Theory of Rectangular Plates-Part 1
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Lecture – 09
Levy Solution

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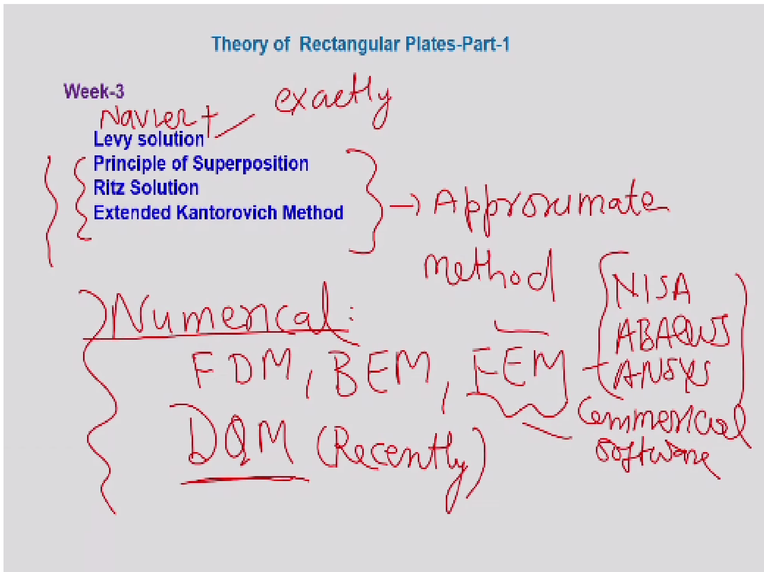

So welcome to our week 3 lectures. So in this week I am going to cover some part of the Levy solution, Principle of Superposition, Ritz solution, and the basic information that how to use the Extended Kantorovich method. Basically, all three methods are known as Approximate methods whereas sometimes we called as an exact boundary conditions are satisfying exactly, Levy solution or you can say that as well as the Navier solution both.

But in most of the time or in real life plate is not subjected to only single supported boundary condition, it may be subjected to clamed, free or a combined all edges. So for that how to find out an analytical solution. I am putting a stress here that for analytical the approximate methods are preferred but out of principle superposition, Ritz solution or extended Kantorovich method.

In these days some new techniques also discovered or developed like a synthetic method, simplistic approach, or some other techniques which are also use to develop the approximate solutions when plate is subjected to some arbitrary boundary condition or loadings. Then another is we have some numerical solutions. So there are number of numerical techniques like FDM – Finite Difference Method, BEA- Boundary Element Method then FEM, Finite Element Method, then recently DQM- Differential Quadrature Method very recently developed.

So out of it Finite element or which that some commercial software are there, like ABAQUS, ANSYS or NISA or some other software they are basically technique is called Finite Element method. So there are some advantages, disadvantages. So in our; in this course I am going to cover only the Approximate techniques. But definitely if somebody is interested in developing a numerical solution, I am may provide all hint and idea that how to develop specifically FEM solution for a particular problem.

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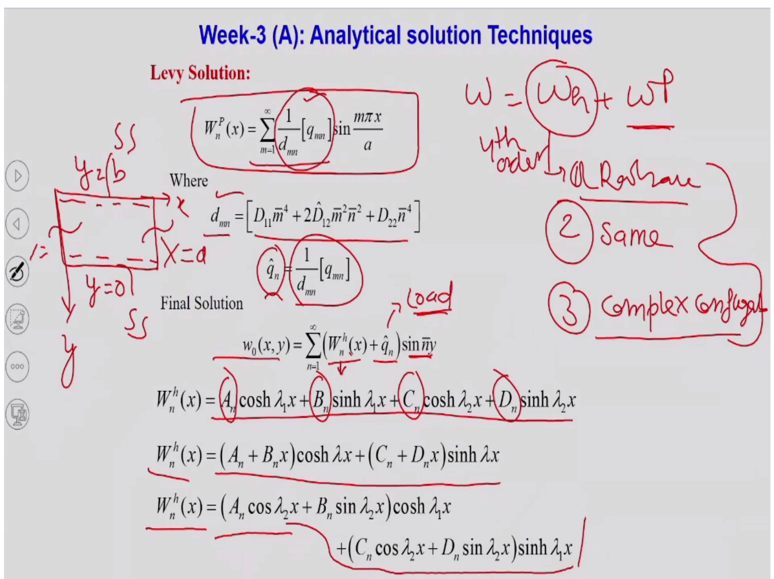

Now in the Levy Solution in the last week we have developed the particular solution particular part of that. In actual this solution can be divided into two parts, homogenous solution and particular solution. So based on we have obtained three special cases homogenous solutions we say that this is a 4th order equation for route, so we will say first case route are real and distinct and second case same, and third case complex conjugate.

So we have developed that what is the kind of the homogenous function for; if the our routes are real or distinct if routes are same or linear combination of that or if routes are complex conjugate. Then, the particular solution is also dried, so for that case particular solution can be written as like, where dmn is nothing but this. Again, we can say that this term qn hat in terms of basically a loading kind of thing, load vector.

So I can write a final solution W0 Wn of h which is corresponding to the homogenous function and qn is basically load and n hat y direction we have chosen our geometry box that this was x; this was y; and y=0 and y=b simply supported. Generally single supported symbol is by dash in some of the book or just putting some SS. So x=0; x=a can have any support condition. So now you see the final solution, if I talk about let us say first case.

My routes are real and distinct for that case the homogenous solution will look like this. If my routes are same or a linear combination, then homogenous solution will look like this. And if my routes are complex then my solution homogenous part will look like this. Now you see here An, Bn, Cn and Dn, these are the arbitrary constant, we have to determine these constant then only a solution will be complete.

Just writing this those Wn and Wh is not going to solve the purpose. So we have to identify An, Bn and Cn.

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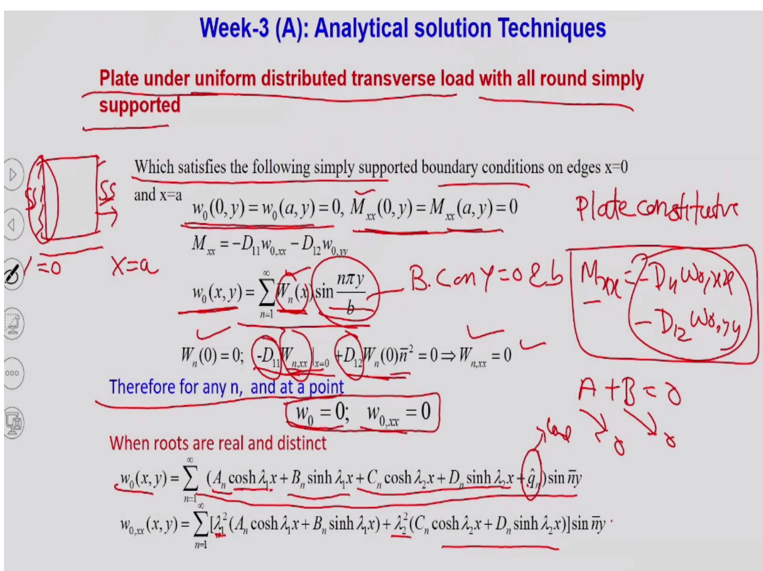

Let us say we are going to solve a plate under uniform distributed transverse load with all round simply supported case. We have already in Navier solution for all around simply supported plate.

Similarly, we can develop a Levy Solution in which we can say that our xx is also along x=a x=0 is also simply supported. It maybe turns, free, but I am taking a case where this is simply supported.

So the reason is behind that I am going to tell you that this solution when you are developing a solution for a Levy type and Navier what is the difference. First, identify the boundary conditions whether at x=0 and x=a, what are the variables need to specify. So if my plate is simply supported I am going to say that deflection has to be 0 at x=0 and x=a. Then, the normal movement on SS has to be 0.

If I another case if say let us say this is my clamp this is my free so add this as deflection W as well as U and V will also be 0 but we are talking about only a one case so deflection as well as the rotation will be 0. Now you see that this written in a general form W0 and Mxx what is your function W0, W0 is Wn m of x sin n pie y/b. So basically this is satisfying the boundary condition on y=0 and b. And this Wn need to satisfy the conditions along xx.

So when I am saying W0 is 0 it means Wm of x has to be 0. When I am saying no one need to be 0, so what is the definition of the movement if you use plate relations, plate constitutive, so my Mx is nothing but – of D11 W, xx -of D12 W, yy this is my movement. Since Mx is not our primary variable, so boundary conditions in that form it has to be 0. So I have written –DW1.

Now you say W0 is capital W mn sin so double derivative of that; similarly, double derivative of y that has to be 0. So familiarly you see that any two terms let us say a+b=0. So one condition will be that a independently 0 and b independently 0 or maybe sometimes it maybe a combination of kind of things. So you say that from this equation it implies that this needs to be because D11 is constant, D12 is a constant, this cannot be 0. So Wn, xx has to be 0.

So the, now over boundary conditions W0 has to be 0 and W0, xx has to be 0, W0 we know we express like this, so whenever roots are real and distinct so W0 can be written as An cosh hyperbolic lambda hat x + Bn sinh hyperbolic x and so on + particular part loading. This is my solution. So what will be the my double derivative of Wm, xx? So you will differentiate twice so lambda 1 square, similarly it will common out here from here lambda 2 square.

And qn is constant if you are saying that it is a uniformly distributed loaded it is not a function of x, so going to vanish to W, xx is 0. Now what is the next step?

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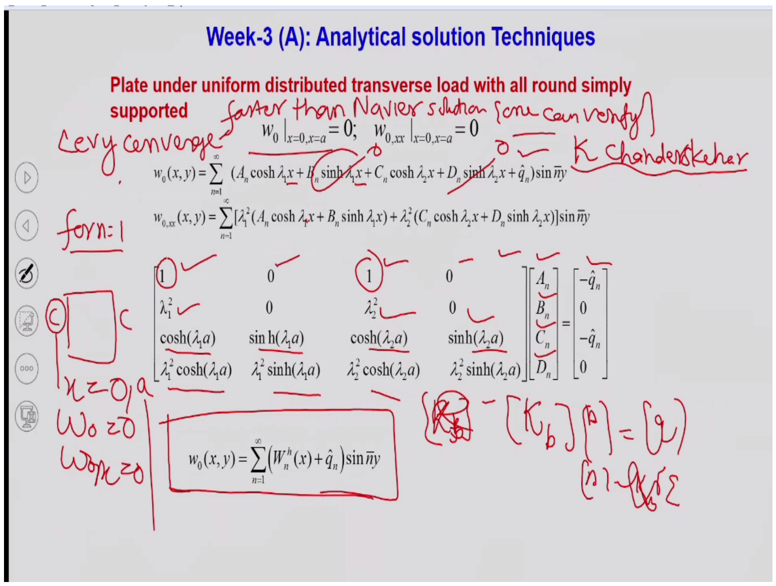

Substitute x=0. So when you are going to deflection put x=0, so sinh hyperbolic going to be 0 here again so only cos 0, cos hyperbolic 0 will be 1, so An and Cn contribution, others will be 0. Similarly, the movement or W, xx when x is 0 so contribution from here. Then, W0 at x= put x=a here so it reduces to like this lambda a, lambda 2, lambda b and constant. Similarly, from W, double derivative x so you can put it lambda 1 square, 2 square.

So you see it is reduced to a beautiful four ball four matrix where this is a constant matrix, An, Bn, Cn are the constants and this is the load vectors. So if you know that if you write a program then you come to know that these all the numbers or if you invert this matrix let us say my matrix I would like to say K final, Kb or I would like to say that K boundary and this An, Bn, Dn = q.

So A matrix will be, K b of inverse or the; Because for the present case even you can find in terms of a symbolic form you put A, B, C, D. And in mathematically you just make the inverse of that so some of the terms can be written down. So in this way you are able to solve An, Bn, Cn, Dn. So for each value of m, you will have a sort of equation so you are applying to for loop n=1; A0, A1, B1, C1, D1. When n=2, A2, B2, C2, D3 because this is a series solution, so we need number of terms

Then we add it together and check whether our solution is converged or not. So some cases we can write directly. Now, our solution is known. If you change the boundary condition instead of a simply supported you are putting some clamp here and clamp here so you are going to say at x=0 my W0 is 0 and W0, x is 0. Similarly, x=a. So for that you will have some different kind of constant and you can evaluate.

So in this way you have learnt that how to apply a or how to develop Levy solution for a plate. For every boundary condition one can develop so one need to know this matrix.

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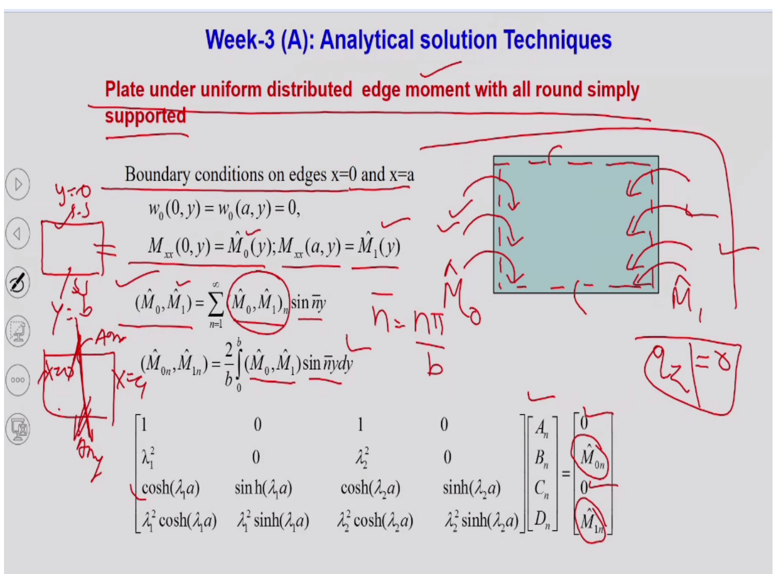

And then next which is I would like to say is a very important or a specific problem which is covered in every book of theory of plates, whether it is a book of Prof. Bhaskar or Prof. Chandrasekhar or Prof. J. N. Reddy you take any book you will find the standard problem is solved. Before going to there I would like to come back that Levy solution converge faster than Navier solution, one can verify, anytime and it is beautifully given in a K. Chandrasekhar book.

That how many terms that you require 10 terms; when you go for Navier solution you may require 21 terms or more than that. So I am talking about the plate under uniformly distributed edge moment with all around simply supported. So instead of a qz now plate is having this kind of moments. Qz is 0 but it is applied some moment is here. Now can you solve that? You see that these two opposites are simply supported. Other two can have any boundary conditions.

So Levitate approach we can go for that. So only difference where it comes, so you setup a boundary condition of x=0 and x=a. Deflection is 0 but you have a applied moment at x=0 let us say M hat 0. When we say that, apply iteration we denote by a hat or some distinct that the resisting moment. Then x=a; M 1 hat. So these are function of y. So like qz you have x in a single sinh series for a Levy case; similarly, you will express M0 hat and M 1 hat like this.

Where M bar is nothing but M pie/b. Then what are these? M 0 hat and M 1n like qn same 2/b 0 2 b and integration over that. So when you are saying this boundary condition is stayed of putting moment 0 we are putting the applied value and qz is 0. Now anybody can solve this problem and find out this An, Bn, Cn and Dn? So this solution is very important solution. It helps us basically to develop or applying a principle of superposition where we need some clamp; where we need

the clamp the moment will built up. So for that kind of solution we require such analysis. Further, I would like to say that, that Levy solution in the present context I have taken y=0 and y=b simply supported. Similarly, one can close x=0 and x=a simply supported, they may have any boundary condition y=0. So you can switch over. It is not that only always that y=0 has to be simply supported y=b has to be simply supported.

It may be other two as x0 or xa. The main condition in that to opposite as it, either this or this need to simply supported. So that reverse of that will also be useful for developing a solution for arbitrary support condition. I think with this basic background now you can develop a solution or write your own code in a MATLAB or Mathematica and find out the deflections and moments for a plate.

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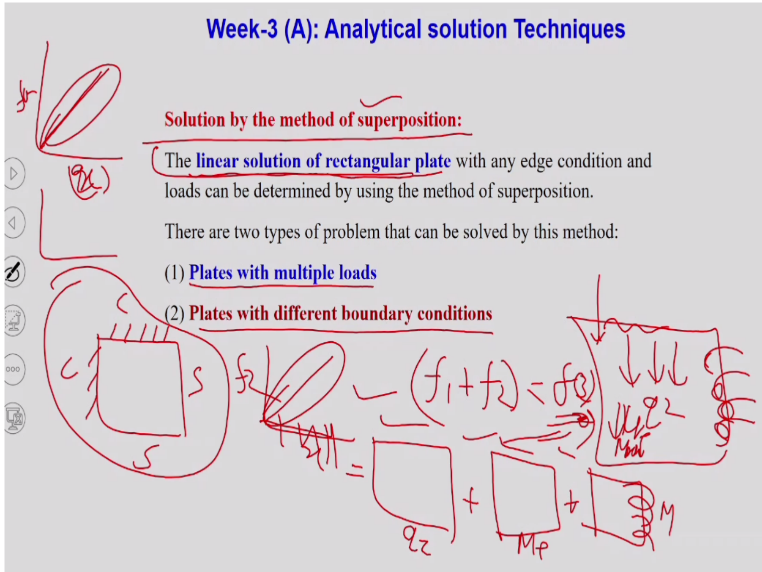

Now I am talking about a method of superposition. This is a very, I would like to say that powerful method. Let us say, we are interested to find out all round or some say like this Clamp, clamp and S, S. Can we solve with the help of Navier? No. Can we solve with the help of Levy? No. Definitely we can solve it with the help Principle of Superposition. So you have to know that where we apply a principle of superposition.

The very first start in a linear solution of rectangular plate, whenever your deformation is small and governed by the linear equation then only you can use the principle of superposition. The concept behind that, let us say n function under some x is going and some other function let us say f2 or on the same x is going some linearly, so what will be the result of F1+F2. So if we add those things over that reason proportionality at our every point it gives a the same result.

Let us say any function F3, you call it F3. So we can solve Plates with multiple loads. It is very, very important. Let us say our plate is having mechanical loading qz, already thermal loading I have explained some temperature also there Mx t + maybe some edge moments or maybe some concentrated node not udl or maybe some different kind of loading. So we will do that same geometry, let us say under only qz + same geometry under mt + same geometry under some edge moment. If we add it together it will give you, so this plate analysis.

So how can you the like—like I have done like in structure engineering, in civil engineering suppose a building is subjected to the wind load, dead load, live load and some other kind of loading then how do you analysis, so we say that, let us say building is under only live load, building is under only dead load or under only wind load, and then you add the solutions, for whatever the solutions you will get you can add at simply.

If you are talking about a plate, if you are; but reason is that, that should follow that deformation should not be large, means it should not be under the non-linear range, it should be under the linear range then only you can add the solutions. Next, Plates with different boundary conditions as I have told you that a plate; can we divide such that using Navier and Levy solutions we will get the solution of a plate this kind.

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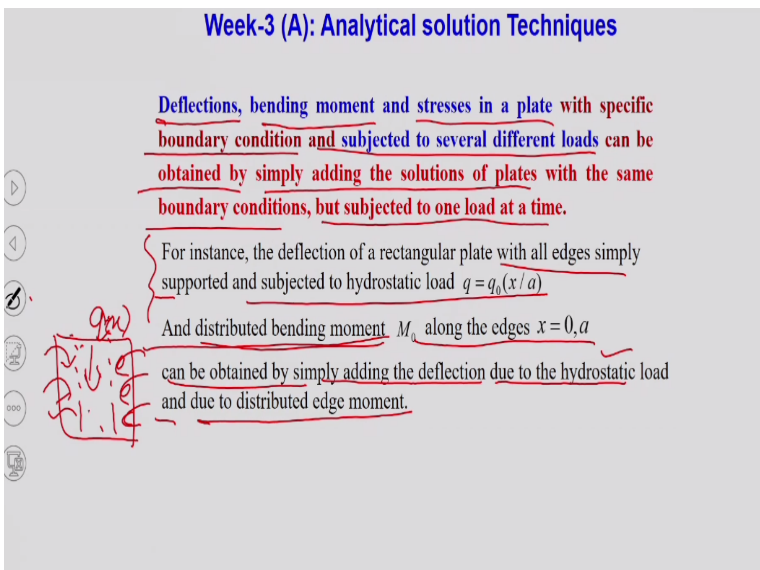

So I am going to explain that deflections, bending moments and stresses in a plate with the specific boundary condition subjected to different kind of loading can be obtained by simply adding the solution of plates with the same boundary conditions, but subjected to one kind of load at a time. Like that I have said that you just apply qz then you applied some hydrostatic pressure or later on wind load then thermal load whatever.

In the next, any distributed moment further for the example is that a plate is subjected to that having all edges simply supported and subjected to hydrostatic load as well as distributed bending moments along the edges. So let us say my plate is such all that all simply supported but x=0 it may have some bending moment + some hydrostatic load which is a function of x. So I can analysis that I just apply under qx then under M0 which I have just recently explained in the slide.

So can be obtained simply adding the deflection due to the hydrostatic load plus due to the distributed edge moment. So got the deflection due to qz got the deflection due to the edge moment and add it together at any point whatever you want.

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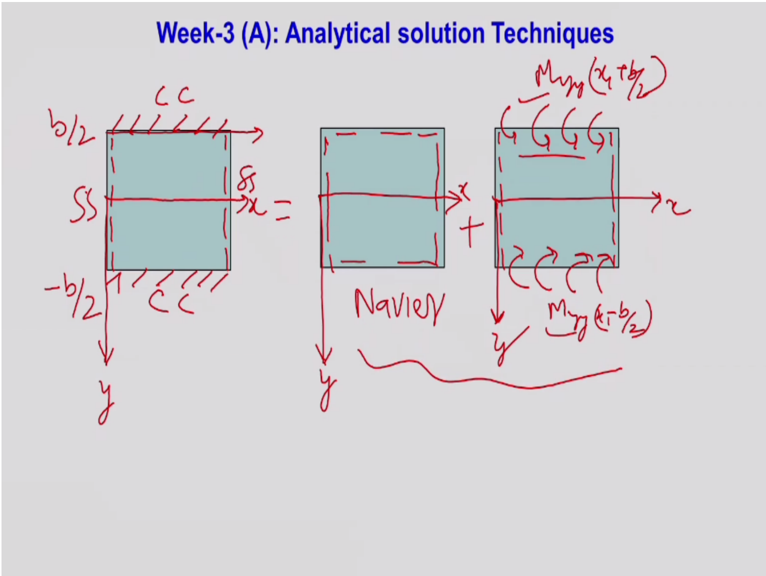

So your plate like clamp here, clamp here so how do you solve that let us say divided into two part, so this is basically a Navier solution, all round simply supported and this is the plate all round simply supported but having some moment, built-up moment Mx and My. So we can divide it into two part and we can solve it.

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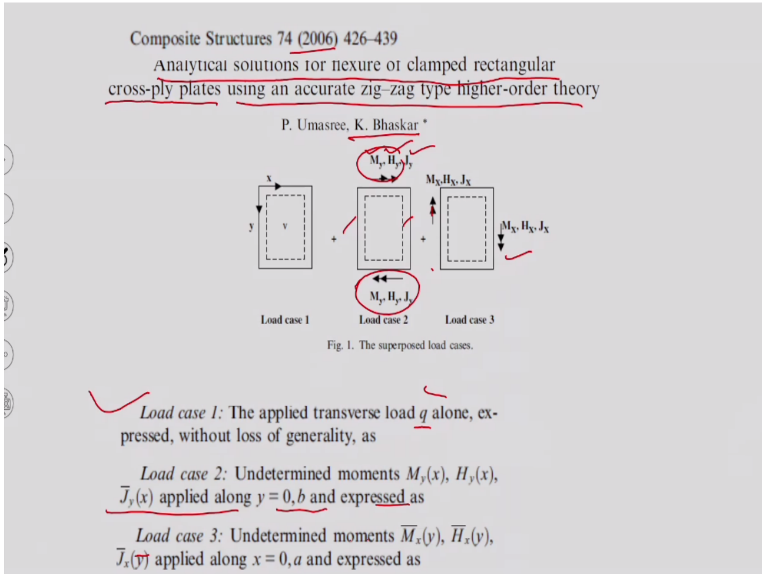

So I would like to refer a general paper in 2006 Prof. Bhaskar with Umasree presented a analytical solutions for a flexure of clamped rectangular cross-ply plates using an accurate zig-zag type higher order theory. You see that a plate all round clamped as a resolved by two Levy solutions basically, it takes simply supported and clamp so there will be an applied moment, similarly you will take moment on this edges.

So this is the superposed load condition where they have three load cases, one applied transverse load then second undetermined moment My, Hy because they have use the higher order theory, so instead of only My they may have some higher order moment Hy and Jy along y=0, b and along x=0 and a. So one can go through this paper, so this is way a; I would like to say that beautiful example of a superposition method.

So it is a very beautiful research article you can go through and understand the basics behind that.

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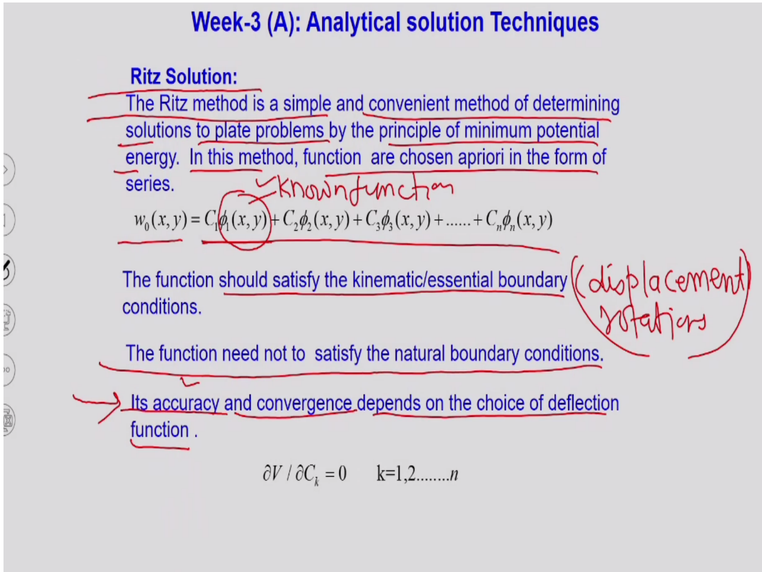

Now comes to the Ritz solution. I would like to say that Ritz solution is one of the very famous technique use to develop the solutions. So Ritz method is very simple, any technique which is easy to apply definitely the undergraduate students, post graduate students or the programming of that will be easy. So a convenient method of determining solutions to the plate problems. So basically this technique uses the weak form of the problem, like in the form of minimum potential energy form.

In this method, functions are chosen apriori in the form of series. So we choose a series like that and where these are the known function which satisfy the kinematic or essential boundary conditions, kinetic or essential boundary conditions means displacements and rotations. This function need not to satisfy the natural boundary conditions. Suppose you have a three edge you have a boundary condition in terms of fours.

So you need not to satisfy that function, you will satisfy that fours boundary condition, it will satisfy only displacement based boundary conditions. I would like to say that major limitation or disadvantage is that its accuracy and convergence depends on the choice of deflection function. For simple cases, one can find out a function which satisfy the boundary conditions.

For a difficult one, suppose you do not know the exact function which is going to satisfy the at least displacement boundary condition, you are not able to make that function, then you will not get the converge relation or accuracy will be very weak, so how to proceed for such kind of solutions?

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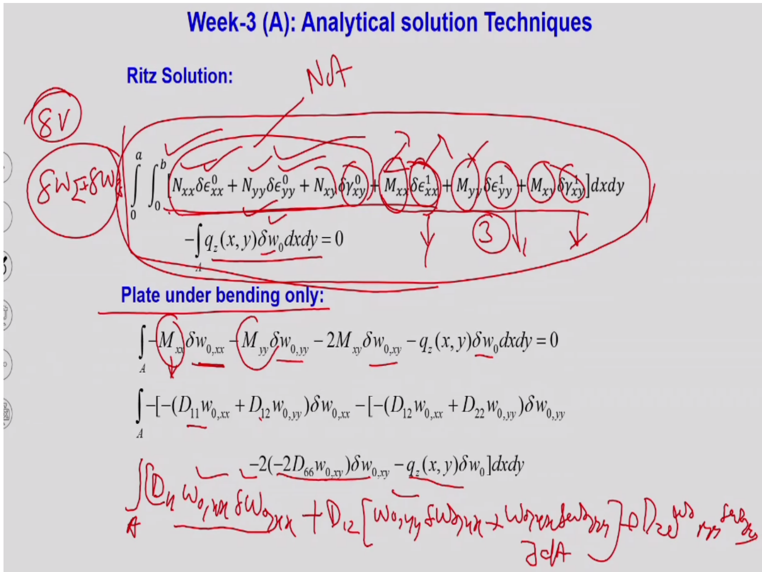

That you must know the weak form of the equation. So most of the time in the books you will find, write out any one potential on g and some other kind of things, but recently we have developed Del Wi, Del Wd what are these, basically variation in the potential energy. So we can use those directly this equation, no need to cramping like what will be the my weak form if I want to apply a Ritz solution or a Galerkin or a Finite Element solution.

Basically, in all these solutions we are going to use this form. Previously, for Navier and Levy solution I have use the strong form of the solution, ordinary differential equation but since then you are applying for a Ritz solution or a finite element you have to come up, up to here using the same whether suppose you are interested to develop a Ritz solution for a functional graded plate, for a Piezoelectric plate, or some different kind of a material, then at least you have to come up here. Know that strains these things.

Then, in the next step instead of developing a governing equation partial differential form we are going to use this and using the plate constitutive relations and displacement forms directly substituting those things. So if generally in the some of the books it is just solve for a plate under bending, so only then plate under bending all 3 terms will contribute; these are the stretching terms not contribute.

But if you are interested to develop a Ritz solution for a complete whether stretching as well as bending then you have to consider this, and this is the loading one, you see. So first of all, substitute the value of del epsilon 1 xx; del epsilon yy; gamma epsilon y. What is that Del W, xx; Del W, yy; Del W, xy and this Del W0. The next step substitute the value of D11 Mx – D11 of W, xx + D12 of W, yy. Similarly, M xy, D12 W, xx + D22 W, xy and M xy twice of D66 W, xy and this thing.

Then, you can write -, - become + so basically D11 W, xx Del W, xx + D12 W, yy Del W, xx + W, xx Del W, yy + D22, W, yy Del W, yy and this contribution and this contribution. So this is the form which we required when we are interested to apply Ritz solution. So sometimes students get confused that whether this how to come up with solution. You just go to our internal work done and external work done principle, develop these things and substitute this M x, M y in term using the plate constitutive relations.

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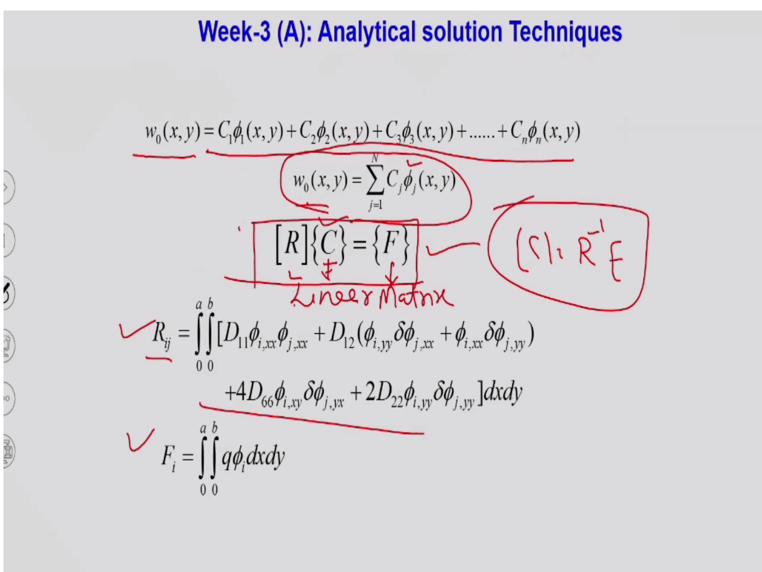

So next step is that you assume a solution W0 in this form and or you can write n a convenient form like this. If you substitute this W0 and took these equations this leads to give an algebraic equation where your R can be determined like this, ij is a function of xy and C are the unknown constants even C2, C3, and F is the load vector. So here you are solving an algebraic equation. So C can be i inverse of f or sometimes using some Gaussian elimination techniques or some techniques one can find out.

Sometime finding out the inverse maybe difficult or it may be iterated, so for that point of view one can use another that linear algebra or linear I would like to say that matrix technique solving.

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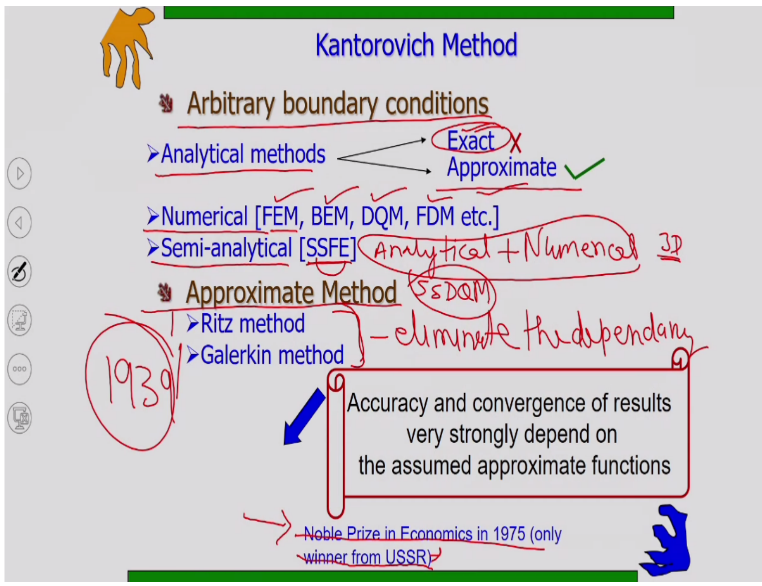

Now I am talking about an old method but not widely taught, or it is not really given in most of the books. But it is very, very elegant method or strong method. So I am going to tell you slight history that, arbitrary support conditions. If you are interested in developing analytical solutions there maybe some Exact or Approximate. But as you see that for arbitrary support conditions even the Levy have some limitations.

Only two has to be supported other two has to be any boundary condition. So we go for approximate techniques. Then there will be some Numerical technique already I have discussed Finite Element, Boundary Element, Differential Quadrature and Finite Differential etcetera. Then Semi-analytical techniques are also developed recently. What are these, basically these are the combination of Analytical + Numerical.

Basically, when we are talking about a 3-dimension solution 3D solution This-- the partial differential equation meaning x, y and z. So for that case you solve vernally alone z direction analytically and x and y using the finite element, so that technique is known as SSFE – State-Space Finite Element or sometimes SSDQM. So recently some of the researchers are using these techniques. So now come about the Approximate method. So there may be a Ritz or Galerkin method.

So in I would like to say in 1939 so basically the name of Krylov Kantorovich developed a solution to eliminate the dependency. But since that solution was in Russia, Russian language, so 1958 that book was published in English. So later on that it was known to the, there is a solution which is better than Ritz or Galerkin method, so I am going to explain and that Krylov Kantorovich got a Noble Prize in Economics in 1975 using the linear programming in an economic field. Basically, linear Kantorovich.

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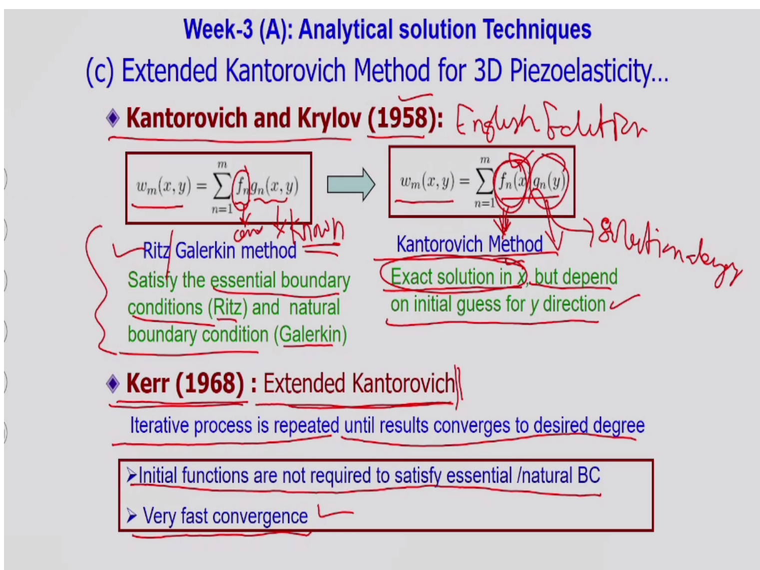

So Kantorovich and Krylov in 1958 this is an English edition of the book, you see that a general function any deflection function, if you use a Ritz or Galerkin method a function is assumed like that Fn is constant, and Gn is a function of x and y, but this function is known function, priori known, percept satisfy the essential boundary conditions for the Ritz case and natural boundary conditions for the Galerkin case.

I would like to say to eliminate this dependent because there solution accuracy depends on the initial choice. So Kantorovich propose in that book they proposed a method, let us say, you divide a deflection into a two bi-varied function, fn is the function of x only and gn is the function of y only. Then, you will get an exact solution along x direction so in that we have use the one iteration. Let us assume a solution along y like in Levy solution, solution along y, okay.

And then for x. So this accuracy is a high the reason is that because they are solving an ordinary differential equation, they use a weak form and finally will get to ordinary differential equation, so we are going to solve that so it is more accurate than the Ritz and Galerkin solution. So in this way we get an exact solution along x direction but it depends upon the initial guess for the y direction. So right now you can say that like Levy type you are assuming a simply supported some 1x opposite 2x assuming in a Fourier series and solving exactly along x direction.

So that in 1968 Kerr, extended this technique, so after that this method is known as Extended Kantorovich method. And in it one you solve this, now you say you know this, I am going to solve g y in iterative fashion. First you assume a g y, solve for fn then you take this fn as a known function and solve gn, so in this way iterative process is repeated until the converges is desired.

So what are the; I would like to say two basic things that initial functions need not to required any boundary conditions. Even in Kerr in original paper in 1968 he has spoke with the examples. If you take just a constant or any function it will satisfy just in 2-3 iterations boundary conditions exactly and solutions are more accurate and very fast convergence. So after 1968 this method has been applied to study a various kind of plates, shelf, under different boundary conditions and loading.

I would like to say that static, dynamic, buckling, transient, and recently we have further; I would like to say that extent to the 3-dimensional case in our recently paper if you see that.

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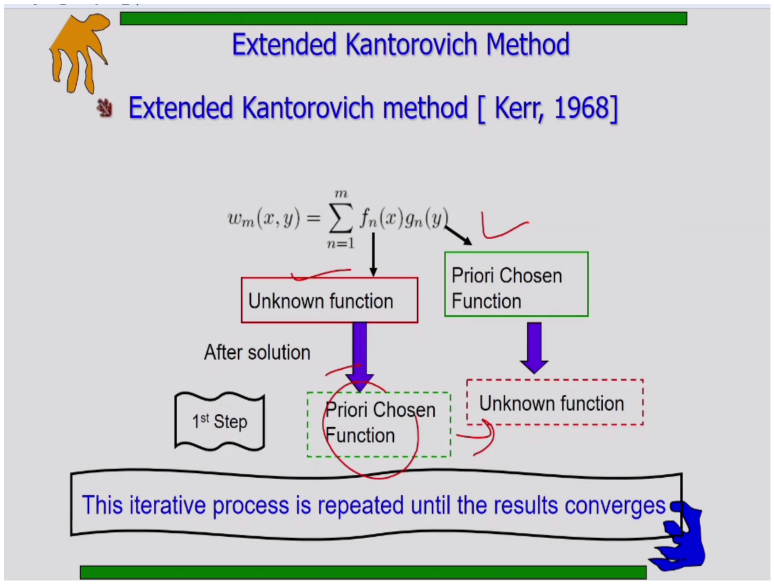

So this I have explained, you choose Priori and then Unknown and then you take it and you solve it.

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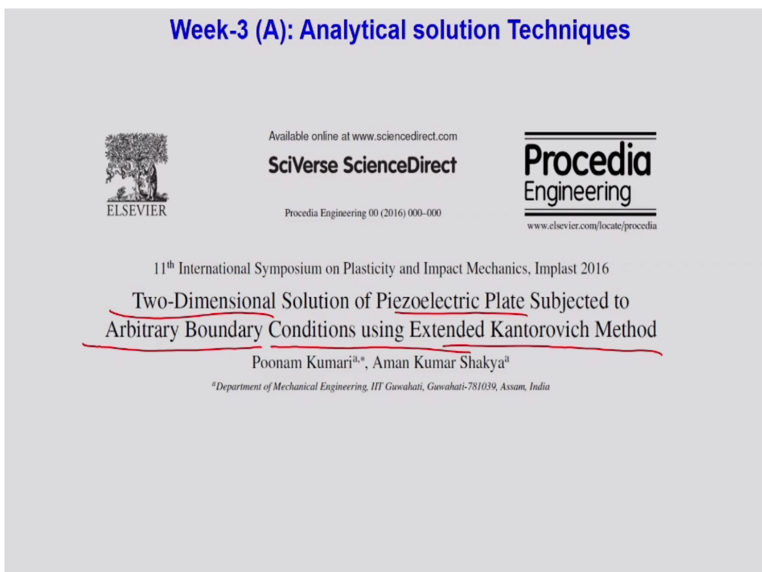

So in this paper, in our paper we have solved, applied this EKM and solved a Two-Dimensional problem, Piezoelectric Plate using Extended Kantorovich method. So in this, if somebody is interested they can go for that or you just extend it Kantorovich method around 200 papers are there, some recent review article is also there; one can go through. This is the very good technique one can apply to analysis a problem with arbitrary support conditions.

And it need to be further, I would like to say assist or develop. So let us say for different kind of geometry or different kind of loading, so that it we come to know that whether this technique gives for each and every case accurate and fast converge solutions.