Theory of Rectangular Plates - Part 1
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Lecture - 01
Basic of Solid Mechanics

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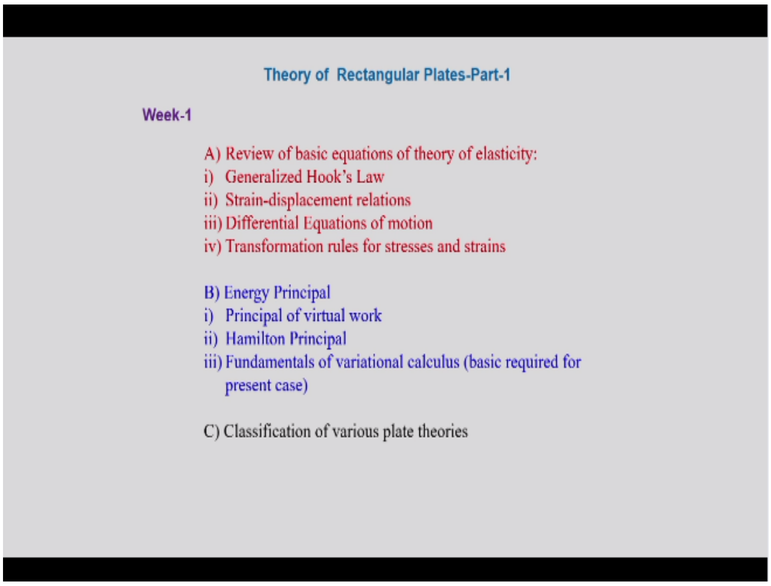

Hi everyone. So the first week topics we are going to cover the following topics in week 1 like reviews of basic equation of theory of elasticity, under that head we may cover that how to represent a generalized Hook's Law or the constitutive relations, then the strain displacement relations, differential equations, transformation rules for stresses and strains and part B will be the energy principles.

Basically, these energy principles are useful for developing the plate or shell theories. Then we will classify the shell theories based on various approaches whether it is a thin plate shell theories or the thick plate shell theories.

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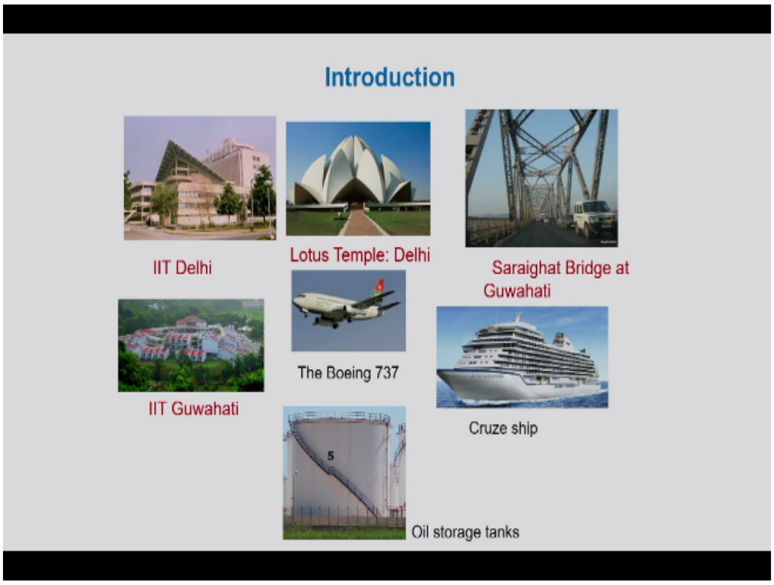

Before going to the main topic, I would like to explain where these theories are helpful or applicable. Like you see the many structural elements like the building of IIT Delhi that is a triangular shape roof or you see that Lotus Temple in which this shape is on a curved shell shape and you see that this is also a picture where the tapered element of a shell not only in the structures but also in the bridges you see that various stress elements, in aerospace industries, in ship building cruise and oil tanks.

If we can divide that all structures into small elements either they will be a beam or a plate or a shell. So any structure, any machine, any component can be subdivided into 3 basic elements beam, plate and shell. If we are able to predict the behaviour of that basic element with subjective to different kind of loading, different kind of boundary conditions definitely we can analyze the structures.

Because the structures are made up of this so applying the continuity conditions or some relevant assumptions we can analyze the structures.

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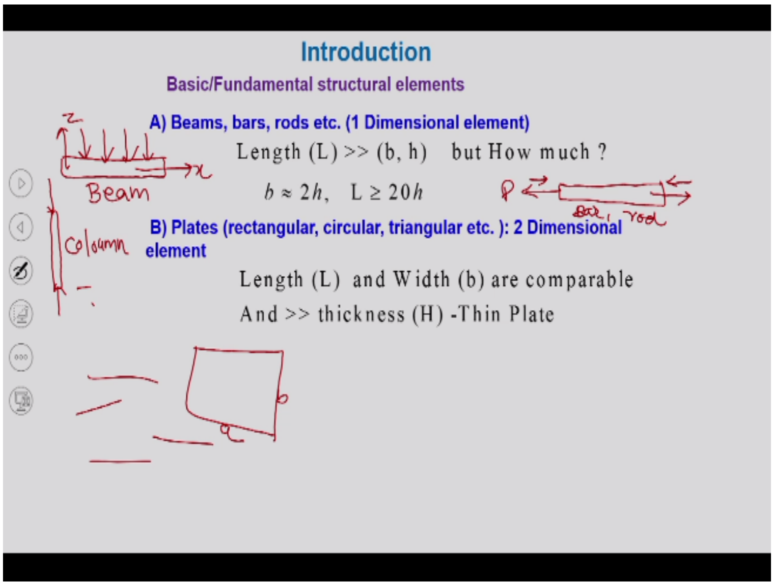

So very first thing comes that beam, the whole or the qualification of a beam that in a structure if you find a component whose length is very, very long compared to its width and thickness. Then, we called we can treat as a beam or a bar or a rod but how much? It cannot be infinity, in theoretically it should be infinity but in practical application that how much that minimum should be.

So if we take that width is twice of h, the length should be greater than that 20 times of h. If you find any element whether it is in a bridge or in a building or in a ship or element if that qualify this criteria, you can treat as a beam. Further the classification of a beam that if an element like this and subjected to a transverse load then we treat as the beam if this is my longitudinal axis x and z.

But in a case of rod or a bar if same element is subjected to an axial loading whether it may be tensile or it may be compressive, it depends upon the situation then we treat as a bar or a rod element. If the same element is subjected like this and in vertical direction, compressive loading, then it will be treated as a column element. Similarly, we are coming because our subject is the theory of plates.

So the definition of a plate, it is a 2-dimensional element, so its length and width are comparable and thickness is not generally we take, thickness must be small, it may be higher. So if you define a plate when a and b are the thickness and width and it is less to h, very, very less than then we can treat as a plate.

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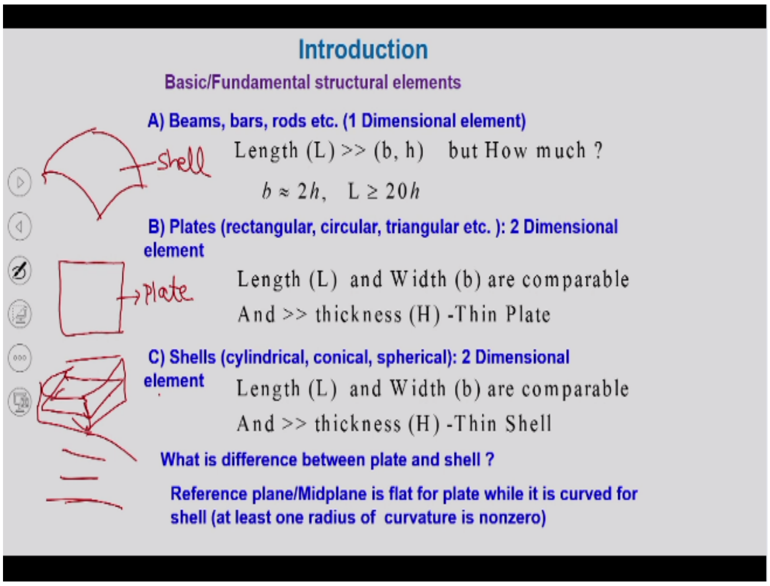

Then further shell or cylindrical shells, just in the shells there may be cylindrical, conical, spherical, so 2-dimensional cell elements that length and width are comparable and thickness is very small but how do you differentiate between a plate and a shell? That differentiation is that middle surface of the plate is flat whereas the middle surface of the shell is curved. So I am just going to give you example.

If you have a curvature, then we will treat as a shell. If it is flat, then we will treat that as a plate. I have used a term middle surface. Generally, we define a body or its component let that let us say a middle surface and at the thickness of h/2 if we add another plane similarly like this, so this will make the surface, complete plate. So this surface defines the geometry of the plate.

If this is flat then it will be treated as the plate, if this is having some curvature then this will be treated as a shell.

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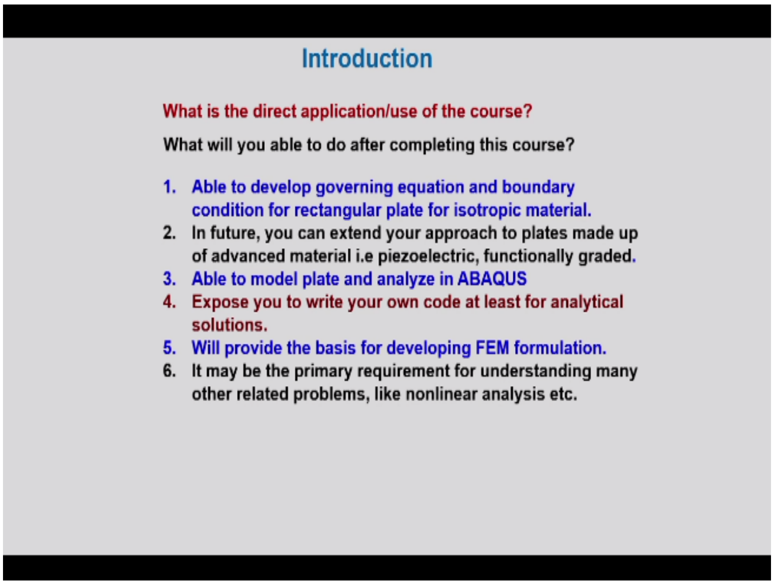

So what is the direct obligation of this course? Where we can use or the knowledge which we were gaining through this course we can apply? Next that what will you able to do that after completing this course? First, you will be able to develop the governing equations and boundary conditions for the rectangular plates for isotropic materials. Then, in future you can extend your approach to plates made up of advanced materials.

They maybe piezoelectric or the recent functionally graded materials. During the course, wherever possible I would like to expose you that how you can expand the present theory to the advanced material. Then, you will be able to model plate and analyze in abacus or any finite element software basically in which I will prepare a one short video for you so that you will learn how to model a plate in abacus, 3D modeling of a plate or a 2D modeling of a plate.

So that in future if you develop your plate theory, you have to compare your results, this will be helpful. Next, you will be exposed how to write your own code at least for the analytical solutions. Further it will provide the basis for developing FEM solutions. It may be the primary requirement for understanding many other related problems like if you understand a linear analysis of a plate then you can go for a nonlinear analysis of a plate, buckling of a plate or a transient analysis of a plate or doing the control of the plates.

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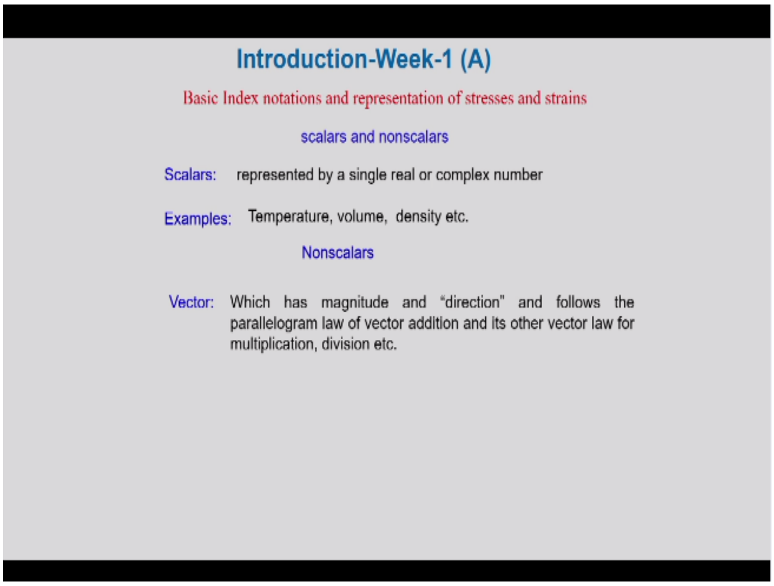

So from here our basic terminology starts. So basic index notation and representation of stresses and strains. So in undergraduate levels, you may be exposed to this kind of scalars and nonscalar quantities. So I am just going to give basic informations that we are going to use during this course. So first is the scalars. So how do you define a scalar quantity? A scalar quantity is represented by a single real number or the complex numbers. The examples are temperatures, volumes, density etc.

Then, the nonscalar quantities which has magnitude as well as the directions. There you have understood I think up to class XII, even during the undergraduates we use to do number of analysis using the vector analysis like in basic course engineering mechanics. So whole engineering mechanics solution is based on the vectorial approach in which direction as well as the magnitude taken care.

What are the examples? The basic examples are the displacement vectors, force vectors, accelerations and so on.

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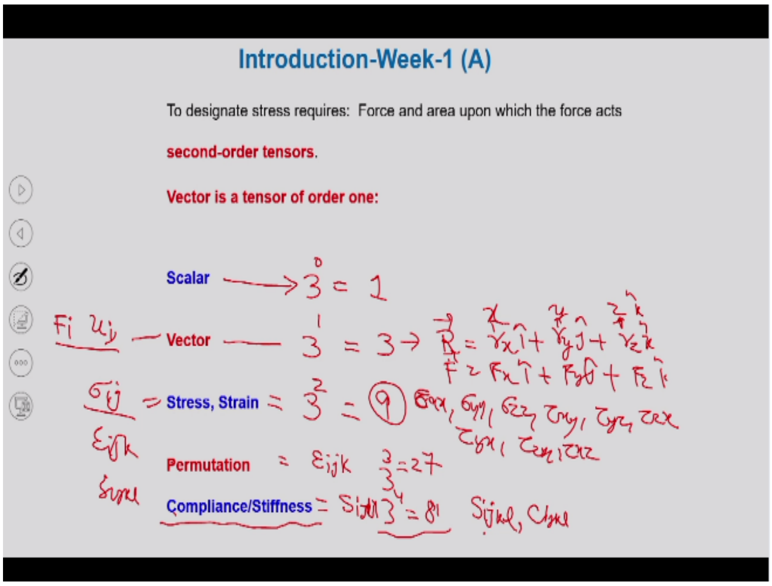

But there are some other quantities like the stress and strain which required two directions. So then how do you define? What do you call it? So we cannot call it vectors. So we call it second-order tensors. Then, vector is known as tensor of order one. So in that direction, I would like to tell that if we have to define mathematically 3 raised to power 0, it means scalar will have only 1 quantity.

For the case of vector, 3 raised to power of 1, it will have 3 components. For example, if you talk about a position vector R, it may be rx of i or whatever ry of j rz of k or sometimes you write just x y and z. Similarly, the force Fx of i+ Fy of j +Fz of k. So you have 3 quantities, when you are talking about the second-order tensor then it will be 3 raised to power 2 which is equal to 9. So stresses or strains will have 9 components.

Then, epsilon ijk some permutation kind of things or sometimes we use this kind of symbols in the compatibility relations. Then, the stiffness or compliances Si or I will write 3 raised to power 3, 27. If we talk about compliance and stiffness, this is 3 raised to power 4, 81 components. So for a case of vector, if you want to know the state of a displacement at least you need 3 components to completely define the state of that displacement vector.

If you want to know that state of stress at a point in a body, then you need 9 components. For example, if we talk about sigma xx, sigma yy, sigma zz, tau xy, tau yz, tau zx and three are yx and zy and xz. So in this way, we required 9 components of stresses to define a state of stress. Then, there are some symbols permutation or computations that they require 27 components and if you talk about the compliance and stiffness, they require 81 components.

So basically, we define Sijkl if we talk about in terms of index. So whenever you find generally in advance theory books like theory of plates, you will find that these are written in terms of index 1 if you find a four index, it means it is a fourth order tensor and it will have 81 components. So we can define a stress second order it means sigma ij. If we talk about vector, it means ui or Fi only one index, 2 index, here 3 index, here 4 index.

So you may be aware so our aim is just to tell you that this kind of things are available, so we define it as second-order tensor, then third-order tensor and fourth-order tensor. So they are addition, multiplication or they are further transformation follow some different rules.

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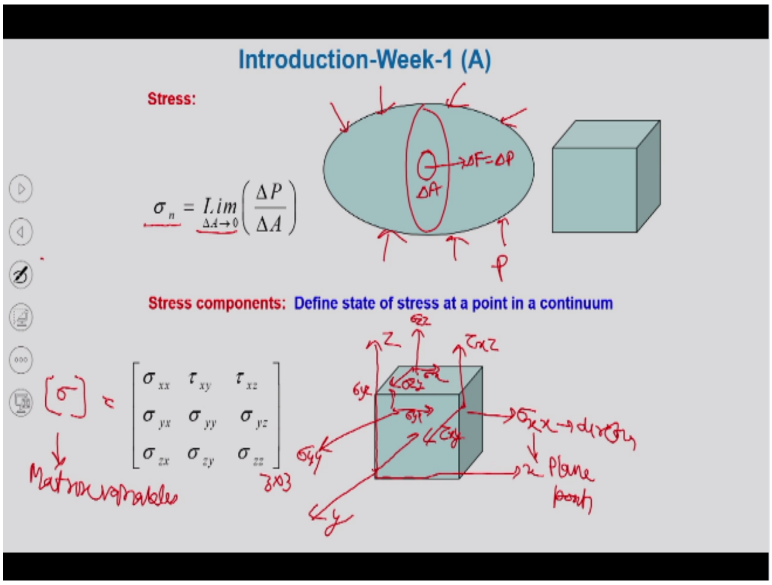

So how do you define a stress? So definition of a stress if this body is subjected to number of forces, let us say p, then if you take over that body small area and the resultant force will say that delta F working over in delta A or delta P equivalent to delta P no problem. Then, add the point. If this area reduces to a point, area tends to be 0 then will say that stress on that point acts in a normal direction delta P/delta A.

S in a cube how do you define these 9 component of stresses over the cube? Basically, this part is covered in advanced solid mechanics in undergraduate courses or sometimes in postgraduate courses under the head of theory of elasticity. So if I define, this is my x axis okay if I say this is my y axis and this is my z axis, then this phase will have a component that this phase is a positive phase, this will be defined sigma on a x plane, first index tells you the plane position.

And second index tells you the direction over that plane. If my component is like this then it will be tau xj along z direction and the component along this direction, it will be tau xy. So in this we did define the traction, vector or on a plane 3 component of stresses. Similarly, on the top phase of this block, there will be sigma zz along x direction, sigma zx and along y direction sigma zy.

Similarly, over this phase it will be sigma yy, then sigma yx and sigma yz. So we can write our 9 components in the matrix form. So we write that sigma in a matrix form 3x3. Most of the time, we use this kind of notations so that is why sometimes we call stresses and strains are the matrix variables because we can write.

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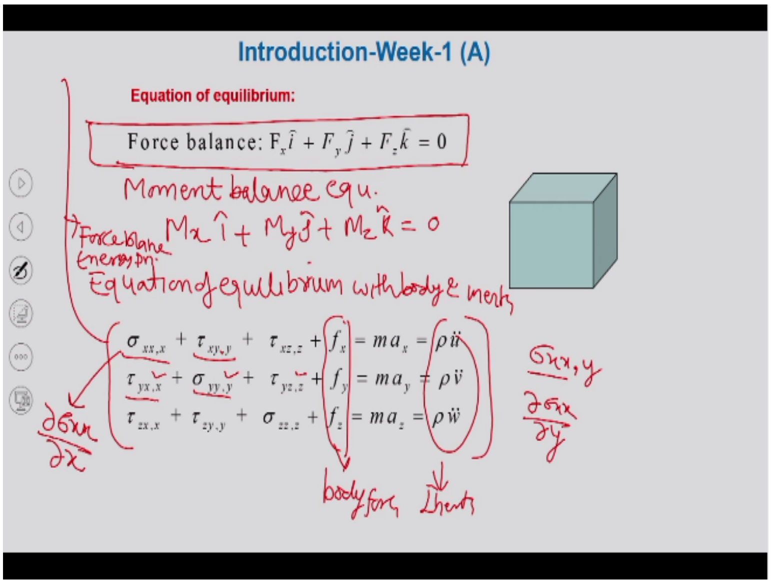

Then, the equation of equilibrium, equation of equilibrium I think in undergraduate you are aware about that force balance equation. Similarly, you have a movement balance equation which is equal to Mx of i+My of j+Mz of k=0. If we talk about a continuous system and in terms of using the stress kind of thing, so we will write this is basically equation of equilibrium with body force and inertia.

This terms are body forces, these terms are inertia terms where sigma x, x is nothing but partial derivative of sigma xx with respect to x. So this is also a standard notation when we use a comma like if I say sigma xx, of y, it means sigma xx derivative with respect to y. So whenever you encounter in theory of elasticity, in theory of plates, a comma by default it is assumed that okay this is the differentiation with respect to the quantity.

Otherwise they will clearly specify that it does not be in the differentiation if it is other than that. So this is the equation of equilibrium which can be obtained by the various ways, the first is force balance, sometimes we can find out from the energy principles.

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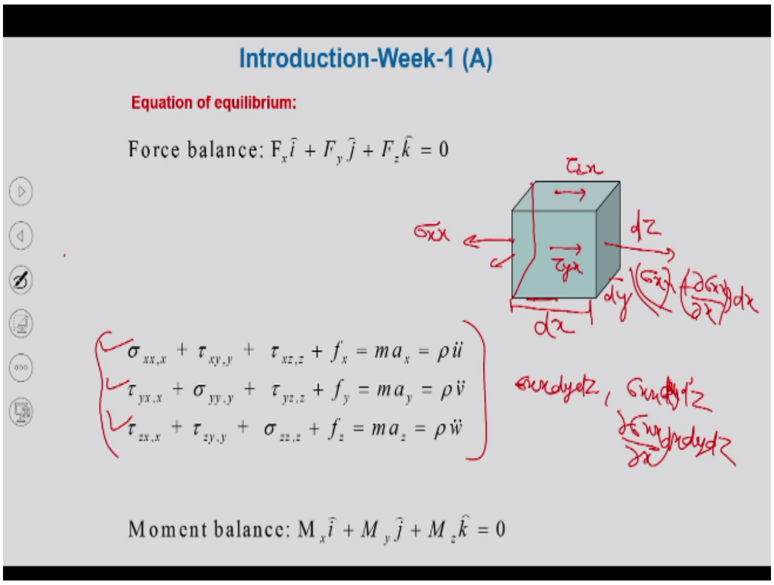

So with the help of force balance how do we find out? Let us say along only one direction I will explain, so next two directions one can any term rewrite or try the set of relations. Let us say this is taking a differential element if this is my dx, this is my dy, this is my dz. So if our this phase in a negative direction let us say sigma xx is acting, so over this phase how much it will be a sigma xx+del of sigma xx/del x*del x dx or that length.

So using this concept similarly from this top phase along the x direction, the tau zx over this direction tau yx so on the back phase also so along x direction if you say and total force working over this area will be sigma xx dy and dz and here sigma xx dy and dz and sigma xx/del x*del x dy and dz. So if you do some mathematical simplification, the first equation can be obtained.

Similarly, the second equation can be obtained and third equation but in our course we have to just remember these set of equations. We are not going to try these relations because if these relations the derivation is given in undergraduate courses where you can try that how to try this equation of motion. For our case, we have to just use this set of relations.

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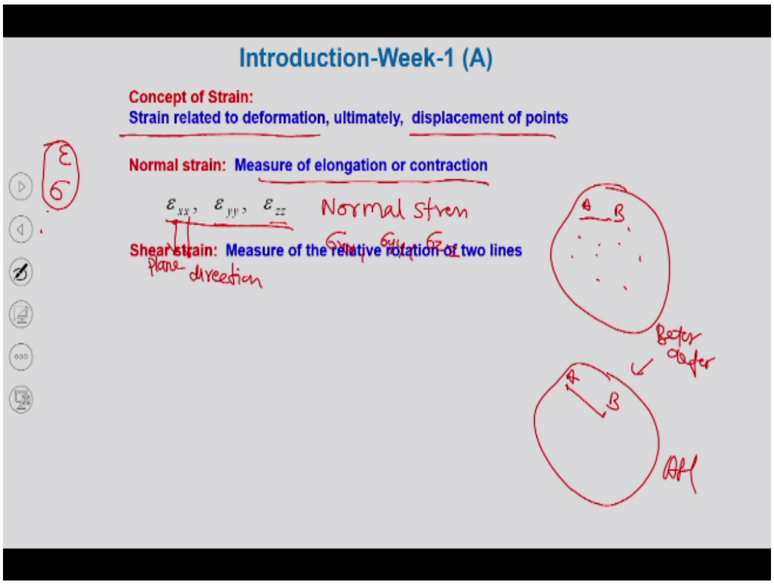

Now the concept of the strain, so basically strain related to the deformations. Ultimately, if we talk about in terms of measuring point of view that displacement of the points. Let us say in a body there are points, material points so after the applying forces so before deformation let us say this point is A, this is B before deformation and after deformation let us say this point is A and this becomes B.

So that distance so relative change the strain is related to the deformations. Further we can define that normal strain which measures the elongation or contractions along x axis, y axis and z axis. Similarly, like if this index first tells you the plane and second index tells you the direction similarly you have a normal stresses. Sometimes some of the students or persons are not aware in undergraduate levels just we denote that strain by epsilon and stresses by sigma.

So they are not aware about their components, so I am thinking that I will just review the basics. Let us say these are the components, epsilon xx, epsilon yy and epsilon zz and these are known as normal strains. Similarly, the normal stresses will be sigma xx, sigma yy and sigma zz. Then, we will call about shear strain.

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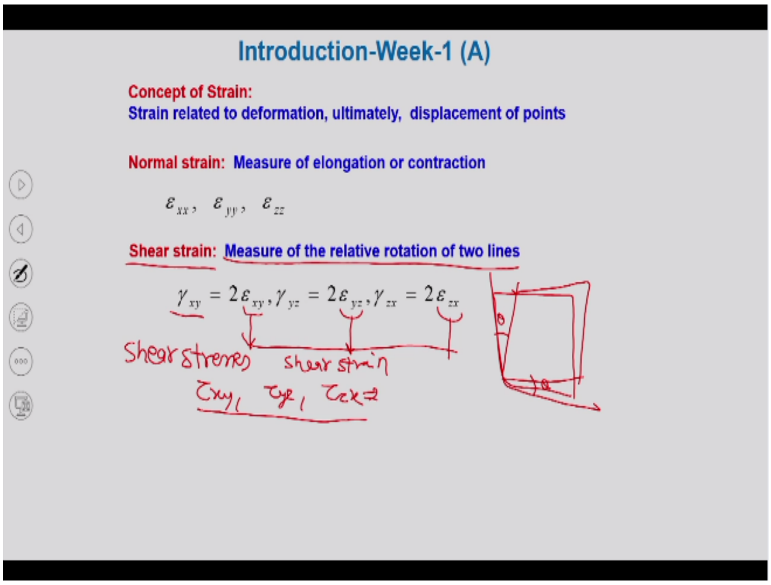

So in this shear strain, measure of the relative rotation of the two lines, so if we talk about a plane before deformation if this is a reason so after the deformation how it would look like that, that angle, so they are denoted by if we talk about epsilon xy, epsilon yz, epsilon zx or these are known as shear strains. I think you are also aware that we relate that gamma xy so basically this is the twice of epsilon xy.

So we relate that gamma xy is the twice of angle of that epsilon xy when this and gamma yz epsilon twice of yz and gamma zx twice of zx. Similarly, you have a concept of shear stresses what is that tau xy, tau yz and tau zx. These are also known as shear stresses.

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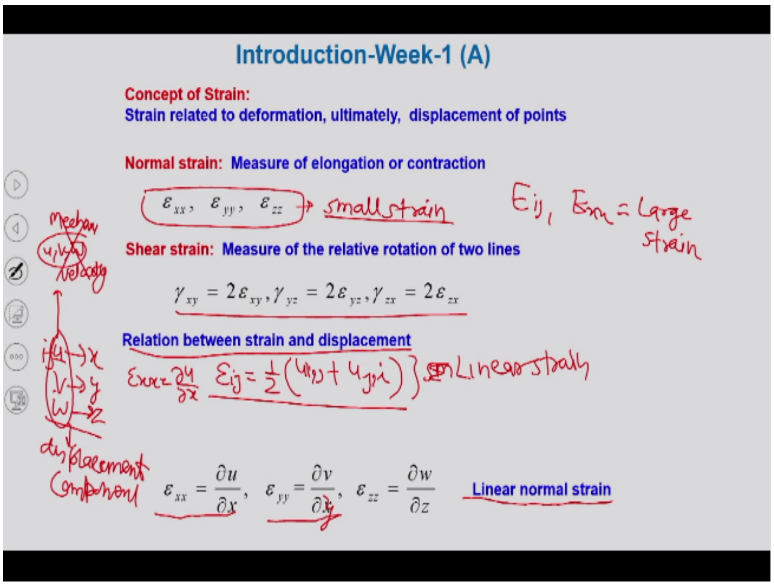

Then, the most important part relations between strain and displacement. Before going to there, I would like to tell you when we denote small epsilon this is known as small strain, small deformation or strain gradient. These are all small strain. If we denote by Eij or if I say Exx then it will be large strain. So strain displacement relations so you may be aware that epsilon xx nothing but del u/del x.

If u is the displacement along x axis, v is the displacement along y axis, w is the displacement along z axis. So do not confuse, in undergraduate level sometimes u, v, w are related to velocities in mechanics but here when we talk about strain these are the displacement component. So this is the displacement component if u is along x axis, displacement y x axis then we can write epsilon x axis is nothing but del u/del x, epsilon yy is nothing but del v/del y and epsilon zz should be del w/del z, so this is called linear normal strain.

In general, we can write epsilon ij=1/2 of ui, j+u j, i this is the normal or I will say that small strain, linear strain definition in index form.

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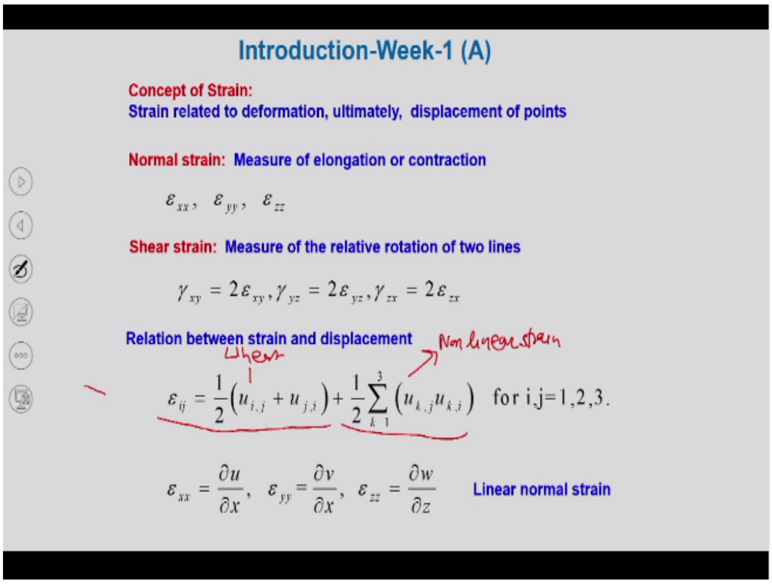

So this is the definition of linear strain and these are corresponding to nonlinear strain that is why I told you if you are able to develop set of relations or theories for considering the linear strain then definitely you can develop if you consider this part nonlinear part in the strain the nonlinear theory of plates.

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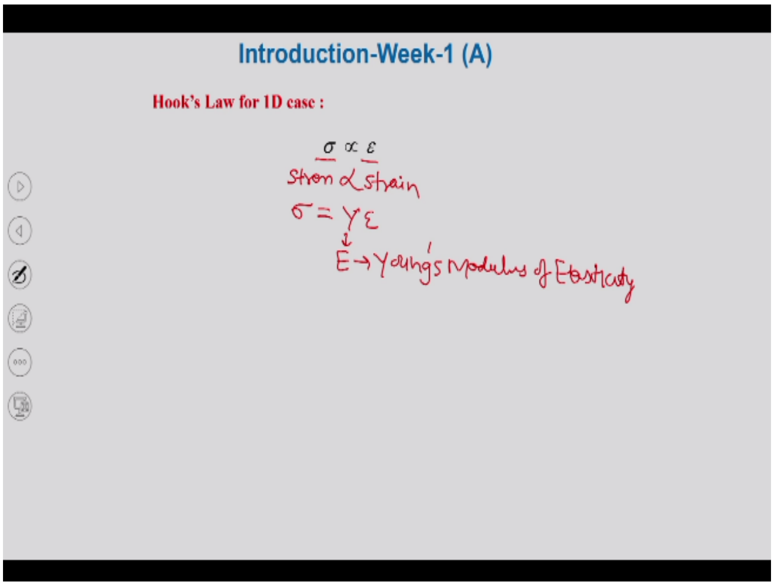

Now coming to the Hook's Law. So generally you have gone through that Hook's Law for 1D case that stress is directly proportional to the strain. If you remove this proportionality constant, it becomes y epsilon or sometimes we call it E, so this is known as Young's modulus of elasticity but what about the 3-dimensional case, if you have plate or a shell in which you applied stresses in all three directions and then you will have strains in all 3 directions so we require the more general definition for this constitutive law.

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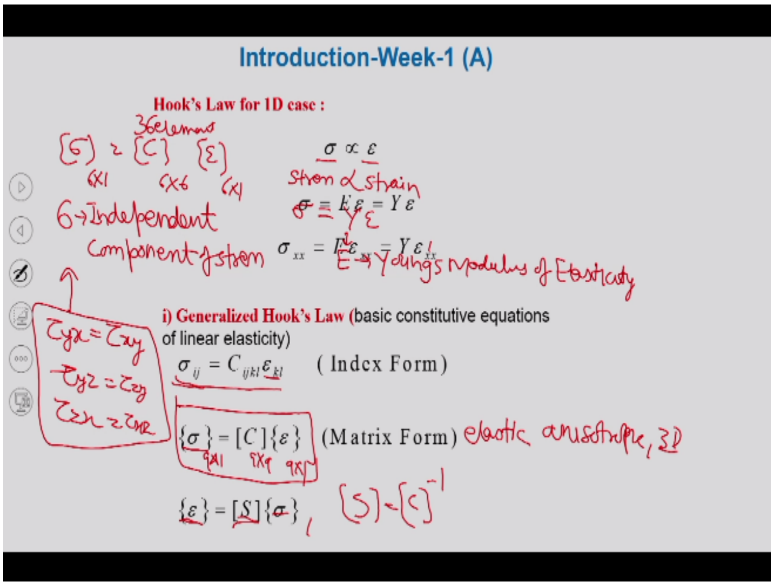

So general definition can be related treated as sigma ij second-order tensor then stiffness and epsilon kl. So this is the standard form of writing constitutive relations for 3-dimensional case that sigma is the Cijkl epsilon kl or in a matrix form you can write sigma=C epsilon, so I would like to here say that here sigma in generally it is 9x1 and this matrix is 9x9 and 9x9 because we have 9 component of stress.

But as you know that by using the moment balance, so we have proved that sigma tau yx is nothing but tau xy. Tau yz is nothing but tau zy, tau zx is nothing but equal to tau xz. So if we use these relations then we will say that 6 independent component of stresses. So in that case it will be sigma, it will be 6x1 C which will be 6x6 and epsilon 6x1. So now you see that if you use this symmetry then this Cijkl reduce to 36 elements.

In inverse form, if you want to know what is the strain in terms of stresses so it will be related by through A matrix S where S matrix is nothing but this is the inverse of C. So basically this now is valid for elastic material, elastic anisotropic or we can say that 3D which are known 3-dimenisonal case so basically it is the elastic material.

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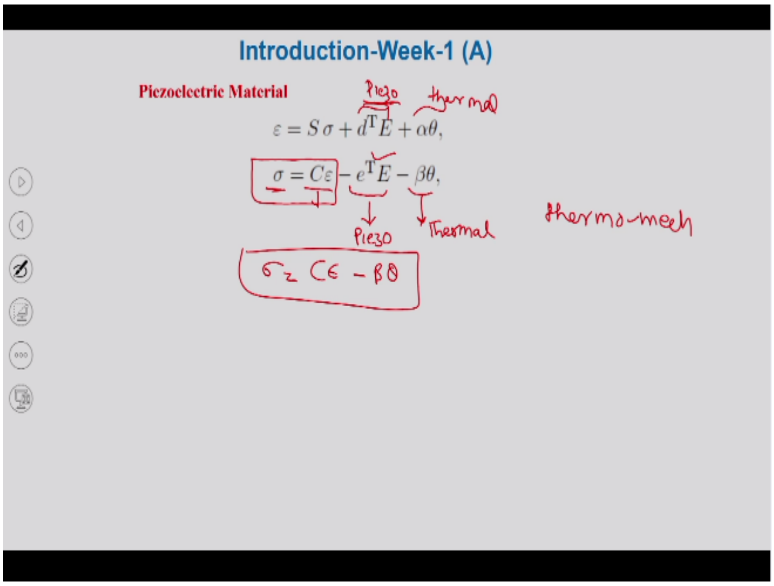

If you talk about the piezoelectric material but extra things will come up here, so you see the definition of stress Ce same then extra, this is contribution to the piezo component, this is contribution to the thermal component. Similarly, here due to piezo material behaviour and this is due to thermal. So suppose I have explained you how to develop plate equation, plate theory using these relations and if somebody is interested to develop thermo-mechanical case.

In that case, he has to take the constitutive relations, this and this beta if further is developed want to develop for a piezo case then they have to consider this extra term.

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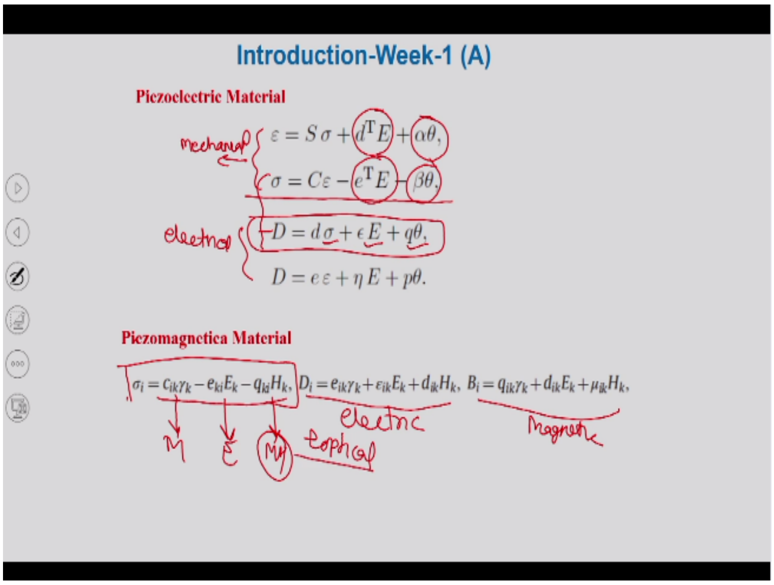

This is okay up to here, as soon as these terms comes into the picture so they are corresponding new relations also comes into the picture that electric displacement in terms of stresses in terms of piezoelectric constant in terms of thermal. So when we are interested to develop set of solutions let us say piezo electric material, you have to take care mechanical constitutive relations and electrical also.

These are the two or if we talk about the system two new set of equations will come into the picture. If we talk about kind of piezo magnetic in which mechanical piezoelectric, magnetic and thermal then the relation will look like this. You see that this is due to M for mechanical, E for electrical and MH for magnetic. So this constitutive relation is further extended, so sometimes students get afraid oh I have to do for functionally graded or shape memory alloys or some advance material.

So you have to just find out those set of relations. As mechanical engineers, we have to just get these relations that what is the constitutive relations available for this kind of material. If this relation is valid definitely we can analyze, we can do the stress analysis or vibration analysis or any transient analysis similarly. Now you have added a magnetic, so this is due to electrical component and this is due to magnetic equation.

I think now you have get the idea that the basic relations is extended from elastic to piezo, piezo to magnetic and later on maybe some new material may exist, this may have a some other kind of property, optics or some other so may be some optical parameters may also may be added.

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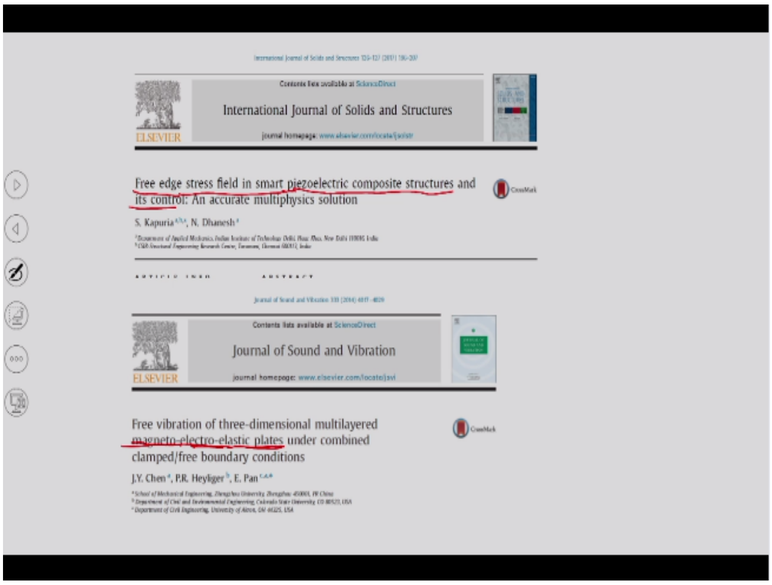

And the relevance whatever we are doing in the course or why we are studying these theories, are they are really useful to the current area of research, whether peoples are working in a current area of research, I will say yes. So you see that these days, you see that free stress as in smart piezo composite structures basically it is a plate.

They have done its control analysis so the relevance whatever we are studying in this course whether this material is relevant to current area of research I will say yes. These days using the advance materials like piezoelectric, so they have done that free stress analysis for a smart composite, piezoelectric composite structures so using a different approach. So you see this current paper.

Then, the next they have done the very recent that free vibration analysis of 3-dimensional multilayered magneto-electro-elastic plate whatever I have relation given here, so these are valid for different boundary conditions clamped or free boundary conditions.

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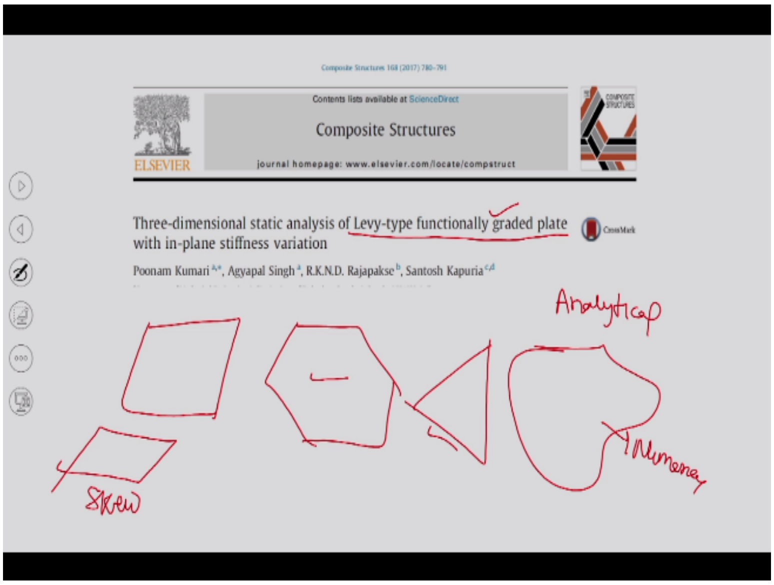

Then, you see that for the Levy-type functionally graded plate, this is the plate which is having boundary conditions, two edges simply supported other edges too but the material is different, so we have to develop altogether new approach to analyze that plate. So these whatever we are teaching going to teach in this course that is having application in the recent research if different kind of material or different kind of boundary or maybe different kind of shape sometimes these days we are just able to analyze a rectangular plate.

The plate maybe of any hexagonal shape or plate maybe of a triangular then you have to think their solutions. Even that solution of arbitrary any shape plate is I would like to say that analytical solutions are not available. We have only up to that numerical solutions for arbitrary supported plates. Sometimes we are having just very basic solutions for the case of triangular plate or some this kind of plate or some I will say that skew plate.

But the detailed analysis or detailed behaviour using some analytical technique or the more accurate still is required.

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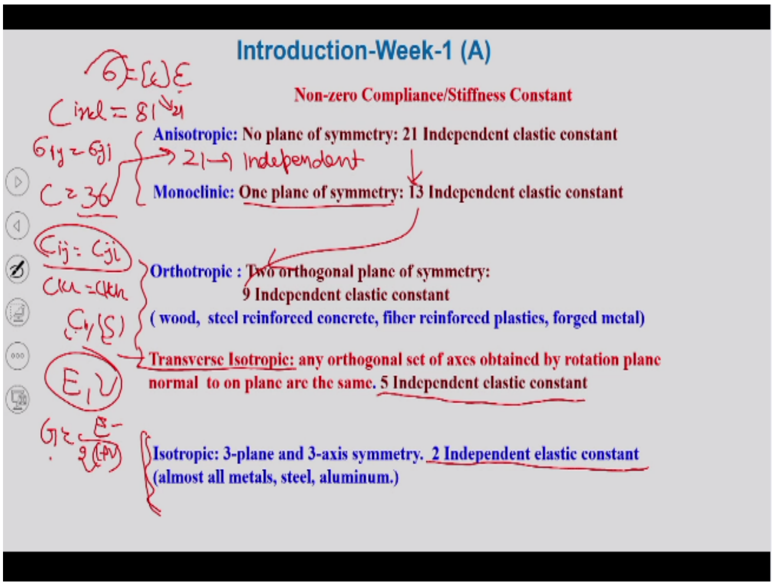

So we are talking about that compliances Cijkl. So there I told you they have 81 components but if you use the symmetry sigma ij=sigma ji then this reduces to 36 components. Further if you use that symmetry Cij=Cji and some Ckl=Clk something like that whatever, so basically that reduces to further 21 independent elastic constant. So if you have a material which is having no plane of symmetry then you have to give 21 constant to find out the stresses relations with the strain.

If you want to know the stresses and you have obtained some strain, then you need here 21 constant then only you can obtain the stress or the strain behaviour. Then, there are materials in which we have one plane of symmetry then it reduces to 13 and if we have two plane of symmetry then it is further reduces to 9. Being a mechanical engineer or I think in this zone more physics or chemistry kind of things or even that some relevant material also there.

But we talk about composites or sandwich plates they are generally treated as orthotropic materials or if you talk about a piezoelectric materials. Then, the other is transverse isotropic material, so most of the time you will find in the literature that people have studied using the transverse isotropic material in which you have only 5 independent elastic constant and the last the most famous and most ancient material is isotropic material in which we have 3 plane symmetry in which only 2 independent elastic constant.

What are they? E and U where G can be found it out like this, so G is not an independent constant. So whenever isotropic plate is analyzed so solution is written in terms of E and U but if you talk about transverse isotropic material, orthotropic material then we write the relation in terms of C or in as complete components C11, C12 like this.

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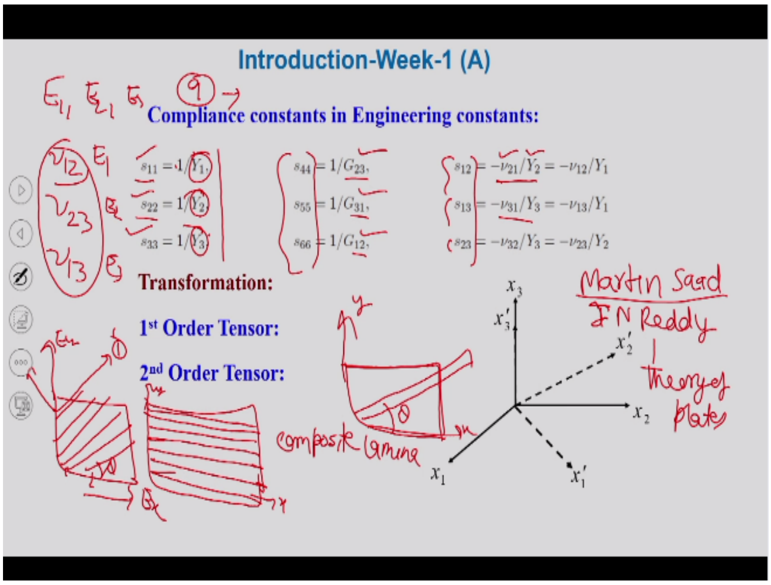

So compliances S11, S22 can be related to the engineering constants basically E1, E2, E3 Young's modulus or sometimes we call it Y1, Y2, Y3. So if you know that engineering constants because students are generally aware about the Young modulus not about these compliances, so we can obtain these compliances if we take 1/Y1, 1/Y2, 1/Y3. Similarly, if we know the shear modulus we can obtain the compliance s44, s55, s66, s12, s13, s23 can be obtained using the Poisson's ratio.

So I think some of you maybe first time encountering this kind of notations that mu 21, mu 31, mu 32 or you may be aware about that or for the case of orthotropic materials so mu 12 is an independent constant, mu 23 independent constant, mu 13 these are the independent constant like E1, E2, E3. Then, G23, G31, G12 so 9 independent constants, so if you know E1, E2 you can obtain S1, S2, S44, S55, S66 and so on.

Further if you want to transfer suppose you know a material is made of this composites or some fiber and the geometry let us say this is your x and y align like this but you have aligned that material axis at some angle with respect to this or very suitable example is that let us say a composite lamina, here fibers are aligned like this okay parallel to the shape x axis. There is no angle but if fibers are at an angle some angle then we have to obtain the effective or the transformed elastic constant along x direction Ex along y direction.

Material is oriented in this direction, so this direction is 1 corresponding perpendicular direction will be 2. So one can obtain how to find out this transformed one, so this transformation rules are given in very basic book, I would like to say that Martin Saad or in some Bhaskar book also that J.N. Reddy theory of plates. So their transformation is clearly explained how to transform a first-order tensor, how to transform a second-order tensor, how to transform a third-order tensor.

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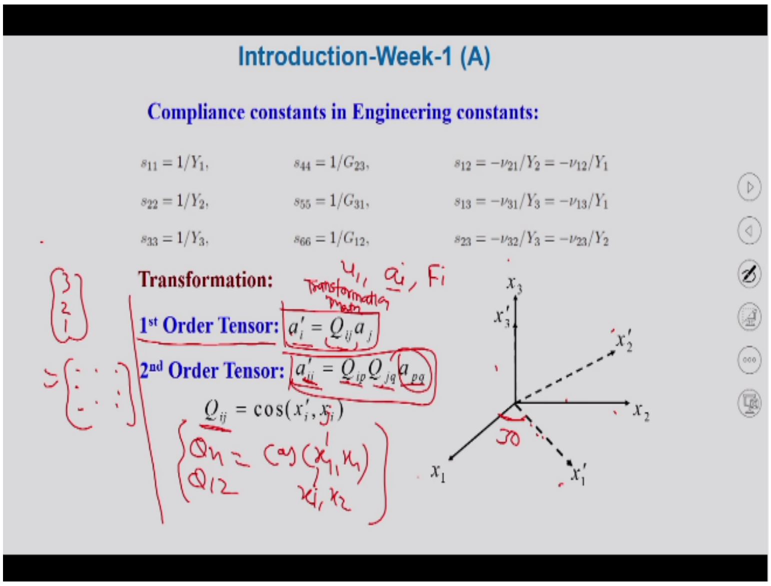

But for our present scenario point of view, I would like to say that first-order tensor if we define by ai let us say ui or Fi, so then Qij is known as transformation matrix. It will contain basically cosine angles. Similarly, the second-order tensor double index then it will be two transformation matrix and that basic elements, so Qij basically components inside that matrix Q11 is nothing but cos of x1 dash, x1 what is that?

If let us say x1, x2, x3, these are our 3 basic axis and material axis is rotated to x1 dash and x2 dash and x3 is matching to this. So basically the angle between this x1 dash 2, x1 then x1 dash to x2, x1 dash to x3. So Q11, Q12 will be here it will be j basically i and x2. So in this way we can obtain the components. So details are given in J.N. Reddy book theory of plates or Martin Saad book, one can go and very basic example also given.

They have given like a component 3, 2, 1 some A, this angle is given let us say 30 then what will be the ai dash? So just we have to obtain x1 dash to x1, x1 dash to x2, x1 dash to x3 and then x2 dash to x1, x2 dash to x2, x2 dash to x3. So similarly you can obtain this cosine components and multiply with that that will give you the transformed vector, transformed second order tensor if you talk about the stresses of strains.

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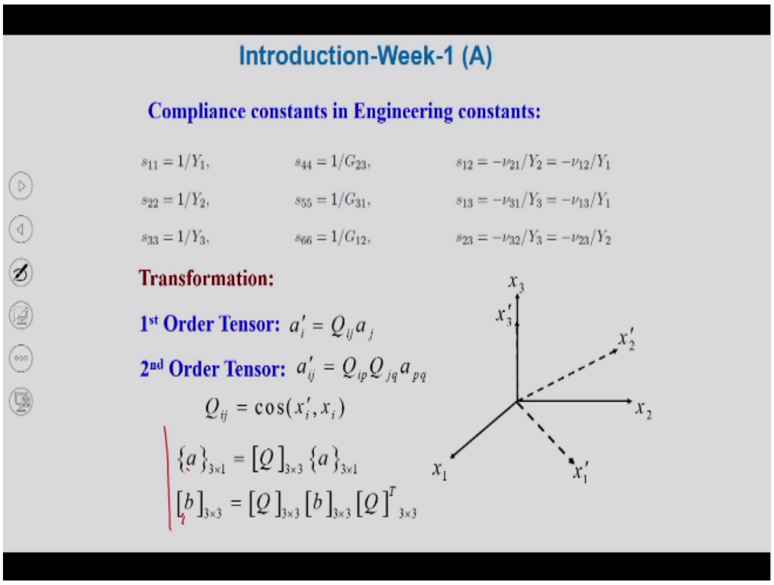

So in matrix form, it can be written like this. Let us say b which is a matrix it maybe stress, it may be strain, 3/3 components, so Q b Q transpose. If we talk about a vector 3x1, it will be 3x3 and 3x1.

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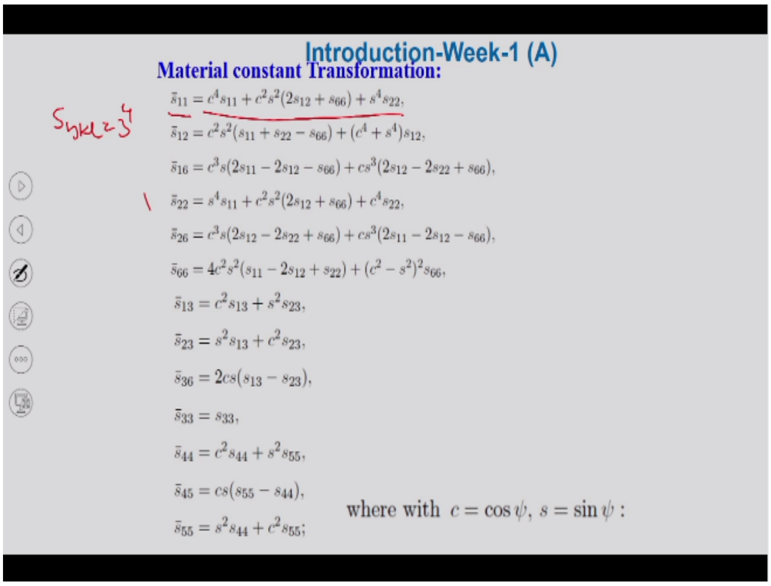

Then, you are talking about compliances Sijkl which is a fourth-order tensor. So you can transform like this. I am just exposing you that these are the relations are available. You need not to remember. Only you must be aware that this can be transformed and this kind of relations are available. These relations are available in the basic theory of elasticity books. So we are not going to remember these relations.

We are just going to use these relations later on wherever we require. So just I am giving you that basic informations that this you required for developing the plate theories.