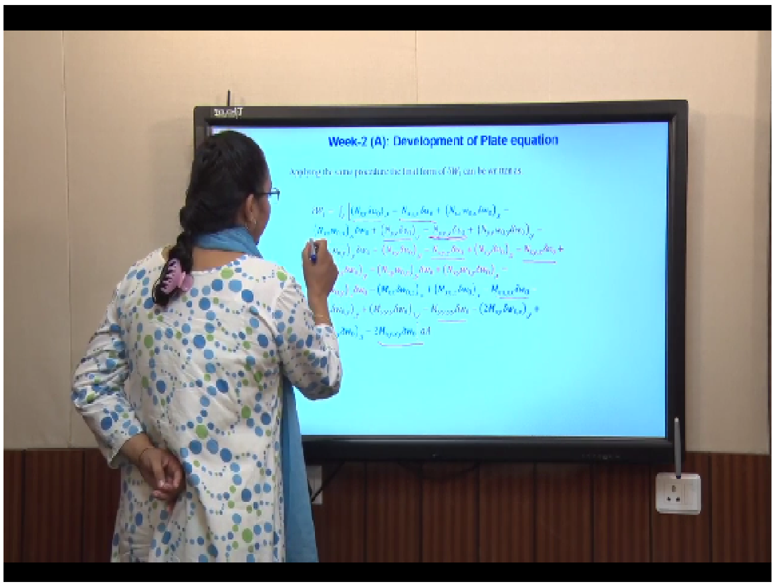
Theory of Rectangular Plates-Part 1

Dr. Poonam Kumari
Department of Mechanical Engineering
Indian Institute of Technology - Guwahati

Lecture – 06
Governing Equation for Plate - 2

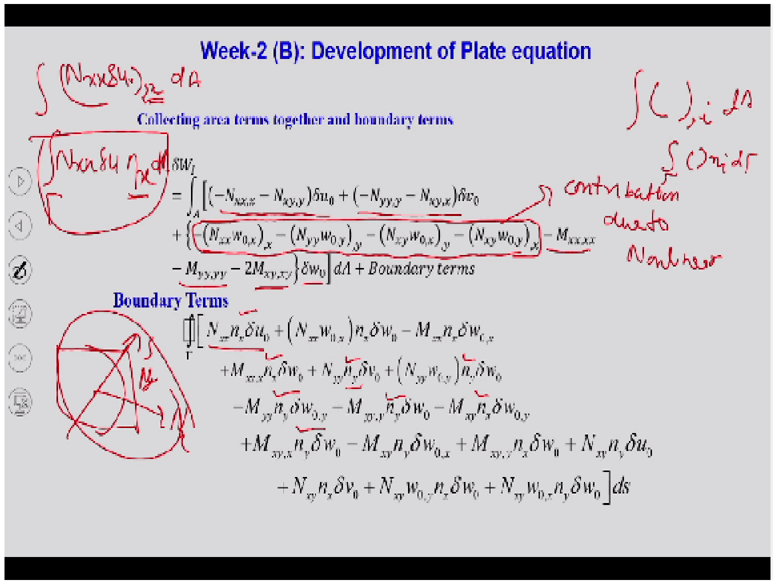
(Refer Slide Time: 00:26)



Say in the last lecture, I have obtained the terms of variation in internal work done. So in there we see that there are number of terms. The terms like this Nxx, x del u0 underline terms. These terms will go to the contribution towards the boundary and the terms like this whole derivative of x or the whole derivative of y will be on a line integer or in general term if you say that at least the derivative.

And this area is integration 0-x a on 0-b, so if you take the integration along x axis, so it will be integration remains along y direction and if you integrate along y direction, it will remain integration along x direction. So these terms will contribute or will go to the line integers. But the underline terms will go to area.

(Refer Slide Time: 01:33)



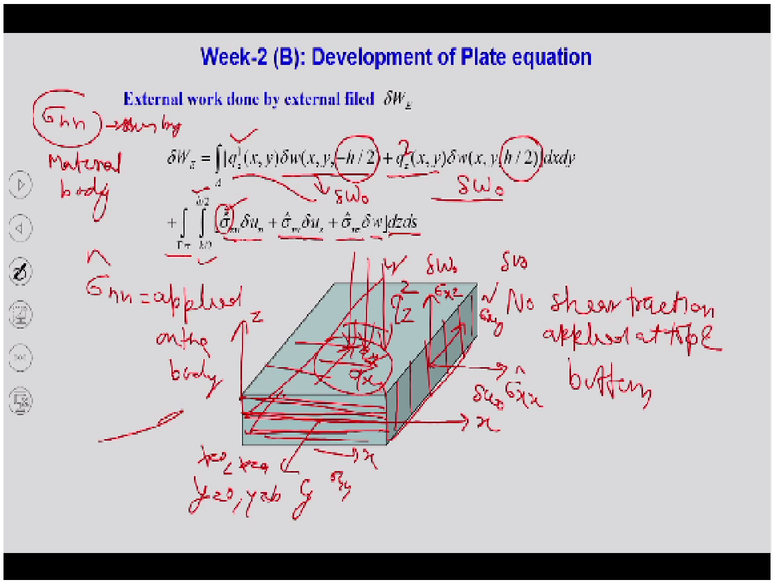
So in this slide, I have just collected the terms or I would like to say the coefficients of del u0 which contribute on an area. Similarly, the coefficients of del v0 and then the coefficients of del w0. Basically we have 3 primary variables, u0, v0 and w0. Similarly, we have here 3 variations, del u0, del v0, del w0. So the terms like this, these are basically a nonlinear terms, contribution due to nonlinear.

So if you do not consider the nonlinearity, so these terms will vanish. So you will have only Mxx double derivative of x, My double derivative of y, Mxy mixed derivate of xy, this contribution from the internal work done. Similarly, the rest of the terms will contribute over the boundaries. So I have used that cos theorem that if you have equation like this Nxx del u0 and this is a… So using the Gauss theorem, we can say that it will be on a line integration and derivative becomes Nx cosine vector of that d gamma.

So it is a standard theorem of mathematics. The purpose, I can directly integrate and put it at along the y direction. The purpose of writing like this that later on if we evaluate in a general sense that suppose you have a plate, now just like this or maybe some, you cut that, then you have a normal and shear directions. So there you may put normal x normal along y and shear and so on.

So this will help to develop the general formations. So wherever we have a derivative x, it will be Mx. Whenever we have a derivative y, we will replace by Ny. So no need to afraid. You just put it systematically or consistently Nx Ny Nz. The concept in this or in a Gauss theorem if any something is i, then dA, then it will be curve ni and d gamma. So we are following the same.

(Refer Slide Time: 04:34)



Now the contribution by the external work or the external field. So let us say a plate where this is your x axis, this is your y axis, this is your z axis. First we talk about top surface and bottom surface of the plate. Let us see at the top surface you are applying only a qz along that direction, pressure. Similarly, we can also apply the shear forces like along x direction, qx; along y direction, qy.

But most of the theories, you will find that we do not take qx and qy, the shear forces. But from manufacturing point of view, there are some cases where the, at the top surface, shear force is applied, then you have to consider this. But 90% theories which we are going to develop in this field, we consider that shear stresses are 0, that there is no shear traction applied at top and bottom.

So qz1 is nothing but applied pressure at the bottom of the plate. Here it should be qz2, is pressure applied at the top of the plate. 1 as the bottom, 2 as the top. And then the deflection, see, -h/2 and +h/2, that location. For present case, del w is a constant throughout the thickness. It is not changing. So ultimately this leads to only del w0. Whether you take at the reference surface or you take at the top of the surface or bottom of the plate, it remains del w0.

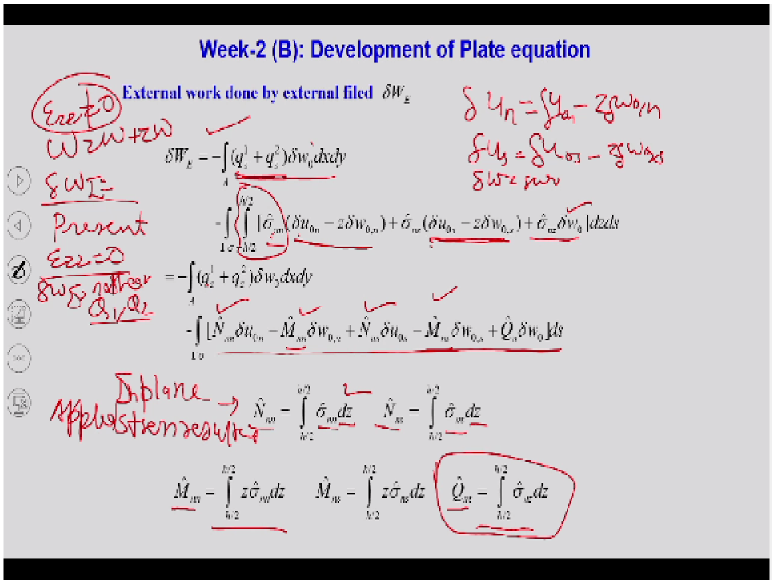
So qz1 del w0 is work done by the bottom pressure and qz0 del w2 is work done by the top pressure. Now these are the faces. What? You see I am putting here something and something. We have now 4 edges, one x02xa and another is y02ya. So that is why we have written in general form sigma nn, applied traction boundaries at these 4 edges. So sigma nn and through displacement del u1 sigma ns del us sigma nz del w0.

So I am going to explain what are these? Let us say you are talking about this face. So the normal direction of this face is this, x. So we will get sigma xx of hat and then the shear direction of, is this sigma xy of hat and then nz sigma xz. Are you getting that x is telling you the normal direction of this face? then sigma xx xy and xz. So along this direction, what is the displacement? del u0 along this direction, displacement is del v0 along this direction displacement in del w0.

So del un for a specific case if you are talking about this edge, it becomes del u0. If you are telling about this edge, so the normal direction outward will be y direction. So it will be sigma y1, shear will be x. So accordingly, so this is a general formation. So we have written the boundary. In a general sense, it may be along x axis or it may be along y axis. Thickness remains -h/2 +h/2 and the special purpose putting a hat.

So hat shows you that, that it is externally applied traction boundaries. So it is the outside force. So initially we were having nn that with the stress by the material, inside or a body but nn hat is applied on the body, okay. So we can write sigma nn hat del u0 hat sigma ns hat del us+sigma nz hat del wz, where it will work, dz and ds.

(Refer Slide Time: 10:15)



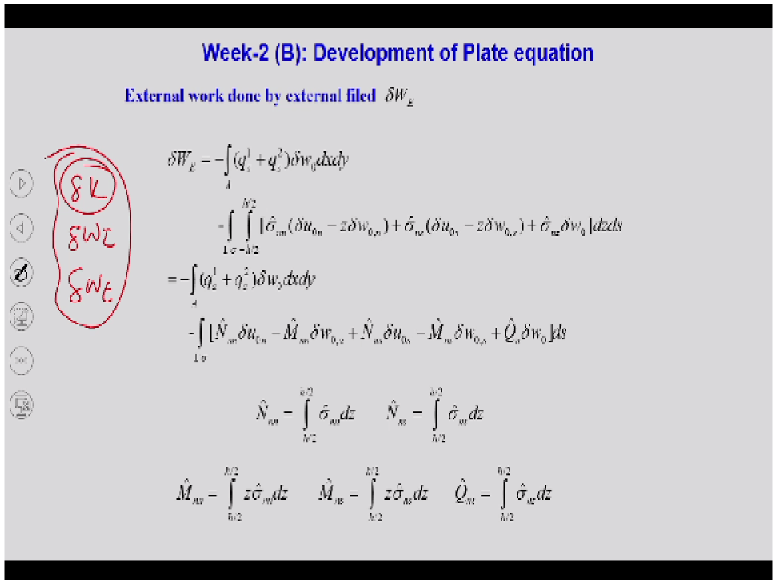
So del w0 was same, so we have taken common. So work is the external field. Work done on the system, so it will be negative if we consider, negative work. Then similarly, this will also be a negative. So now you have un is u0-z*w0, n and us will be u0s-z*w dot, s. So there first variation del u0, what will be that, put del here. Similarly, del us will be, put del here. So we have replaced del un by our main assumption.

Similarly, del us and del w is just del w0. Now again using the definition of the stress resultant, so we will say that nn hat, the resultant normal force applied will be equal to, I would like to say that inplane resultant first, inplane stress resultant. But what are the applied stress resultant nn hat, sigma nn hat dz. Similarly, sigma ns will be sigma ns dz. So the quantities, multiplying this and this will contribute nn and ns.

Similarly, z time of nn will give you the applied resultant moment, Mnn. Similarly, Mns. Now what about sigma nzwz? So that tells you the shear force. So resultant of shear force applied, it will be nzdz. So previously when we are talking about the stress resultant in the case of internal work done, we have a contribution (()) (12:50) but we do not have contribution of this. Because there epsilon zz was 0, so there was no sigma nz term was there.

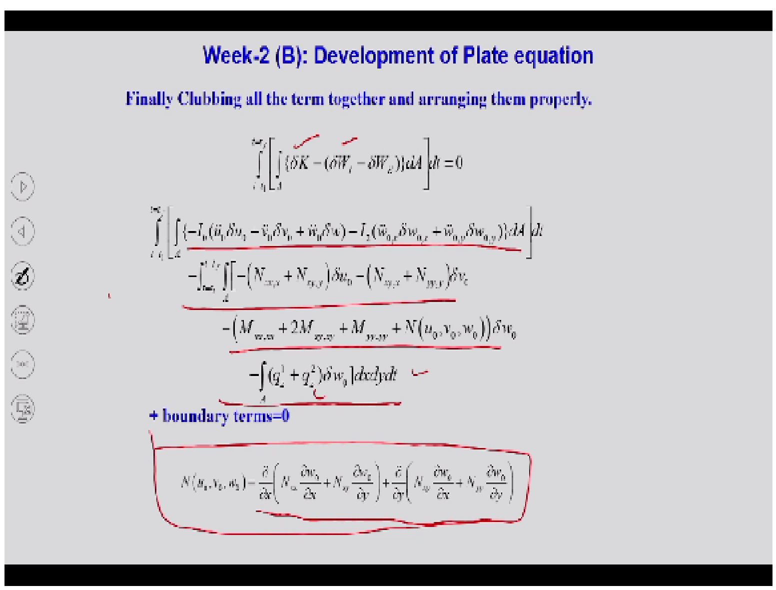
But for higher order theories, if you talk about consideration of w is just not w0, you take w=w+z time of something or you say that if you are not neglecting this, then internal work done will also have a contribution of shear stresses. So but for our present case, we have assumed epsilon zz is 0. So in del wi, no shear, concept or I will say that q1 and q2, no concept. But higher order theories may have this concept. So now this is our modified one. So you see this is on the boundaries, this is on the area.

(Refer Slide Time: 13:57)



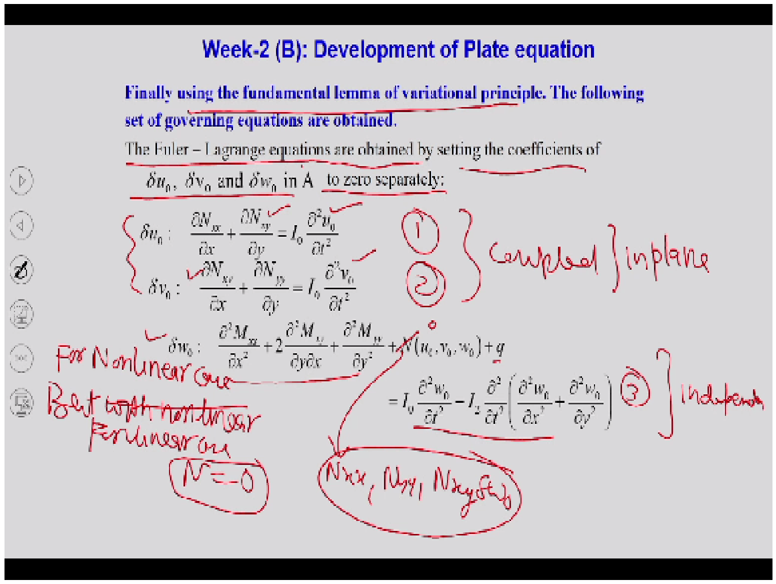
So till now we have obtained del k, we have known. Now del wi we know and del we we know. So all the terms we have calculated. Now you are going to put back into the Hamilton principle.

(Refer Slide Time: 14:12)



This is the Hamilton principle, substitute all the terms. So this is the contribution due to del k. First I will concentrate on the terms which are on the area. Then this is the contribution due to the internal work done and this is the contribution due to the external work done. So these terms will be remained on the area. Now rest of the terms under, comes under the boundary. So basically, this N is nothing but a contribution of nonlinear terms.

(Refer Slide Time: 14:57)

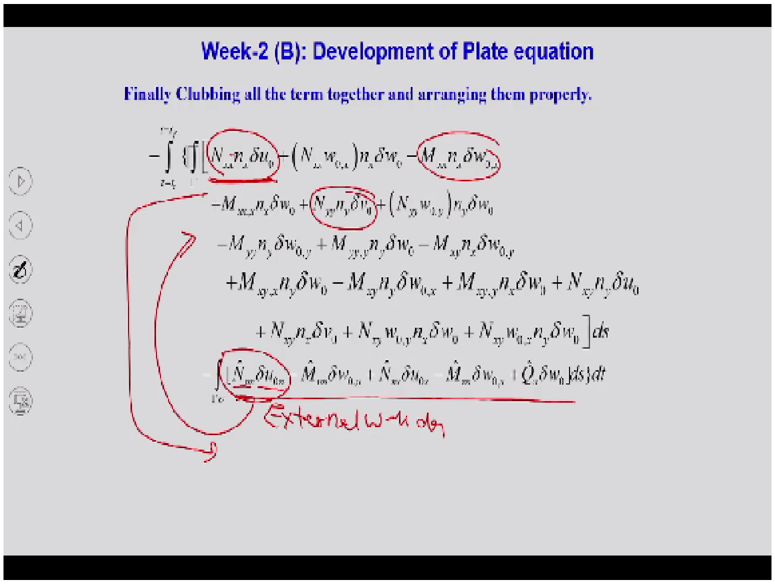


So finally, using the fundamental Lemma of variational principle, following set of governing equations are obtained or sometimes we can say that Euler Lagrange equations are obtained by setting the coefficients in area or under integration=0. So first equation, second equation and third equation where this N contains Nxx, Nyy and Nxy terms. Basically with the del w0 something.

If you see here for nonlinear case, that this N contains Nxx, Nyy, Nxy and equation 1 and 2 also contains Nx, Nxx Nxy. So these are the coupled equations that some of the terms are here and some of the terms are here. But without nonlinear or I would like to say for linear case only. N term will be 0. So equation 1 and 2 Nxy is here, Nxy is here are coupled inplane mode, inplane stretching and equation 3 is independent.

If we say only for linear case, then there will be, this term will be 0. So it will have Nx Ny. No contribution, Nx Ny. So under the pure bending or bending case, only third equation to be solved independently. So most of the time you will see that plate under bending or a pure bending, we solve only third equation. We are not considering these 2 equations. But if we say that plate is thick and we are interested to find out u0v0, then we have to solve all the equation together.

(Refer Slide Time: 17:33)

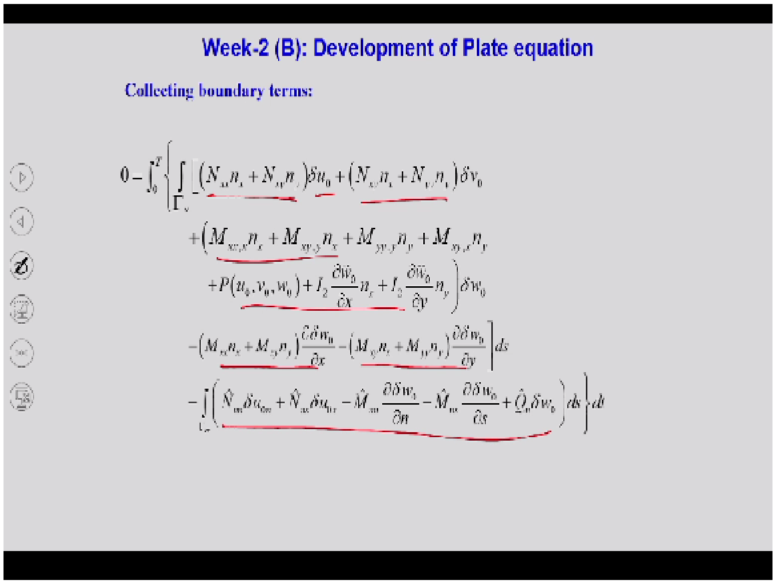


Now comes about the arranging the boundary terms and their coefficients. You see first difference that which is from external work done. Work done is having Nnn hat del u0n, this term and this term is like this. Can we club it together? Someway might. This Nn hat is not same, so we cannot put these things together in this form. So first there are 2 was, either you convert this into this form or you convert this into this form.

That this form is converted into a suitable generalized normal and shear components and then both of the terms will have same coefficients and then we can club it together. This part is very important step while developing the theory. Specifically, if we are interested to develop a generalized program or a generalized methodology, then we require these steps.

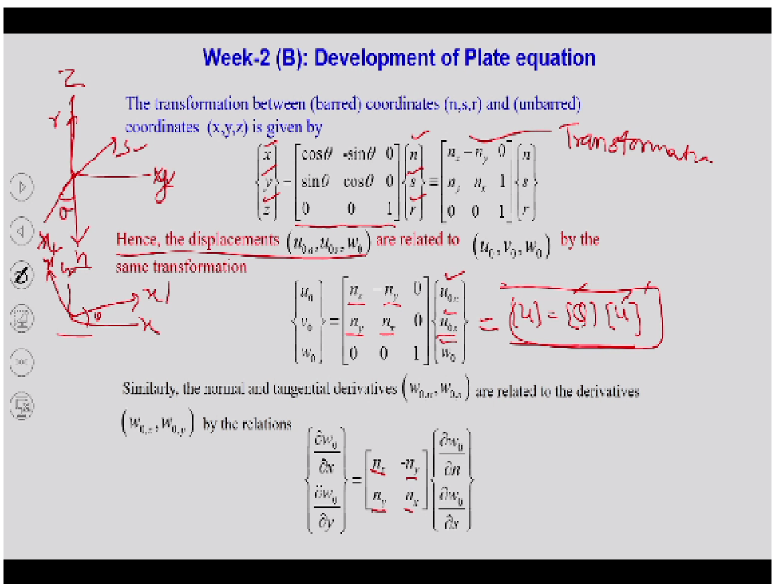
If you have only just a simple problem, one specific case, then you can go for a direct substitution like along x axis, do the integration and similarly, external work done. Sometimes we do not consider that there is no externally, for that equal to 0. So only this contribution will come out. So we are going to develop for a generalized case. Later on it can be used. So we are going to convert this thing into this.

(Refer Slide Time: 19:25)



So first of all, we are going to del u0 coefficients together, Nxxnx Nxyny del v0 and del w0 del w,x del w,y and the contribution from the external work done.

(Refer Slide Time: 19:42)

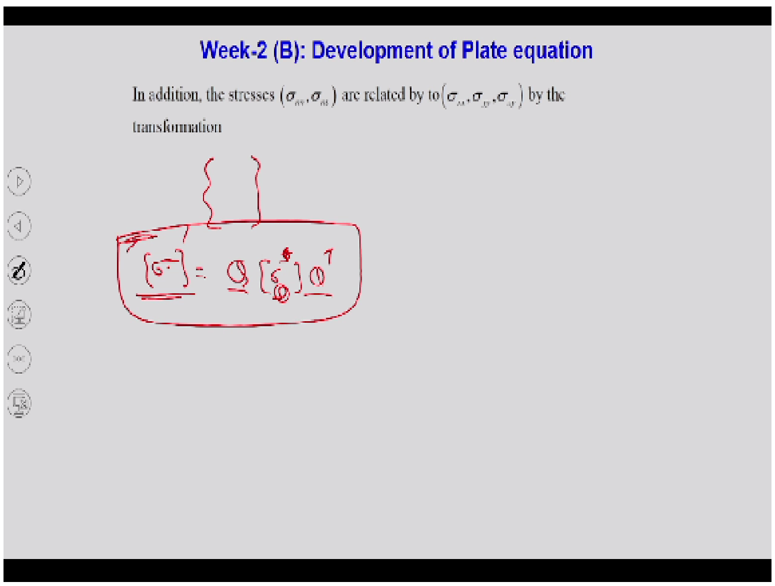


Now see the transformation. Let us say we have a coordinate system x1 x2 or you say that x y or z, this we have a coordinate system. Now a normal another coordinate system which is making an angle theta s and z remains r. So along the z axis if you rotate xy and it becomes nns and r is z, same. So how to transform? This is your transformation. This is the standard that even if you have that component, x will have a component along n direction, n cos theta and x cos theta.

Similarly, along y direction, -ny sine theta. So you say that ns transform coordinates our original coordinates. So this is your transformation matrix that how to transform from a xyz coordinate to normal and shear coordinate system. So displacements, now, so u0 v0 w0 can be written as nxny in terms of u01, nynx in terms of u0s and w is w0 since it is rotating about third axis, z axis. Similarly the derivatives of w, x w, y can be written like this, just 2D matrix, w0n and w0,s.

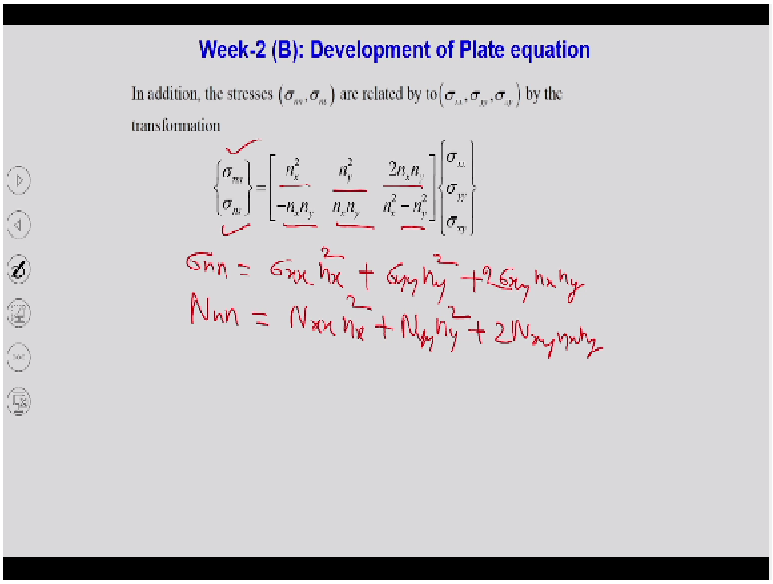
This is the standard form I think in second semester or third semester if you go to an AutoCAD course or any Matlab program, then you will see that how to transform a 2-dimensional or xy just xy is this and this becomes x dash and y dash and +2 level also if this is theta, so instead of that you say in that cosine vector will just, cos theta is nothing but the cosine term x. Sine theta is nothing but cosine of term y.

(Refer Slide Time: 22:14)



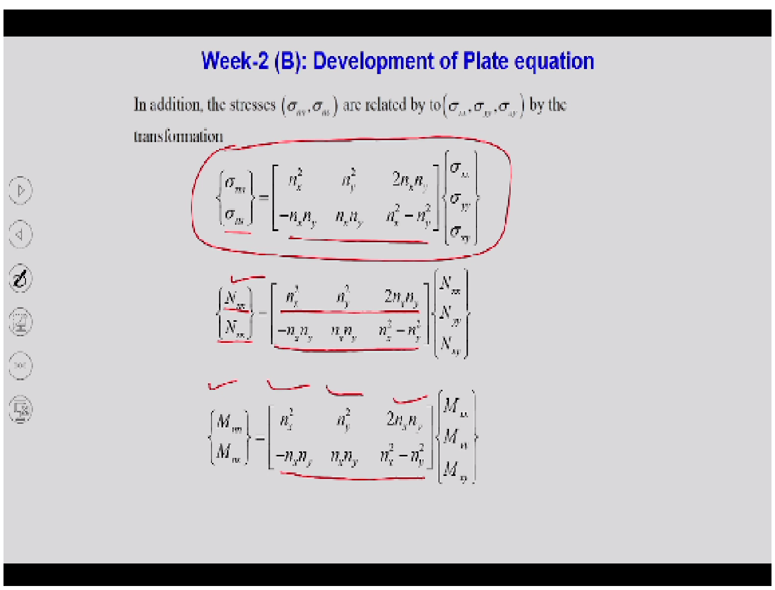
So now the transformation for the stresses. I would like to say that previously this u0 v0 are the vectors. So in that if you remember in index form, Q and u dash, this was the Q0 transformation matrix, okay. Now we are talking about the stresses. So we will say that sigma is nothing but Q sigma dash and Q transverse over, so basically. Sigma dash is nothing but Q whatever sigma and sigma transverse.

(Refer Slide Time: 23:14)



Following that way, we are expressing sigma n and sigma s. That is why ns square, ny square, 2nxy, this terms comes into the picture. So if you, somebody tells you what is that sigma nn? Sigma nn is nothing but sigma xnx square+sigma yyny square_sigma xynxny. So corresponding Nn, (FL) Similarly, one can write, it will be Nxxnx square Nyyny square+2Nxynxny.

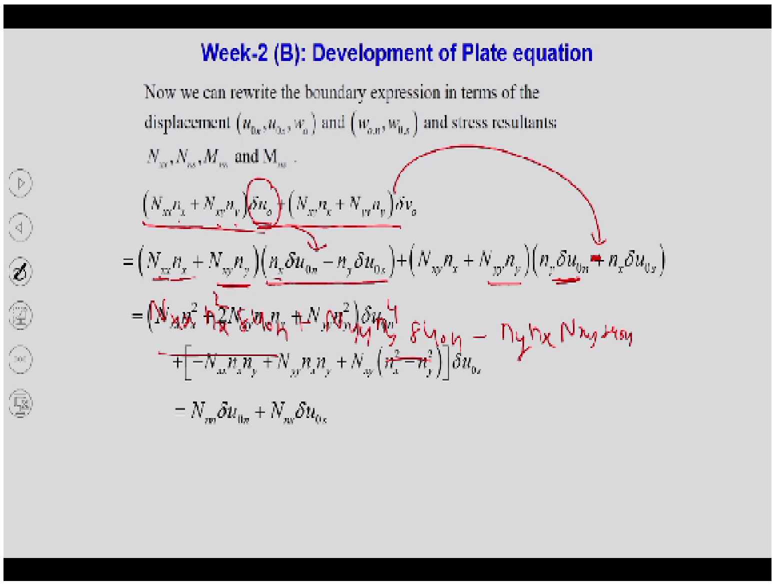
(Refer Slide Time: 24:03)



So you see Nxx is nothing but this. Similarly, Nns as per this following this. These things are given in the book. You need not to remember that what will be my, or this is very easy, even you can remember this. But sometimes people get afraid, okay this is the so big equations, how to remember all these things?

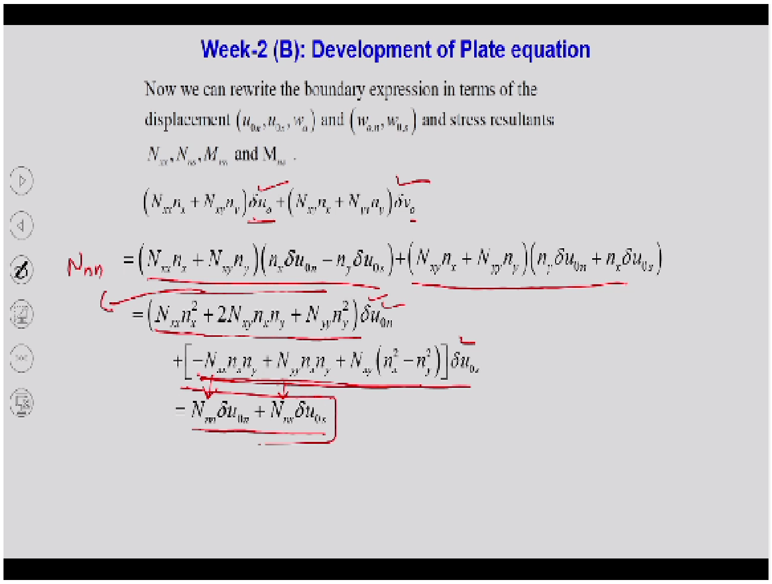
These are the standard formula given in most of the Theory of Elasticity book, whether you talk about or specifically you go J N Reddy Theory of Plate and Shell book or the Theory of Elasticity Martin and Sadd book, there you will find the standard formula or the Mechanics of Composites book, there also the formulas are given. Next in a moment, we will also follow the same rule.

(Refer Slide Time: 24:57)



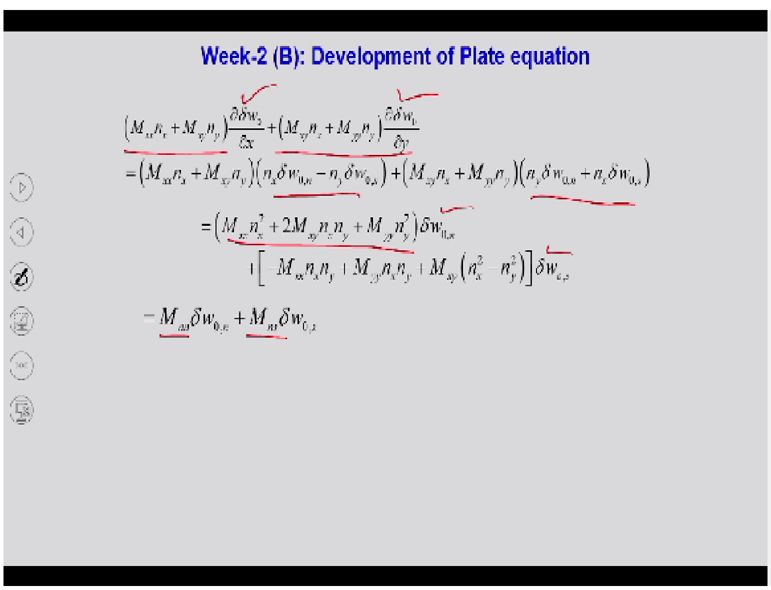
So now you see, how to convert this term, Nxxnx Nxyny del u0+this thing. So we are going to replace with it nn ns, okay and similarly this to this, okay. And then multiply it. So what will be that? Nxnx square del u0n. The next I will try that Nyny square del u0n. Next contribution, -nynxNxy and del u0n and then here we will also some, I have just written, minus things will come up here or let me just see. Yes, here is +, okay. So similarly, del u1Nxyn.

(Refer Slide Time: 26:21)



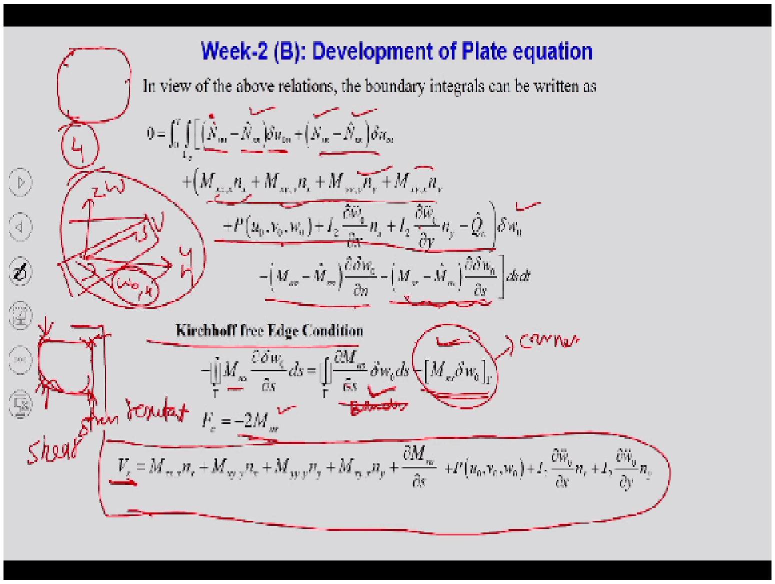
So basically if you put these things together, you will get this contribution del u0n and this contribution Nxyny square Nxxnx Nxxnxy. So this is nothing but the definition of Nn and this is nothing but the definition of Ns. So one can check or verify, just it is simply extension of del u0 and del v0 and multiplying together and taking the coefficients of del u0 and del u0. So you can write like this.

(Refer Slide Time: 27:12)



Now we have terms like this. So substitute those things here and then multiplying and arrange the terms coefficients of del w0, n and del w0, x. So you see that definition of Mnn and definition of Mns.

(Refer Slide Time: 27:40)



Now finally, our constant or the coefficients are equal. Now we can club together. So by the system, on the system, contribution of del u0, ns and here it they have written this, this term together which is a contribution of del w0 and this is the contribution of del w0, n del w0, s. So here I would like to point out that till here you find that 1 2 3 4 5, there are 5 boundary conditions, need to specify.

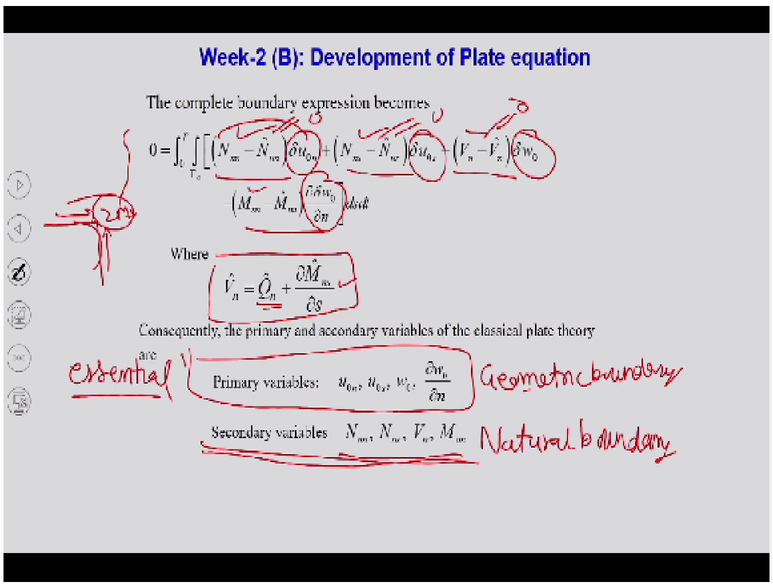
You see. But on an edge, if you are talking about in edge of a plate, so how many variables can you specify? Along normal, along shear, along z direction and 1 is the slope. So basically if you talk about 4 variables maximum you can specify at 1 edge, 1 slope and 3 basic variables. They may be mixed, they may be like if you talk about all displacement, u v w and then w, x. That you can specify.

So if you talk about a clamped boundary condition, this is your condition. So the Kirchhoff for the first to rectify this system, so what is this that we say, let us say to work with this. So further del w0, s again you do the partial integration that first of all as it is, differentiation of that and so on. So basically this term remains on boundary and this term goes to the corner. So I would like to put it that it will be the twice contribution because Mns from there, you will get.

So at the corners of sharp edges, you will have some shear stress Mns and xy from that. So whenever we develop the theory, we used to put assumption that for smooth corners, fill it, then it will not have the shear. So this will be taken as 0 for a plate which is having a smooth corner and contributed further to the boundary conditions. So now again defining a, so basically one contribution from comes to here.

Now you see what is this Mnnx Myny. So putting together, we are saying let us say this term is nothing but a Vn, shear force along normal direction or shear stress basically, Kirchhoff's shear stress. We used to call it Kirchhoff's shear stress, shear stress resultant along normal direction. So all these terms+contribution from Mns, we are putting together and making a Vs.

(Refer Slide Time: 31:43)



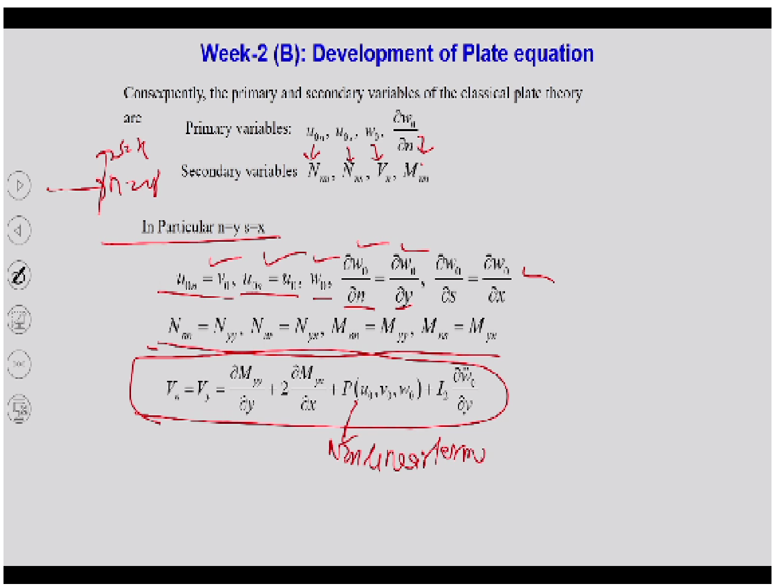
So now our modified boundary conditions 1 2 3 4. So this concept of Vn, if you do not go through the variational principle method or energy method, then it is very hard to understand, okay from where this contribution is coming. So here mathematically these things are coming and we are saying that okay physically at the sharp corners, there will be a resultant shear stress, one from, if this is a sharp corner, one from this side, one from this side, so at this point twice of Mns will be there.

But if you say that these corners are round, so this will not come. So we can write in standard form, Vn hat is nothing but Qnn Mns. So u0, you know as w0 and w0, n is our primary variables. Sometimes we call is geometric boundary conditions or sometimes you will say that essential boundary conditions, that displacement condition should satisfy exactly. Then the secondary variables Nn Ns Vn and Ms, secondary variables.

So there may be known as natural boundary conditions. In most of the literature, you will find that natural boundary conditions are not satisfied exactly or the essential, so when we talk about 1 element formulation, there we say that that essential conditions have to satisfy then the safe function you choose that they should satisfy this boundary conditions. They may not satisfy this or may be satisfying in average sense.

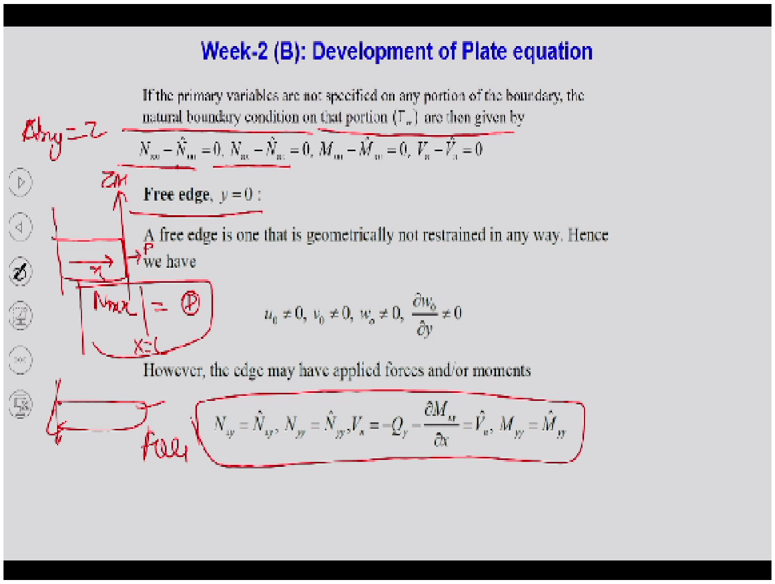
So these are the 2 primary variables, one is the geometric boundary conditions, another is a natural boundary conditions. So if this term=0, so ideally this whole has to be 0, has to be 0.

(Refer Slide Time: 34:19)



So we will have a combination of this, either and or concept that either un are 0 or Nn as 0, us is 0 or Ns is 0, w0 is 0 or Vn is 0, wo, n is 0 or this is 0. So in a particular case, if you say my normal direction is y and my shear direction is s, so un is nothing but v0, u0s is nothing but u, w is w w0 and similarly you can find out for a particularly, what are the different variables and Vn will be look like this. Where this is the contribution due to the nonlinear terms.

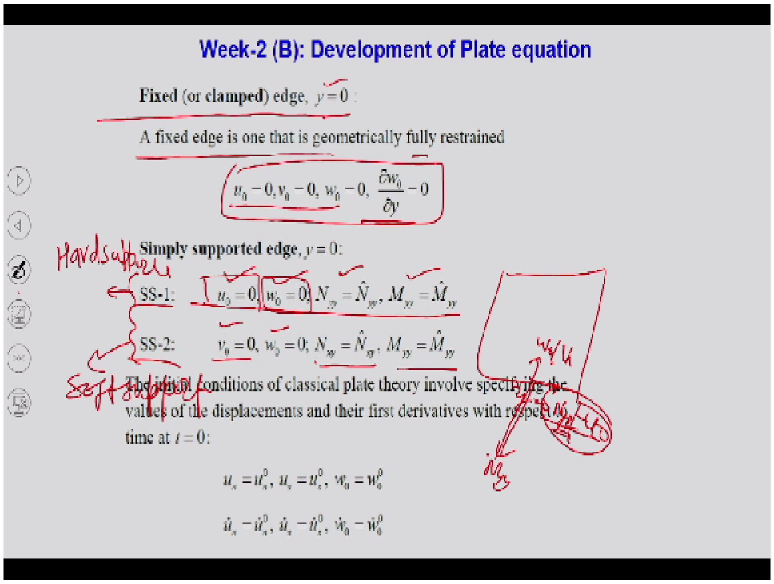
(Refer Slide Time: 35:09)



So if the primary variables are not specified on any portion of the boundary, then the natural boundary condition on that portion has to be specified Nn=Nn hat, Ns=Ns. for example, if we talk about a beam and this edge is having, let us, let us say an axial force (()) (35:34) the P, so I will say that N and n or if this direction is S, so Nxx at x=L=P, that is the boundary condition, it means.

Similarly, if you are applying just a shear force tau here, so I will say that Nxy=some tau applied or we will say that. Similarly, you can take a moment resultant on this edge. If this is free, so that is equal to 0. So we are just further specifying for a free edge, it means a free edge is when where geometrically not restraint, like this, as is free, cantilever beam. This edge is free. So there you have to satisfy the natural boundary conditions, all these stress resultants.

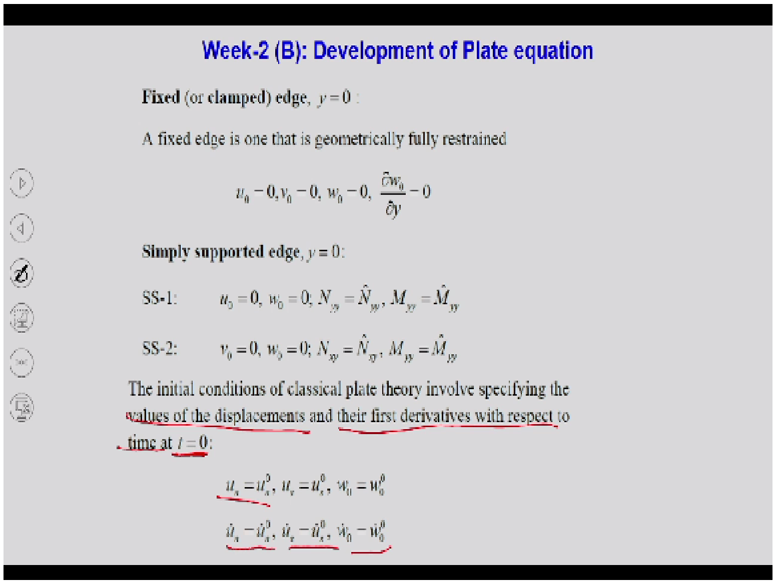
(Refer Slide Time: 36:35)



For the case of clamped edge, where edges are restraint, so all 3 displacements and 1 rotation or I would best, or we would like to say that slope is 0. So basically here at y=0. So y is your normal direction. And if you are talking about a simply supported case, people think that simply supported case is a very very simple case and so on. It is not so. It is a mixed boundary condition case.

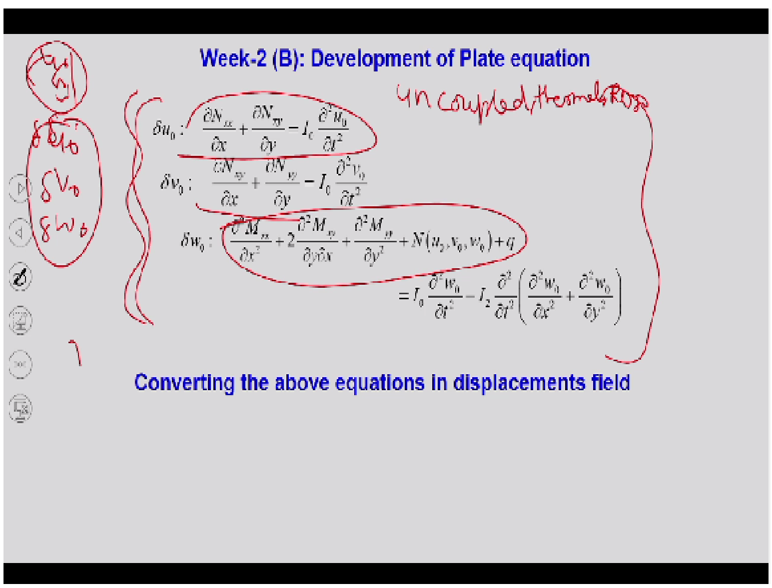
Here in literature there are 2 type of cases. One is that SS-1, another is that SS-2. Sometimes we call this hard supported and this is called soft support. So in the hard support, let us say this edge is my y=0. So Nyy simply supported (FL). You see first variable here Nyx and here you say that w0. So at simply supported from the top, this is clear. That either w0 or vn we are talking about some.

So deflection cannot, plate cannot deflect downward. So w0 will be 0. And what about u0? So u0, either u0 or Nxy. So in that variable, we choose is plate is not allowed to bow along x direction, u0 is 0 and that normal force Nyy. Similarly, the normal moment, so this is not allowing to move along x as well as along the z. Then we have a condition SS-2 v0 w0 Nxy Nyy.
(Refer Slide Time: 38:50)



Similarly, since we have assumed that all our variables having time dependent, then value of displacements and their first derivatives with respect to time at t=0 will be like this, initially conditions.

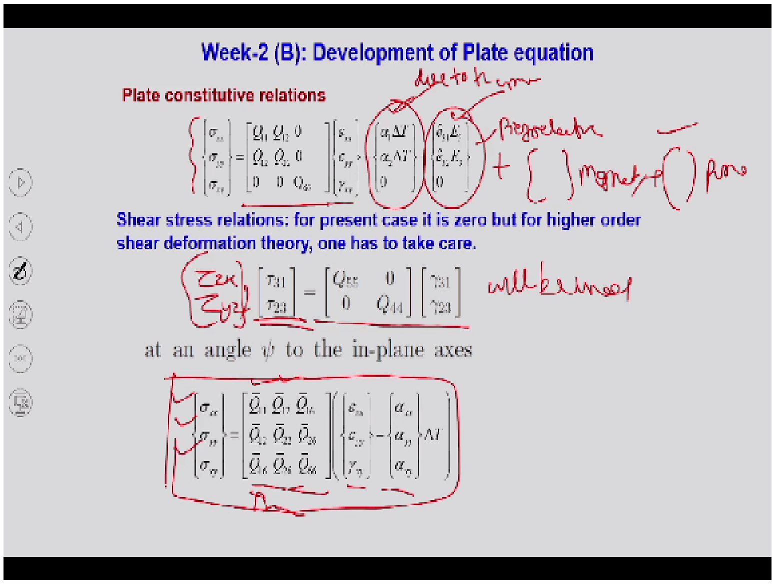
(Refer Slide Time: 39:12)



Now you are going to see that we have this set of governing equations. Can we solve directly? What are our primary variables? u0 v0 and w0 or it's variations? But the equations are in the form of Nx Ny or Nx My. These are not our variables. In this approach, we have assumed our u0 v0 and w0. If we know them, then only we can know that Nx Ny Nw. So we cannot directly solve this set of equations.

We have to convert to first this set of equation using the constitutive plate relations into primary variable form. Then only we can solve it.

(Refer Slide Time: 40:00)



So what is the, our constitutive relations for plane stress case, sigma xx sigma yy sigma xy is Q11 Q12 like this and some epsilon xx epsilon yy gamma xy and this you may say that contribution due to thermal, due to temperature, thermal load. Further if you say that okay my plate is made of smart material or piezo, so you have to take consideration of piezo.

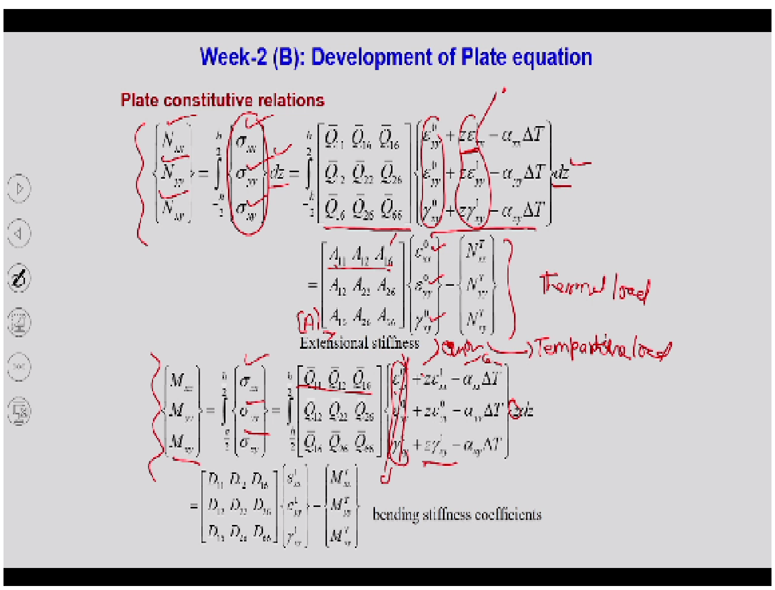
This I have just for your exposure point of view that this set of equations will remain same whether, if you are talking about a developing of a classical plate theory for a piezo electric plate or a plate under thermal loading or a plate under mechanical loading. So this set of governing equations remain the same. But if you say that for a coupled one, that is for uncoupled I would look side an uncoupled field, uncoupled thermal and piezo field.

So this remains same. Changes come here only. So while making the constitutive relations, you may consider a thermal load as well as electrical load. So if you know that the governing set of governing equations, so you directly even these, those governing equations are given in the book, standard books. And for curiosity point of view or for a small project in masters or a Btech, you can just put it here, some or later on some magnetic contribution or some, if you want to take some porosity effect, this goes on and on and on.

So those kind of effects you can consider at this stage. So next for the present case, our shear stresses tau zx and tau yz we have taken 0 but for other theories, they may not be 0. So for that case, these constitutive relations will be used. Again if we say that for a composite plate, my fibers are aligned at an angle or my geometrical axis is not with the plate axis, so in that case using the transformation law, that transformed reduced stiffness you will get and present like this.

So I am also putting stress here that these things need not to remember. Just you say that okay, if my geometric axis is not with that plate axis, material axis is not aligned with the plate axis and it is at an, at some angle, then we are going to use these constitutive relations. So these are the standard relations, are given in any Mechanics of Composite books or the Theory of Elasticity book.

(Refer Slide Time: 43:15)

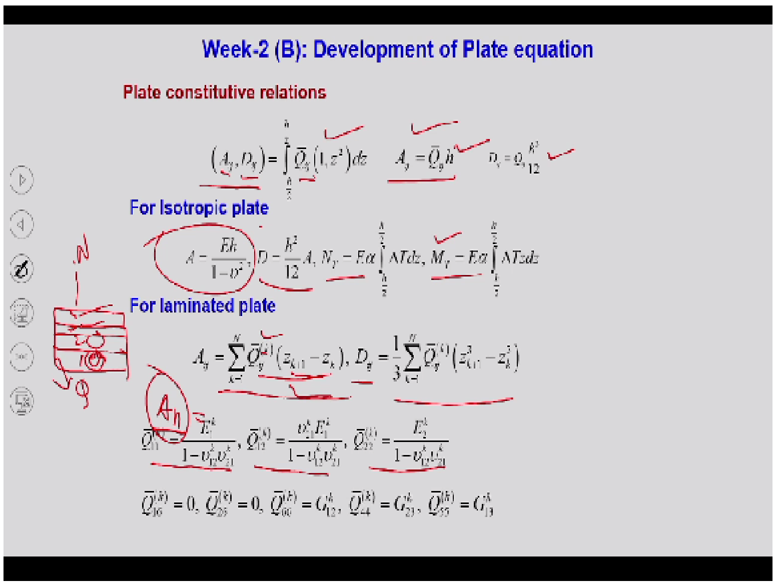


The real things come here. What is the definition of Nx Ny Nxy? So definition is sigma xx sigma y sigma xy dz. So put that constitutive relations here, okay and do the integration 0 to, sorry -h/2 to +h/2 dz. You see they z times of something will come but here is z, so z square. So there will be no contribution from this side. So that is why I have written finally, it reduces to some matrix A11 A12 A16.

So we call matrix A which is known as extensional stiffness matrix which contains epsilon xx0 epsilon yy0 gamma xy0 and the contribution due to the temperature loading or sometimes we call a thermal load. Similarly, the moments Mx My Mxy, moment or resultant basically. So we will say that sigma xx sigma yy, so again, so this having z times, so it will not contribute, only z square becomes z cube.

So this will contribute and this will contribute and this will also contribute. So finally one can write D11 D22 D16 D12 D22 D26 like bending stiffness coefficients.

(Refer Slide Time: 44:56)



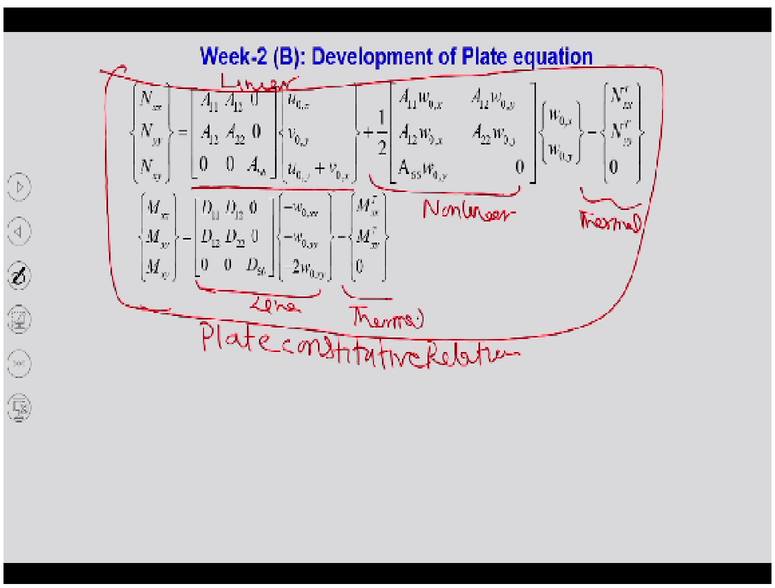
So now we can say that what is Aij or A11 A12 in the index form. In the short form, we can say that Q bar ij, Ai (FL) this we multiply it 1 and when we are interested to find out that Dij elements, then we will put z square over this. So finally after the integration, this term will be there. If it is an isotropic plate, so A becomes Eh/1-mu square and D becomes h square/12*A and the thermal load and or the thermal load and t and Mt, contribution to the moment in the, due to the temperature, so Mt will be there.

If it is a composite plate or a layered plate, when I am saying a composite, it means a number of layers 1 2 3 4 and so on and each layer will have a different material property. So the basic difference comes here. When we are talking about isotropic, this is a single material. So our A becomes like this. But when we are saying it is a layered plate, then you have to take summation of this like it says that, first layer data, then second layer material property K, third layer, fourth layer up to nth layer.

So you can find out an equivalent extensional stiffness. Sometimes when we use classical plate theory to analyse a composite plate, we say that equivalent single layer plate theory. That term equivalent comes to here that Aij is combining of this, giving only 1 constant but taking consideration of all these things. Similarly, the equivalent bending stiffness coefficients, we can obtain like this.

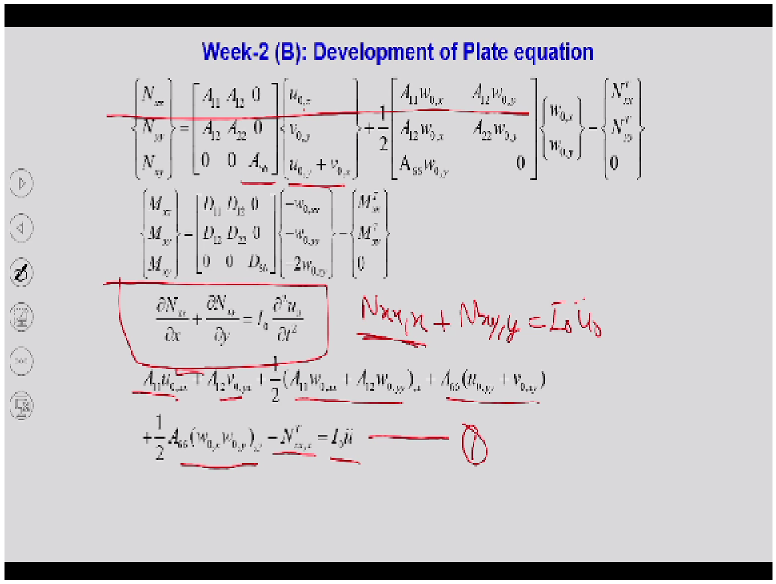
And in general, for case of composite Q1 Q2, you know that if E1 E2 mu1 2 mu3 are different, these are the standard relations.

(Refer Slide Time: 47:11)



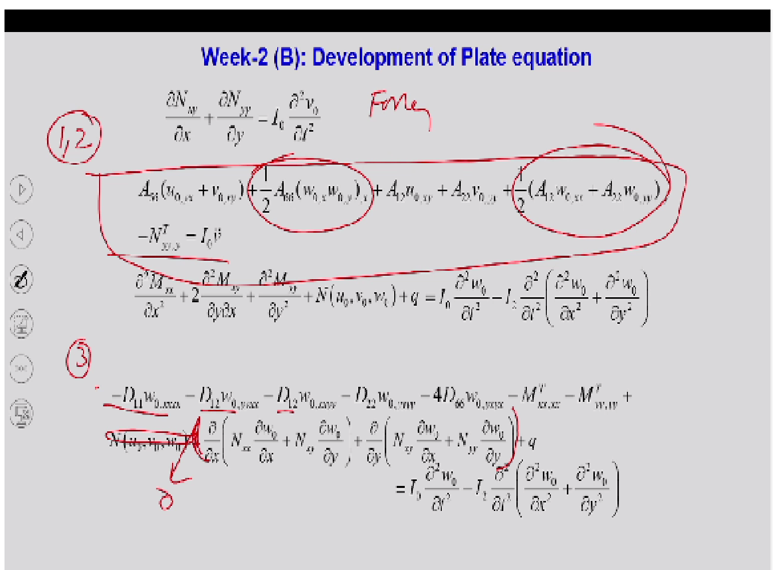
So finally for the case of a cross ply plate or an isotropic plate or I would like to say when the material axis is aligned with the geometrical axis of a plate, so the following equation of Nx Ny as is. So these are known as plate constitutive relations. So I have arranged systematically, so this contribution due to the linear terms, this contribution due to the nonlinear terms and this contribution due to the thermal load and here it contains only linear part and thermal part.

(Refer Slide Time: 48:10)



So we have this first equation or I would like to say that Nxx, x+Nxy, y=I0u double dot. So you substitute this thing from here. So it looks like that. A11 of u0, x. If you take derivative second derivative A12v0, yx second derivative, then this along x and NxyAxx and this and nonlinear contribution and thermal contribution in this. So our first equation becomes like this. So now this equation is in the form of primary variables.

(Refer Slide Time: 48:59)



Similarly, the second equation can be converted into this form and the third equation also using D11 D12. Basically I have to put some brackets here. So this is your third equation. You see that second equation and third equation, if you talk about only first equation, second equation and third equation, if you consider only a linear case, for linear case, so this contribution will not be there. So these 1 and 2 are not coupled with the third equation. So we can solve it directly. Here even this contribution will be 0. But when we have a contribution of a nonlinear terms, then we have to solve all the equations together.