Theory of Rectangular Plates-Part 1

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Lecture – 05
Governing Equation for Plate - 1

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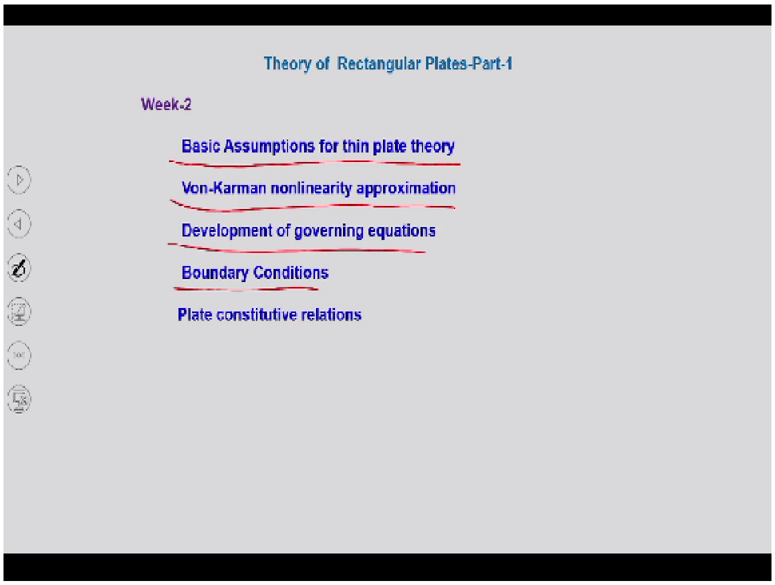

Welcome to our second week lecture. In this week, we are going to cover that basic assumptions of thin plate theories, Von-Karman nonlinearity, development of governing equations, boundary conditions and then plate constitutive relations. Till date, I have explained you that what is the virtual work displacement, what is the Hamilton principle and how to apply that? Now we will learn that for a particular, for a specific case, how we are going to use the Hamilton principle to develop plate solution, governing equations?

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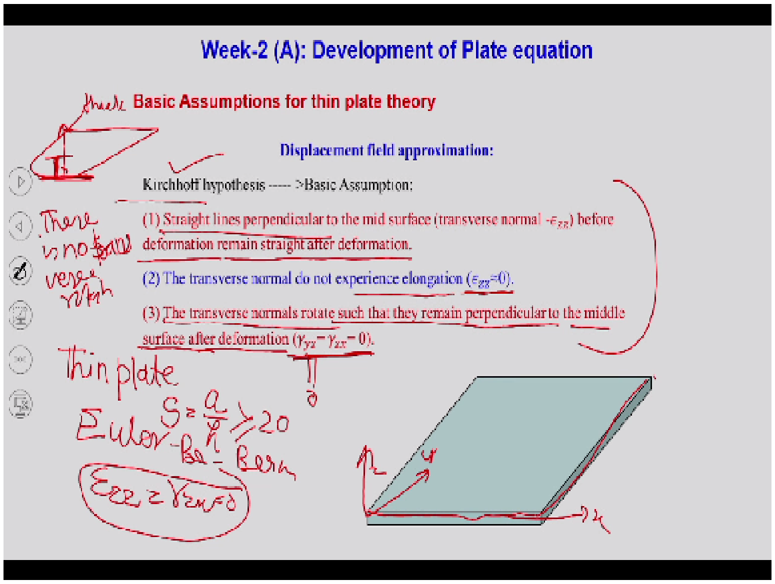

So first the basic assumptions for a thin plate theory. As I have already told you thin plate means when S (()) (01:30) to thickness ratio is greater than or equal to 20, then plate is considered thin. Like your thin beam Euler Bernoulli beam. So Euler Bernoulli beam is assumption for 1-dimensional case. So Kirchhoff's first time I would like to say extended the 1-dimensional assumption to the 2-dimensional assumptions and what are those assumptions? Like first straight line perpendicular to the mid surface before deformation remains straight after deformations.

If this is my straight line perpendicular to the middle surface. So this line remains straight before and after deformations and the transverse normals do not experience any elongations, means there is no stretching along the thickness direction. It means thick strain along the thickness direction is considered negligible and the last, the transverse normal rotates in such a way that they remain perpendicular to the middle surface after the deformations.

So mathematically I am saying that there is no rotation, no shear rotations; basically, transverse rotations. So shear strain which is the measure of the rotation basically. When you talk about a normal strain which is the measure of the elongation or a stretching, whereas shear strain is a measure of deformation in the angular positions, the angle, rotation. So there is no shear strain or way of neglecting.

The concept or physics behind this assumption is that since plate is thin along the thickness direction, so, and it is subjected to the small deformation case, means the deformation or the deflection is very small. So it will not cause any shear strain or any stretching, thickness stretching in the body. So these are the basic assumptions if you would like to compare with the Euler Bernoulli beam.

Then there you have epsilon zz and gamma zx as 0. When you are talking about a plate, so gamma yz is also 0. So that Kirchhoff's extended the 1-dimensional hypothesis to the 2-dimensional case. So this is my x axis, this is my y axis, this is my z axis and to the center of that will be the mid plane.

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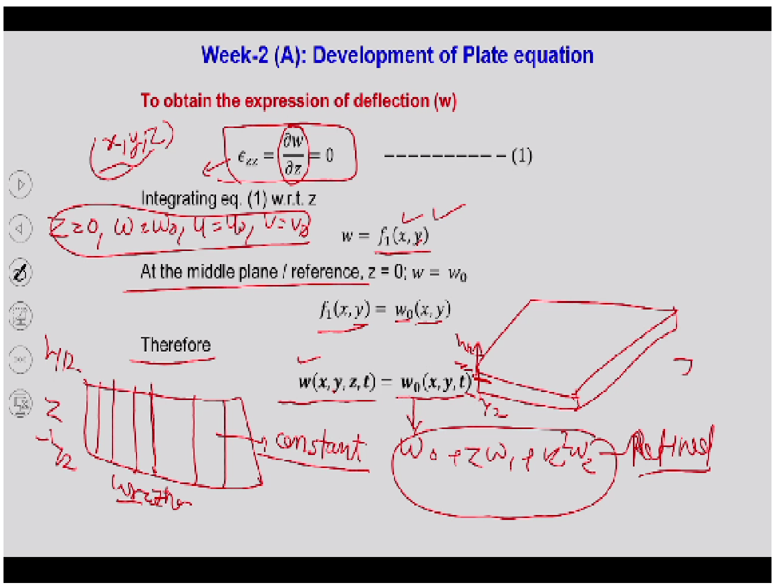

There are number of ways to assume the displacement field that how to assume a displacement field. The very first way using the mathematics as in the hypothesis it is said that epsilon zz is 0, normals strain along the transverse direction is 0. So if you put it=0 and integrate this equation with respect to z, since this is a partial derivative along z and we have a function x, y. if we talk about the special functions and z, so the constant, integrating constant, it will not just a constant.

It will have a function in x and y. So at the middle plane or reference plane, when z is 0, we is w0. So basically a coordinate of z is taken as central and -h/2 and +h/2. So when z is 0, w=w0, u=u0 and v=v0, generally this is the assumption. So if you put f, integrating constant is nothing but 0 and which is a function of x y because at the mid plane only x and y, z=0. So therefore, we can write w or you can say also time if you want to take, so w in general is a function of x, y, z and t and is equal to w0, a constant along the thickness.

It means if you want to plot, if this is my thickness, -h/2 to H/2 and you want to plot a w, what will be that? Just a constant line. Let us say my w is coming something 7, if I say the non-dimensionalizing note, just 7 of that, so it will just a straight line, like that, any number. So along the thickness, it is a constant, not a function of z. So 90% theory is, we consider w is a just w0.

But when we talk about a refined theories or if we are interested to accurately model the behaviour of a thick plate, then or even these days we can consider some more functions, z*w1+z cube of w2 later on. So but that is the feature of a more refined theories. So in the latest paper, in the literature, you may find that w is just not w0, w maybe w0+z times of w1+z cube of w2, etc.

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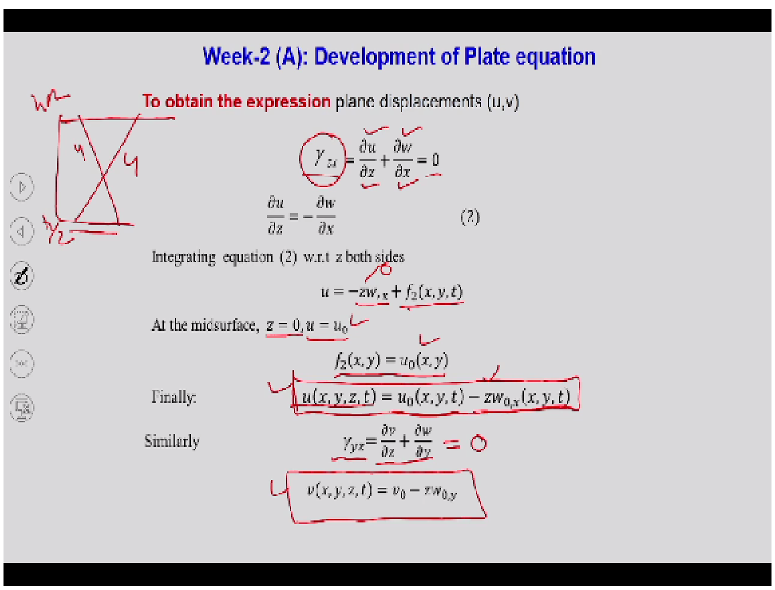

Now the second assumption that shear strain is also negligible. So definition of shear strain is del u/del z+del w/del x=0. So from there you write that del u/del z is nothing but -del w/del x. If you integrate the equation to both sides, so z*w, x and f2, integrating constants, again at the middle surface, putting z=0, so this term will vanish, u=u0, so f2 will come u0. So we can directly write that. u can be written as u0-z*w0, x.

So this tells you the variation of u along the thickness. So what is the variation of u? Linear variation like if this is -h/2 +h/2, it will mean very linearly like this or like this, it depends upon that value whether it will be at the bottom, it will be negative or positive, it depends on that. So linearly varying across the thickness. Similarly, gamma yz definition and if you equate to 0 and integrate with respect to z and putting the middle surface at 0, v=v0, so v can be found out v0-z*w, y.

So you see here, based on the assumptions, we can find out the variations of displacement variable across the thickness.

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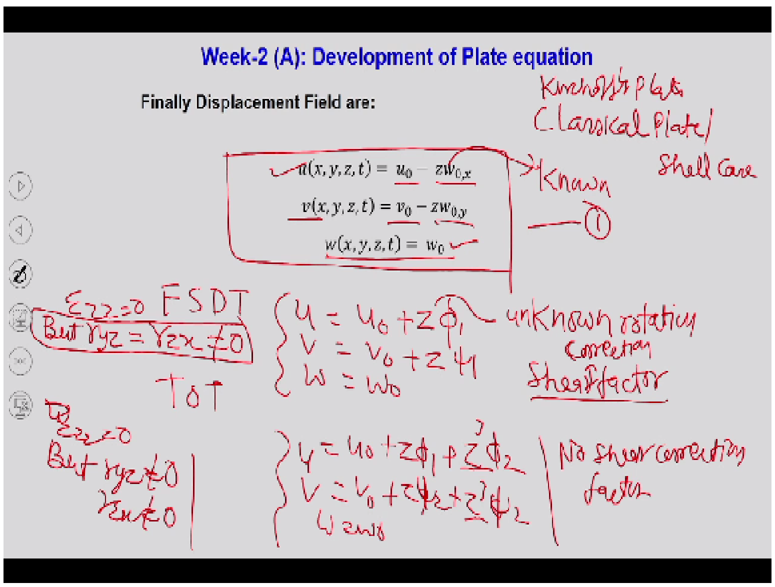

So finally we have this assumed displacement field but you call sometimes you may say assume, sometimes you mathematically derive, but this mathematical derivation is only possible for classical plate or shell care. If you are interested for a first order, third order, then that, those kind of things cannot be done. There you have to assume a series. So but for a classical or I would, let us say Kirchhoff's plate, you can also derive them mathematically.

That u is nothing but u0z*w, x; v=v0-z*w, y where w, x w, y is nothing but the partial derivate of w with respect to x and y and w=w0. I think in the first week lecture, I have already explained if you are interested first-order theory or third-order theory, so what will be the assumptions? That this u will be u0+z*phi 1 where phi is unknown rotation. Here rotation is this much, known rotation in terms of w. v will be v0+z* you may say psi 1. w is w0.

So this will be the first-order shear deformation theory. The, in the assumptions that we say in first-order, epsilon z is 0 but gamma yz, gamma zx not equal to 0, that shear strains are not 0. So if you do not follow, if you have this kind of conditions, then this will be unknown rotations. Similarly, in third-order, you have some third-order terms. There also that epsilon zz is 0 but gamma yz != 0, gamma zx != 0.

Here I would like to tell you the limitation of FSDT that whatever shear stress you will get, they are not very accurate, so we use the shear factor, shear correction factors. So that is what people prefer. In case of FSDT, third-order theory, here no shear corrections factors are required. Further it contains a cubic variation, so it may accurately predict the behaviour of thick plates.

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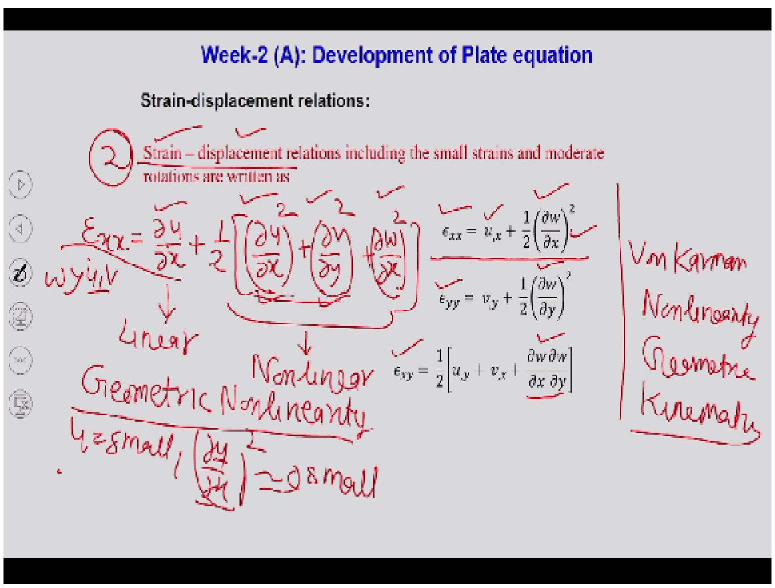

Then what is our next step? The next step is to choose suitable strain displacement relations. This is also a very very important step that you have to choose as per your requirement whether you are going to consider a nonlinear part or linear part, nonlinear part or not, or some component or not or you want to take fully. So I would like to just write the full expression of a w, epsilon xx started will be del u/del x+1/2del u/del x whole square+del v/del y whole square+del w/del x whole square.

So this corresponding to the linear and this corresponding to the nonlinear contribution. So if I have said that Von-Karman nonlinearity or sometimes we call geometric nonlinearity. So here you are saying that in epsilon xx, I have considered a trump u, x and 1/2del w/del x whole square, what is that corresponding? Why I have considered only this? Why not this, these 2? The assumption behind that since plate is thin and we have applied only transverse loading, so we will say that this deflection is more comparison to u and v.

And there derivatives, since u is small, so there derivative will also small, further square of derivative is also small negligible to 0 or I will say small. Compare that del u is small, so there derivatives will also small compared to w and another thing that rotations, we say that moderate rotation. There we assume that there is slight rotation we assume. That is why this term has been considered.

If you see in the basic relation, we are also that w, x that belongs to a rotation. So a small term which is equal to w, x rotation that is adjust, so we take care of this. So if you assume, so here in epsilon xx, you have 1, 2, 3, 4 terms, out of that we have chosen 2 terms. May be later on or these days you may choose all. If you have a, plate is thick, deformation is large, rotation is large, then you have to consider all the terms.

So you have to, as per your requirement, you have to choose or you have to consider. So initially that Von-Karman considered this because in olden days, there was no very big computing facilities. So even solving with this kind of term taking a single term was a huge task. So solving a simultaneous set of ordinary equations or doing some further calculations. So I am also giving a first course in this area.

So I am just limiting myself considering only this term. So if we consider del w/del x or del w/del y or the mixed derivative, then it is known as Von-Karman nonlinearity or Geometric nonlinearity. Why we call, or sometimes kinematic, because they are taken in terms of kinematic relations that strain, all this, so this was the major step that what strain displacement you are going to follow. If you are going to follow a linear or a nonlinear or a partial like us.

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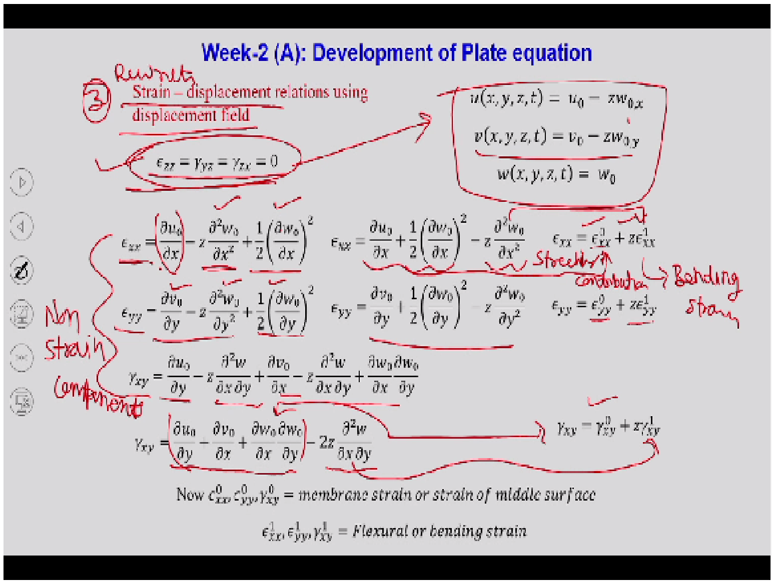

Now in the third step, you have to obtain the, I would like to say, derive or rewrite whatever you want to say, strain displacement using the displacement field. What is that displacement field? This is our displacement field. If we have used u v w like this, so what will be my strain? Epsilon xx. So del u0/del x del w, double derivative del x and this term. Now you can arrange, this is first derivative.

This is first derivative. So we are putting it together and this is double derivative. So basically this terms are putting together, denoted as epsilon xx0 and sometimes we call it is a membrane strain or middle surface strain. Then double derivative term is denoted as epsilon xx1. So we call it is a bending strain or a flexural strain contribution and stretching strain. Similarly, the components in epsilon yy.

So epsilon yy will be del v/del y. So from there in double derivate along y, del y and nonlinear component, we have to rearrange epsilon yy0 epsilon yy1 and gamma xy. Gamma xy is del u/del y+del v/del x. So if you substitute those things and arrange those terms, first order terms, so you will see this is gamma xy0, sorry this one and this one is gamma xy1.

So why we are arranging like this epsilon xx, epsilon yy, gamma xy? We can proceed even without rearranging also. But for programming point of view, for further development point of view, if we can arrange these terms in a shorter form, we can recall, we can remember the things easily and programming also.

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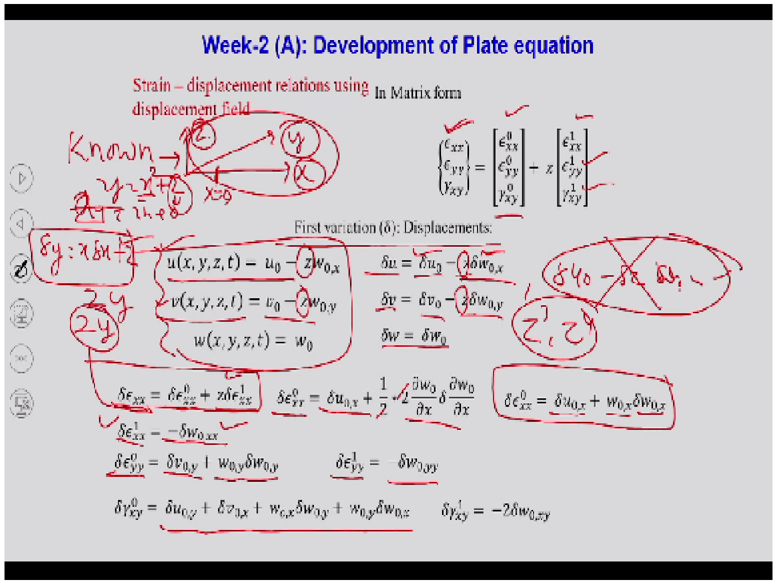

So in the programming, you have to just like see, a column vector epsilon xx, epsilon yy, gamma xy and first will be epsilon xx0 epsilon yy0 gamma xy0+z*epsilon xx1 epsilon yy1 gamma xy1. If you already based on these assumptions, we have developed this, again you can verify also if you substitute these things, so our 3 strain components will be 0 epsilon zz gamma yz and gamma zx.

So we have only 3 non-zero strain components. Since we are going to use a Hamilton principle where me, we may need a variation, variation or I would like to say that first variation in the displacement or first variation in the strains, those terms we require at some stage. So I am going to find out if this is my displacement field, what will be the del u. Del u will be nothing, del u0-z*del w,x.

Similarly, what is my virtual displacement or the first variation along y direction in displacement? Del v=del v0-del z del w,y. Similarly, del w0. Now del epsilon xx will be del epsilon xx0+del epsilon xx of 1. Very easy to remember. Then in particular or explicitly del epsilon xx0 is nothing but del u0,x. So from there, 2 gets cancel out, first function del u0,x and del w0,x.

So finally you can write del u0, x+w0, xdel w0, x, first variation of this. Similarly, what is del epsilon xx1, -. So first variation is nothing but it actually is a differential operator, one can go through internet or a Google or a book of J N Reddy of Theory of Elasticity. It is clearly explained that it acts as a differential operator and its rules are also same. If you divide or multiply or add, follow those rules.

Similarly, del epsilon yy0, del v0w0, y and del w0, y. Similarly, del epsilon yy of 1 and gamma xy of 1. Sometimes, it looks very easy, then it is on the presentation or on the board but when I give you actually to solve or to find out the variation, it may be difficult. Sometimes people may get confused. You see here z kind of thing. Most of the time when I used to take a class of Theory of Plate and Shell, students used to ask, we want to take del u of this, this is fine but it remains same, z.

You have taken only variation in this. Why not in the z? Or sometimes to 3vx. So known quantity like your references, coordinate system x y z, you know coordinates at particular point that what will be my z coordinate, what is my x coordinates? There is no ambiguity. So no variation. When you do not know and you want to extrapolate something, okay what will be the my value?

So then you have to take, so the coordinates which are known or the values which are fixed, which is not changing, means at a point x coordinate is known to you. So if these things come, do not take variations. Sometimes my students do that, del u0-del z w0,x blah, blah. This is the wrong. Coordinates. In the quadrants, there will be no variation or may be some just you will put 2, constant. it remains as it is.

When you go for a differentiation, let us say yx+2 or x square+2. So if you take differentiation, I will have to do twice of x+0 instead of this. But if you say that del y, it will be x del x+2, simple. You need not to change it. Or sometimes in the multiplication, 2y, you just put 2y as it is. So when you are taking the variation, you have to be careful. So maybe for higher order theories, may be some more terms z cube, z, z square or z 4 times, they will remain the same, there will be no variation in that. So I think this point is clear.

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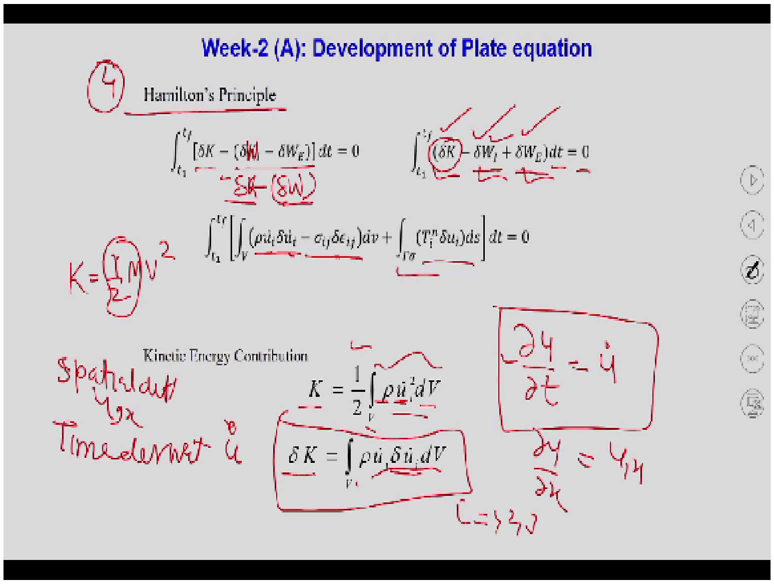

We can move. Now our fourth step. Framing of the principle. Which basic principle we are going to use? If we are interested only in the static analysis, then go for principle of virtual displacement. If we are interested to find out the dynamic behaviour of that plate, then Hamilton principle. So I have also would like to say that you see that t, here, time, that u is also a function of time.

The 3 spatial coordinates and 1 time coordinates, that is why I am going to use a Hamilton principle. So in the Hamilton principle, kinetic energy-, okay it is written wrongly, wi, it should be potential or whatever say work done. So if you substitute like this, so del wi+del w, external work done-internal work done, kinetic energy, dt tends to be 0. So first variation in kinetic energy, first variation in internal work done, first variation in external work done.

Explicitly or in index form, what is that kinetic energy, first variation, that rho ui dot del ui dot, what is that? Internal work done. Sigma ij del epsilon ij. What is that external work done? Tin del ui/surface*dt=0. So actually if you talk about a discrete body, your kinetic energy is nothing but 1/2MV square. I also again discuss these things.

So for a continuous system, if mass is distributed over volume, then instead of this, we will put (()) (30:56) remains same, rho*dv over that volume that will become a mass and u is a displacement and u dot, so when you have a spatial differentiation, then we will put the comma that the u, x. When we have a time derivative, then we will put a dot over that quantity. It means del u/del t=u dot.

So this is the standard notation followed in the textbook of Plate and Shell that u dot means, upper dot, that variation, differentiation with respect to time, if you say u, x, it means del u/del x. So our kinetic energy is this. If you take the first variation, then you say that twice will come up here and ui and del ui dot. So basically first variation in kinetic energy becomes like this in index form where i can go 1 2 3.

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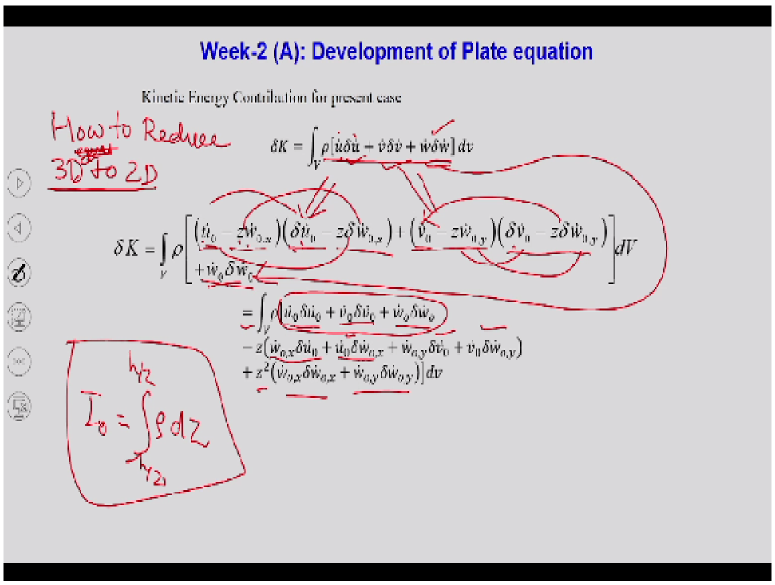

So for our present case, del k can be written like this u dot del u dot v dot del v dot w dot del w dot. Further what is u dot? U dot is u0 dot-z*w0, x dot. Similarly, their variation, del u dot-z*del w dot, x v0 dot-z*w, y and del v0-z*w0, then. And here only one term w0 and del w0*dv. So this we have for our present case. So first if you have a third order theory, may be some more terms required.

If you have some other theory, maybe some different kind of combinations may be there, but the process remains the same that whatever you have assumption, you have to put it here. Now next you multiply, open the brackets, first you multiply this to this, so u0*del u0, first contribution. Then I have already arranged in such a fashion that constant and z* and z square, then again v0del v0, second term, and del w0 is third term.

Three contribution. Then the function of z. So -zdel u0w0, x, this one. Then -zdel v0, again this and this, sorry u0del w0, x. So from there, 2 contributions, one from this and one from this. Then one from this, one from this, z contribution. Then z square, z and z. So it will be +, only one contribution and one contribution. So now you have a 3 terms inside that constant case, 3 terms, 1 2 3 4 terms inside that z and 2 terms, the coefficient of z square.

So we are on the volume integration. Now the concept of moment resultant comes into the picture. Basically any 2-dimensional theory, this is the very very important step that how to reduce 3D to 2D? Or I will say the 3D case, now it is on volume. If we say the moment resultant, mass moment of inertia, then it will be, so the definition of mass moment of inertia here is -h/2 to +h/2 rho*dz, we say that let us say it is equal to a some constant which is denoted by I0 and equivalent to the mass moment of inertia, okay. Equivalent to that?

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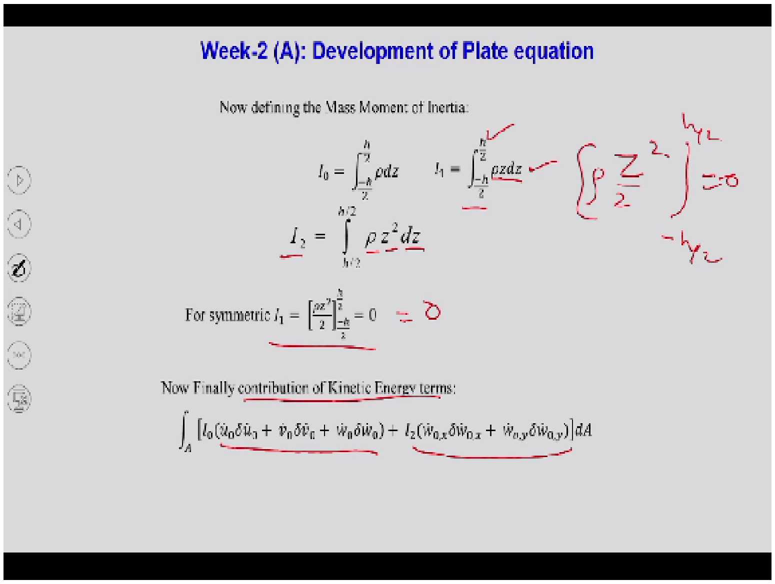

Then you define I1 -h/2 +h/2 rho*zdz. Similarly, you define I2, rho z square dz. So for a symmetric case, when you are talking about a single layer and symmetric plate that, then this if you do the integration, what will be that? Z square/2 and rho times and you put the limits, so it becomes to 0. So generally I1 will be 0. So the final contribution in the kinetic energy terms, there are 3 I0 and 2 I2 contributions.

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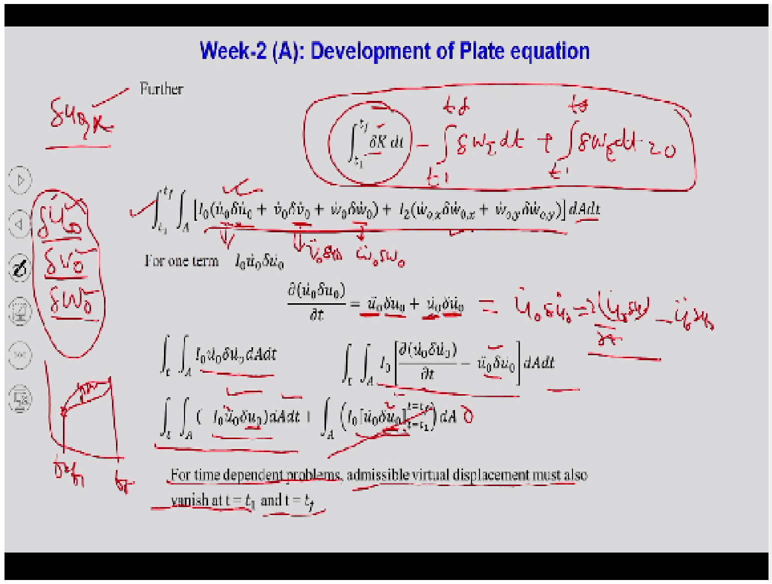

So here I would like to proceed it further in the kinetic energy. In actual, you can write... =0, "theek hai." So we are going to evaluate now this one. Del k now we know what are the terms in the del K and you put time integration over the and put the terms dAdt. The concept further evaluation is this because we know our primary variables are, variations, del u0, del v0, del w0. Other than this, if there is a derivative, time derivative or the space derivative, we have to get rid of that because we have no information about that.

So we have here del u dot. So this is not going to be there in the main set of equations. So we have to get rid of that. You know only the information or you have taken the arbitrary variations in u0, v0 and w0, not in their time derivative. So their time derivative must not be there. Or we have to get rid of their time derivative. If there is also space derivative, then also we will have to get rid of that space derivative.

So but here we see in kinetic energy, time derivative, so like as I explained in the first week of lecture for the case of beam, you can write like this. u0 del u0 and time derivative. So what is the rule that differentiation or you can say that second function as it is, differentiation of first function, +first function as it is, differentiation of second function. So from here, you see that del u0 is nothing but u0del u0 time derivative- of del w0del u0.

So I will explain only for 1 term and similarly apply for other terms. So I have taken this 1 term.

So this term can be replaced with this. Now this will remain on area and it is fine with us that del u0 variations in primary variable not seen is derivatives space or time, then again this derivative with respect to time and time integration is there. So it will be integrated over time and putting the limit over there.

Now that from time dependent problems, admissible virtual displacement must also vanish at the initial time and final time. You have a function and you have a variation. If this is my initial time and this is my final time, so it is exactly matching with, there is no variation, that should be vanished. So del u0, only in between is del u0, but here is equal to 0. So this will vanish. So this term will not contribute at all.

So you see from this term, you have a contribution of u0 double dot del u0 dA/dt. So there they have written slightly dot over there. Similarly, this contribution will be, one can tell, d double dot and del u0. From here, double dot del w0 contribution and from here also.

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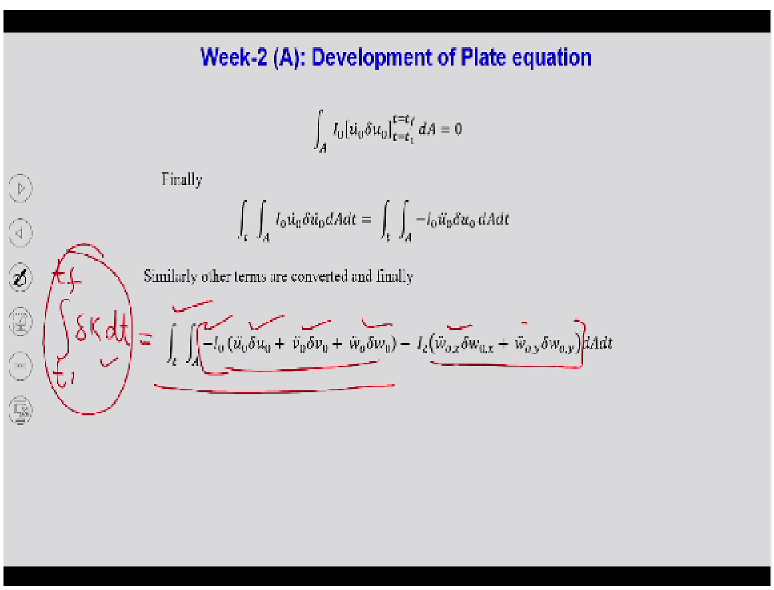

So finally we can write this contribution and this thing, dA/dt. So you can put it from outside bracket. So this is very important step that obtaining a contribution of kinetic energy terms. So for a particular case of theory whenever you are going for an advance material, there may be some more terms or some other contributions may be mechanical other than some may be some, some more contribution we come.

So this you have to derive properly. If one is interested to develop a program or develop a solution for any plate or any structure, very very first requirement is that theoretically or by mathematically formulations would be absolutely correct. There should be no mistake, no errors with sign or these things, we have to check whether my signs are accurate, whether my this assumptions are accurate or integrations I have written, you must verify at this step.

So first you find out for a particular structure for a particular advance material structure that what are the contributions or the terms inside my kinetic energy region.

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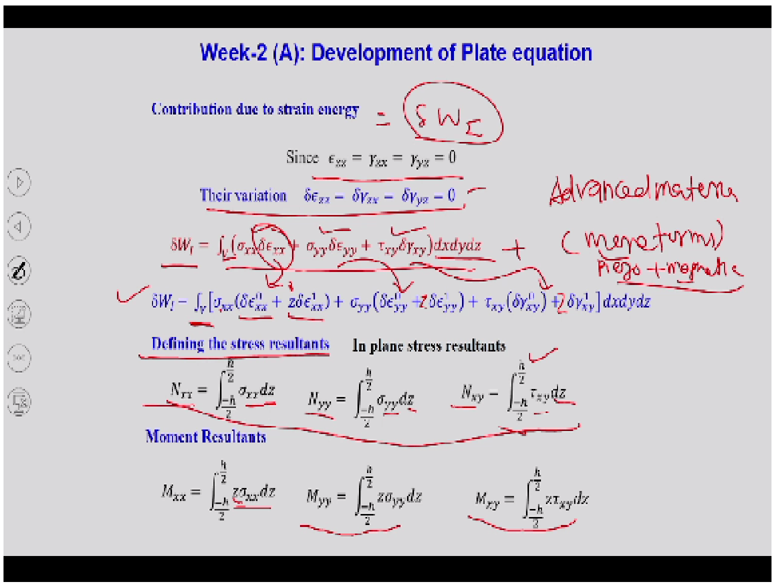

Next, now we are going to obtain the contribution due to the strain energy or I would like to say that internal work done, first variation in the internal work done. Since 3 strains are 0, so their variations will also be 0. So only contribution in internal work done will be from sigma xx del epsilon xx sigma yy del epsilon yy tau xy gamma xy and volume integration. Suppose you are going for some different theory may be, some more terms will be there which may be corresponding contributing to the internal energy of the body.

Some may be magnetic terms; some may be piezo terms if we talk about advance materials. Some may be piezo contribution, magnetic contributions. So if you have understood this concept and if you are interested to and if you read any research article in which a piezo or magnetic kind of thing is done, you can directly understand, okay this is your mechanic contribution in the strainer, internal work done due to the mechanical+due to piezo+magnetic, corresponding part of that.

So I think with this introduction, one can develop for a different kind of material, for a different kind of geometry. Now what is epsilon xx? Del epsilon xx is nothing but del epsilon of xx0 and +z time del epsilon of xx1. Similarly, epsilon yy and gamma yy. Now again the very important step, it is a integration over the volume. How to convert it to 2-dimensional case, x and y?

So we define the stress resultants. So basically Nxx, -h/2 to +h/2 sigma xx and dz. Sigma yy and dz and Nyy. Nxy, -h/s to +h/2 tau xy/dz. These are known as in plane stress resultant, Nxx, Nyy, Nxy. Similarly, you see that z term is here. So z*sigma xx, (FL) Movement, Mxx, Myy, z*sigma yy, here z times, they have did not put. Then Mxy, z*tau xy dz.

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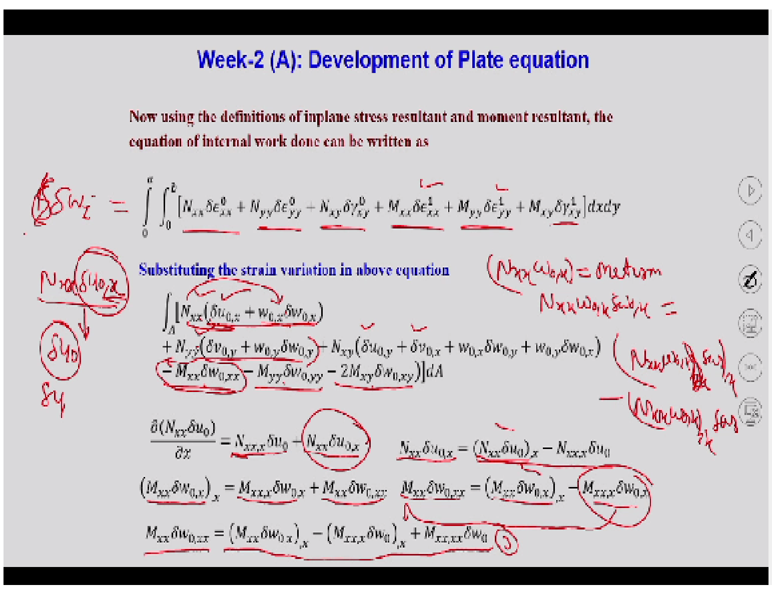

So using those stress resultants, we can rewrite del wi, "time to nahi," like this, clear. Now substituting explicit strain values, epsilon xx0. Now we may recall that, what is that? Del u0, x w0, x del w,0. Similarly, Nyy. Then Nxy. Then epsilon xx1 epsilon yy1 and del w, xy gamma xy1. Now you see Nxx del u0, x, I have already put a stress that we know only del u0. This is our first variation or our primary variable.

This is not. We do not have any information about their derivatives. So we have to get rid of this. So the basic step you put derivatives, so Nxx, x del u0, and Nxx del u, x. So basically this term can be written as Nxx del u0 whole (FL) derivative, x-Nxx, x del u0. So this term, this term, this term, you can proceed in a same way. Similarly, you have a contribution due to this. Here again you have to put it Nxxw, x as a one term and then you have to proceed.

So you can write Nxxw, x del w, x nothing but equal to Nxxw, x whole (FL) derivative, x and then w0-, oh sorry. Nxxw0, x, del w0. Then you have a double derivative here. So from this you have 2 terms contribution. So this one term will give you these 2 terms. Now Mxx del w0, xx, how many terms it will give? You see, first you take this term Mxx, x Mxx del w, x. So this can be written as like this. This is okay. But this is not okay.

Again you have to split it up and finally, this term gives 3 terms. So basically 3 3 3 and here 2 2 2, like this, you have number of terms.

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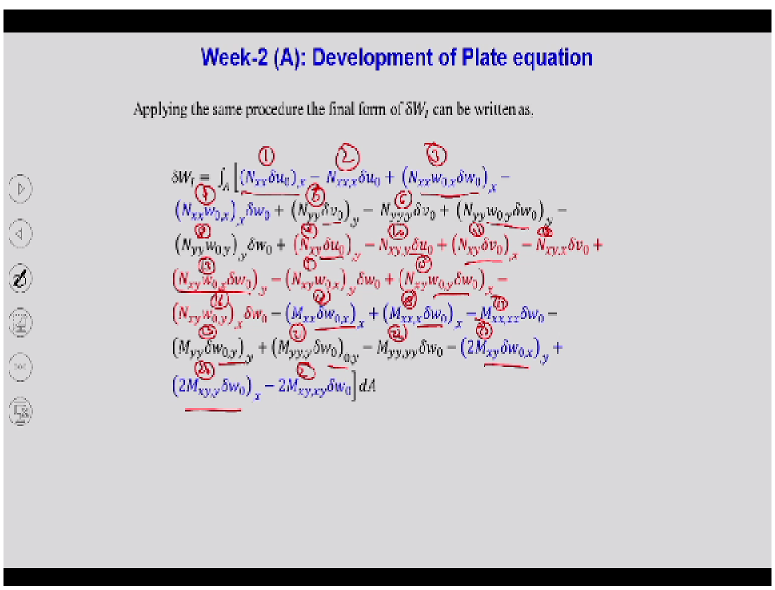

So finally, you see 1 2 3 4 6 7 8 9 then 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 and 25. So now we have 25 terms. Out of these 25 terms, some of will go to the area integer and some of them will go to boundary integer. So it is very much important if you are developing a theory. So at this stage, you mark all these things. How many terms? And you have to rearrange those terms. Once, some terms under the area and some terms under the boundary.

You see this whole term, derivative, x will go to the boundary. The underline terms will go to the boundaries and this and this and this and this and this and this, again this this this this this and this, this will go to the boundary and the rest of the terms, second number terms will be on the area.

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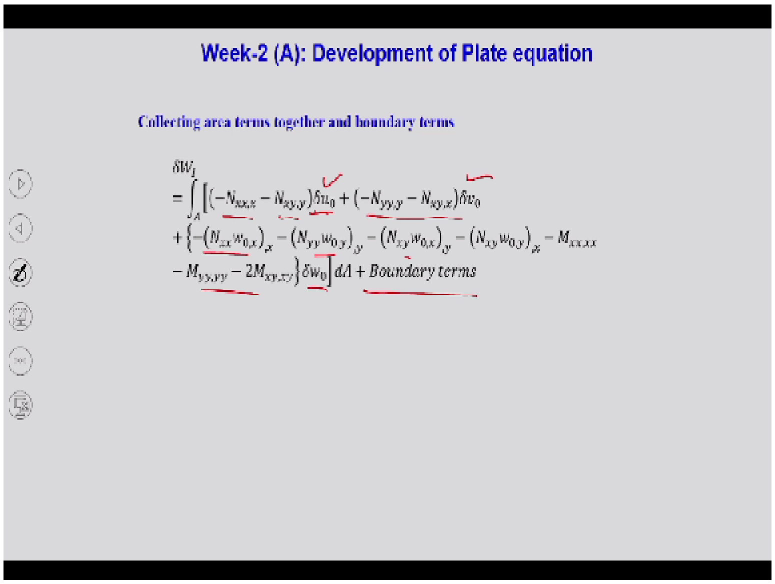

Some going to see that area, Nxx and Nyy and clubbing, its similar kind of coefficients of del u0, coefficients of del v0 and coefficients of del w0, putting together and some boundary terms. So in the next lecture, I will explain rest of the boundary terms and we will proceed further.