Mechanical Vibrations

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Module - 3
Single DOF damped free vibrations
Lecture - 3
Coulomb damping, other damping models

In previous lecture we have already seen that how the viscous damping model we can able to a use for modeling the damped system. And in this particular case you have seen that whatever the mathematical expressions we got for free vibration and energy, those were simpler, and we could able to get the closed form solutions. And then now we will be considering some other kind of damping model especially when there is a direct contact between two surfaces, and there is a dry friction between two moving bodies. One body is moving related to another, then due to this friction the energy of the system is getting dissipated in the form of heat. And we call this is as Coulomb damping.

And today we will see in detail analysis of the Coulomb damping, how the motion can be analyzed when Coulomb damping is there. Coulomb damping it is very difficult to model, but we will be taking some simplification in that, and we will try to analyze the motion of a body having Coulomb damping which is in contact with another body and having friction. The dry friction force is generally parallel to the surface, and it is proportional to the weight of the body; if that particular body is sliding on a surface generally it is proportional to the weight of the body.

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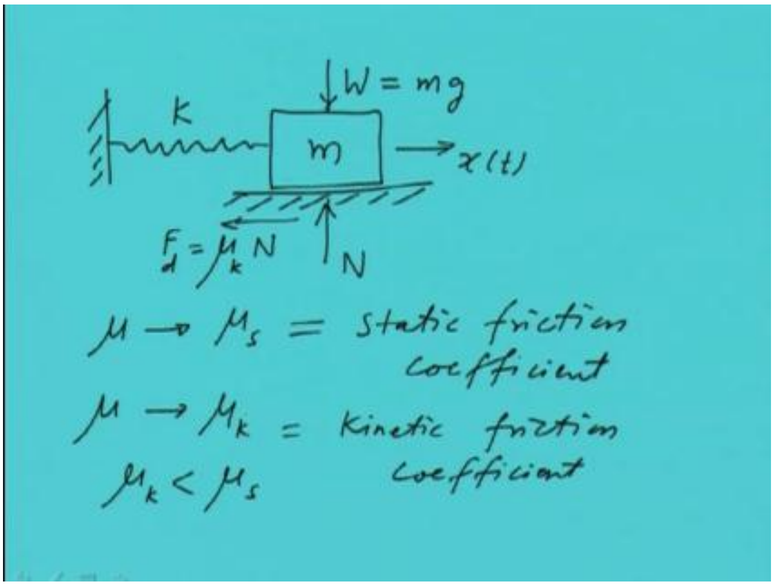

This particular Coulomb damping let us we try to see through one example in which one object is there in the ground; this is the ground, and this is connected by spring. It is a mass of the object, and weight of this is acting downward which is m into g. And on the ground, there will be a force, normal reaction force will be there which will be opposite to the, I can write this as the weight. And this is the normal reaction force, and when this body is moving in this direction, the friction force will act opposite to this. And the magnitude of this will be friction coefficient into the normal force.

When this particular object is not moving is just at the same position, then we will be having the coefficient this as mu s; that we call it static friction coefficient. And once the body start moving, we have the friction coefficient which is called mu k; that is the kinetic friction coefficient. And generally, the kinetic friction coefficient is less than the static coefficient. So, you can see that once the body starts moving, we will be having the friction force less than as when it was in the stationary position. So, the friction force will decrease, but when the body is moving, the friction force will not change, it will remain constant; that is one of the conditions in this particular dry friction case.

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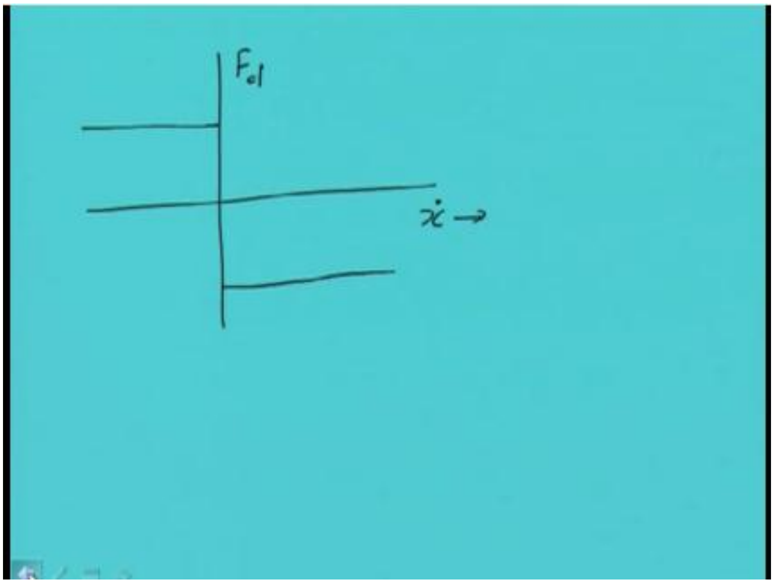

Now let us see how this particular friction force changes with speed as we know that when body starts moving it. So, if we have positive display or positive velocity, the force will be acting in the negative direction like this. And when body is going in the opposite direction that is toward the negative, it is having negative velocity, then we will be having force here. So, we can see that they are constant but changing depending upon the direction of velocity.

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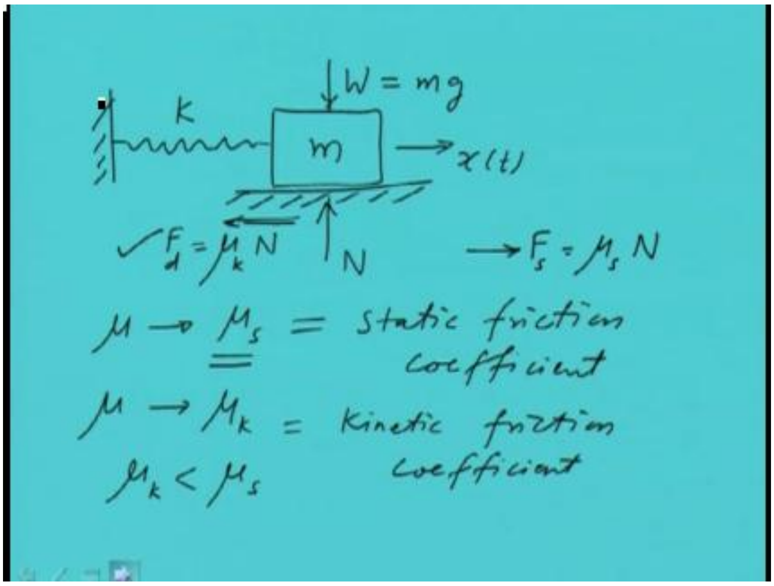

And in the previous example let us see how the motion takes place of this particular mass when we are giving some disturbance through this mass. And if whatever the force which we have given to this particular mass is sufficient to overcome the static force that is static friction force that is mu s into the weight of the body or the reaction force at the bottom. If whatever the disturbance we have given whatever the force we have given if it is more than this, then the motion of this starts. And once the motion of this mass starts it is governed by the friction force which is equal to this one in which the coefficient of friction is kinetic.

You can see that once we have disturbed, it is going in the positive x direction. It will go up to certain distance, but as it is going toward the right, the extension of the spring is taking place. So, it will try to push back the mass toward the negative x direction. So, whenever the inertia force of the body because of the initial disturbance is less than the friction force and the stiffness force. If they are less than that, then this particular body will stop at some extreme position, and it will try to return toward the left side.

And at this point you will see that once the body is start moving from right to left, the friction force direction will change. Then it will be opposite to this one, which is here, it will be opposite to this. So, that will be from left to right; friction force will act from left to right. And because once we have given the disturbance, this particular mass is coming from right to left; it will cause the equilibrium position, because it will be having some inertia. So, it will go other side of the equilibrium position till because then this spring will get compressed, and it will try to push the mass toward the right side.

So, once that particular force and the damping force is more than the inertia of the mass, again this mass will try to go from left to right side. And sign of the friction force will change, and this motion will continue till we have the inertia force and the stiffness force less than the friction force. So, once this condition occurs the mass will stop. This we will seen in detail how this motion take place and what is the period of this particular mass when it oscillates like this.

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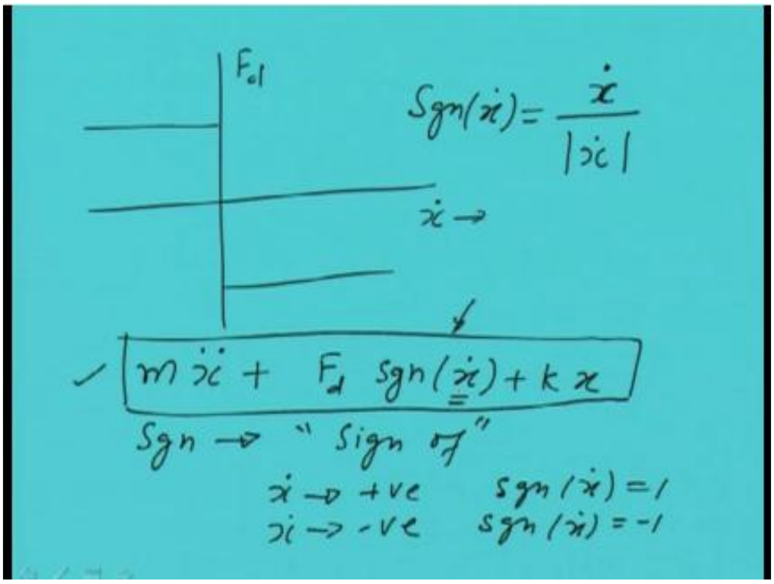

So, this particular motion which is described can be written like this. This is the friction force, and I am defining one function sign x plus k x. So, you can see that sign x is this is a standard function which represent the sign of x dot. So, whenever x dot is positive sign of x dot is 1 and if x dot is negative sign of x dot is negative, we can able to combine these two like this. Here I can write sign of I can combine this here, sign of x dot can be written as x dot that is velocity and modulus of that.

So, you can see now when x dot is positive, the sign of x is having value 1. If x dot is negative, then it will give value of minus 1, because this is negative, and the denominator always remains positive. So, finally, sign of x will be negative. So, this particular equation if you see it changes its sign depending upon the direction of the velocity. So, basically it is a non-linear equation; it is a bilinear equation. There are two conditions. So, we need to analyze this particular equation in two regions. You can see here in the graph when we have velocity negative or positive, in this two regions will this equation be having a different form. And we will be analyzing this equation in two parts.

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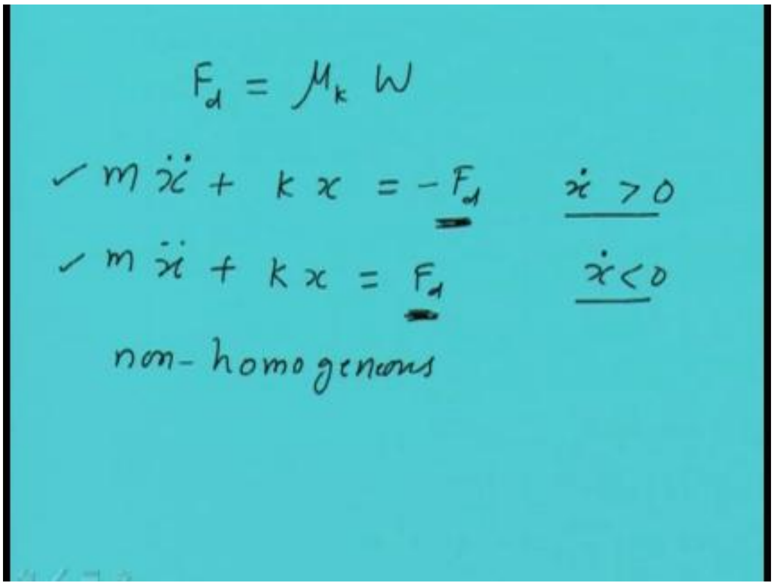

Here that F d which is the damping force, we are calling as kinematic friction coefficient into the weight, because body is in motion. So, we need to take the kinematic coefficient. So, now let us write that previous equation in the linear form this when x dot is greater than 0. So, you can see that when it is greater than 0, the sign of x dot was positive. So, that is why we got this expression, because when we transfer this to right hand side, we are having negative sign here.

Similarly, when x dot is negative will be. So, we will be having two equations which are linear, but we need to satisfy these conditions. We have to keep watching this particular condition in which the velocity is positive or negative; otherwise these equations are now linear. If we see carefully this particular equation is as such not homogeneous; they are non-homogeneous equation, because we have these constant terms here. So, they are making them as non-homogeneous, but these terms are constant.

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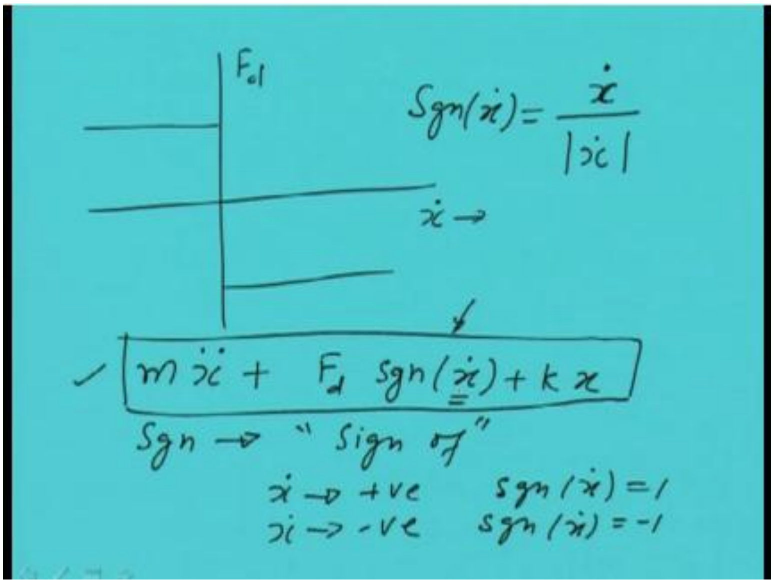

This friction force we have already seen that in the previous plot, they remain constant. So, there is some kind of static force. So, still we can able to analyze these equations as free vibration case, because forces are static in nature. So, we will give some initial disturbance, and we will try to see how the mass get this various form of the displacement with respect to time. So, once we are saying we are giving some displacement. So, we need to have some initial condition, and once we are giving this initial condition then if we go back to this.

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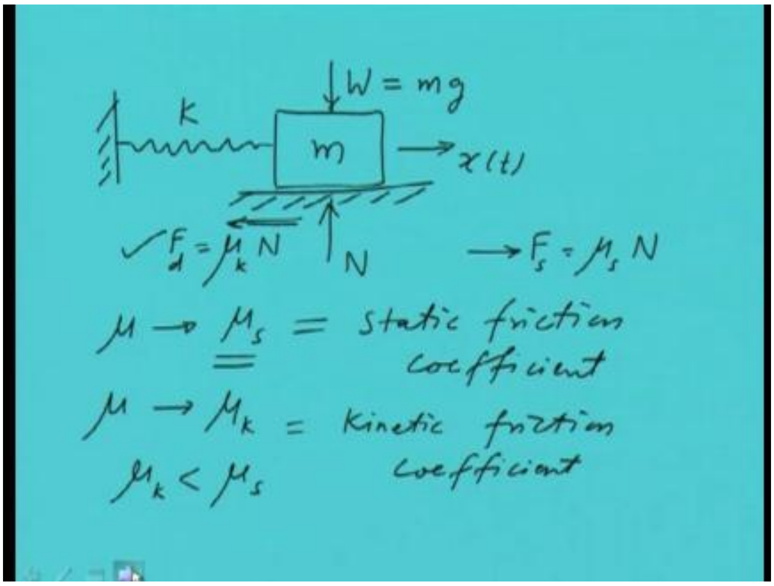

So, we are pulling this mass toward let us say right up to some distance and we are leaving it to osculate. But whatever the displacement we are giving that should give the restoring force in the spring large enough to overcome the friction force, then only the motion will start in this mass.

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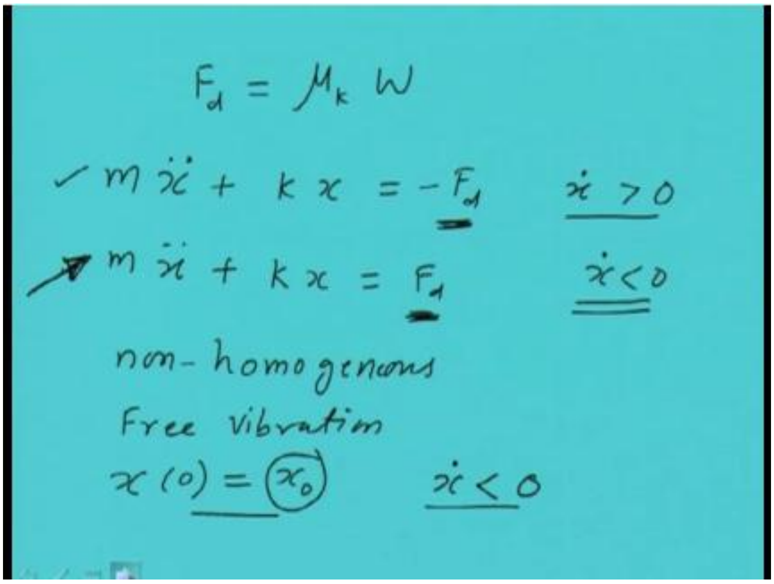

So, that is the condition we will be having that we will be giving sufficient initial disturbance, so that the restoring force is more than the static friction force, so that the oscillation of the mass begins. And now we will see that once we have given the positive displacement and we are leaving the mass at that point; obviously, it will try to move toward the left side; that means the velocity will be less than 0 that is in the negative direction. So, velocity will be 0.

So, initially we need to take this equation in which the velocity is this to find out how the response is getting changed. So, initially when we are giving the x naught displacement negative is 0. So, we need to consider the second equation first.

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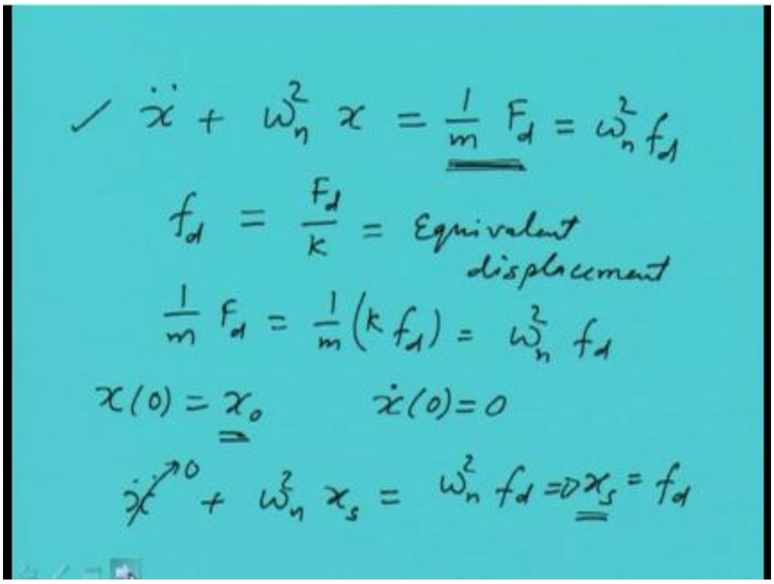

So, let us the write the second equation in more convenient form. This is m by F d, and this particular term I am defining a small f d which is something like equivalent displacement equal to F d by k. This is equivalent displacement, because force divided by stiffness is some kind of displacement, and this is the friction force. So, there is some kind of equivalent displacement related to the friction force. So, with this if we substitute here and we know that 1 by m F d, then we can able to write it is k f d and k by m is we know undamped natural frequency square.

So, we are writing this as omega square and f d. So, now, we are applying the initial condition. And when we are just leaving the mass by giving this displacement, initially it will be zero, but once we are leaving the mass then the speed becomes negative. So, that is why we are solving this equation. And for this particular case we can see that this particular equation is having one static part and then because of this force we can able to analyze this system.

So, when we have static force, obviously, what we are using? We are using the superposition principle in which first we are considering that we are applying this much force, what is the displacement because of that. So, once we are applying a static force, we will be having acceleration as 0, because there is no motion plus this term is equal to omega square and small f d. So, this will give us the static displacement, and this can be written is omega; omega n square is common. So, we will be having f d itself. So, this particular displacement is due to the static force.

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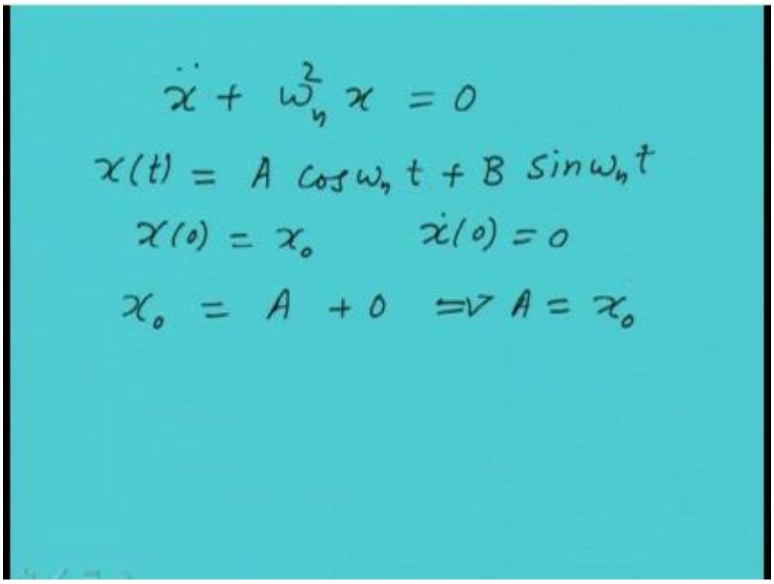

Now we will analyze this remaining part that is X double dot plus omega n x is equal to 0, and for this we know that we have the general solution like this. And if we apply the initial condition x at time t is equal to x naught and x dot as 0. So, first initial condition will give us x naught is equal to A and this cos will become 1 plus this is 0. So, A is equal to x naught.

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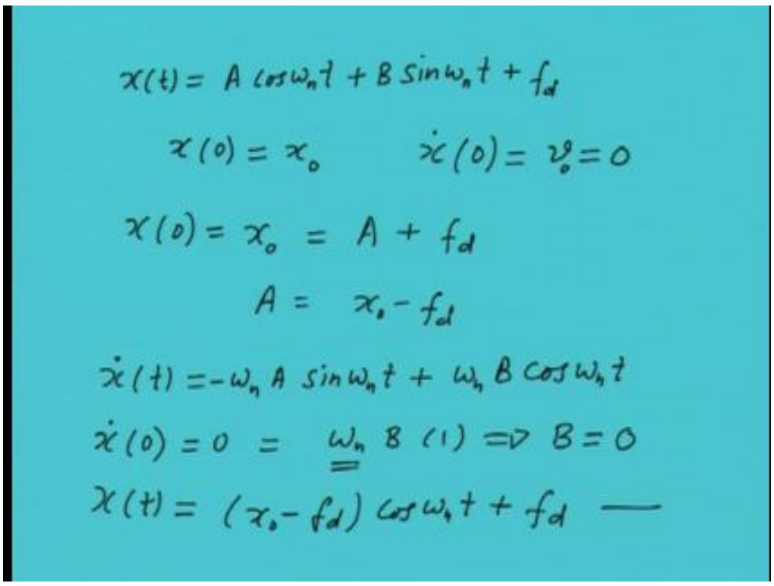

So, the solution for this differential equation will be having harmonic component in which we will be having two constants and because there is a constant term, so that will also come in the general solution of the differential equation. Now we have the initial condition at time t is equal to 0, the displacement is x naught and velocity is v naught but that is 0. Because we are giving initial displacement, and we are releasing the mass at that time velocity is 0.

So, these are the two boundary conditions. If we satisfy the first one, we will get this as x naught A plus f d, because sign term will be 0 for this. So, you can get from here A is equal to x naught minus f d. Now to apply the velocity boundary condition, we have to differentiate the first equation with respect to time. So, we will be getting this term which is having minus sign plus, and then constant term will be 0. So, if we apply the boundary condition here which is 0 is equal to first term will be 0 sin, then second term will be 1.

So, this gives us B because omega n cannot be 0. So, B is equal to 0. So, our solution we can able to write general solution as A which is x naught minus f d cos omega n t plus f d. So, this is the general solution for the previous differential equation.

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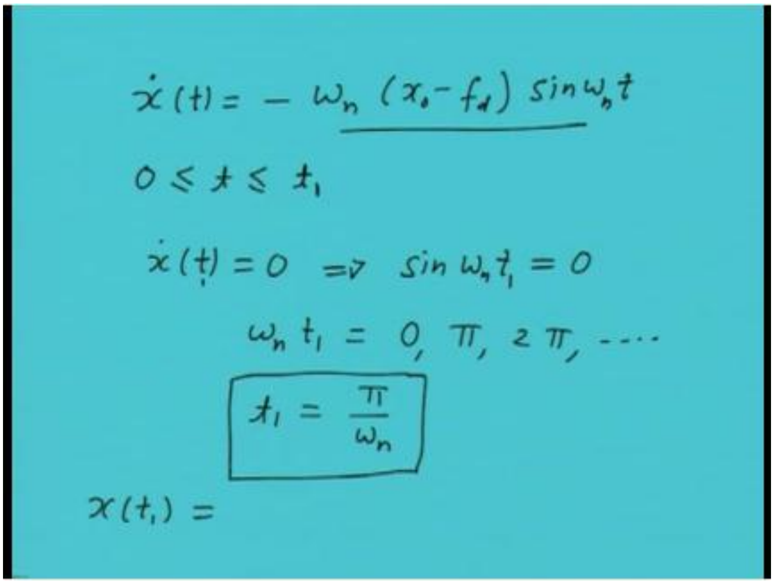

Now we can get the velocity also after differentiating the pervious expression. So, velocity will be this one. Constant term will be eliminated from this. These solutions they are valid between time period from 0 to let us say time t 1. So, once we have released the mass that is at time t is equal to 0, t 1 is corresponding to that time when mass is again becoming 0 at the other extreme. So, this can be obtained t 1 by substituting t x, this is equal to 0. So, that time will be t 1.

So, from this we can able to see that because first two terms are constant. So, this gives us sin omega n t is equal to 0 that will be t 1. And we know this that it can have either 0 value which is nontrivial solution; apart from that we can have pi, 2 pi, etcetera. So, this gives t 1 apart from the 0, the next one is pi by omega n. So, here corresponding to this particular time, the mass will be having again 0 velocity. So, once we release the mass, it comes to the rest at this time, and corresponding to this we can get the displacement.

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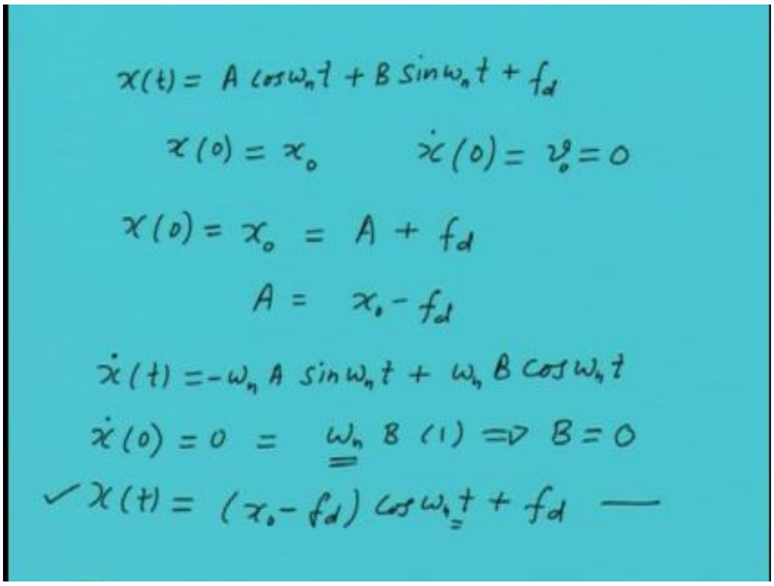

So, the displacement will be from previous, from here we can get the displacement after putting that t is equal to t 1.

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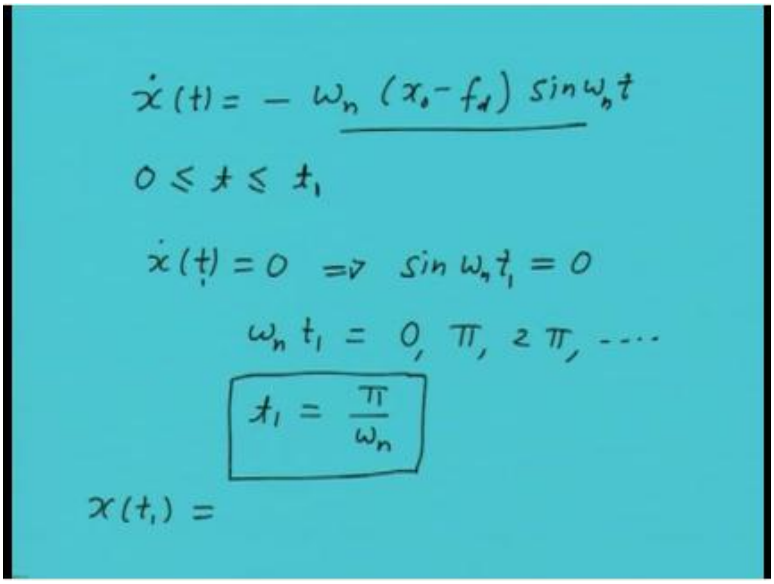

Or here we can able to see that this quantity will become 1; oh, sorry this quantity I am again repeating the sentence. This quantity will become minus 1, and because of that we will be getting X t 1 as x naught minus 2 f d. So, this is the displacement of the mass at the other extreme. So, once we have released the spring from one extreme position that is the initial position and we are leaving that mass to oscillate, it comes let us say from right side to the left extreme side, and there again its velocity is becoming 0.

Now the motion of that particular mass from left to right will take place only when it has sufficient restoring force to push the mass against the friction force. We are assuming that it has sufficient restoring force to push the mass toward the right side for the motion against the friction force and then velocity is now positive direction because as we have taken in the previous case.

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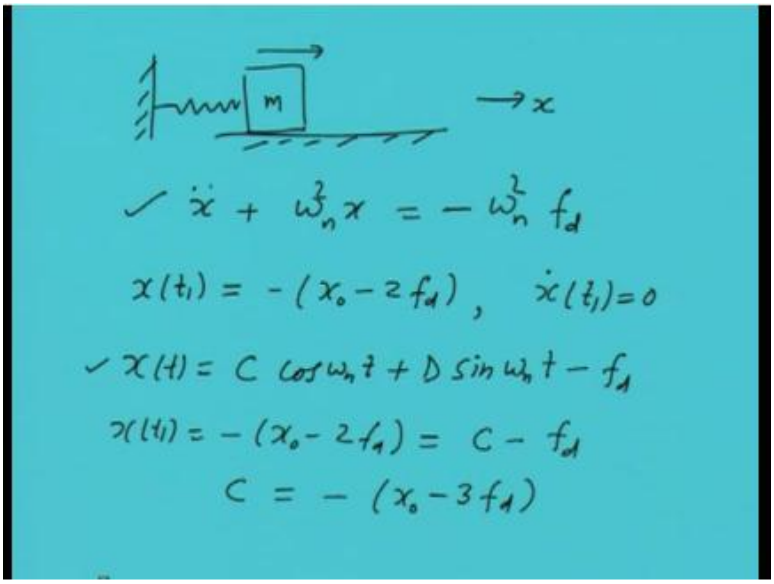

Now the spring is compressed; the mass has reached one of the extreme here. Now it is trying to move toward this direction. So, the direction of velocity is now positive because x direction positive we have taken like this. So, the motion when it is going from left to right, the equation of motion will be this. So, negative sign is coming here. The positive sign was there when the motion was from right to left. So, now we have to solve this equation, and the initial conditions are now different because we know the displacement at t 1 is x naught minus 2 f d. And velocity at this point is 0, because this is the one of the extreme position of the mass. So, velocity is 0 and displacement is this.

So, now we have to solve this differential equation for these new initial conditions. So, as in the previous case let us say we have the general solution like this. Now this is f d because this is corresponding to the term, which we have in the right hand side negative. So, it is coming from there. So, it is superposition of the harmonic component and a constant term. So, if we apply the first boundary condition we will get x t 1 is equal minus x naught f d is equal to C then minus f d, second term will be 0. So, C we will get from here. This will be minus 3 f d, and for velocity we have to differentiate this once.

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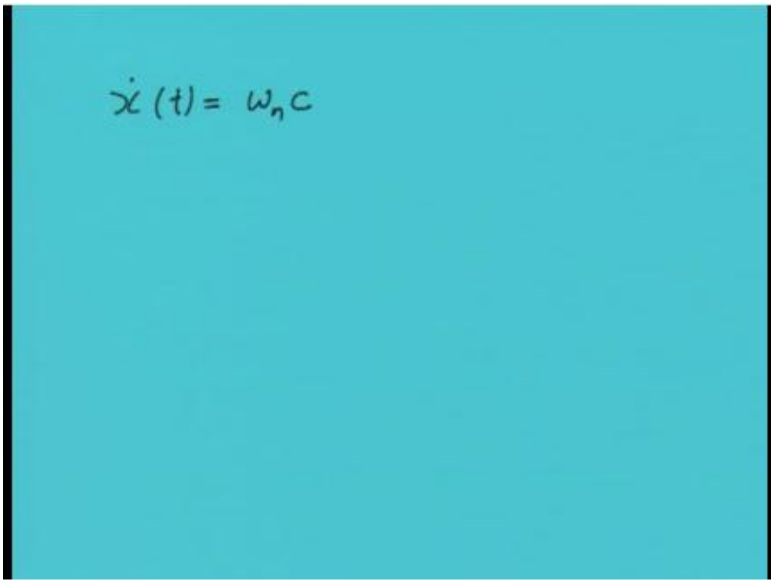

So, we will get velocity that is omega n C; I am just repeating this.

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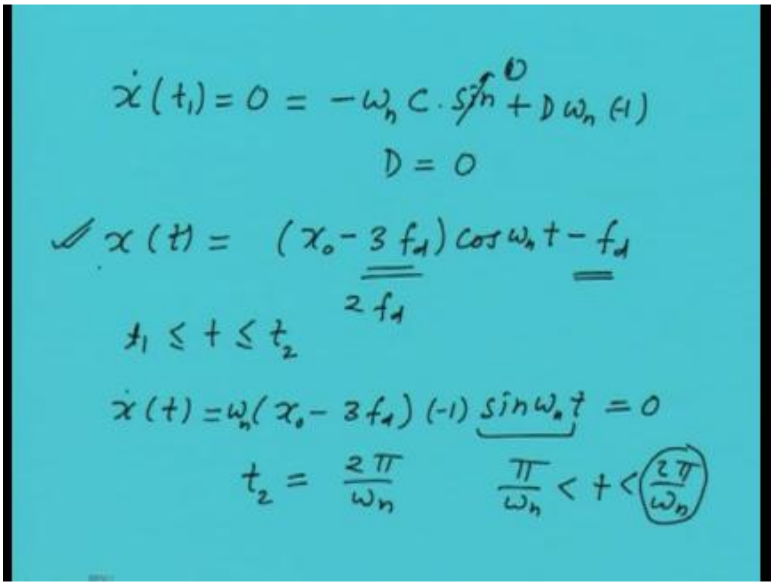

So, from velocity we will get if we substitute the boundary condition 0, this we can able to differentiate or here sin term is there that gives the 0, and we have the cos and we have the cos term which gives minus 1. So, this gives D is equal to 0. So, our general solution for the second half of the oscillation becomes this cos omega n t minus f d. So, as compared to the pervious solution we can see that now the harmonic component has decreased by 2 f d. Earlier it was x naught minus f d, and here it was earlier positive f d; now it has become negative f d.

So, for over one cycle of the mass oscillation we have obtained the solution, and this particular solution will be valid between time t 1 to t 2. So, t 2 can be obtained by differentiating this with respect to time and equating it to 0, because that is the condition when this mass will again become stationary. So, the similar condition this one we are getting. So, you can see that earlier we had the solution pi by omega; next one will be 2 pi by omega n. So, this particular oscillation is the solution is valid between pi omega n to 2 pi omega n. And at that time when this velocity is becoming 0, we can able to get the displacement by substituting t time, time as 2 pi by omega n.

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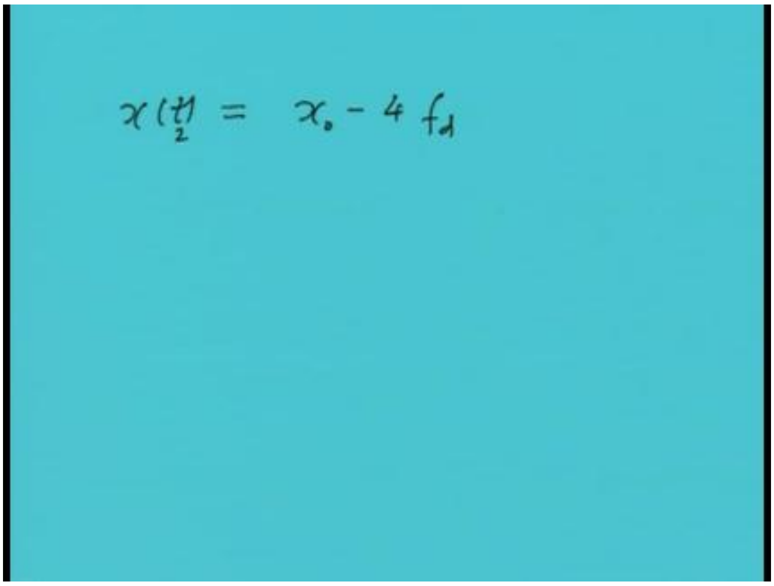

So, let us see in this equation if we substitute T 2 we will get x naught minus 4 f d, and yeah, this is the solution when again the mass is becoming stationary.

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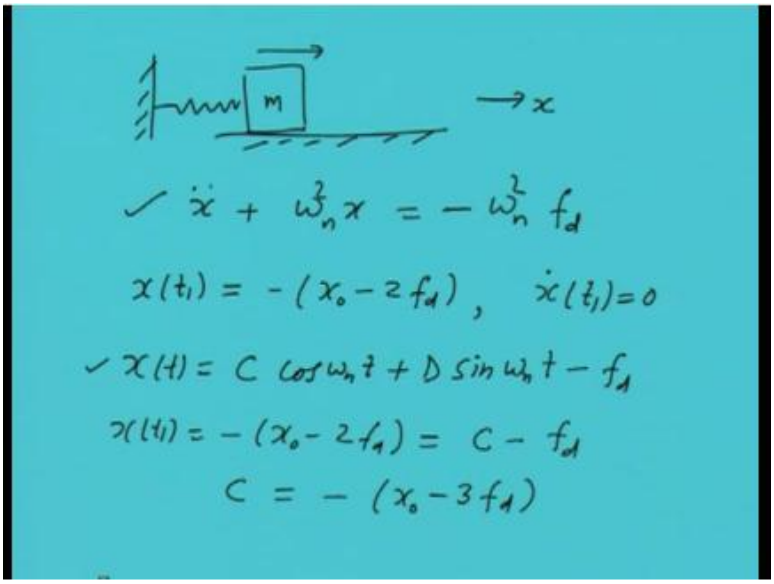

When this particular mass is going toward the right and becoming stationary that is the time t 2.

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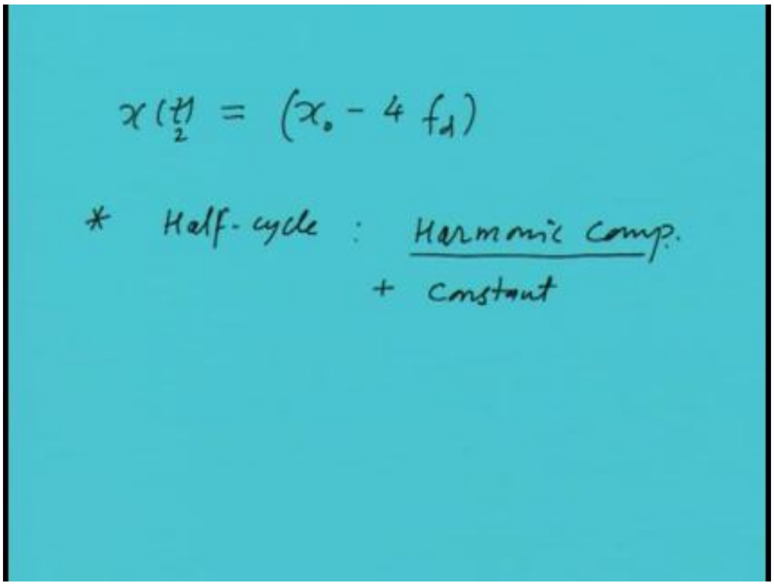

This particular analysis, which we have performed over two half cycle can be repeated for subsequent cycles, but now whatever the solutions we have got; from there we have got the general trend. And we can able to extend these particular results to obtain the analysis for subsequent motion of the mass. So, let us see what are the different kind of trends we have got from these two half cycles and how we can able to extend this analysis for the subsequent motion of the mass. So, first observation is that each half cycle which we analyze the motion is having harmonic component plus some constant component of the motion.

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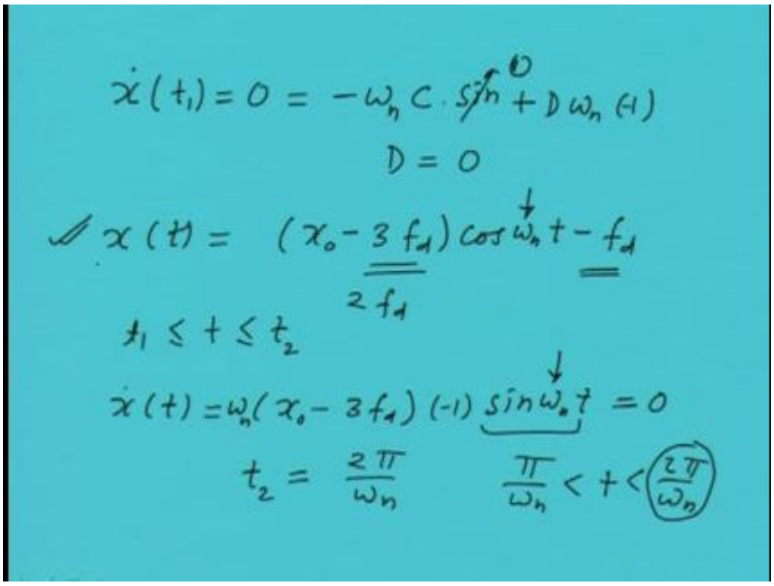

And this particular harmonic component if we see in the solution is always omega n. So, that means the frequency of this particular oscillation the natural frequency because there is no external force. So, the natural frequency of this oscillation is omega n of this particular spring mass system. And each half cycle is having duration of pi by omega n, and if we see the constant term, in constant term it is fluctuating between plus f d and minus f d. And after each cycle, this particular magnitude of the displacement we have seen that it decreases by 2 f d.

Once it is completing the one half cycle its amplitude is decreasing by 2 f d. And for complete one cycle, the total reduction in the displacement is 4 f d. This expression you can see this was the initial condition, and this was the reduction in the displacement over one complete cycle. So, we have seen that over one cycle, the reduction in the motion is a constant; each and every half cycle we have seen that reduction is of 2 f d. So, that means the decay in the response is linear; it is not exponential as we had in the viscous damping case.

So, now we will see the condition for which this particular motion will stop, because this particular mass when have disturbed it, it will be oscillating, but there will be a point when it has to stop, because friction is constantly acting on this. So, in this particular case when the motion is taking place, so at whenever the velocity is becoming 0 in one of the extreme. If that particular at that point if the restoring force, it is not sufficient to push or pull the mass away from the equilibrium position or I should say to push the mass toward the right side or left side of the equilibrium position. If that restoring force is not sufficient to pull the mass or push the mass against the friction force, then it will stop.

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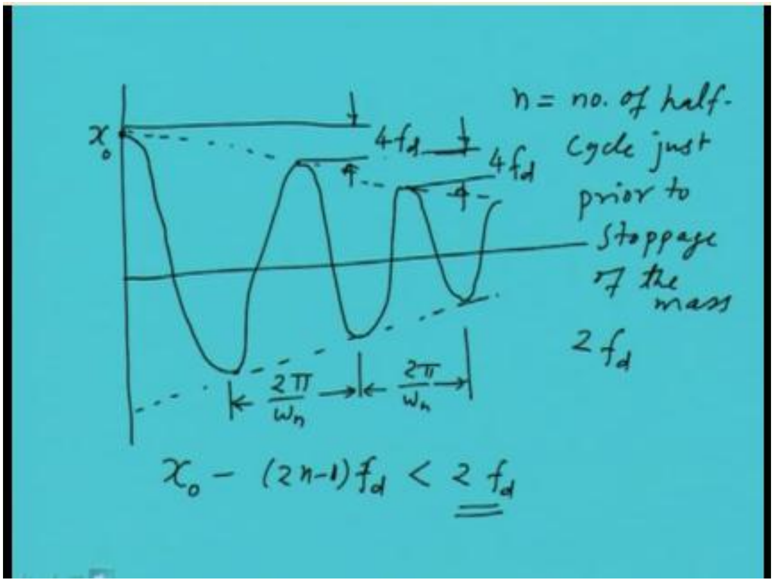

And we can able to see this particular graph. So, we have this as initial displacement and as we have already seen that the decay is linear. So, let me draw the linear envelope inside which the motion will take place. So, from initial position it is having oscillation like this. This is a one half cycle; this is another half cycle, so third half, forth half. So, it will continue like this, and if we see from here to here because it represent a complete one cycle; so, decrease in the displacement is 4 f d. And if we see the time period that is given as 2 pi by omega n, and it remains constant; here also it is 2 pi by omega n.

Similarly, the decay if we see in the subsequent another one cycle it remains the same because of the linear decay, okay. And now let us see the condition at which this particular motion will stop. So, we know that we have the initial condition. And we know that over one cycle we have decrease in the displacement by 2 f d, and let us say n number of half cycle I am defining n. These are the number of half cycle just prior to the stoppage of the mass. So, x naught minus 2 n minus 1; this is corresponding to the just one before when it is stopping into f d.

This should be less than to f d; that means because over one cycle it has to decrease by this much, but if we found that that decreases in the displacement is less than this. Then this particular mass will stop oscillating, and from here we can get the n; let us see how we can able to obtain the expression for n.

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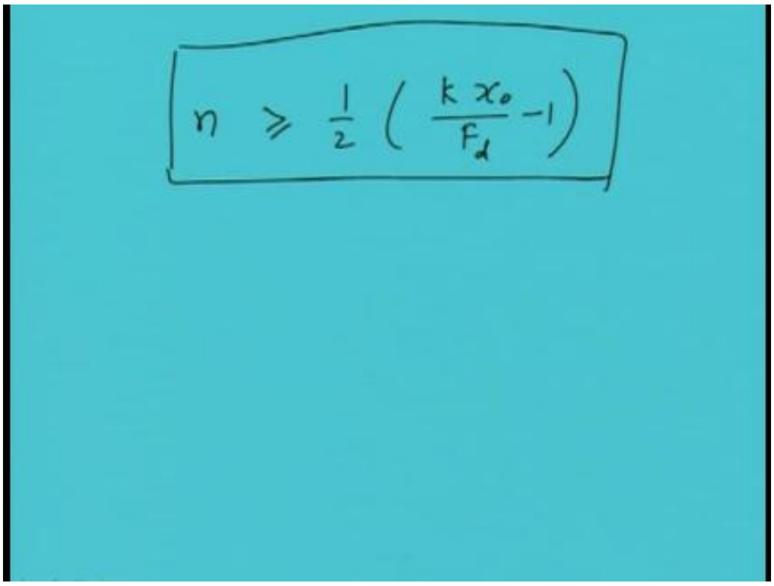

I think n will be if we simplify that half k x naught by F d minus 1.

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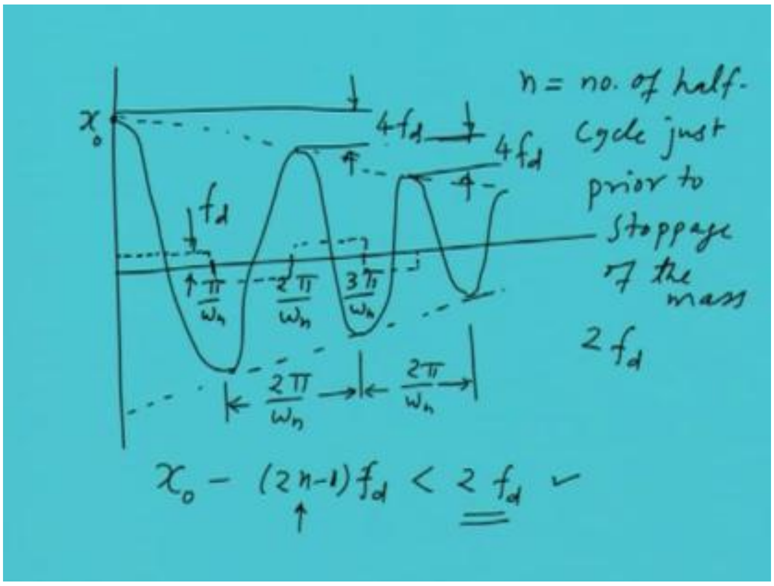

So, this is the condition which we can get from the previous expression, this one after rearranging the term and solving for n. In this particular graph, again if we see carefully this point is corresponding to pi omega n because that is corresponding to half cycle; this is corresponding to another half or full cycle. And here this will be 3 pi by omega n, and if we want to see the constant term due to the friction force whatever the equivalent displacement we define will be positive in this region. Then it will be negative in the subsequent region; again positive as we have seen in the solution, it will fluctuate between the positive and negative, and this is corresponding to the F d. So, let us take one example, so that the concepts are clear.

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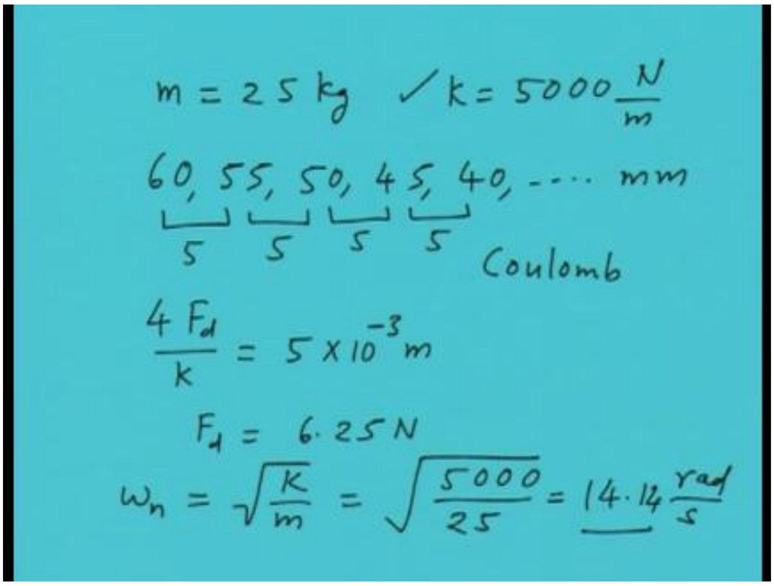

So, in this particular case we have one spring mass system as we analyzed earlier, I am repeating again; I am repeating again that sentences can be taken. So, in this particular example we have the mass as 25 kg, stiffness as 5000 Newton meter, and this amplitude of the successive cycle are found to be 60, 55, 50, 45, 40, etcetera in millimeter. We need to find out what kind of damping is there in the system also the damping force and natural frequency of the system.

So, if we see the decay in the first neighboring cycle, we have 5 unit of decay, from second to third again 5, subsequence are also 5 decay. So, from here we can able to conclude that this is a Coulomb damping. And for this particular case we know that the decay per cycle is given as this where F d is the friction force, k is the stiffness equal to the 5, and we are converting into meter. So, k can be taken this much. So, F d will come out to be 6.25 Newton from this; this is the damping force.

Now the damped natural frequency in the Coulomb damping case we know it is same as the undamped case. So, this stiffness is 5000, mass is 25 kg. So, this gives 14.14 radiant per second as the natural frequency of the system. In this Coulomb damping case, we have seen that over when the motion of the mass is taking place, the friction coefficient of the kinetic coefficient of friction remain same. And because of that the friction force remain the same, irrespective of the amplitude it remains the same; that is why it is called constant damping force model.

And now we will see how we can able to simplify the Coulomb damping model using the equivalent viscous damping model. Because if we can obtain how much energy is getting dissipated per cycle as we did for the viscous damping case and if we can calculate that energy, then we can equate it to that energy to the viscous damping energy model. And from that we can get the equivalent viscous damping coefficient, and then we will be having a simpler model of the viscous equivalent damping. So, let us see how we can able to get the energy and how we can able to get the equivalent viscous model of the Coulomb damping.

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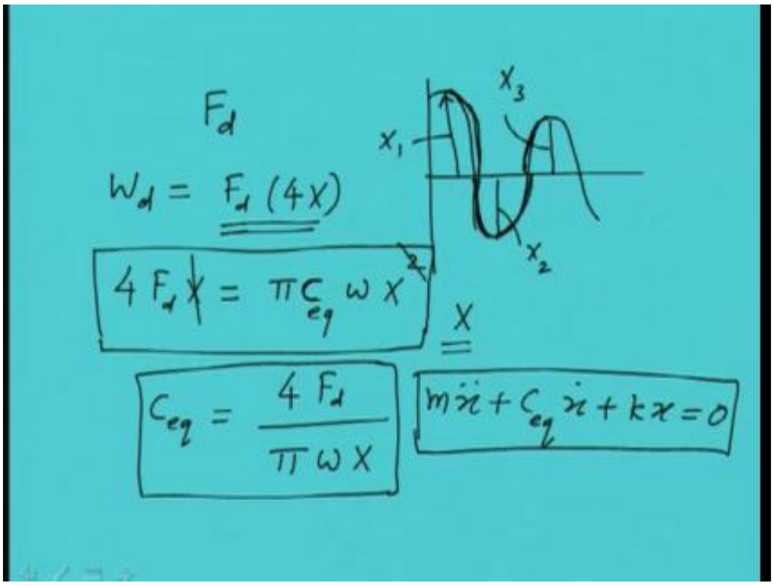

So, as we know that the damping force remains the constant and when we are talking about the motion decaying, this is the decay, and let us say this is the x 1, this is x 2 and this is x 3. And let us say because over the cycle whatever the displacement is taking place; if let us say the average of these amplitude is x, and a friction force is constantly it is acting on this. So, we can have the total energy dissipation equal to F d, and this displacement will be acting in the four places will be there. One is this region; another is this, two, three, four. And we are calling this as an average displacement.

So, for this damping this friction forces are acting against these displacements. So, we are calling this as the total energy dissipated over one cycle. And this energy now we are equating it to the viscous damping energy which we calculated earlier, but instead of C we are writing C equivalent. So, we have seen what we are doing? We are equating the energy of two different damping models, and our aim is to obtain what is the equivalence of the viscous damping coefficient.

So, this will be given as pi omega x; this will get cancelled. So, this is the equivalent damping. Now once we have this, we can able to use our conventional equation. And we have already analyzed this particular equation in great detail. So, all the equations are valid for this case. So, you can see that using this we could able to simplify our analysis in great detail.

Now we will be considering another kind of damping which is called structural damping. This is nothing but the material damping also we call it, and when that particular body deforms, then at the molecular level the molecules they interact with each other. And because of that the heat is getting dissipated, and that is why it is called structural or material damping. And as I have already given the example if a piece of wire is given a cyclic stress compression tension, it gets heated up because of whatever the energy is getting dissipated in the form of heat.

So, anybody if we give a cyclic stress to that, the energy will be dissipated in terms of the heat. And most of the experiment shows that most of the engineering materials like steel or aluminum, they dissipate the energy which is in effect non-proportional to the frequency at which we are doing this particular cyclic change in the stresses. But it depends upon the amplitude all and the square of the amplitude of the motion. This particular internal damping is also sometime called solid or solid damping also.

As we have already seen how if we can plot the damping force with respect to displacement. Then we get a hysteresis loop and this hysteresis loop help us in finding how much energy is getting dissipated per cycle. And here also we can able to use this hysteresis loop for finding how much energy is getting dissipated per cycle. And because whatever the mathematical model is there for this internal damping or material damping is very complex.

So, sometime we simplify this using the similar concept of the equivalent viscous damping model by finding the energy of the hysteresis loop and equating it to the viscous damping model energy. This is one of the way of analyzing the material damping.

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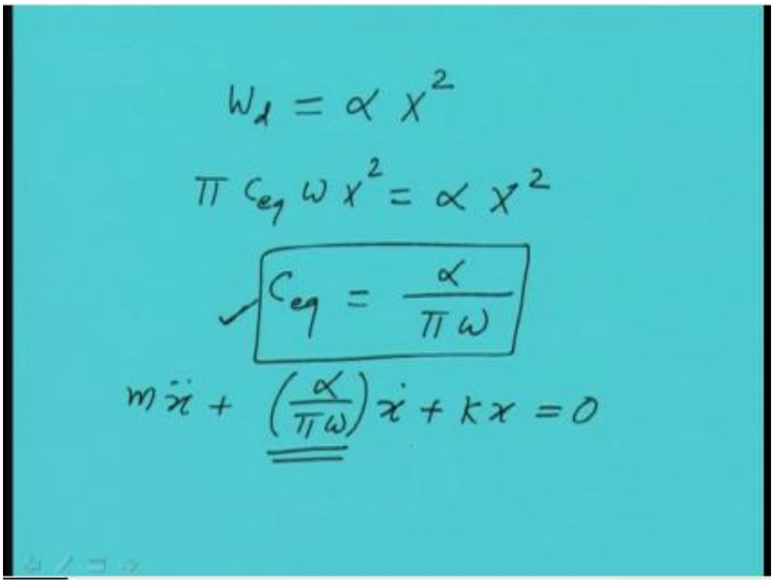

This as we mentioned that work done is proportional to the square of the amplitude of vibration and alpha is a constant. And if we equate this to the equivalent viscous damping model, we can have this. And you can see that now we got the equivalent damping, and it can be used for once we obtain the viscous damping we can able to use whatever the analysis we have done for the viscous damping model. So, the equation will take this particular form like this. So, you can see that we have simplified the analysis by using the equivalent viscous damping model.

So, today we have seen that other kind of complex damping model like Coulomb damping model, and we have analyzed that in great detail how the motion can be analyzed for that particular case. Also then we analyze some kind of simplified model of these Coulomb damping and other like structural damping, or we can able to obtain the equivalent viscous damping coefficient. And for which kind of damping we have the mathematical expressions and the procedure of analysis are simpler. In the subsequent lecture, now we will be dealing with the force vibration especially for the single degree of freedom system.