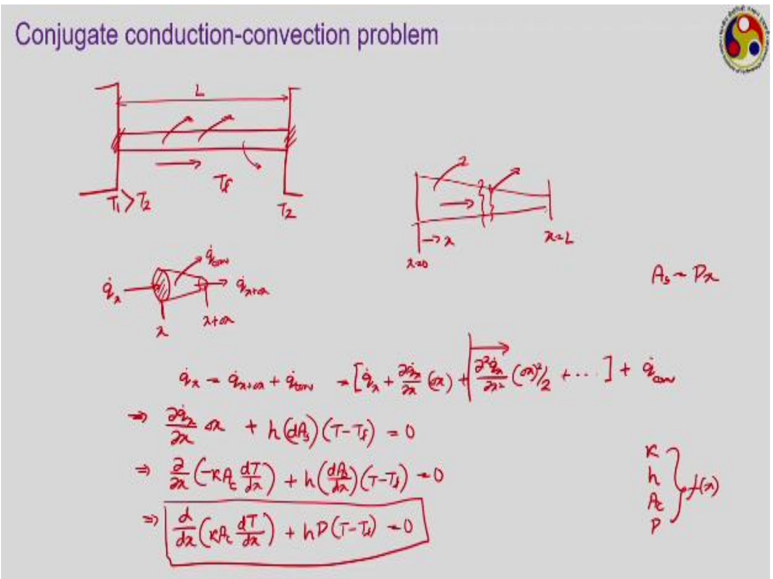

Let us take the situation where we have 2 plates, 2 surfaces rather. This surface is maintained at temperature T_1, this one is maintained at temperature T_2 and these 2 surfaces are connected by a long solid bar or rod like this. So this bar is being subjected to temperature T_1 at this end, temperature T_2 at this end. Let us assume T_1 greater than T_2, so there will be a conduction heat transfer in this direction.

If we assume the length of the bar is L, so this L is too large compared to the dimension in the other 2 directions. Accordingly, conduction in this direction from T_1 to T_2 from the left surface to the right surface will be the prevalent mode of heat transfer, hence we can virtually treat this one as a 1D conduction problem.

But if this bar is open to the surrounding which is having a temperature of T_∞ and if this T_∞ is significantly different compared to this T_1 and T_2 then this bar is also going to lose energy via convection to the surrounding. This convection can also be quite significant if you are talking about large difference between T_1, T_2 and this T_∞ and in that case though we are still talking about a 1D conduction situation we have to go for a conjugate heat transfer approach.

Let us take a general situation. Let us take the bar is also having a variable cross section area as we are moving in this direction the cross section area is also changing. These are x direction, this is x = 0, this is x = L. The conduction is happening in this direction, but convection losses are also happening.

Now as conduction and convection both are present and it is not 1D, we cannot go for the thermal resistance concept. Uniform cross-section area or variable cross-section area that does not matter either, and in fact we also cannot go blindly for the generalized heat diffusion equation. Rather it is more suggested to develop the equation for this situation on it is own starting from the first principle.

Let us take an infinitesimally small segment of this rod. This position is x, this position is x+Δx. This segment is receiving $\dot{q}_x$ amount of energy by conduction and $\dot{q}_{x+\Delta x}$ amount of energy is leaving, and $\dot{q}_{conv}$ is amount of energy leaving via convection.

Then if we write an energy balance then

$$\dot{q}_x = \dot{q}_{x+\Delta x} + \dot{q}_{conv}$$

Now if Δx is sufficiently small we can assume this entire block to be more or less at the uniform temperature and assuming Δx to be sufficiently small we can expand that first term using Taylor series

$$\dot{q}_x = \dot{q}_{x+\Delta x} + \dot{q}_{conv} = \left[\dot{q}_x + \frac{\partial \dot{q}_x}{\partial x}(\Delta x) + \frac{\partial^2 \dot{q}_x}{\partial x^2}\frac{(\Delta x)^2}{2} + \cdots\right] + \dot{q}_{conv}$$

So we can cancel out $\dot{q}_x$ from both side, and neglecting tasks from second order onwards assuming Δx to be sufficiently small, we are just left with

$$\Rightarrow \frac{\partial \dot{q}_x}{\partial x}(\Delta x) + \dot{q}_{conv} = \frac{\partial \dot{q}_x}{\partial x}(\Delta x) + h(dA_s)(T - T_\infty) = 0$$

Convection of course will be equal to the heat transfer coefficient h multiplied by $dA_{s.}$ which refers to the surface area through which the convection is taking place.

Remember conduction is taking place or rather conduction heat transfer is associated with this cross section area, which is also varying in this direction, whereas convection is happening from the surface from the peripheral area. Now conduction heat transfer can be represented using the Fourier law of heat conduction.

So you can write this one as

$$\frac{\partial}{\partial x}\left(-KA_c\frac{dT}{dx}\right) + h\left(\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

A_c is the variable crossection area and as Δx is sufficiently small, we can write dA_s/ Δx to be dA_s/dx.

Now how we can relate this A_s, the surface here to x? If it is a uniform geometry very easy to calculate this, if it is not an uniform geometry then not so easy to calculate, but still A_s can probably be written as

$$A_s = Pdx$$

Where P refers to the perimeter, of course this perimeter is also changing, as we are moving in this. So this dA_s/dx can be replaced with the perimeter. Now instead of using the partial derivative notation actually as it is a 1D steady state situation, so we can go for the ordinary derivative notation

$$\frac{d}{dx}\left(-KA_c\frac{dT}{dx}\right) + hP(T - T_\infty) = 0$$

Remember here we have not assumed anything, this K, thermal conductivity h, convective transfer coefficient cross section area A_c and perimeter P. They all can be function of x, they all can vary or they can remain constant giving a special situation. However, this is the equation that we get for such conjugate conduction-convection situation. More application of this one will be coming in the next week when we shall be going for the discussion of extended surfaces or fins, which we will start from an equation of this particular form.

So this is where I would like to finish on module 4 where we have discussed about some special situation of 1D steady state heat conduction.

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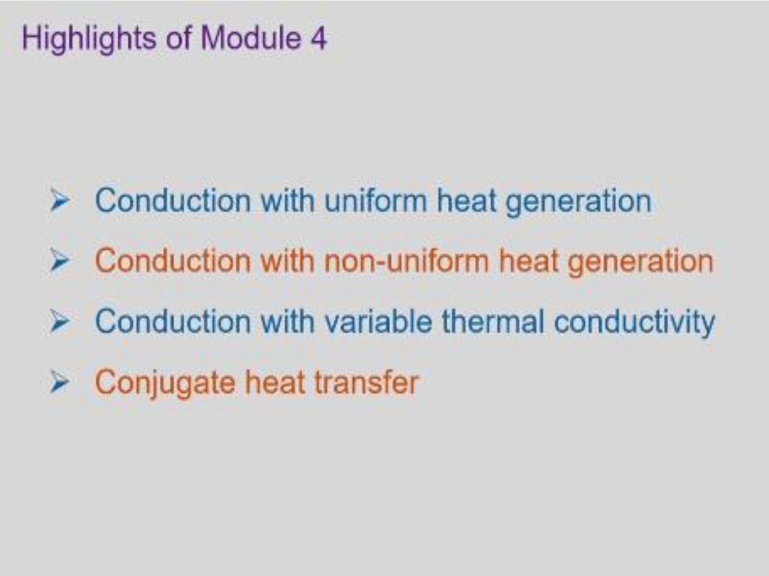

We discussed in detail about conduction with uniform heat generation in Cartesian and cylindrical geometries and also we have discussed today a dimensionless approach to solve similar problem particularly when we are having either dirichlet or convective boundary conditions or maybe Neumann boundary condition, then we discussed one case of conduction with non-uniform heat generation.

Then very quickly, the situation of conduction variable thermal conductivity and conjugate conduction convection problem has also been discussed. So I hope you have been able to understand what we are trying to communicate here, if you have any query please write back to me because I shall be very happy to respond to you.

That is where the module 4 is finishing and also that is where my role in this course is finishing for the moment. I am going to take a break from the next week onwards professor Amaresh Dalal will be coming in and he will be taking your lecture for the next 4 weeks covering the remaining part of conduction heat transfer. I shall be back again in week number 9 with radiation heat transfer. Till then see you all, take care, bye bye.