Theory of Rectangular Plates - Part 1

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Lecture - 02
Energy Principles

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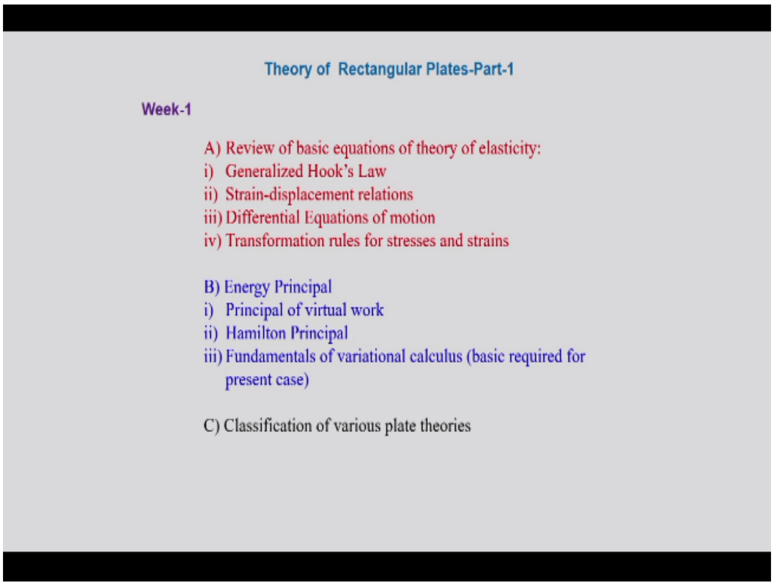

Welcome to our second class. Already, we have covered review of basics, generalized Hook's Law, then strain and displacement, then differentials, transformation rules of stresses. Today, we will study or I will explain that energy principles first that principle of virtual work, second Hamilton principle and fundamental of variational calculus.

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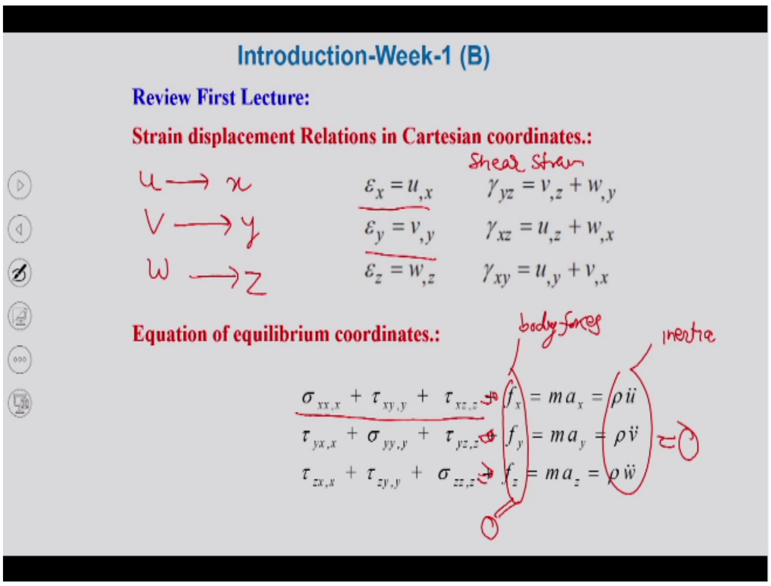

So in the very first lecture, I have given you the relations between strain displacement relations in Cartesian coordinates. They are epsilon xx, epsilon yy, epsilon zz are represented like this, del u/del x, del v/del y, del w/del z where u is displacement along x direction, v is displacement along y directions and w is displacement along z directions. Then, corresponding the shear strain gamma yz, gamma xz and gamma zy.

These we have to remember or these are the standard notations given in the book of advanced solid mechanics. Then, I have given you the relations for equation of equilibrium where this is the body forces. What do you mean by the body forces? They maybe the gravitational forces, electromagnetic forces such that which acts over a volume. Then, these are your inertia terms.

So most of times when we are not interested in the dynamic behaviour of the body, we are interested only in the static behaviour then inertia can be taken as 0 and further sometimes we are saying that body forces neglecting, so most of the time when you say look at the equation of equilibrium you will find this set of relations=0.

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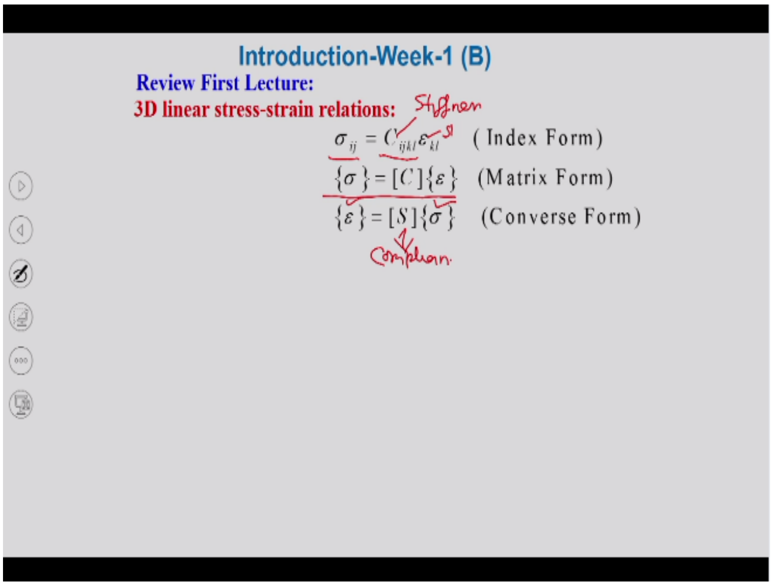

Then, I explained that linear generalized Hook's Law where sigma ij is equivalent vector. This is we call stiffness and epsilon is also a column vector strain so in matrix form it can be written as and if you are interested to find out the strain in terms of stresses then this is denoted by S and we call that as compliance.

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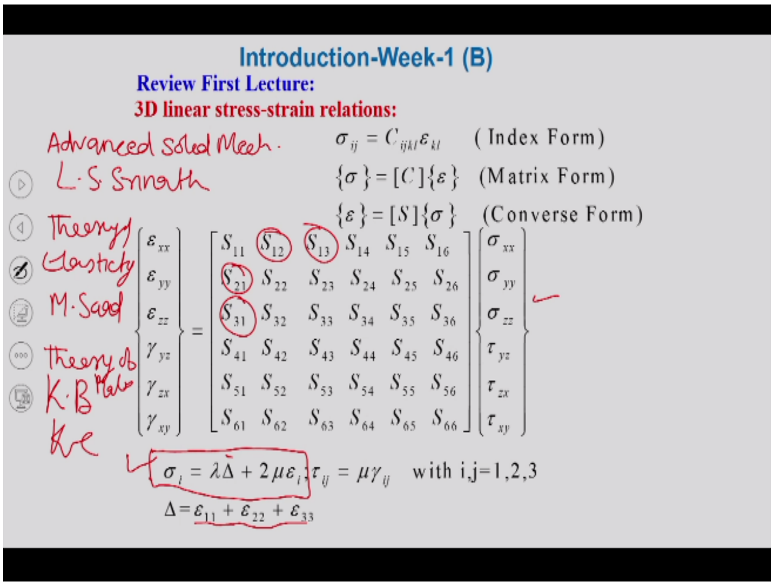

So I would like to say inexplicitly we can write strain column vector can be represented, this is the S matrix and stress matrix. This is valid for a anisotropic material if you talk about orthotropic then there will be some similarity 12 and 21 will be same, 13 and 31 will be same like that. Then for the case of isotropic materials, we can write in this form, so in the basic solid mechanics book like advanced solid mechanics by L.S. Srinath or theory of elasticity by M. Saad.

So for case of isotropic material even in theory of plates book also whether it is a K. Bhaskar, K. Chandrashekhara. So theory of plates there relations between stresses and strains for case of isotropic material where delta is basically the sum of diagonal strains and it is related to G.

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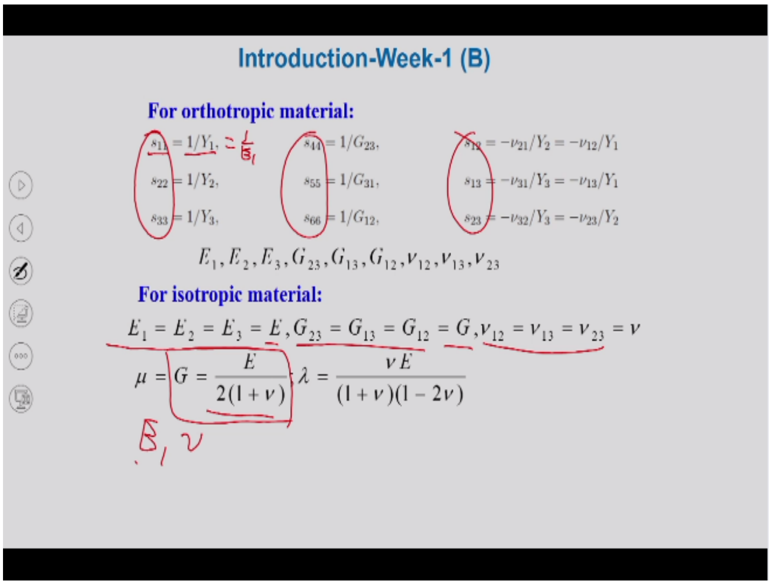

But you may be aware that we use to write aware about the material property in terms of Young's modulus, shear modulus or the Poisson's ratio. So how to find out these compliances. So it is a typical example for a orthotropic material, if you know that Young's modulus in one direction then compliance is S11 will be 1/Y1 or 1/E1 if you say that Young's modulus/E1 s22 similarly.

Then, the shear modulus is related to s44, s55, s66 and Poisson ratio is related to s12, s13 and s23. For the isotropic material, E1, E2, E3 will be just E. All shear modulus will be denoted by just G, all Poisson's ratio will be denoted by just mu. So it looks like 3 constants but actually G can be written like this with this relation. So for the isotropic material we have only two independent constants E and mu.

If you give a material which is isotropic like steel or ammonium, we have to just give the value of E and mu and G can be calculated.

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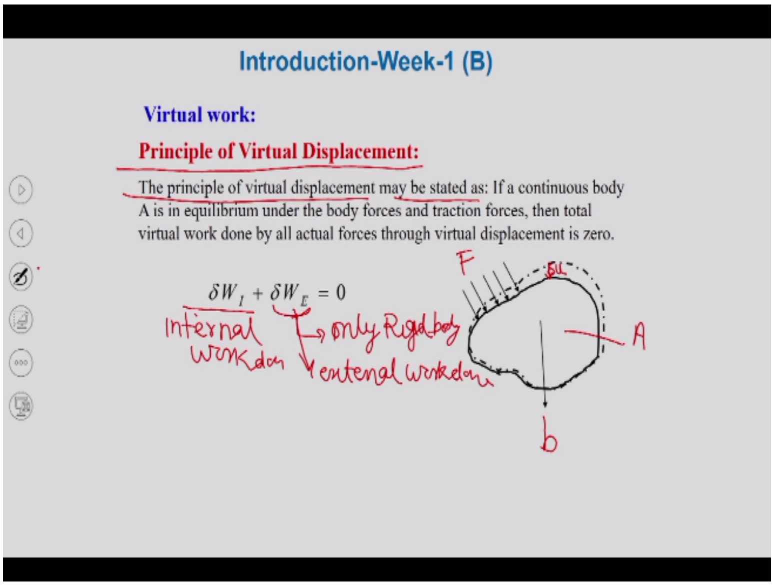

Now our today's topic the principle of virtual work, so basically principle of virtual displacement first I would like to state this principle and explain its basic terms. The principle of virtual displacement maybe stated as if a continuous body if you say this body A, A is in equilibrium under that forces let us say this forces F and some body forces b then the total virtual work done by all the forces through virtual displacement it will be 0.

Let us say this is your virtual displacement delta u or del u, so F*del u+b*del u will be=0 because body is in under the equilibrium. So total work done since displacement is 0 so total virtual work done will also be 0 so we can write. Now we are discussing about a deformable body. In engineering mechanics, generally we talk about a rigid body so there only we discuss about only the external work done.

The forces which is acting on a body externally let us say F1, F2, F3 and that del u1, del u2, del u3, they are virtual displacement, external body for if you talk about only rigid but we are talking about elastic body, deformable body then it will have 2 components what is internal work done and external work done.

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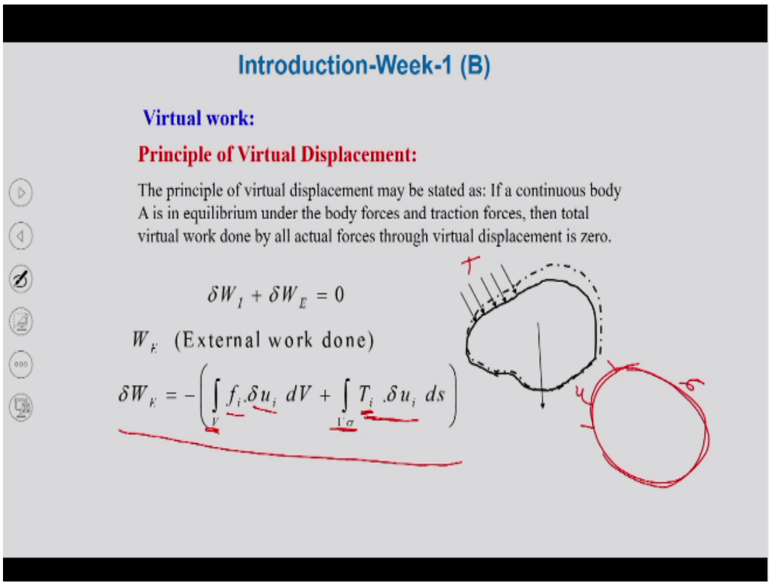

Then, how can you define that external work done for a continuous system for like a beam or a plate, it will be this is the body force into the virtual displacement working over a volume. Then, the traction forces, it works over a boundary where traction is applied, where displacement is 0, there no work but where there is traction forces are applied, so over that boundary only.

For example, a boundary there may be some displacement prescribed and over this boundary stresses may be prescribed. So the boundary where stresses maybe prescribed over that attraction*virtual displacement. So that will be your external work done that traction forces over that surface area and body forces.

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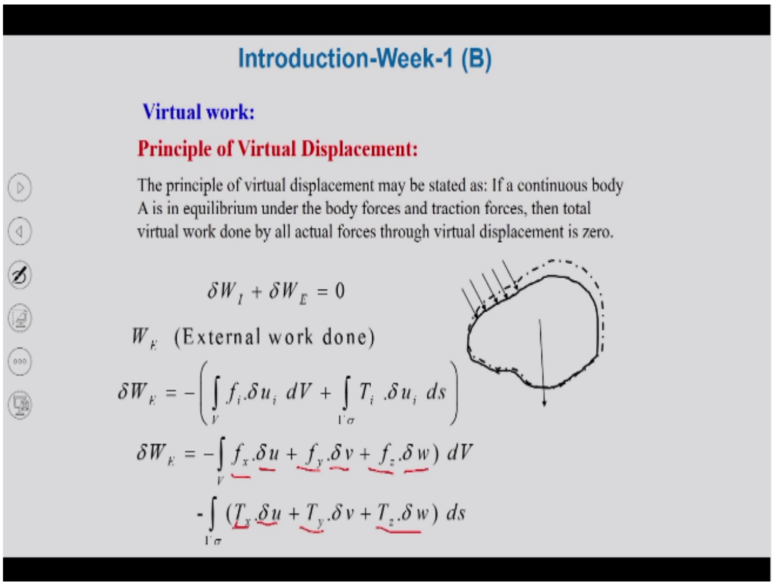

Further inexplicitly if you would like to write in terms of a Cartesian coordinate, then it will be fx along x direction, fy along v direction and fz along z direction. Similarly, traction along x directions del u, along y directions and along z directions. Why it is negative sign because work done is on the system by the external forces.

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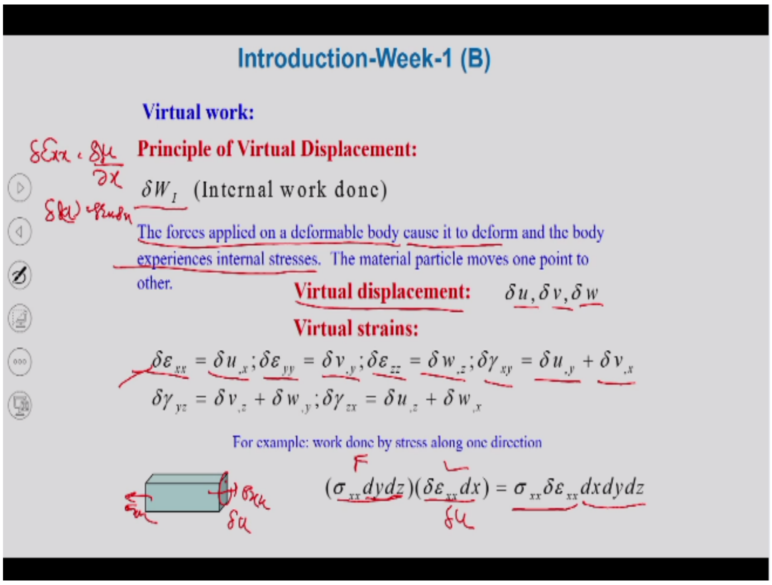

Now you are interested to find out what is the internal work done. Since there is a body and we have applied some tractions or body forces due to that insight the registering stress system is generated, so basically the forces applied in the deformable body causes it to deform and the body experience internal stresses and the material particle moves one point to another point.

So work done by this stresses when inside the body, the material displaces material point. So that is work done by the system that will be positive, so internal work done will be positive. So virtual displacement del u, del v, del w. So what will be the virtual strain? So del epsilon xx is nothing but del u, x, del yy will be del v, y and del epsilon zz will be del w, z. Similarly, the shear strain del gamma xy will be del u, y del v, x.

Now if we say that in a one-dimensional body only sigma xx is acting means the stress generated and the displacement along that direction is del u. Then, the work done will be over this area the resultant force sigma xx dy*dz. This much will be the force and the displacement will be this much. May be you can write in properly epsilon xx will be del u/del x, so del u can be written as del epsilon xx del x so that will be equivalent to del u.

So this will be your work done under one dimensional cases.

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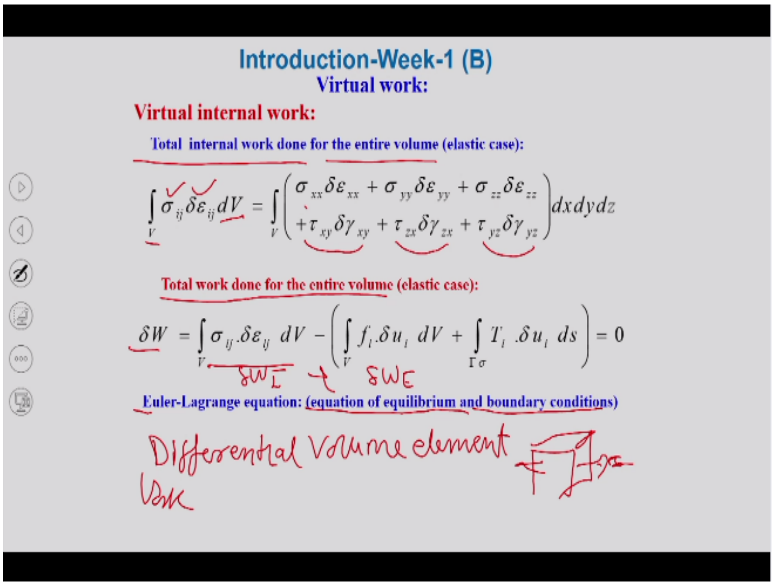

So we can extend it to for 3-dimensional case, so you have sigma xx, del epsilon xx+sigma yy del epsilon yy sigma zz del epsilon zz and work done by the shear forces, shear stresses, shear strain. So we can remember that sigma ij del epsilon ij works over a volume. If we talk about total internal work done for entire volume, then we can write volume integration of sigma ij*del epsilon ij.

Now total work done for entire volume del w will be this will be the del w I and this is del w E external work done. Now interesting point is that we can obtain equation of equilibrium using this virtual work done. Previously, at the undergraduate levels you have already obtained that equation of equilibrium using the differential volume element.

There are number of techniques like you have a cube and applying some forces plus but with the help of virtual work done we can also obtain the equation of equilibrium+the boundary conditions. This is the advantage of the energy principles. You will get a system of equilibrium or governing equations plus the associated boundary conditions whereas in other approaches like differential volume approach or any other approach which is not based on the energy methods, you cannot found the associated boundary conditions.

So we have to just own our physical interpretation that what should be the boundary conditions, what should be the forces or displacement has to be prescribed over that boundary but if you use that energy principle, so for that we you get equation of equilibrium as well as the boundary conditions.

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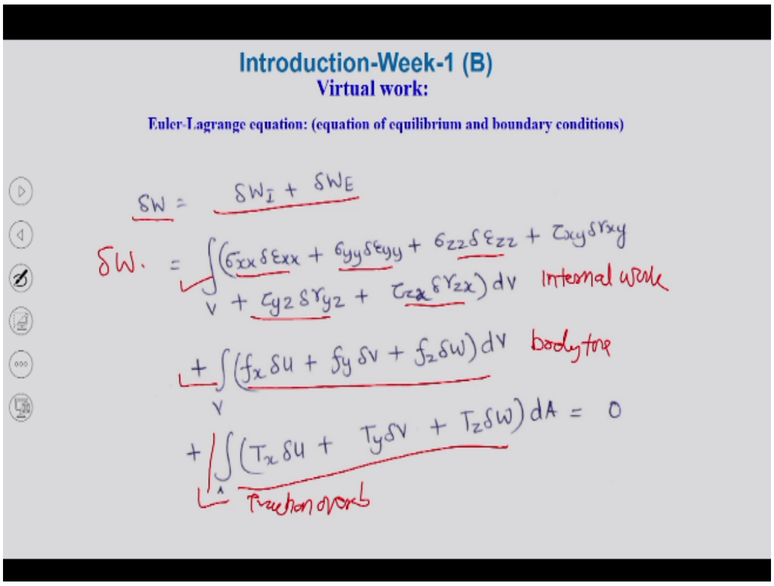

So I am going to prove this. Let us say the total virtual work done will be del w I/del w E and del w I or I will say that inexplicitly in a large bigger form due to the contribution due to the internal work done+contribution due to the body forces and this is the contribution due to tractions over boundary. So this is the area integral, volume integral, this is volume integral. So this is the internal work. So this up to your aware, now this will help you to develop equation of equilibrium. Let us see.

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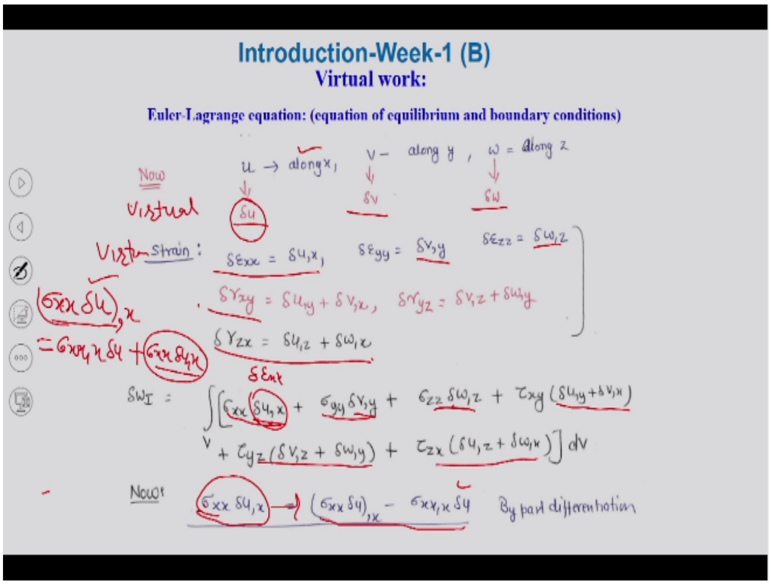

Now displacement u which I have already explained along x axis and its virtual displacement will be del u, virtual displacement along y direction, virtual displacement along z directions. Similarly, the virtual strains can be written like this. Then, you substitute the strains into the internal work done instead of del epsilon xx I am representing this is del u, x. Further del epsilon yy is replaced by del v, y, del epsilon zz is replaced by del w, z.

Similarly, del gamma xy is replaced by del u, y+del v, x. Till this step is clear that you have to just replace virtual strain in terms of virtual displacements. Now this term you have only idea about the virtual displacement that body we are providing a virtual displacement. We do not know, no information what about the derivative of that virtual displacement whether it will be 0 or not.

So we can write this expression like sigma xx and del u. If you take the derivative along x axis, it will be written that differentiation of first function, second function as it is+first function as it is+differentiation of second function. So which is this? So this can be written as summation of this whole derivative, x and –of this. So this term can be replaced with this term. This is a very, very important step.

And I will say that this step or this process is used whenever you are going to develop a theory for a beam or a plate or a shell we are going to use this step. Similarly, sometimes there may be some other entities not like this stresses or displacement maybe something else but we use to try u because we are interested to only in terms of del u not in terms of del u, x not derivative in this. Similarly, we can replace this one this thing, this thing.

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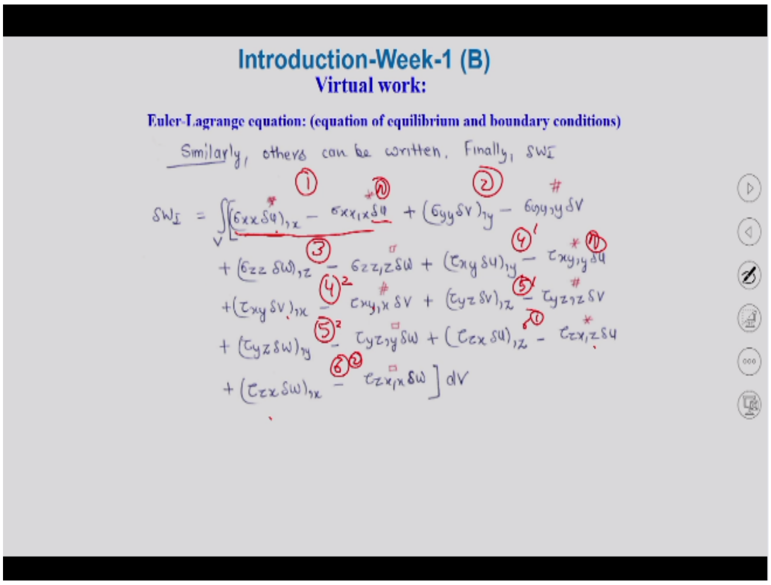

So finally you see one contribution. First term contribution, second term contribution then third term contribution. Then, it is a shear stress contribution then we have a two displacement terms. Then, there will be two contributions up to the fourth of one and fourth of two. Then fifth of one, fifth of two similarly sixth of one we have six basically terms.

Now you see you have to clump the coefficients of del u 1 star and this is the star marks are the coefficients of del u. So I will say that A and further star del u coefficient are this. So we keep these things together similarly we will collect the coefficient of del v which is marked as del v and again del v, similarly 4 del w but what about these. So similarly the derivative of x we will keep it together the derivative of y and derivative of z.

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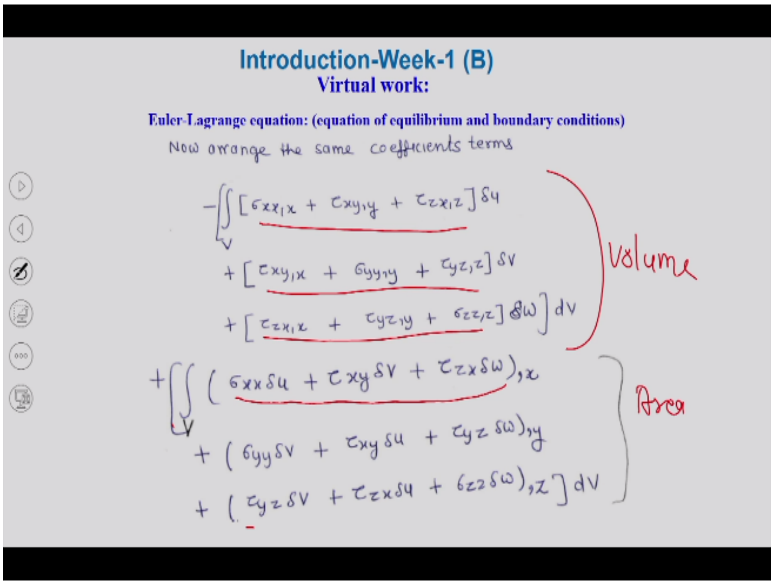

So I am writing separating that coefficients of del u, coefficients of del v, coefficients of del w. You see clearly this is the first equation of equilibrium coefficients, sigma xx, x+tau xy, y+tau zx, z and these are the whole derivative of x, whole derivative of y, whole derivative of z. So if there is integration derivative so one derivative will go so it will reduce to area integral and this remains on the volume integral and it will reduce to area because there is a derivative kind of thing is there.

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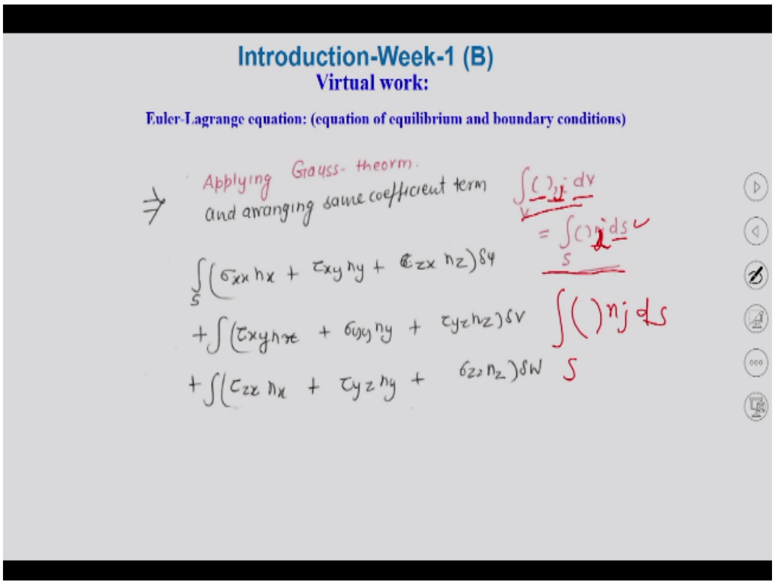

So you see again I would like to tell you, you can do explicitly also that x if you remove, it will be integral over y and z, this will be integral over x and z and this will be integration over x and y. You can use doing the more general term which is that if any term and derivative along y volume integral that can be written like this that entity*normal vector in that you will say that ni basically when it is i or if it is j it will be nj and ds surface integral.

So volume integral can be inverted into a surface integral like that quantity nj and ds over that.

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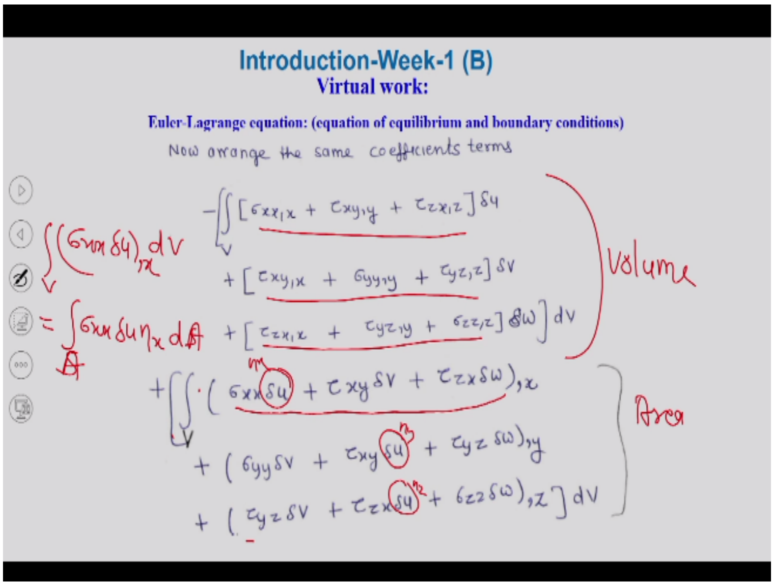

So similarly sigma xx if I go to back here sigma xx del u, x volume integral this can be written as sigma xx del u mx and ds we want to say that over the surface area, area integral or area integral you can say that. Similarly, this will also have nx and x and this ny and y and then accordingly del u arranging together. So this will have nx, this will have ny, this will have nz.

So in this way I have arranged the terms like this, del u coefficients, del v coefficients and del w coefficients.

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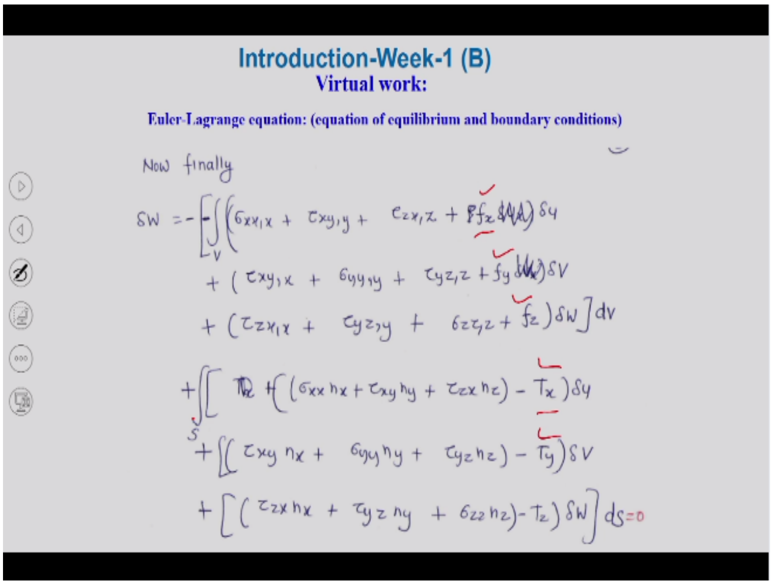

Now finally one can write over the volume, we have body forces and over the area or over the surface we have tractions.

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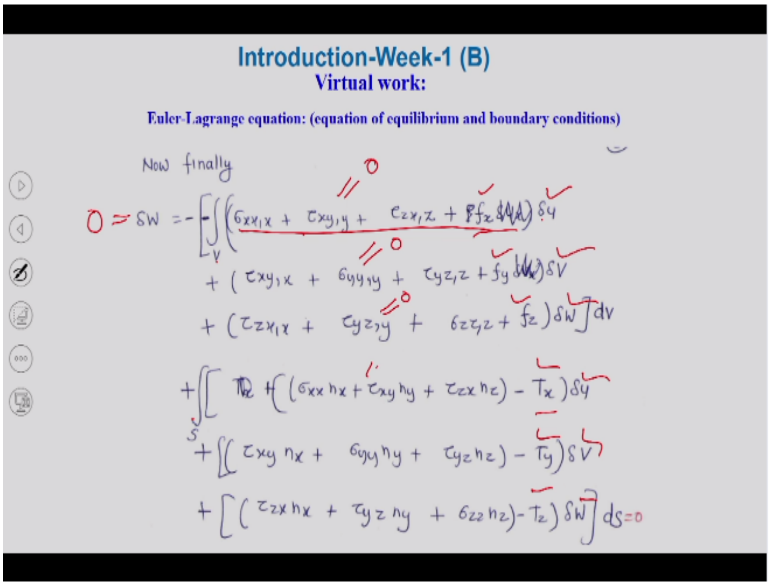

Next, since displacements are virtual, they are arbitrary; their coefficients must vanish, so this displacement are arbitrary. Ultimately, this has to be 0 so it means this will be 0. It was on under the integration but del u, del v, del w are arbitrary this thing so their coefficients must vanish.

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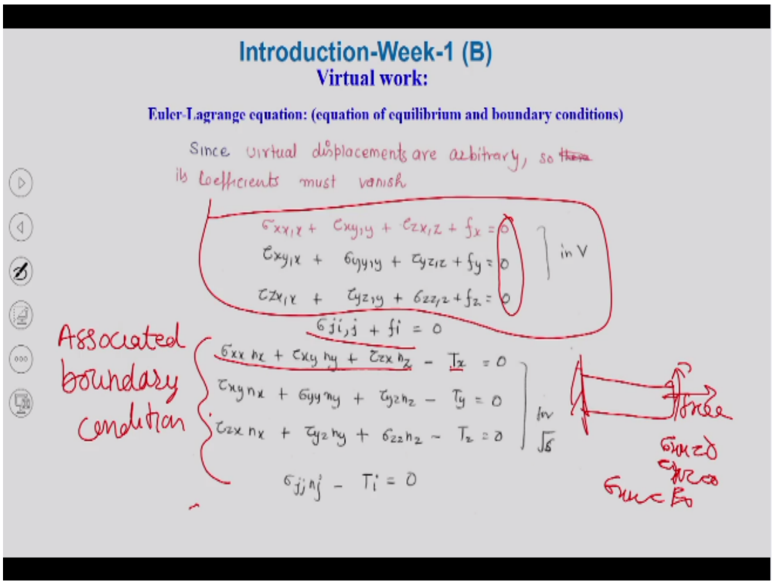

So it leads to you first equation of equilibrium. We can write index form like this and the associated boundary conditions. So this term you will not find anywhere, if you apply a differential volume element or something it tells you that this term is equivalent to, this is due to the internal stresses and this is due to the external. So externally applied loading, if that surfaces free let us say if we talk about a beam and this surface is free cantilever.

It means there is no stresses, so this quantity has to be 0 that is why we say over this what are the quantity, sigma xx has to be 0 or tau xz has to be 0. So these things come from there. If it is applied, then we say that whatever the component of that let us say there is axial force then we will say that sigma xx=Fx applied one.

Now the application of virtual work, okay one more thing I would like to say it here that right hand side are 0, there is no inertia terms. So this principle is used whenever you are interested in the static loading, no dynamic case, no vibration case. So for that you can obtain the governing equations.

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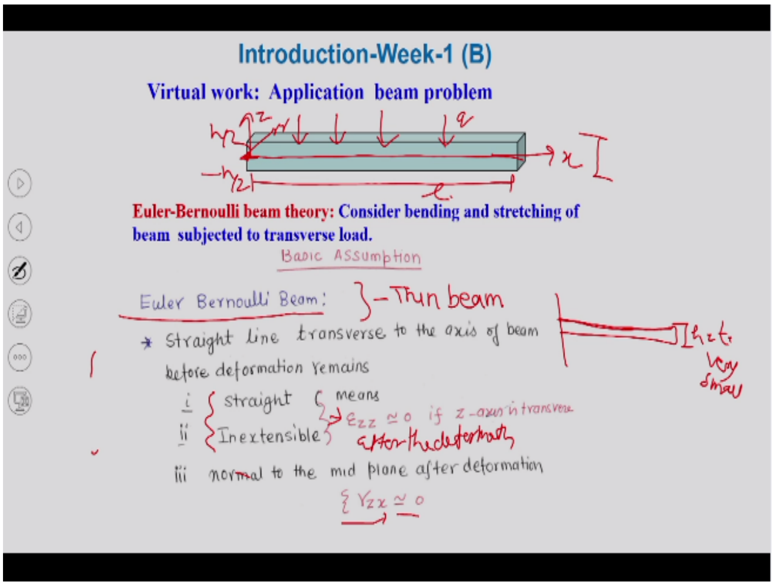

So I am going to explain for that how to apply this method to analyze a bending of a beam if let us say some q load is there and this axis is x, this axis is z and this you can say y. This length may be taken as n, this height so basically at this –h/2 to +h/2 at center 0. So most of the undergraduate books you will not find that how to develop the equation the governing equation of motion for the case of beam.

They are using the strength of material approach that the bending moment and other concepts finding out the governing equations but using the energy method, using the virtual work, I am going to explain how to develop set of governing equations. Basically, this is the first step or if you can understand this step definitely you can able to develop a theory for the case of plate or for the shell.

So I am going to explain the basic steps. The very basic step of any theory suppose if you are talking about a Euler Bernoulli Beam theory, there are some assumptions that straight line transverse to the axis of a beam before deformation remains straight inextensible after the deformation. Then, it is normal to the mid plane, remains normal to the mid plane after deformation.

So basically these physical assumptions help you to find out that there is no strain along z direction and there is no shear strain along the z direction. There is a purpose of writing epsilon zz equivalent to 0, gamma zx equivalent to 0 that we are neglecting the strain that is why this theory is valid for thin beams because of thinner that along z direction your thickness is if you say h or you say t is very small.

It does not take any shear strain or normal strain along zz direction.

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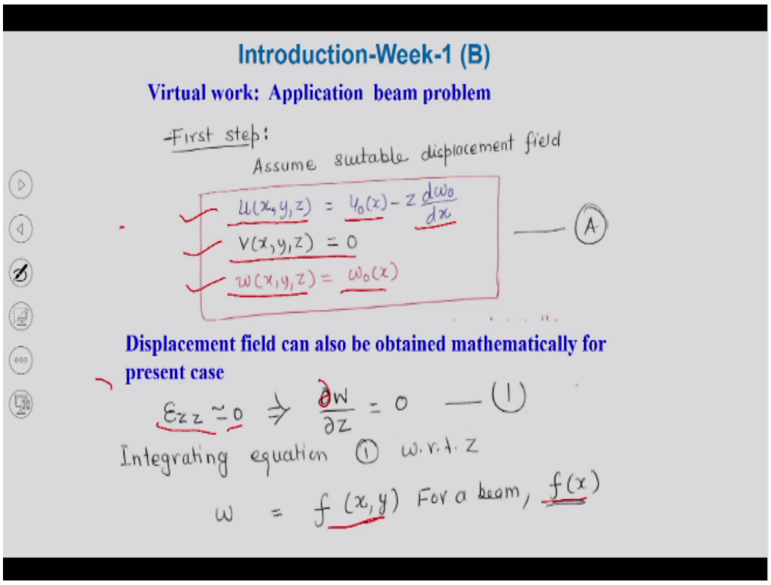

So very first step if we talk about a displacement based theory, I will come up that there are theories which are based on the stresses. There are some theories which are based on mixed displacement as well as stresses there are theories but in the literature or in the structural engineering you will find 90% of the theories are based on the displacement based assumptions.

In that we assume displacement first and based on that assumptions we solve the set of governing equations. So here assume displacement for the case of Euler Bernoulli. This is the neutral or mid plane displacement, this is due to the curvature effect and w, along y direction displacement is considered 0 because that direction y is very, very negligible compared to length directions and we considered that beam is subjected to the transverse loading.

If this same set of displacement field, the one way is that based on some guessing or some based on experience or some based on experimental data one can assume this displacement field but as we see that in the basic assumption epsilon zz and gamma zx are 0. So from there we can also obtain the set of relations. Let us say epsilon zz is 0, what is that epsilon zz? Del w/del z.

So if you integrate with respect to z, it will be function of x and y or for the case of beam it will be a function of x only.

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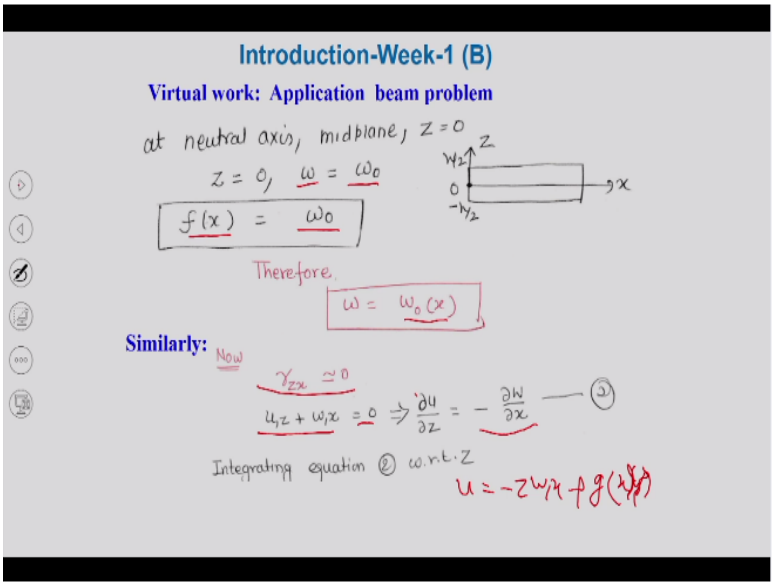

Then, at the neutral axis when z is 0, w will be w0, so fx is nothing but w0 so we can write w is w0 and w0 is a function of x only. Similarly, if you take the shear strain is 0 and then equate to 0, so from there del u/del z can be written as like this if you integrate with respect to z. So here it will become u-z time w, x+let us say some function g which will be a function of x and y or just an x in case of b.

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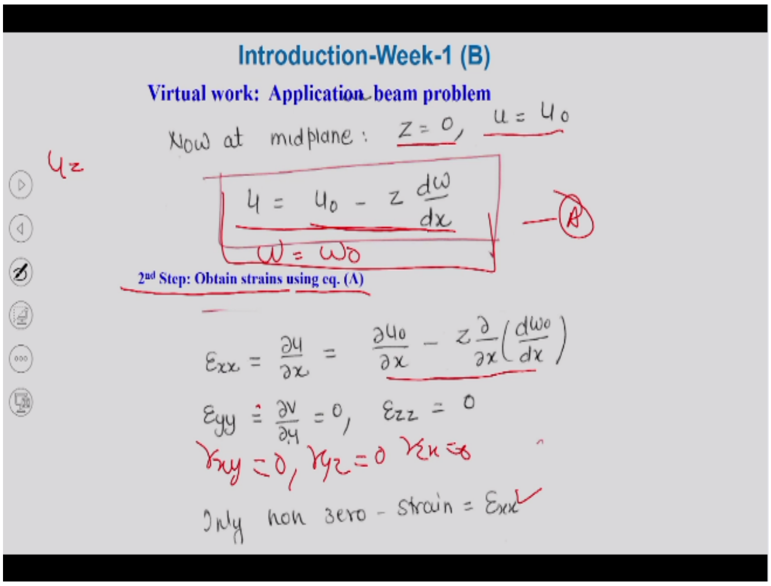

So finally when at the neutral axis z is 0, u is u0, so u can be written as u0-z times del w/del x. So that was our first step either we have to assume u, v, w or we have to initiate based on some experiment or based on some experience. Now we are with that u and w=w0. We have these two displacements. The next step second step, obtain the strains using these equations. I will say that let us say epsilon xx is del u/del x where u is this, take differentiation with respect to x.

It will be coming like this and if you obtain epsilon yy, epsilon zz and all other strains whether you talk about shear, they are coming to 0. So you have only nonzero strain is epsilon xx for the case of thin beam.

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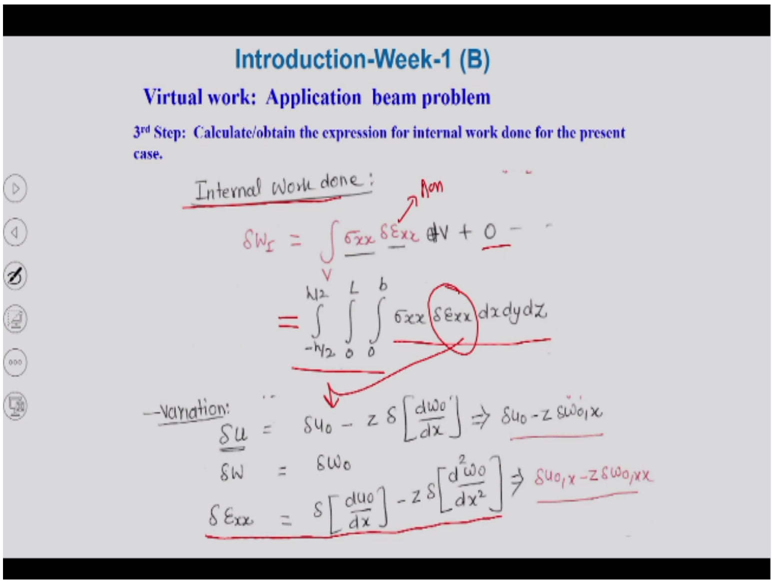

So in the third step, we have to calculate internal work. Internal work done for the case of 3D body epsilon, sigma ij del epsilon ij. So now we have only nonzero is del epsilon xx, others are 0, so we can write sigma xx del epsilon xx over the volume integral dv a so we can write like this. Further there I also told you that replace the strain in terms of displacement, so del u, del w so del epsilon xx is nothing but this.

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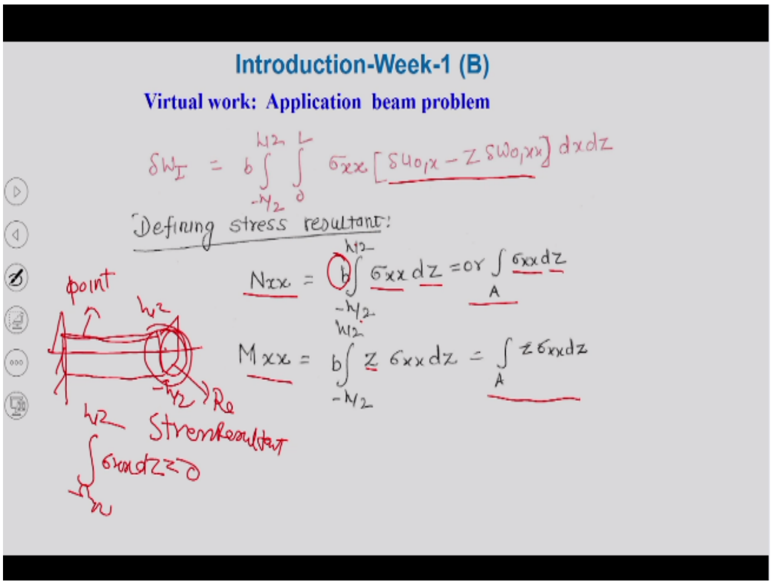

So you can replace del epsilon xx with this where u, x is derivative with respect to x and w, x axis means double the derivative with respect to x. Now there is a time to define the stress resultant. This is also one of the very, very important step for developing the theories that you have if you do not define the stress resultant, this volume integral will not converted into area integral or a line integral.

So along the z direction or if you have gone through any advance courses theory of elasticity or advanced mechanics. So if your this is –h/2 to +h/2 so over this stress boundaries are satisfied in terms of stress resultants. So if you talk about the Airy stress function or any problem, so over these surface boundaries are satisfied point wise but over the thickness zone boundaries are satisfied like sigma xx dz -h/2 +h/2=0.

That boundaries are satisfied in terms of stress resultant. So here we are defining in plane stress resultant Nxx which is nothing but sigma xx dz over that thickness. For the case of the N, this area is also taken considered. So that integration reduces to b or we can write the area integration sigma xx dz. Similarly, if you have a movement z sigma xx dz at moment resultant Mxx.

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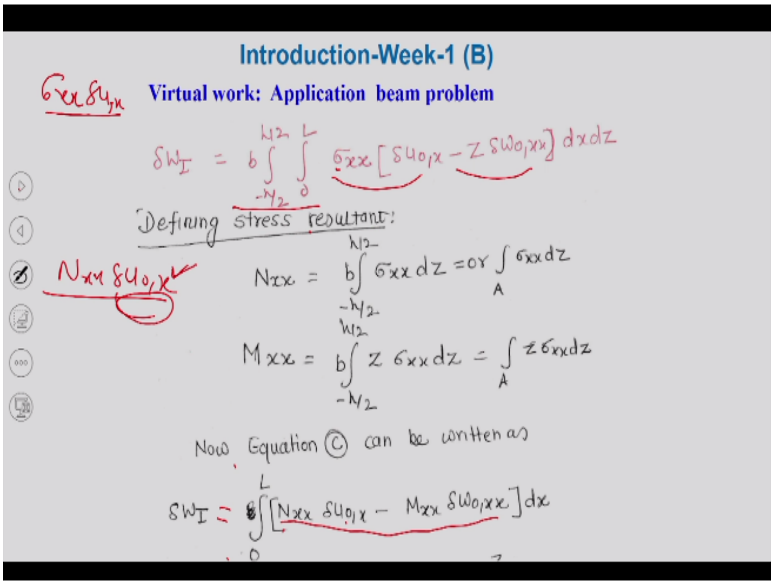

So finally this expression this integration can be reduced to a line integration 0 to l Mxx del u0, x sigma x*z Mxx del w, x. Here you see when I was explaining that how to find out the equation of equilibrium there was a term that del u, x here we have a term Nxx, del u0, x so we are interested to get rid of this derivative of x.

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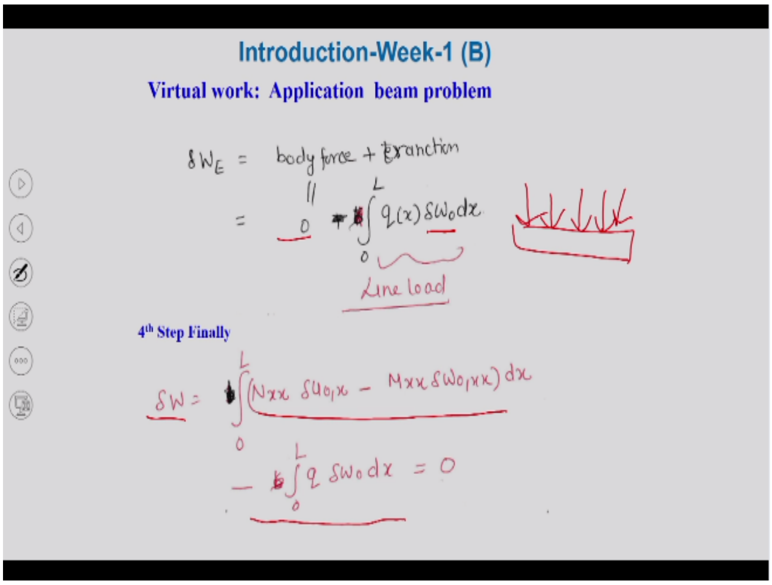

You see okay then again some body forces, we are considering no body forces only external, so beam is loaded only at the top. Sometimes in plane also no problem, so 0 to l qx and del w0. So the total work done will be due to the internal work done and due to the external work.

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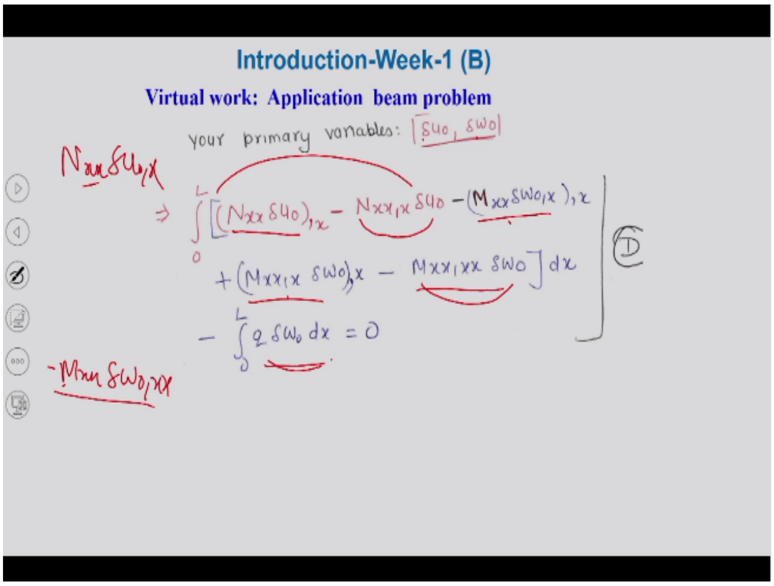

So your primary variable del u0 and del w0, so we are going to express Nxx del w0, x like this. Similarly, you have Mxx del w, xx will be these 3 terms contribution, this we are converting into basic form del w0, del w0 and that form and external work done. Now you have to arrange coefficients of del u0, coefficients of del w0. So basically this integration 0 to x will go to the boundary.

Similarly, this integration will go to the boundary. This term will go to the boundary, only on line integration this and this and this will form.

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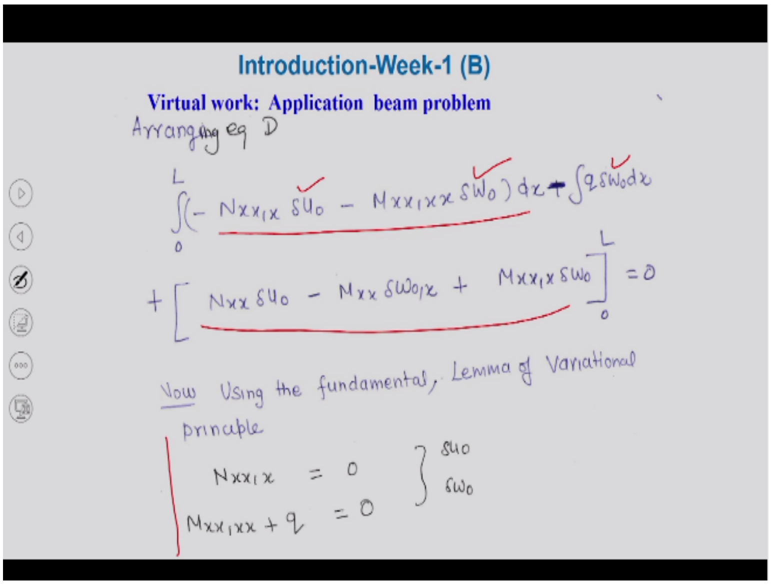

We can see that over a line integration and over the boundary. Now you say that del u0 and del w0 are virtual displacements, they are arbitrary, so their coefficients must vanish. So that is why you get these two set of governing equations. I think you have never encounter that doing this process how to obtain this set of governing equations. In most of the undergraduate books, this is written directly but how can we try this set of governing equations.

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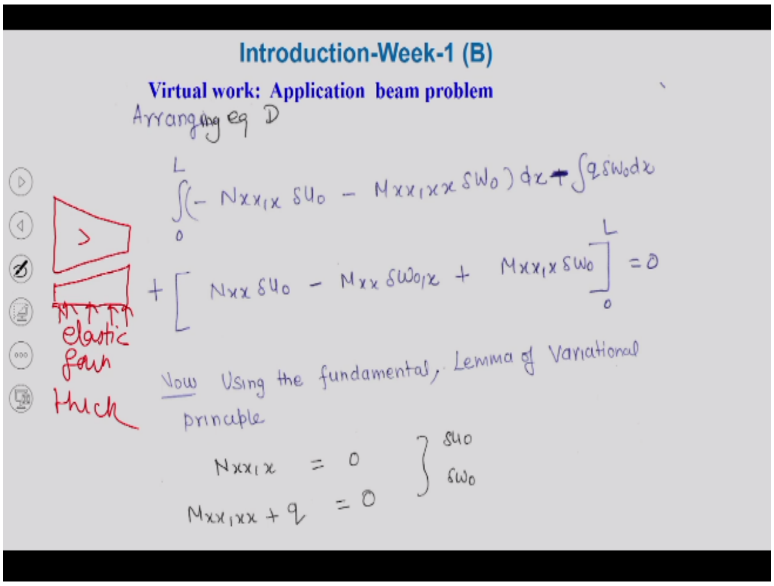

Suppose right now your beam is made of steel material. Later on you say that my beam is made of functionally graded material or my beam is thick or my beam is heavy uniform in varying geometry. Then, this equation does not hold, this is for a special case that a beam is thin, that section is not varying and only loading is transverse so for that cases the governing equation.

But if you know that process that how to arrive the set of relations, so you can say that okay let us say my beam is this shape or my beam is sometimes this shape. I would like to try a set of relations or I am having some resting on some elastic foundation for that case what will be the governing equation or if my beam is thick then this set of equations will change. So we must know the general procedure.

So that later on for advance material, different kind of loading, different kind of geometry, we can develop governing equations.

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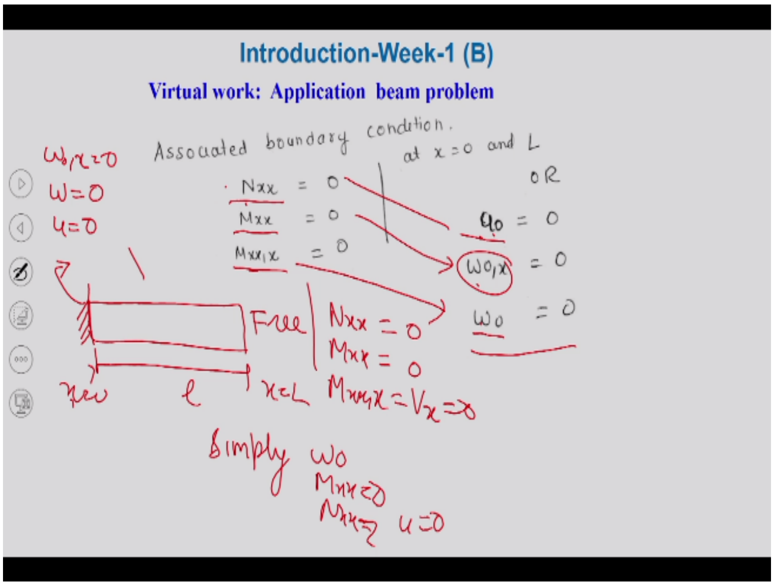

Now come to the associated boundary conditions. These are the boundary conditions Nxx, Mxx. Then, this is the displacement. So if you talk about a beam let us say this length is l and this x is 0 and this x is L. So we can say that either Mxx or u0. If I say this edge is fixed what does it mean? It means at this edge u is 0. Now what about the second combination moment Mxx or it is slow or the shear Mxx derivative or the deflection.

So at the clam test, u0=0, w0=0 and w, x=0. So from this we have to find it. If this edge is free so all stress components, what will be Nxx, Mxx and Mxx, x. Sometimes we call it is Vx has to be 0. If you talk about the simply supported, then deflection will be 0 but slope maybe anything so corresponding term moment will be 0. Similarly, u can be anything so corresponding Nxx can be 0.

If you talk about a hinge term so instead of that u will be 0, so using this principle you have obtained the choices that either Mxx or u0, Mxx or w0, x, Mxx, x or w0. So based on that boundary condition you say if my w is 0, so this can be anything. If my w is not 0, if my w is I say that 0 so this maybe anything. If this is not 0 this has to be 0 combination of this either Mxx or u0, Mxx or w0, x, Vx or w0. So this has given you the choice.

Later on for the case of thick beams or some different kind, so there may be more number of variables not just 3 maybe more number of variables. So from there that you have to choose that for particular set of boundary conditions what are the variables has to specify over there.

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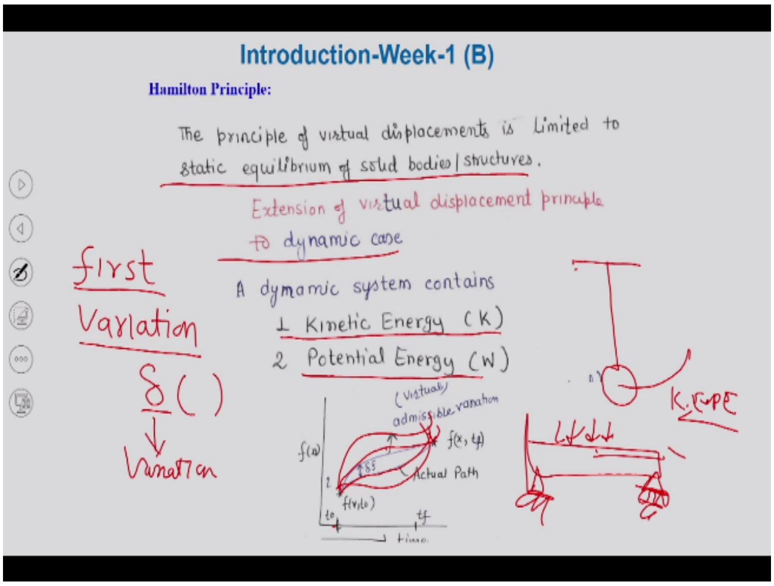

Now as I have said that virtual displacement principle can be used to a static equilibrium of solid bodies or structures if you are interested to develop dynamic version. So I would like to say this is the extension of virtual work done to the dynamic case. So in the dynamic case, what the extra term, that is the kinetic energy. A dynamic system will have kinetic energy as well as the potential energy.

For example, if you take about a pendulum, so it will have a kinetic energy+potential energy, one degree of freedom but if you talk about in the case of a beam, I am interested to find out the dynamic behaviour for a continuous system. Then, you have to use the Hamilton principle. In this Hamilton principle, before that I would like to explain you the variation or specifically for whatever we are using that first variation.

In virtual displacement principle, we said it is the virtual displacement del u del w, so these we call the first variation del operator, del of any quantity. So it is a variational operator. Concept is this, let us say any function which is a function of a time and initially at time t0, it has some value boundary conditions and at final position at time t=f it has some value but in between it may follow this path, this path, or some other path like this.

There may be some actual path which is not known to us. So we will see that we are saying that let us say as deviation from that exact path is delta first variation and we are trying to minimize that variation. So we are going to near the exact path.

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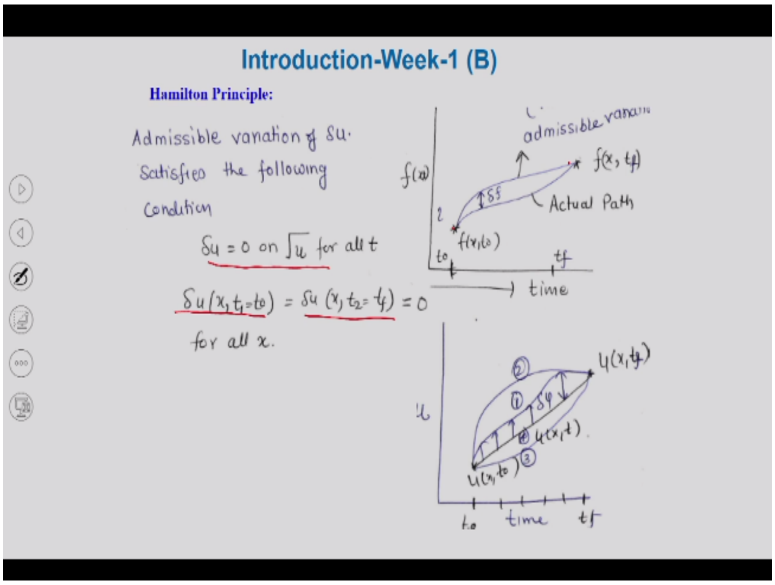

So admissible variation of del u, del u (FL) variation possible (FL) admissible, which satisfy the boundary conditions that on the boundaries del u is 0 or initially and final position that this del u has to be 0 for all values of x. So that is known as first variation.

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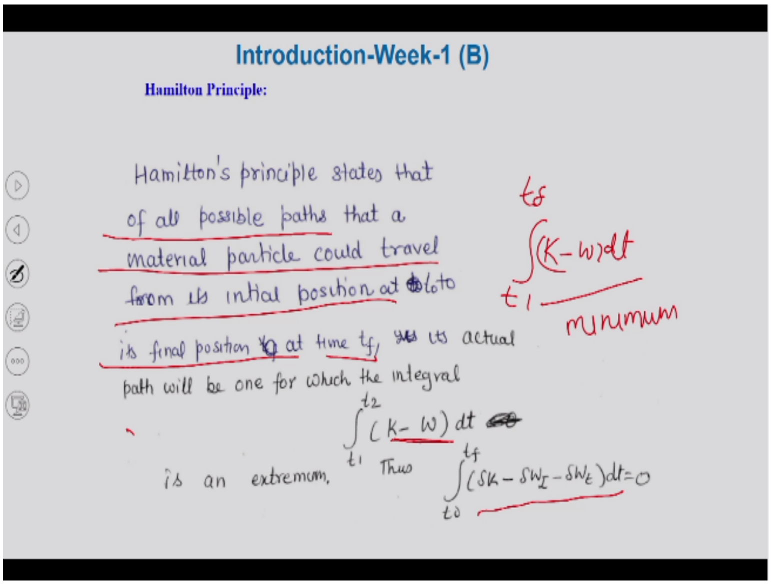

Now I am going to state the principle can be stated that of all possible path that a material particle could travel from the initial position t0 to a final position at time tf. Its actual path will be 1 for which this integration will be extremum. I am going to write it again, t1 to tf kinetic energy, potential energy that this integration will be minimum if I talk about. So how did you get a minimum of that if you take the derivative and equate to 0.

Similarly, we take the first variation and equate it to 0. So ultimately the Hamilton principle gives you this statement del k-del wI-del wE del t=0.

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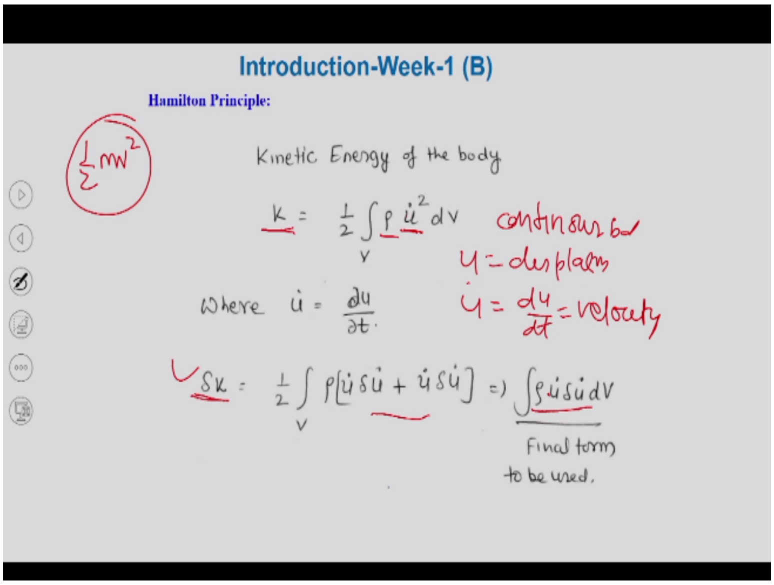

So what is the kinetic energy of a body, you know 1/2 mv square for a discrete body but for a continuous system, continuous body it will be 1/2 volume integral rho u dot square where u is displacement and u dot is the del u/del t, it is related to velocity. So for the case of then if you take the first variation in the kinetic energy, so it will be if you want to take that two times del u dot or first function second function, so it will reduce to rho u dot del u dot del v.

That is your final form of first variation in kinetic energy. We are going to use this form.

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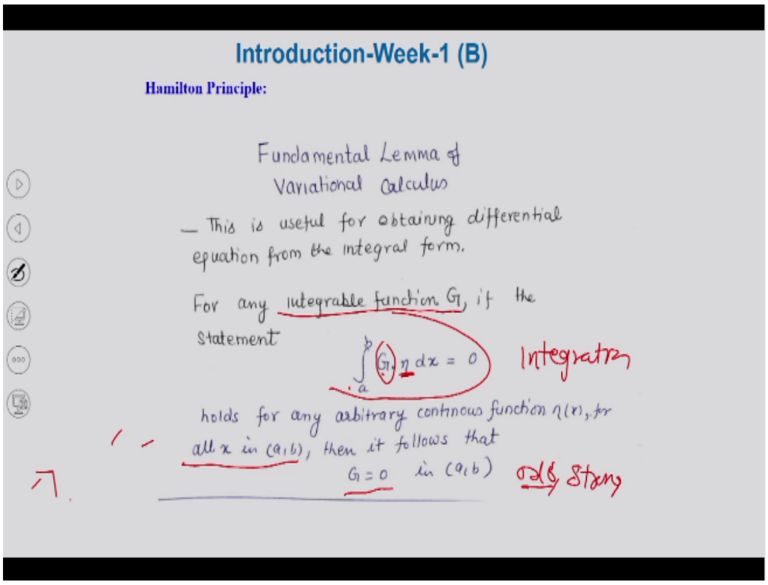

Now like our principle of virtual work done, we have said that displacements are arbitrary. So its coefficient must vanish but in the case of Hamilton principle or some other variational principle, we take this is a first variation. So if we have a system like that for any integrable function, any function like it maybe sigma x or it may be Nx or any function if that is integrable and this statement is there G dot n dx a to b where n is an arbitrary continuous function eta and valid over rho a to b then we can say that G has to be 0.

So from the integration to ODE form or the strong form, so we have to say that if a integrable function and an arbitrary variation, an integration that arbitrary variation is valid over a range a to b and this integration is=0 so we can say that G must vanish for that reason.

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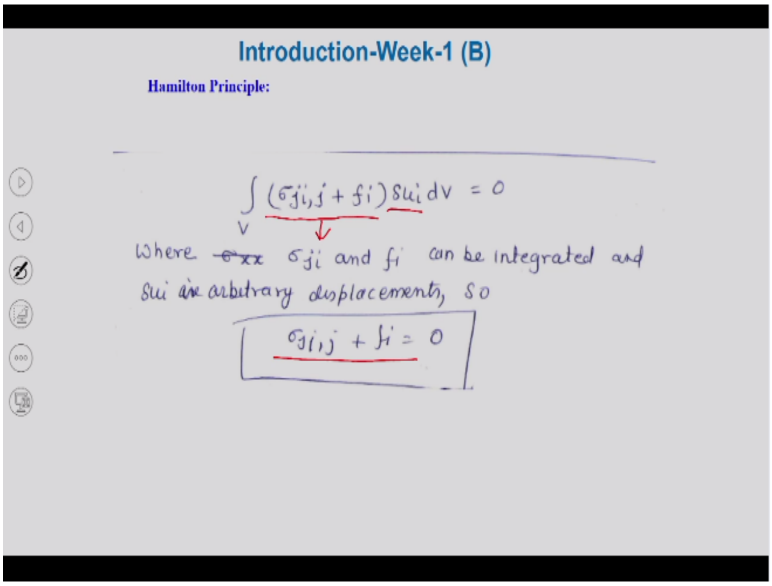

Similarly, I am just going to help you with the example. So in that we have obtained sigma ij, this and this is integrable over that volume and del u is also valid for that volume. So that is the first variation, so we can say that this has to vanish, so it leads to a differential equation form.