Theory of Rectangular Plates-Part 1
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Lecture – 12
3D Solutions

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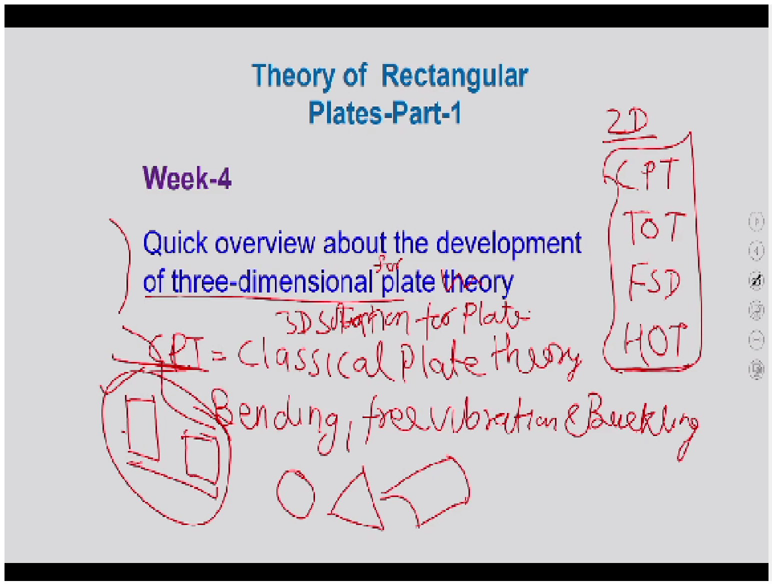

Hi everyone. Welcome to our fourth week lectures. So till now we have covered classical plate theory, CLT or CPT. We have developed a solution, Navier solution for bending case, free vibration case and buckling case. Now with this exposure I may be not go into very very depth but I have given you some exposure that how to develop a bending solution of a plate when you talk about pure bending or you want to take all inplane variables together.

How to develop a free vibration solution or how to develop a buckling solution of a plate. So if somewhere, you with this base, if you go to read any book whether it is a book of Professor K. Chandrashekhara or Professor K. Bhaskar or Professor J. N. Reddy, you can go any of these 3 books, generally I used to follow. So you can understand the chapters or the solutions given there.

And later on you may develop some code of your own in Matlab or Fortran or in any other language in which you prefer and later on in for the research problem, you may go for developing a solutions for advance material or a different kind of boundary conditions or some loading conditions. One main thing what I would like to say that whatever I have solved, the solutions valid for either for a rectangular plate or a square plate, if you are interested to develop a solution for a circular plate or a solution for a different shape, trigonometry, then the governing equations will be modified as per the coordinate system.

If you are interested to develop a solution for a cylindrical shell, then you have to convert those considerations of curving effect into the governing equation. So our solution is valid only thick, or sorry thin, rectangular plate and square plate. For square or any other shape, you have to, from the start, taking the consideration of geometry parameters so that suitable governing equations can be developed.

As you are aware that CPT or you talk about TOT or you talk about FSDT or you talk about higher order theories, all comes under one umbrella that is 2-dimensional theories, 2D theories. In this theories, what the basic thing is that using the stress resultant whether you call NxNyNy and moment resultant MxMyMxy or some higher order moment resultants, we convert that volume integral into a area integral along x and y and then solving those partial differential equations according to our requirement whether we are going for a Navier or a Levy solution.

Now I am going to discuss briefly about the 3-dimensional solutions. 3-dimensional solution of a plate theory for a plate basically or you may say that 3D solutions. First of all, one has to be sure that why we are interested in development of 3-dimensional. We have a 2-dimensional solutions of a plate for our boundary conditions for different loading and we can also as already when you have a thick plate, definitely third order and higher order theories or some refined theories will give accurate results. But still we are interested in the development of 3-dimensional solutions.

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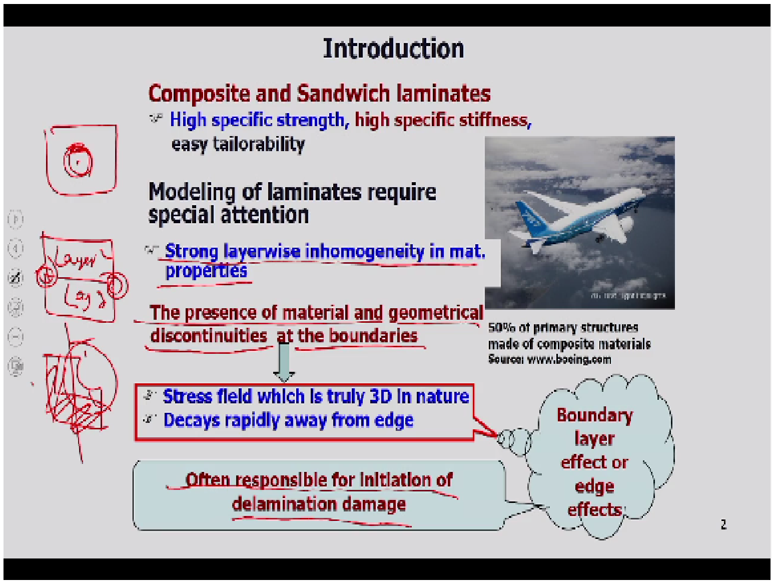

Three-dimensional solutions basically when you have a material like composite or any advance material and striking sequences like that, then most parameter of the strong layerwise inhomogeneity in the material property that at the, that let us say one layer is having some Young's Modulus E1, then second layer may have a Young's modulus of 20 times of that or may be less than 1/10.

So in that case, L layer 1 layer 2, so these interfaces may experience, I would like to say that boundary effect or the edge effects. So the presence of material and geometrical discontinuities at the boundaries or if you talk about a plate with a hole or a plate with cutouts. So generally this relations will have stress singularities on the, where it is cutouts or their discontinuities and in this cases.

So for such cases, 2-dimensional theories cannot predict the solutions accurately. They may give the solution in this interior zone, here or here, hatch zone but near to that, they cannot capture the real effect. So, okay. Something is there. So who, why, why still we are interested, okay? Let may be then some phenomenon is lie there in the world, there are number of phenomenons, you may be interested, you may not be interested.

But reason is that these phenomenons stress singularities often responsible for initiation of delamination damage or some kind of crack propagation. So when we subject these laminates to a real conditions, they fail, at lower strength level, you design for something, but due to these stress singularities, edge effects, or boundary effects, they fail early. So we are interested to find out the exact stress behaviour for those cases.

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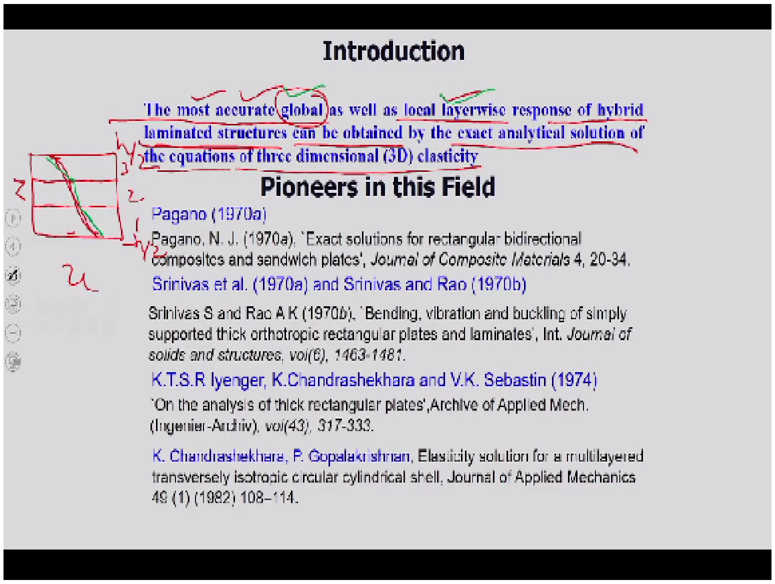

Further the 3-dimensional solutions provide most accurate global as well as local layerwise response. As you talk about 2-dimensional theories, if you consider on the equivalent single layer theories, they can give you a global response accurately but the layerwise response, they cannot predict accurately. Very famous example that let us say this is your z -h/2 to +h/2 and you want to plot some u.

There may be some 3-layer kind of thing, layer 1, layer 2, layer 3. If you obtain a solution using a classical plate theory or using a third-order theory, you may have a variation like this, straight line or it is slightly more that. I will talk about something. But in actual, when you go for a 3D, so for 3D case, there may be some kinkiness that change in, that it's slope changes at layerwise. So that is why we called it zigzag variation of a displacement.

Similar behaviour is observed for the stresses and other variables. So these effects cannot be accurately predicted. So local layerwise response as well as global response can be predicted because I have taken from an advance material, so I am saying that a hybrid laminated structures means a structure integrated with some piezoelectric materials or some smart sensing materials or having some composite and metals and some kind of arrangement.

So can be obtained by the exact analytical solutions of the equations which are based on 3-dimensional elasticity solutions. So before I proceed that how to develop a solution based on the 3D elasticity for a laminated plate or for, or an isotropic plate, I would like to talk about the pioneers in this field.

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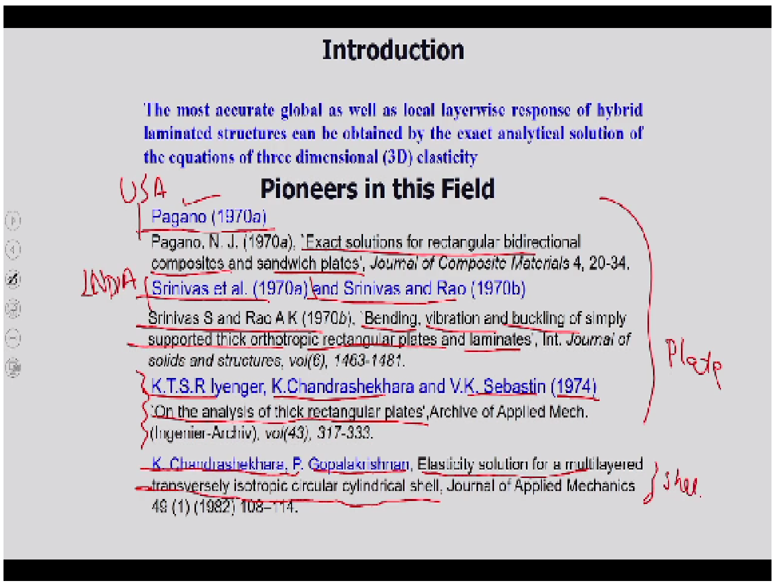

That who has developed the first 3-dimensional solution. So these solutions are not very old solutions like your Euler–Bernoulli solutions or Kirchhoff's plate. Actually they, these solutions were developed in 19th century but if you talk about the 3-dimensional solutions, these solutions are developed in this century, 20th century. So in the literature, it is found that in 1970, Pagano presented an exact solution for a rectangular bidirectional composite and sandwich plate.

At the same time, he is from USA; from India, Srinivas et al and Srinivas and Rao, so basically I have to say that there are 2 papers, one is Srinivas et al, there are 3 authors and in the second paper, there is Srinivas and Rao. So I am quoting about the second paper, that bending, vibration, buckling, or simply supported thick orthotropic rectangular plates and laminates. So simultaneously from the India side, Srinivas and Rao developed 3-dimensional solutions in which they considered bending, vibration as well as buckling of a simply supported laminated plate.

Whereas Pagano only presented the bending solution for that plate. On the same, not very far, that Iyengar and Chandrashekhara and Sebastin in 1974 developed an analysis of a thick rectangular plate which is published in archive of a plate mechanics. And 1983 K. Chandrashekhara and GopalaKrishnan developed the elasticity solution of a multilayered transversely isotropic cylindrical shell.

So these 3 were related to plate and this is related to shell. So these were the solutions, 3-dimensional solutions for an elastic plate were developed in 1970s. But in 1990s or 1985, a material, smart material, we will say that piezoelectric material come up. So people have proposed that if a sandwich or a composite layer is integrated with some piezoelectric layers, then this structure can act as a sensor or can act as an actuator.

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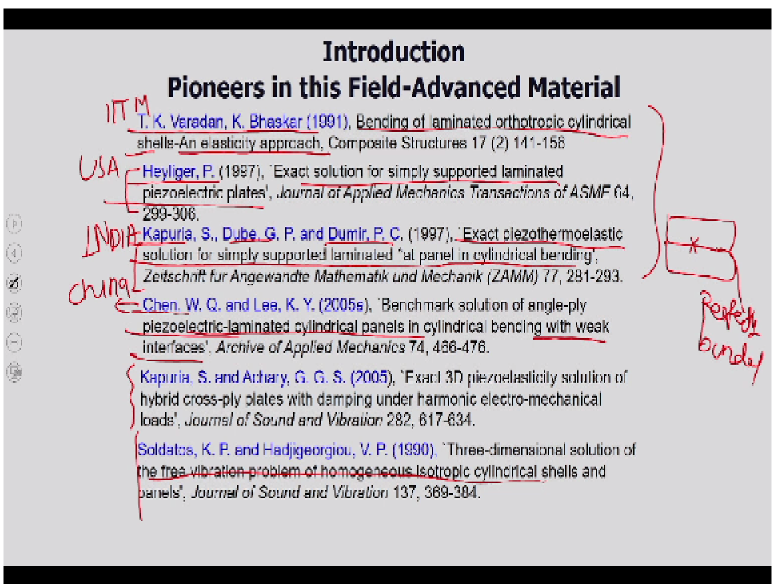

So the 3-dimensional solutions for those cases before going to the more advanced that T. K. Varadan and Bhaskar, K. Bhaskar from IIT Madras, bending of laminated orthotropic cylindrical shells-An elasticity approach developed that. Then Heyliger P., I am talking about piezoelectric plate, exact solution of a simply supported laminated piezoelectric plate from USA. The same time almost similar or same year you see that Kapuria, Dube and Dumir from India, they developed an exact piezothermoelasticity solution for a simply supported laminated panel in cylindrical bending.

So all of these authors, they do not develop only single paper, series of papers in this tat 3-dimensional solution for a shell with buckling or free vibration or some plates with vibration, then shell and different kind of lamination schemes. So basically I talk about Professor Kapuria in series of papers. Similarly, Professor Heyliger, series of papers. Recently they have developed for a magnetoelectroelastic blah blah exact solution for that plate.

So very advance material functionally graded material and so on. So in that sequence also Chen et al from China is also another a reputed author who has developed a benchmark solutions for basically shells or plates with weak interfaces. Actually these have taken that these interfaces are perfectly bonded. But at the interface, you have a (()) (15:44) so there may be slight movement is possible.

So surfaces, interfaces are not perfectly bonded. They are weak in surface. So there may be that, so this kind of, I would like to say that further trying to come, mimicking the real behaviour of plate developed a, with weak interfaces, piezoelectric laminated cylindrical panels. And Kapuria and Achary developed a dynamic solution for a plate, harmonic solutions. And in the same line, Soldatos K. P., they have developed a free vibration solution for a cylindrical shells.

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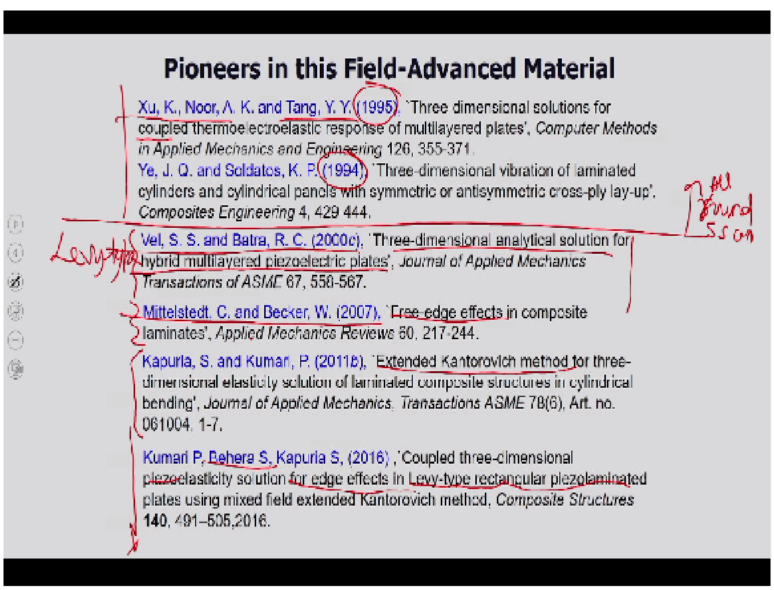

Then one of the famous Noor and Tang Y. Y., they have also developed a 3-dimensional solution.

See the 1995, 1994, 1997. So in that zone, a number of 3-dimensional solutions were developed considering new material, piezoelectric material. So a composite or sandwich plate integrated with piezo layer. Then very famous paper, Vel and Batra developed a 3-dimensional developed a 3-dimensional analytical solution for a hybrid multilayer piezoelectric plate.

So basically this solution is different from the other solutions, in which sense. So I would like to say before that, all solutions for all were simply supported case, SS case. And this solution was for Levy type plate. You see that, and this is the very famous review article and they have done a very nice literature review that find out that people have tried to study the free edge effects using the different classical theories, finite element, numerical analysis, approximate analysis and different type of combinations.

And the end conclusion is that if a 3-dimensional solution can be developed, that can give a real picture of these effects. People have tried, okay let us try, we can do it with the help of finite element. But there were some issues. We cannot exactly mimic the behaviour of the plates or shells or a structure having some geometric discontinuities. So recently Kapuria and myself, using that extended Kantorovich method, try to solve that Levy type problem, arbitrary supported problem based on the 3D elasticity.

And I would like to say that we are very much successful to accurately predict the behaviour near the supports or at the very edge and we are now working on this. So recently with my PhD student, we have developed a Levy type rectangular piezolaminated plate for edge effects and now recently we have developed for a solution for a free vibration and then extend it to the functionally graded materials and so on.

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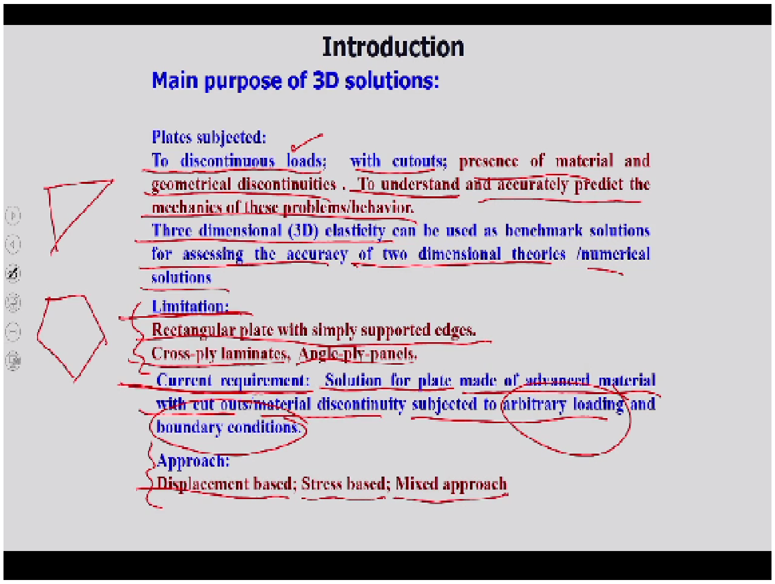

The main purpose of 3D solution development that a plate when subjected to the continuous load with cutouts or presence of material and geometrical discontinuities is to understand and accurately predict the mechanics of these laminates. This I am saying again and again and the next very important that these 3-dimensional elasticity solutions can be used as benchmark solutions for assessing the accuracy of 2-dimensional theories or numerical solutions, that we can assess.

Let us say you have come up with a new idea, new scheme that a refined third order theory or some theory that whether that theory can accurately predict or you have to compare with the 3D solution, how, how near to that solution. So till now, in the literature you will find a rectangular plate with simply supported edges. Similarly, for the shell and only for a cross-ply laminates or angle-ply panels, not plates.

So basically arbitrary shape you may have a pentagonal plate, you may have a triangular plate. We do not have that 3-dimensional solution for that. So current status of research is that solutions for plate made of advanced material with cutouts/material discontinuity subjected to arbitrary loading and arbitrary boundary conditions. So these days, people are working on that and trying to develop some analytical solutions with different shape, different boundary conditions, different loading and trying to get that real picture of the behaviour.

So basically when we are developing 3-dimensional solutions, we have 3 approaches. Like your 2-dimensional solutions, you have 3 approaches that displacement based approach, then you have a stress based approach. Then last one is the mixed approach. So similarly here, we have displacement based approach, stress based approach and mixed approach.

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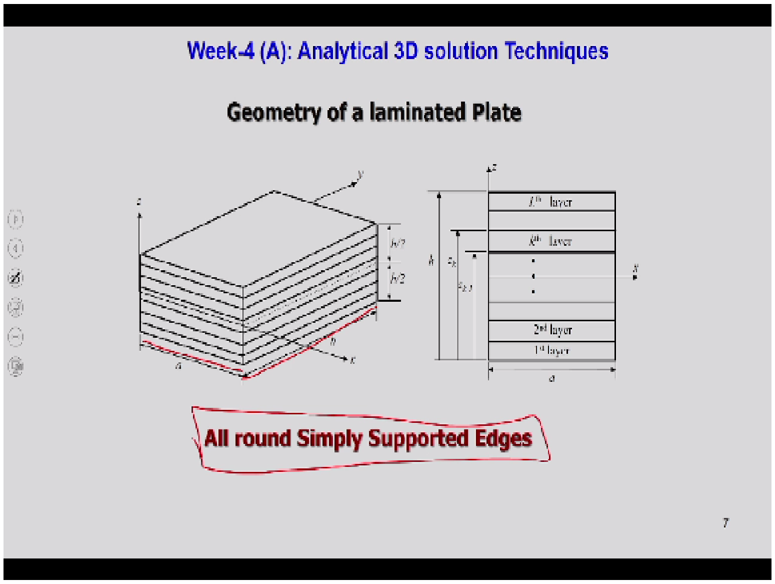

So let us consider a geometry of a plate whose length is a, width is b. This axis is x, this axis is y, this is z and since developing a general solution for a let us say composite plate. So they may have some layers 1 to L. And first condition is that all round simply supported. Recently we have developed a solution for a Levy plate. But I am not going to discuss here. I am going to discuss a general methodology used to develop the solutions.

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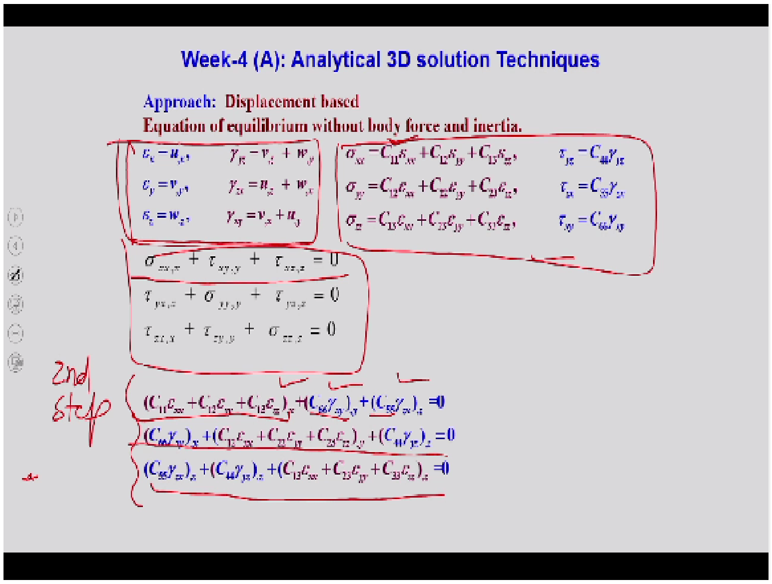

So based on the displacement based approach, what you require? Like you want to do something, so you need, you want to make some box, so you want to make some bench, so what are the tools required or what are materials required. Similarly, for developing the 3-dimensional solutions, you require, in that coordinate, strain displacement relations. Why I am saying? Let us see you are developing for a 3-dimensional solutions for a rectangular plate.

Later on, you are interested for a circular plate. May be after onward you are interested for a cylindrical shell. So very first requirement to obtain strain displacement relations in that coordinate. If it is rectangular coordinate in this form, if polar, cylindrical, or may be some later on doubly curved form, general quadrature form. Then you need constitutive relations. Presently we are talking about only composites.

So we will have those kind of material constant. If you are interested to solve for a piezoelectric plate, then the modified constitutive relations you are required. Then the equation of motion in that particular form in Cartesian coordinate. Or if you are interested to develop for a shell, it should be in cylindrical coordinate. Or in spherical coordinate if you are interested for a sphere or a general coordinate system.

Now you see that in this equation of motion, if you replace all these things with constitutive relations, sigma xx like this, you substitute derivative of x, tau xy derivative of y, tau xz drivative of z. So this is the second step. You will get 3 equation of motion in terms of epsilon xx epsilon yy epsilon zz. Now substitute the value of epsilon x and shear strains, epsilon y epsilon z and so on.

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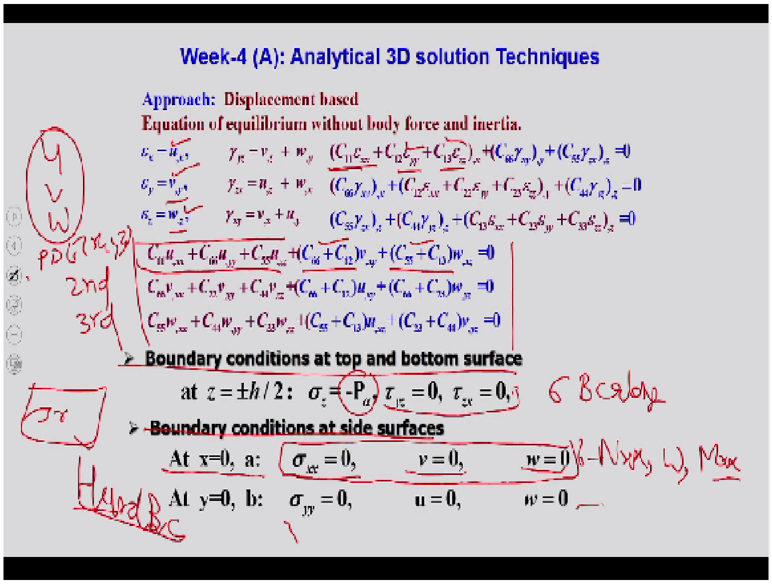

So basically in this equation if you substitute epsilon xx is u, x; epsilon yy is v, y; epsilon zz is w, z derivative of x. So you will come up with this key. C11 of u, xx derivate u, yy y, zz and v, xy w, xz; these are the constants. Similarly, your second equations and third equations. Now you see we have unknowns u v w, 3 unknowns and 3 set of equations and these are the partial differential equations in coordinate x y and z.

You find that derivative of x, derivate of y, derivate of z is there. So now we have to frame up the boundary conditions. First that boundary conditions at the top and bottom of the plate. That we are going to apply at the top only pressure load and shear traction free boundary condition, tau yz and tau zx as 0. Then the boundary condition at the side surfaces. At x=0 and a. It should be simply supported.

So those variable sigma xx, v and w need to be 0. If you remember for the case of 2-dimensional theories, what about that? It was Nxx, then w, then Mxx. So here for the case of 3-dimensional, if we choose a boundary conditions like this sigma x, v and w, and along y axis, sigma yy, u and w. If you want to take some other boundary condition, then you may not get the exact solution.

Because at the end of the day, I will explain if you do not take this, this is a special form of boundary conditions. So sometimes we call it is a hard boundary conditions. So we have to choose, now you see that along x axis, along y axis, along z axis, you have a at the bottom of the plate, 3, at the top of the plate, so basically 6 BC's along z. Similarly, 6 along x, 6 along y.

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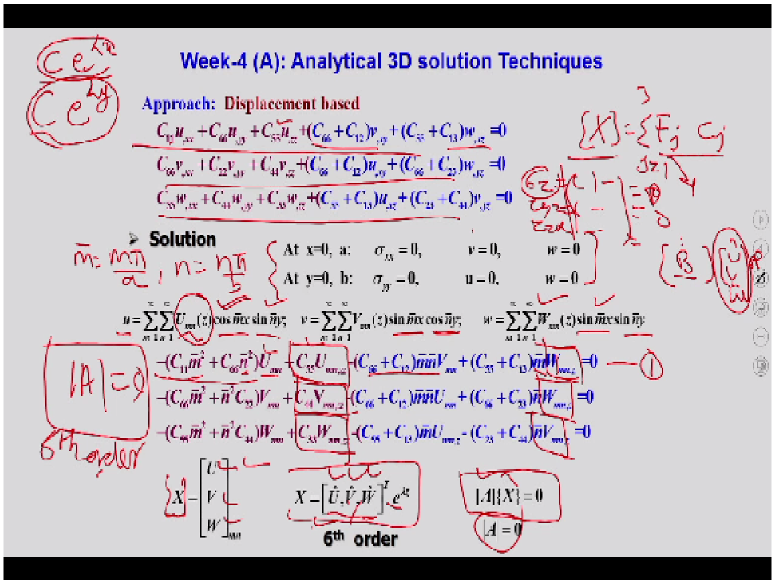

Now we have this set of equations which we have recently obtained and based on this, now we will assume the solutions along x and y. So you are assuming that u is assumed, let us say, Umn which is a function of z and along x axis and along y axis. So along x axis, we, u is non-zero. So it is assumed in a cosine and pi xya. So where M bar is nothing but standard M/a and N bar is nothing but N pi/b.

So cos M sine. Then b along x axis is 0. So in terms of sine, along y in terms of cosine and w along x and y, both it is 0, so it is assumed a double Fourier series, sine series, sine and sine. Now you substitute, you see that u, xx. So derivative along x axis first gives -m bar, then second m bar cos, so basically m bar and then n bar, so this, again these are the constant nUmn which is a function of z.

Similarly, you substitute it here. So Umn, zz and you substitute it here, v,xymnm, then w,xz. So along the x direction, derivative will be there along the z direction, derivative of Wmn. So in this way you get a modified equation. Now you see here only, derivative along z is there. Others we have remote using the double Fourier series in terms of M and N, like a constant. Similarly, in the second equation, this and this, the derivative along z.

So now it is basically a, we can say that ordinary differential equation of second order, double derivative of z. So we can write in terms of some, let us say a matrix a which may contain these constants. So x you say that later on it will come. You define a state vector X in which U V W are 3 variables and you assume a solution that e raised to power lambda z, like we assume like a constant e raise to power of lambda x when we are going for a Levy solution ordinary differential equation along y direction, like this.

We are going to solve. Similarly, we assume a solution like this. U hat V hat W hat, these are the unknown constants and e raise to power of lambda z. If you substitute this thing to these equations, it becomes a matrix a*X=0. So where x I would like to say X hat, not X. U hat V hat and W hat. So trivial solution is that U hat V hat W hat is 0 but we are, there is no use. So next non-trivial solution, a determinant has to be 0 if the solution exists.

So from that we will get a (()) (31:33) expression of a sixth order. So you will have 6 root. Based on those roots, you will write the solutions. I have already explained in the Levy solution if my roots are real, if my roots are real and same, real and different and complex conjugate, 3 base cases will be there. Then you write the solution and from the set of homogeneous equation and then substituting the boundary condition, you will get that, these are known, U0 V0.

If you have this, so let us say I write a solution, F something J or some Cj where j goes from 1 to 3. Fj may contain some that if it is real solutions or complex conjugate according to their cosine or sine terms. So at the top and bottom, you have a sigma z tau yz and tay zx. So you take from there in terms of that, write U1 U2 U3, we know and substitute there and equate to 0. So from there, not 0, so sigma z will be non-zero.

So again you will have, let us say a matrix B and U hat V hat and W hat equal to some matrix B. So U hat V hat W hat can be B inverse of B. So in this way we can solve the problem.

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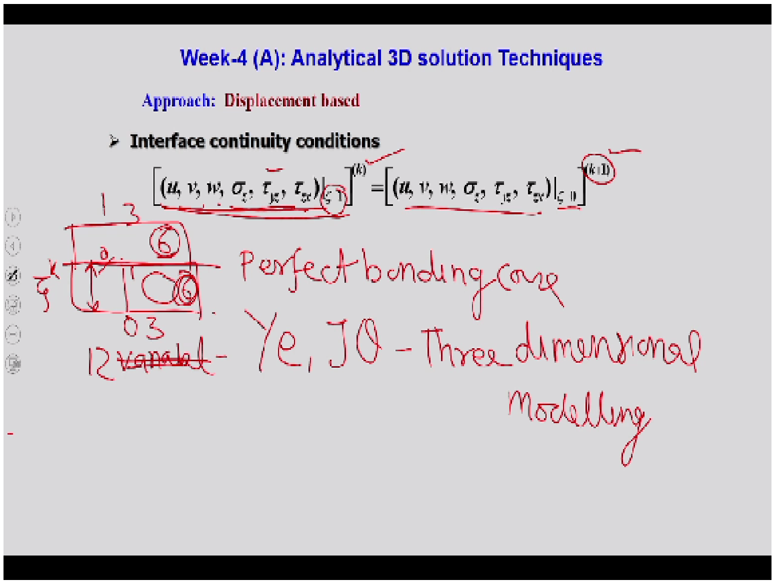

Now we are talking about a laminated plate. So basically at the interface, the extra condition. If it was single layer, the previous solution is valid. But you are talking about a layerwise, so this for a perfect bonding case, following variables obtained from this and obtained from that should match that from the bottom layer xi=, basically I have written in terms of a non-dimensional parameters.

So for that case, xi varies a Kth layer parameter. So here xi is 0, i is 1. Then again the second layer, xi is 0 again 1. So bottom layer top U V W and 3 transverse stresses, top layers' bottom xi=0, K+1th layer, that should match. So using this equation, this we call it interface continuity conditions, we will obtain further set of equations because if you, if you are solving for a single layer, you have only 6 variables, then again 6, you will have a 12 variables or 12 roots I would like to say, not variable.

The after all variables, then you have 6 and 6 and 3 here 3 here, so using those conditions you can write in terms of a transverse matrix. So this can be given in books, specifically if you go through a K. Chandrashekhara book or Ye, JQ, 3-dimensional modelling. You will find a beautiful solution technique is given there. As we go into the details, so you can follow any paper, then you will find that there is a scheme that how to transfer this bottom to top and 1 paragraph is devoted to there, for a laminated scheme.

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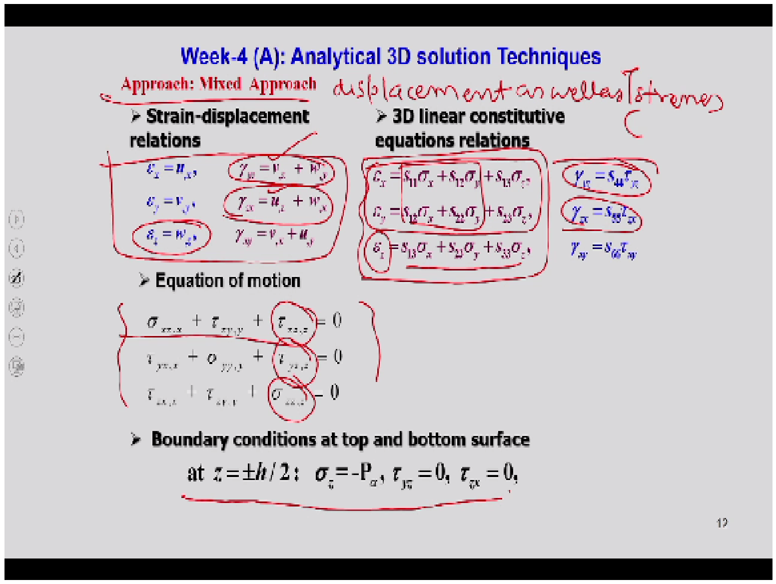

Now we were talking about a mixed approach. So this is a, why this mixed approach is preferred? I would like to say that not here; you have to satisfy this boundary condition but your variables are U V W. So ultimately using those constitutive relations, you obtain sigma x in terms of U V W and then equate to 0. So basically this satisfaction is not exactly. Because stresses are not your variable.

So in the mixed approach, displacement as well as stresses, transverse stresses if you talk about the T, transverse stresses are our variables. So you assume what we need, strain displacement and you see that here I have written in some different way, another way that in terms of compliances, strain-stress relations and then equation of motion in the boundary conditions. So we are going to use all these relations. You first see here, gamma zx, gamma yz, substitute it here and from here, v, z and u, z can be obtained like this.

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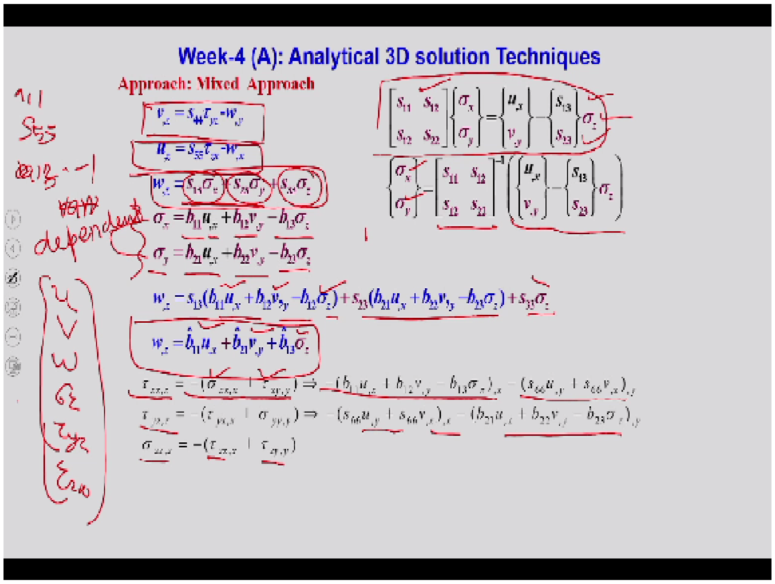

So you see this is the first order differential equation in z. So this, if we are able to obtain that first order differential equations in any coordinate, so solution is comparatively easy, then to go for w,z epsilon z. So from here, substitute this and you see. Now you see sigma 3 sigma z S33 S23 sigma y sigma x. Sigma x and sigma y, we will replace with something, so that all these things will be in your primary variables.

Whatever you are interested. Later on I will explain. You just go for, you can rewrite those equations. What are those? This epsilon x and epsilon y. How can you write? That you put these things together and take that side, minus of that. So Uxv, y S13 S23 sigma z. So from here, sigma x and sigma y can be converted to an inverse of this, multiplied with these things. So basically sigma x can be written as, let us say, some constant will come up, b11u, x b12v,y b13 sigma z and sigma y is b21u,x b22v,y and b23 sigma z.

Now you substitute sigma x and sigma y in terms of there. So you see u v, "w to hai hi" and sigma's are. Now you are getting wz in terms of u v and sigma z and here you will see that sigma x and sigma y. If we know u v and sigma z, we can obtain sigma x. So there is no need to take as a variable. So I would like to say that these are dependent, sigma x and sigma y. Similarly, tau xy will come up.

Then from the equation of motion, this. You get tau xz, z tau yz,z and sigma z,z. So the first equation, tau xz, z is minus of this. Now you substitute sigma xx tau xy like this. So which are again u v and sigma z and similarly tau yz, z which is u v and sigma z and sigma z is where tau zx and tau zy.

So when we are developing a plate, 3-dimensional plate solution or equation in terms of a mixed, so our primary variables are u v w sigma z tau yz and tau z. These are our 6 primary variables. We are going to solve for that. So now you see that first 2 3 4 5 6 ordinary differential equations in the thickness direction we obtained.

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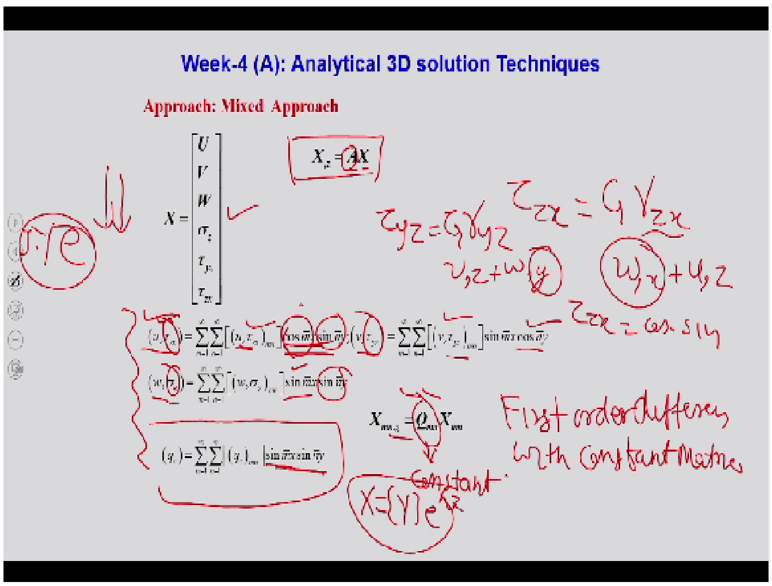

We have also derivatives in along x and y. Later on we assuming in the double Fourier's series, we will convert it to that. So basically that reduced to differential equations along z only. So you write and use let us say X, z is nothing but A of X and now right now you take AA's like this b11 of or you, if you talk about this u, z. So let me say that this is my A11 and if you talk about A13, will be my -1 and if derivative is there.

So substituting, now assuming a Fourier series combination like previous u v and w. Now we have a 3 more variables, stress variables. What is that? Tau zx tau yz and sigma z. Then who told you that whether they have to express like a cosine or sine sine or sine and mix. u v w you can understand along x and y. I have explained that when u is 0, you have to assume cos or sine but I did not tell you that along x and y, what about these variables?

So basically if you write, that is that tau zx? Is nothing but G*gamma zx and gamma zx is what that? Basically w, x and u, z. (FL) assume (FL) sine sine, w,x along the x derivative gives you cosine. So tau zx will be cos and along this will be sine. If you talk about u, z; it is a function of z. So cosine remains the same. So tau zx should be a function of cos and sine. Similarly, you talk about yz.

So yz is what? G*gamma yz. So basically v, z+w,y along y direction cosines function and sigma z. So in this way, we can know that how to express in a double series based on the displacement only. You know that how to first express u v w and then find out using the constitutive relations that expression for tau zx sigma z and tau yz. Then the load function that what, whatever we are applying on the top of the plate.

That should be also a double Fourier series. This are the conditions. If you substitute all these expressions into this equation, then it becomes a matrix Q which is a constant matrix. This is a special case that if first order differential equation with constant matrix. Now you know the solution of that. Assume, let us say a matrices y because this is matrix and e raise to power of lambda z, substitute it here.

It reduces to again an eigenvalue problem or from that find out the eigenvalues and solution procedure is given in any you can go any journal paper of Kapuria and Achary, there it is given or you can go a book of J. Ye, there you go through.

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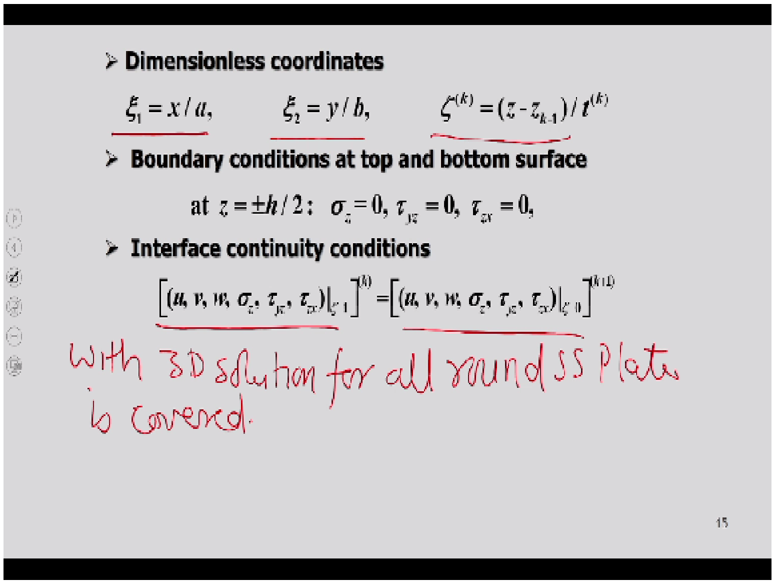

And generally we use a dimensionless coordinate, xi xi1 xi2 zeta thickness and similarly the interface continuity conditions. These are very useful using the non-dimensionless coordinates. So with these 3D solutions for all round SS plate is over. So in the next lecture, I may briefly revise all the contents and may prescribe some for your homework or home assignment, you may after this course, you may go through or you may try yourself.