Mechanical Vibrations
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Module - 3
Single DOF damped free vibrations
Lecture - 2
Logarithmic decrement, experimental determination of damping coefficient, hysteresis loop

In previous lecture we have studied damped free vibration, and in that particular lecture we obtain expressions for the free vibration response. If we give some disturbance to the particular dynamic system may be we can give an initial velocity and initial displacement to the system, and how the particular system behaves after sometime. Today we will see some example, so that the concepts are more clear to us. And apart from that we will be covering torsional damping model, and then we will be going for the hysteresis loop curves that is the energy dissipation per cycle of a particular damping system

In damping so many other models are there. In the previous case, we have considered only the viscous damping, but in subsequent lectures we will be covering some other kind of damping curve like material damping, or sometime it is called structural damping and coulomb damping which comes due to the dry friction. So, first let us see some examples on the damped free vibration.

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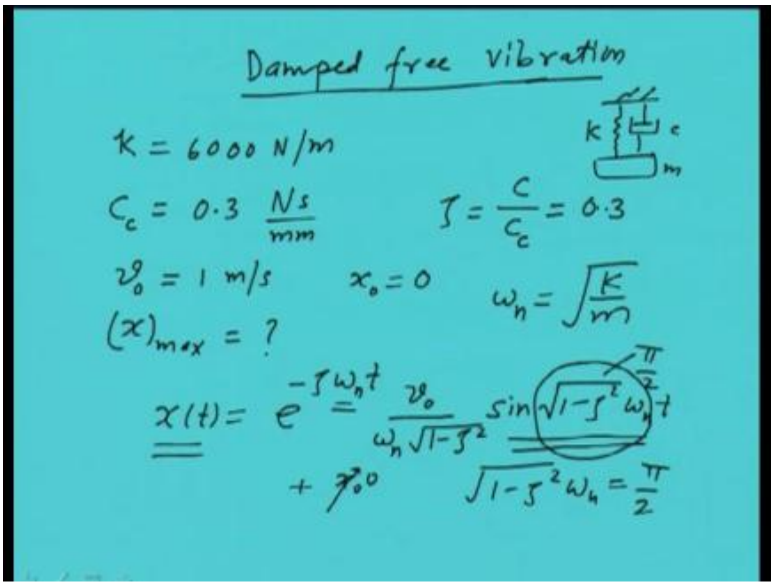

We are dealing with the damped free vibration, and let us take one example in which the system is having stiffness of 6000 Newton per meter. And the critical damping constant of the system is 0.3 that is Newton second per millimeter and damping ratio which is damping coefficient divided by the critical damping is 0.3. Now we are giving an initial velocity to the system. We can consider this as single degree of spring mass system. And we are giving an initial disturbance of 1 meter per second and when the system is at the equilibrium position; this is the initial displacement.

Now we have to obtain what is the maximum displacement this particular mass will be having due to this initial condition. Now because we already know the free response is given as this is the decaying part, then we have the velocity that is the initial disturbance, and this is sinusoidal part of the motion. And there was another term related to the x naught, but because this is 0. So, it is not coming into the picture here. So, now we need to obtain the various quantities like undamped natural frequency and that can be obtained as omega n k by m.

Mass is not given here, but whatever the values are given based on that we can able to calculate the mass, and if you see we are interested in the maximum displacement. So, we will be having this displacement maximum when we have this particular quantity is 1; that means we will be having this quantity as pi by 2. So, 1 minus zeta square omega n as pi by 2. Now let us first obtain the mass.

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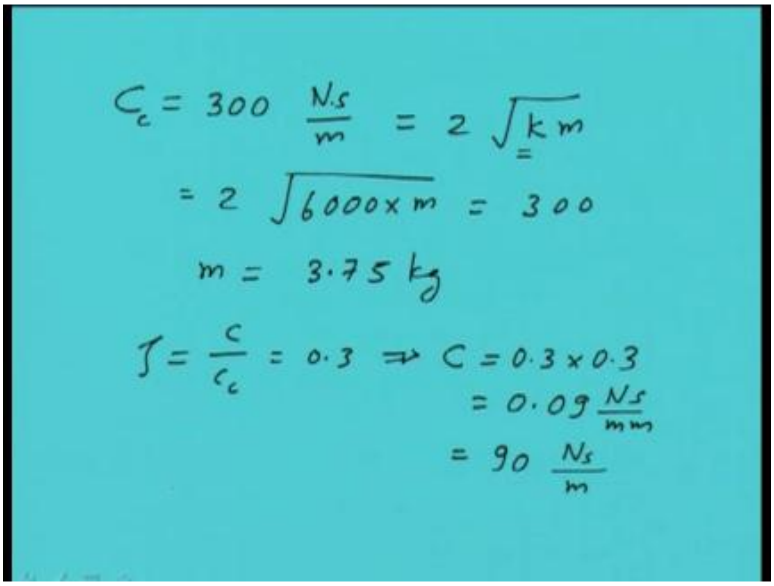

For that we have this critical damping parameter; that is we can convert that was 0.3 Newton second per millimeter. Now we are writing in the terms of meter Newton second meter, and this is given as two times square root stiffness into mass, and we know the stiffness that is 6000 Newton meter. So, if you substitute this here or that is equal to 300, m can be obtained; mass will be given as 3.75. Now once we know the mass zeta the damping ratio we know that is 0.3. So, from here the damping coefficient can be obtained, because we know the critical damping that is 0.3. So, damping coefficient will be 0.09 Newton second per millimeter or if you want to convert this into Newton second per meter.

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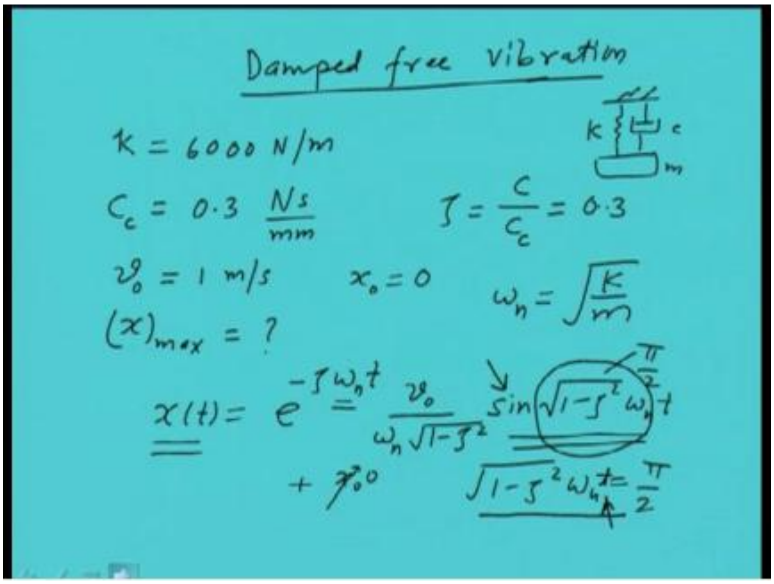

Now again we will come back to the displacement equation which we wrote here. So, when this particular quantity is unity, then we are getting the maximum displacement, and for this to happen we found that this is the condition. So, we can see that now we can able to or this particular t is pi by 2. So, this particular expression we can able to get the time at which it is maximum.

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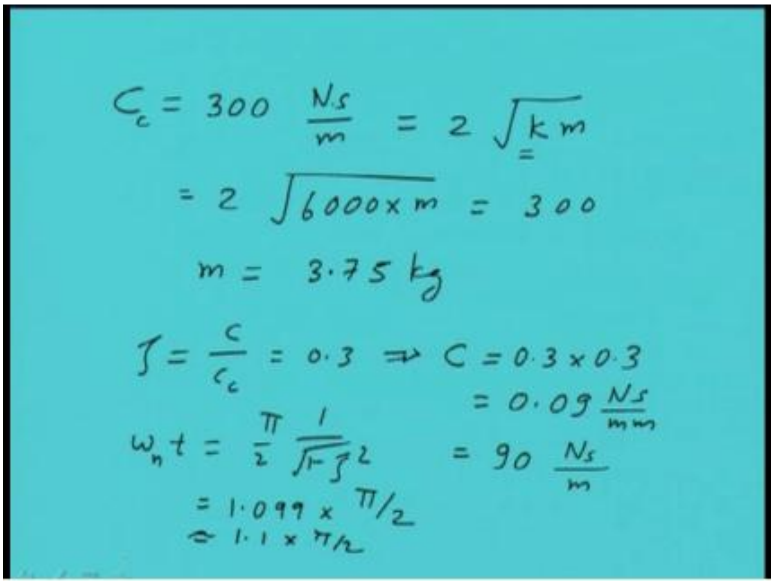

Or we can able to write this as omega n t is pi by 2 1 by 1 minus zeta square. So, this will give us omega n t; that is 1.099 into pi by 2, and this can be substituted in the previous equation here, okay.

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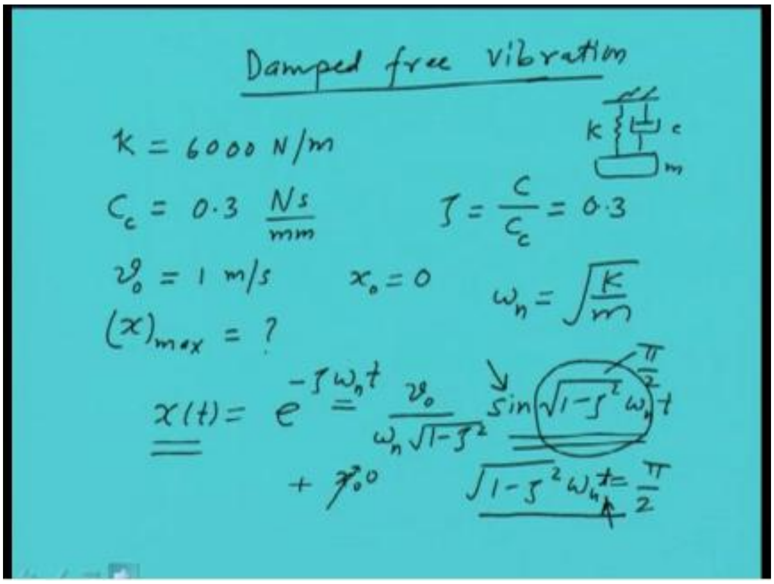

So, if you substitute we can get the maximum displacement that will be e raise to zeta is 0.3 into pi by 2 into approximately 1.1. This 1.1 is coming from here. It is approximately 1.1 into pi by 2.

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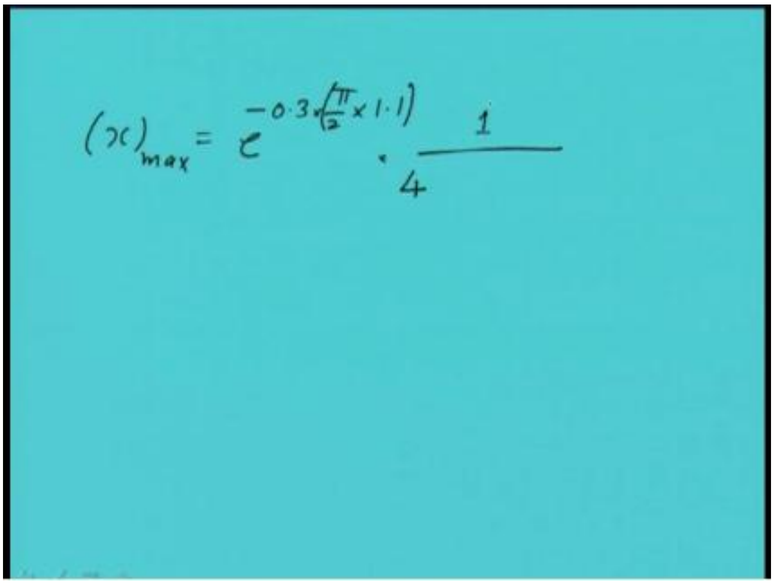

So, that it is pi by 2 is already here and then other terms. So, we have x naught dot that is initial velocity is 1 meter per second. So, this one, then we have the omega n.

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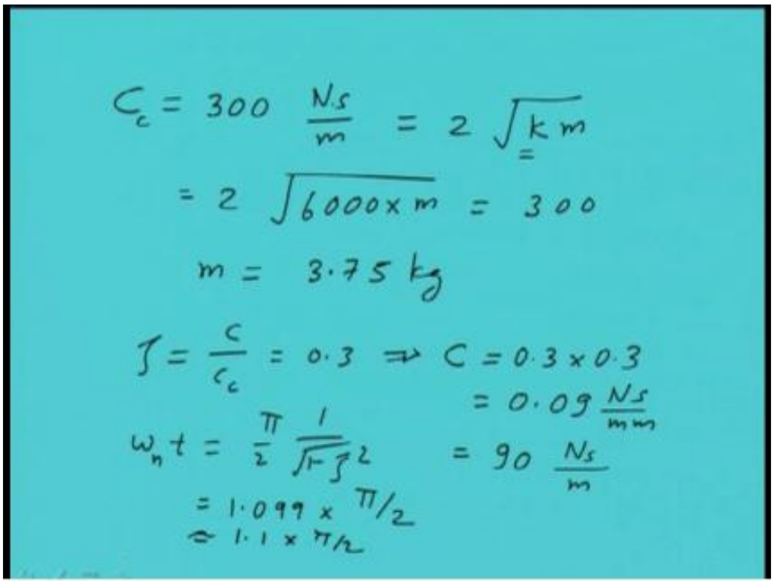

Still we have not obtained the omega n.

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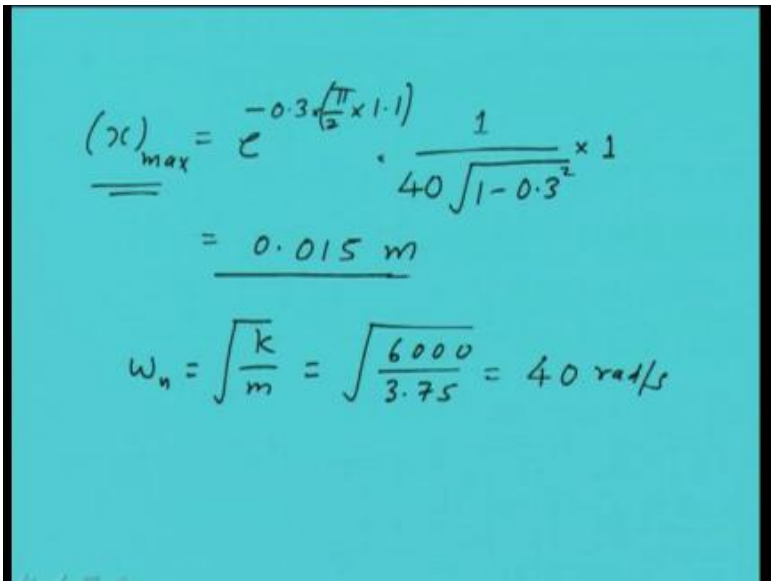

So, omega n we can able to calculate that will be k by m because we know the k m we know then k is 6000, m is 3.75. So, this will be 40 radians per second. So, this can be substituted here omega n. And then 1 minus zeta square 0.3 square into for that sin term we have made it 1, so that we get the maximum displacement. So, this will give us the maximum displacement. That is the result of this particular problem in which when we give an initial disturbance of velocity 1 meter per second, then we got the maximum displacement as this.

Now consider a simple torsional dampered in which a particular disc is in a liquid. So, when it oscillates about its own center, there is interaction between the disc and the fluid, and because of it is getting damping in the form of torsional vibration.

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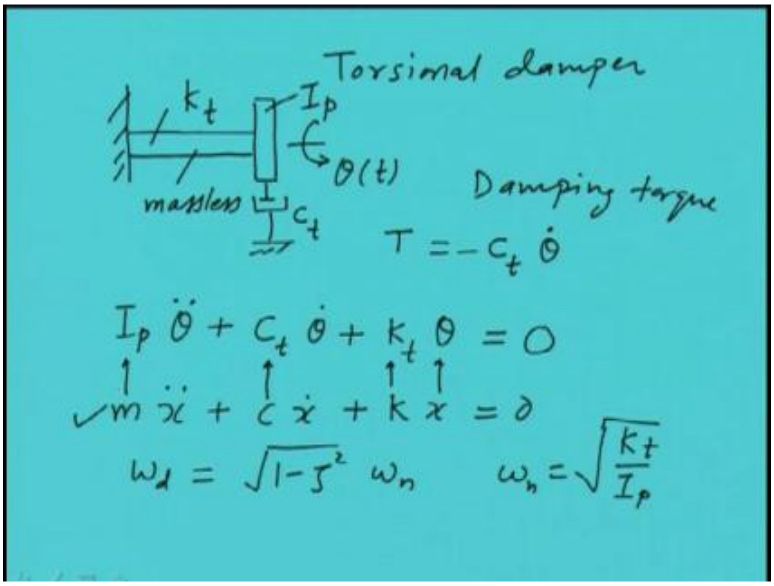

So, let us see how this particular damping is there. This is the torsional damper. So, let us consider a cantilever beam with a disc at the front. So, this particular disc is subjected to when it is oscillating about its center is emerging some fluid. So, because of that we are having a damping. Let us say the damping is ct, t represent the torsional damping. So, in this particular case, the torque which we are getting from this particular damper we are assuming it to be proportional to the angular velocity of this particular disc. And, obviously, it will be always opposite to the motion.

So, this is the damping torque, t is the damping torque and the equation of motion for this particular system, because this is a single degree of freedom system in which we have not considered the mass of this particular shaft. This is massless and only disc is having polar mass moment of inertia. So, based on the previous background, we can able to write this particular equation of motion and k t is the torsional stiffness of the shaft. This particular shaft is having stiffness torsional k t that we already discussed in the previous lectures. So, this is the standard form of the equation of motion for the torsional vibration.

So, you can see this is identical to the previous case in which we had I am writing for linear system how the equation was there, so that we can able to compare. So, we can see that mass has been replaced by polar mass moment of inertia, and this linear damping has been replaced by the torsional damping coefficient. And similarly stiffness and the linear displacement have been converted to angular displacement. So, in the same analogy we can able to obtain the damped natural frequency for this new system that will be this one, where zeta is the damping ratio, and this is the undamped natural frequency omega n, which in turn is given as a present case k t by I p; that is k subscript t I subscript p.

This is the torsional stiffness. This is the polar mass moment of inertia of the disc, and other correlations will remain identical as it is there for the linear system. Now we will take a example of the torsional vibration especially it is having application in the door closer, where the automatic door closer is provided generally we provide critical damping in that system. So, we will see that how whatever we have studied is useful for designing a door closer in a house or in an office, okay.

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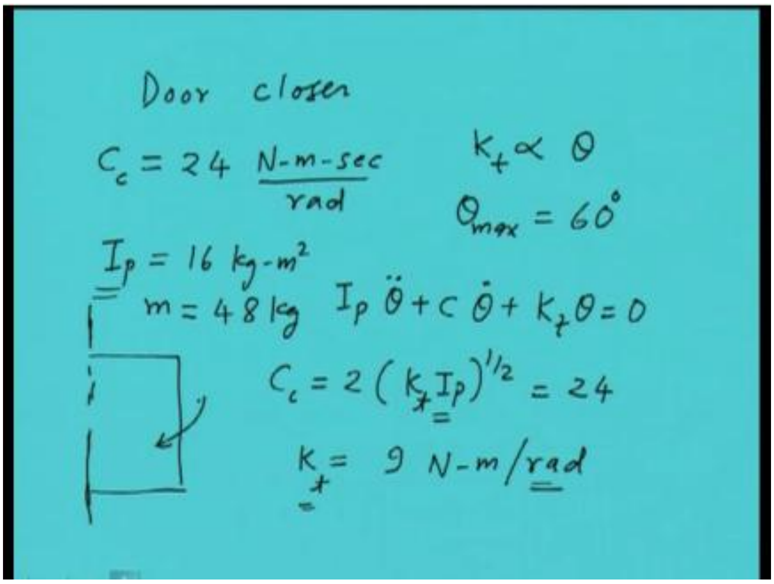

An automatic door closer we are considering and the critical damping of this door closer is 24 Newton meter second per radian, because this is torsional vibration case. Because the door has to oscillate about a particular axis, and we have the model that torsional stiffness is proportional to the theta. So, whatever the springs we have connected in the door closer obey this rule, and the door can open maximum of 60 degree that is the maximum we can able to open the door from its frame the 60 degrees with respect to the frame.

And we have other conditions like we know the polar mass moment of inertia of the door that is 16 Kg meter square. So, with respect to the frame if let us say we have a door like this. So, with respect to this axis because door can have opening and this kind of it can open. So, about this axis is having polar mass moment of inertia for I p that is 16 kg meter square. So, once we have defined the problem, now our aim is to obtain what should be the initial condition or the velocity of the door we should have, so that when we close the door with certain force, it should not hit the frame of the door.

So, let us see how we can able to get this particular answer. So, we have as earlier we obtained the equation of motion for the system is this k t is the torsional stiffness, and we already know the critical damping. So, critical damping can be used to obtain the stiffness in the system, because this we know 24. So, I p we know 16. So, K from here we can able to get that will be 9 Newton meter per radian. So, again because this torsional stiffness. So, it comes in the radiance not in the millimeter. And once we know the k t and the mass of the system is also given that is mass of the system is 48 kg.

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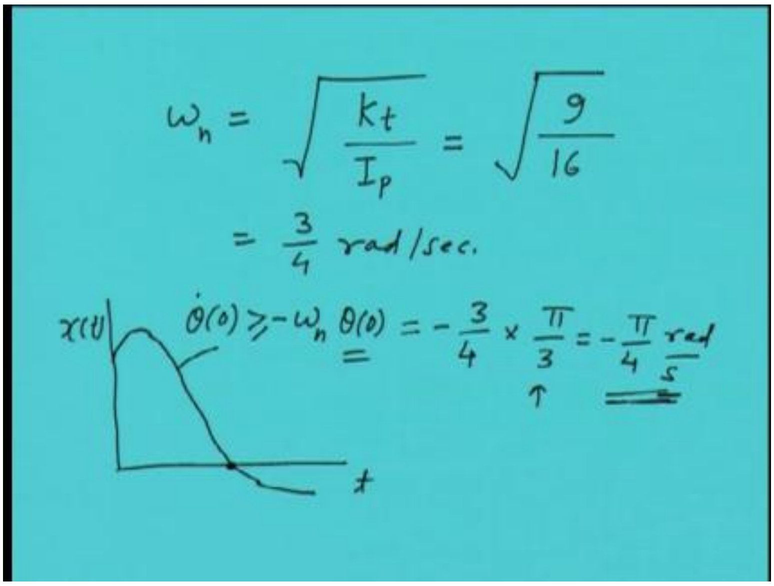

So, the undamped natural frequency can be obtained and then undamped natural frequency is this much; basically mass will not come here, polar mass moment of inertia will come. So, this if we substitute the K t and I p which is with us now 9 divided by the I p is now 16. So, this will give us 3 by 4 radiance per second as the natural frequency; that is undamped natural frequency, and the condition which we derived earlier that maximum velocity we should have to hit the door, or if we recall the previous lectures in which we had the initial velocity such that it crosses this static equilibrium once; that is this condition is that the angular velocity at time t is equal to 0. If it is more than minus of initial displacement then it may cross one.

So, if we have less than this, it may not cross one. So, we are keeping that equal to, so that we just get the limiting case. So, here we know the natural frequency; we know that maximum opening of the door is 60 degree. So, that is pi by 3. So, this gives us pi by 4 radiance per second is the velocity the angular velocity of the door if we give an initial maximum position of the door, it may just not hit the door. So, this is the minimum. If we give more than this, then it will hit the frame which is not desirable.

So, now we have seen some examples of free vibration especially damped free vibration especially when we are talking about the damping and the damped free vibration or the response, we have already seen that these responses decay or they die out with respect to time. And the rate of decaying is related with the damping present in the system, and this particular thing we can able to use it for finding the damping in the system especially experimentally. Because finding damping theoretically especially the mathematical modules are very difficult.

So, generally we obtain the damping experimentally through simple experiments. So, let us see how this rate of decay of the response can be used for obtaining the damping in the system. And this particular term which will be defining that is a logarithmic decrement which will be useful for us for finding the damping.

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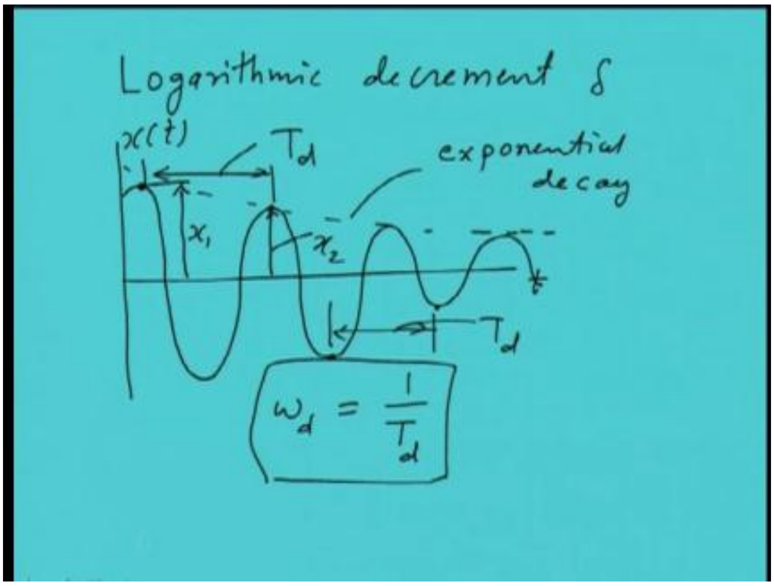

Generally we represent this as delta. So, if we have a free response of a system, if we given any disturbance to the system, it may have oscillation like this and gradually it dies out. And we have already seen that this particular rate for the viscous damping was the exponential decay. And if we talk about the amplitude let us say this is at one location and after one cycle it becomes x 2, and between this if we consider the time that is the one damped period is the period of the cycle; that is the time taken from one peak to another peak to reach, and the amplitude is decreasing. And from here we can get the damped natural frequency which will be 1 by d T.

So, even through experiment we can able to get the damped natural frequency like this, and we will see that in most of the cases these time period between two consecutive peaks is always same, because this is the system property.

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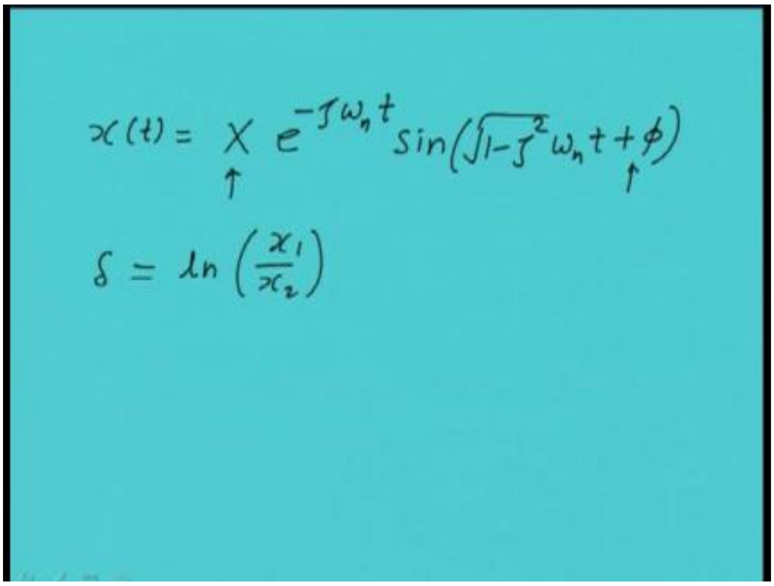

And this particular response we have already represented mathematically earlier like this. This is the response having some amplitude and the exponential decay term and a harmonic function with a phi's. So, this is the phis; this is the amplitude. And now we are defining a term logarithmic decrement which is nothing but natural log of 2 consecutive amplitudes.

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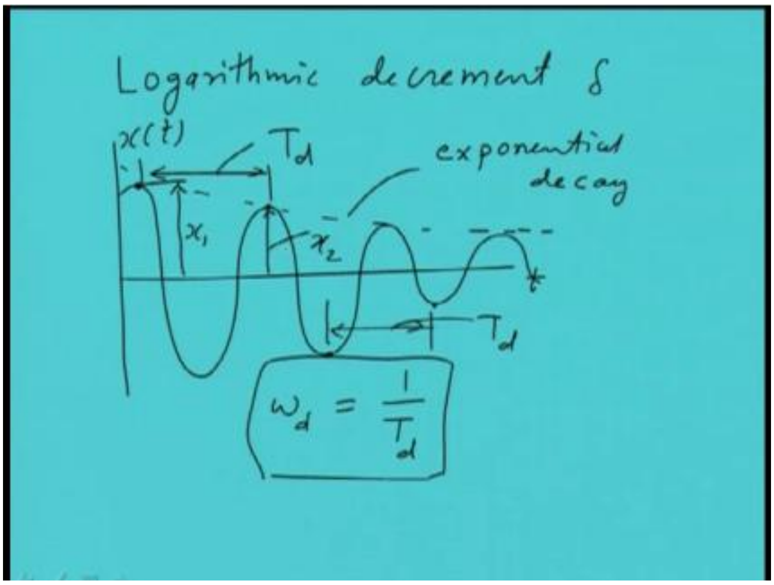

As we have given in the previous figure this is the x 1 x 2; these are the two consecutive peaks.

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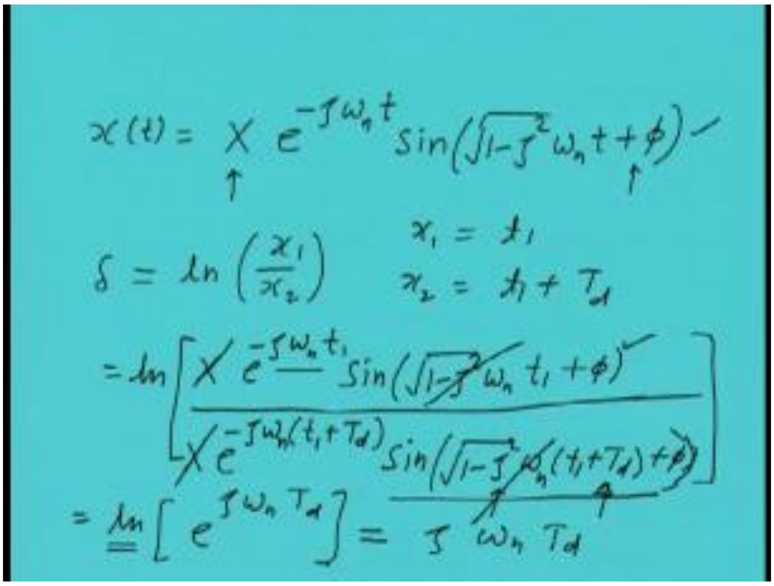

And the ratio of the natural log of that we are defining as a logarithmic decrement. And if we substitute this equation because x 1 is occurring let us say at t 1, x 2 occurs at t 1 plus damped time period. So, if we substitute x 1 and x 2 using this equation, we can write the logarithmic decrement as this remains same; here t 1 will come sin 1 minus zeta square omega n t 1 plus phi. And then in the denominator again amplitude; this is we are writing for time t 2 which is nothing but t 1 plus Td, the damped time period, then sin term, here also t 1 plus T d plus phi.

So, now you can see that some of the terms in the numerator and denominators are common. Also this quantity is we can able to see that because this is a harmonic function, and we are having the time period; this is the time period of this function. So, this itself will be equal to this expression. So, they will get cancelled, because this is harmonic function or periodic function. So, after their time period their magnitudes are same. Also these amplitudes are same and so, finally, we get it a simpler expression this one because some of the terms here, these terms will also get cancelled. And this itself because its natural log will be zeta omega n T d; that is we are calling as a logarithmic decrement.

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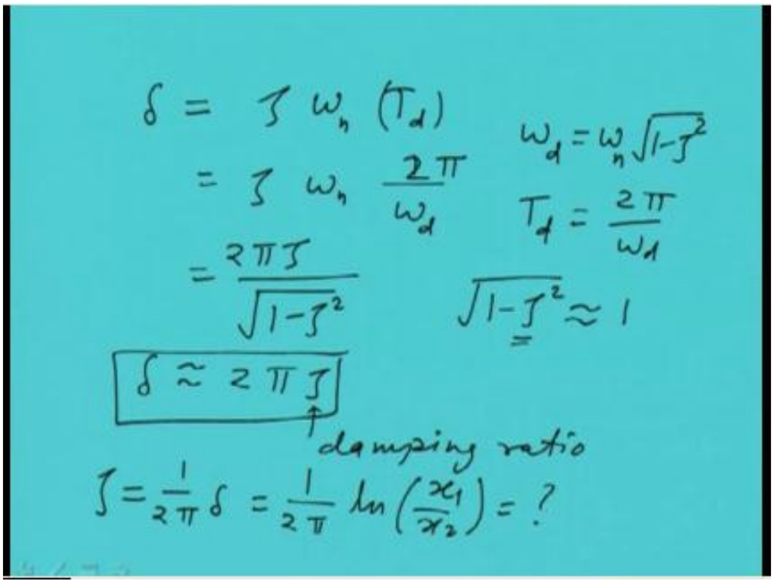

Now this logarithmic decrement again I am writing here zeta omega n T d and T d as we already know is 1 by omega d damped natural frequency is the undamped natural frequency. And this particular we have this relationship also between the damped and undamped natural frequency; this we can use it here. So, we will get and especially when we are talking about this expression, this expression T d is 2 pi by omega d. So, I think I made mistake take here; this will be 2 pi. So, we have this term here, and generally because zeta is having very small value for structurally still is around 0.1.

So, you will see that this will be approximately 0; this will be approximately 1, because this is small as compare to 1. So, this can be approximated as this; logarithmic decrement in the more approximate form can be written as this, and this is the damping ratio zeta. And this will help us in finding the damping ratio experimentally which can be written as like this. So, we will conduct the experiment; we will measure the free vibration, and two consecutive amplitudes we will measure x 1 and x 2, and that will give us the damping ratio. So, this is the way generally we obtain the logarithmic decrement in the experiment case.

Now let us take one example of how to obtain the logarithmic decrement through one experiment. Let us say we have one response which we have measured the free decay; this is the free damped response. And after the six consecutive amplitudes, we found that the decrease in the amplitude is around 25 percent. So, once we know the system property like stiffness and mass, how we can able to get the logarithmic decrement let us see through one example.

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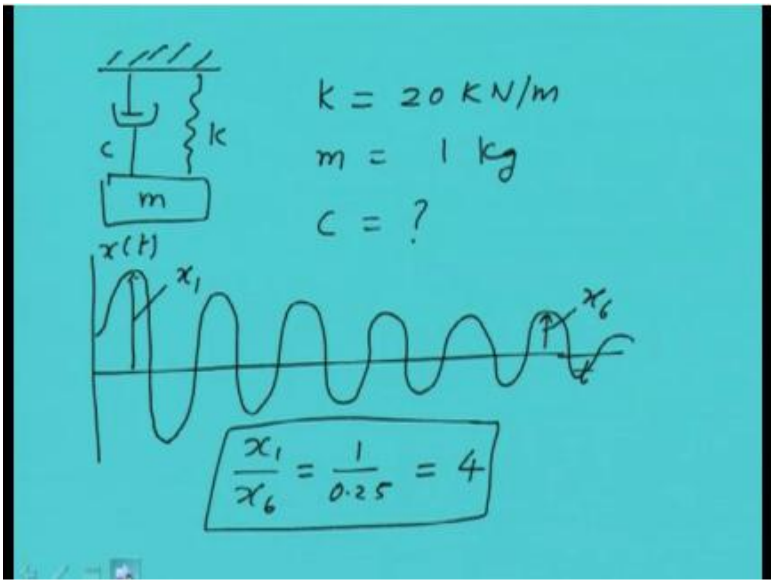

So, we have one spring mass system, and we have disturbed this system with some initial condition, and we got a response of this because of that initial condition. So, something like this; let us say this is x 1 and this is x 6. So, during measurement we found that x 1 by x 6; that is after fifth consecutive amplitude we have that is ratio is 0.25. That means we have this ratio equal to 4, and we have the system property that is stiffness is given 20 kilo Newton per meter, and mass is 1 kg; we need to obtain the damping in the system. So, only information is available in this.

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$$\frac{x_1}{x_6} = \frac{x_1}{x_2}\,\frac{x_2}{x_3}\,\frac{x_3}{x_4}\,\frac{x_4}{x_5}\,\frac{x_5}{x_6}$$

$$\ln\left(\frac{x_1}{x_6}\right) = \underline{\underline{\ln}}\left(\frac{x_1}{x_2}\right) + \underline{\underline{\ln}}\left(\frac{x_2}{x_3}\right) + \cdots + \ln\left(\frac{x_5}{x_6}\right)$$

$$\ln(4) = 5\delta$$

$$\delta = 0.28 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\zeta = 0.044$$

And we know that we can actable to express x 1 by x 6 as x 1 by x 2 x 2 by x 3 x 3 by x 4, like this we can go up to here. So, they are one and same, and if we take the log on this natural log, that will be log of individual amplitude. We can able to write like this up to, and you see that individual these terms are nothing but logarithmic decrement, and we have such five terms. So, you can able to write this as 6 into delta logarithmic decrement. And we know that this quantity was 4. So, this will give us the logarithmic decrement, and logarithmic decrement can be obtained that will be 0.28. And once we know the logarithmic decrement, we know how the damping ratio we are related with the logarithmic decrement. We can able to get zeta from this after solving this equation and zeta comes out to be 0.044.

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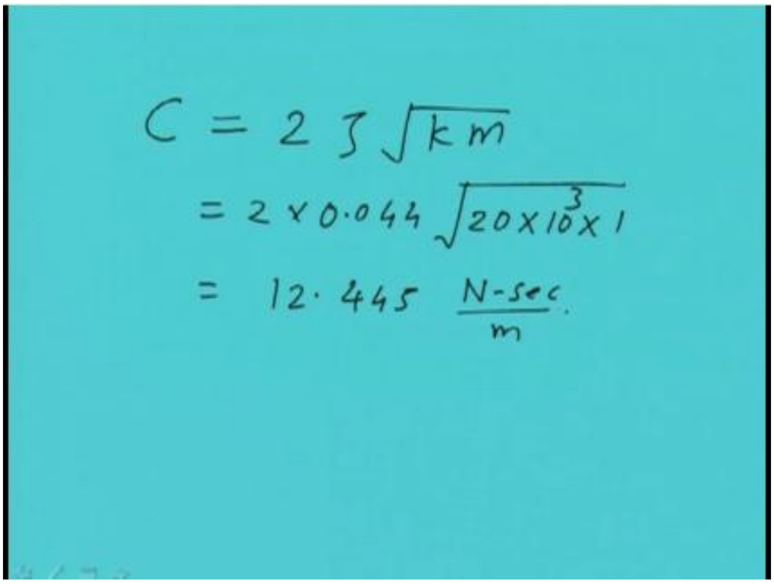

And once we know the zeta we can able to obtain the damping coefficient that will be this. So, if we substitute all the values, we know the stiffness that is 20 kilo Newton. So, this into mass is 1 kg. So, this will give us the damping coefficient value; unit is Newton second per meter.

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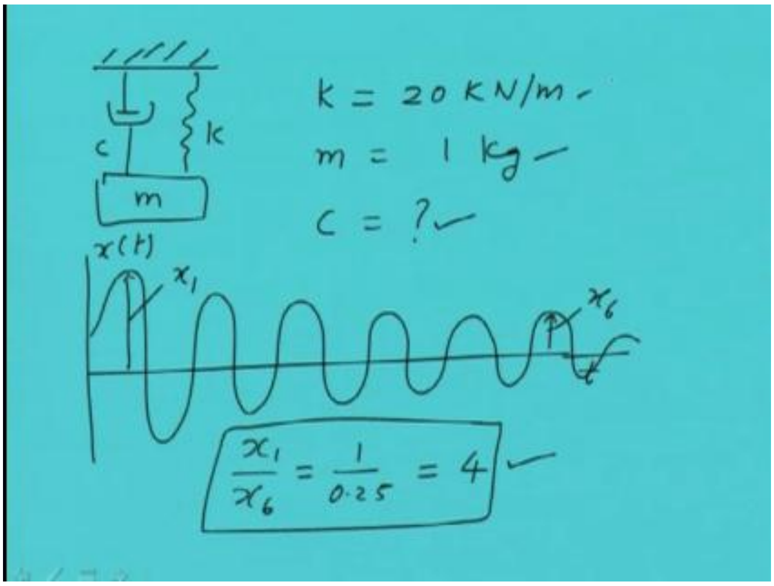

So, you can see that how through measurement we can get the damping coefficient if we know the system property mass and the stiffness of the spring. So, we have seen some of the analysis of the damped vibration especially using the viscous damping. And as I already mentioned earlier, there are several other kind of damping, and their mathematical models are quite complex; also the analysis for free vibration are not simple for these cases. So, people have developed some kind of equivalent system equivalent damping models that we will be seeing in subsequent lectures.

And for that basically because damping is a system by which energy is getting dissipated from the dynamic system. And because of that the amplitude of the vibration is getting decreased, and especially this damping is useful in the force response. So, till now we have not considered the force response. In subsequent lecture, we will be considering those force response, but some of the things is very important in the force response like when we have some forcing in the dynamic system, which is a dynamic forced time depended force. And especially when we are talking about harmonic force, then whatever the external force frequency is present in the system, the response will be having same frequency.

That is one of the assumption we make in the linear system especially the forcing frequency and the response frequency will be same. And especially when damping is there in the system, there will be a phase difference between the force and the response. And these aspects we will be covering in the more detail mathematical expressions we will be giving in the subsequent lectures. But we should keep this in mind when we are dealing with the damping, because damping is very important for force vibration especially at the resonance. At resonance is it decay, especially at resonance it limits the amplitude of vibration by dissipating the energy.

And in free vibration when we are considering, we simply see that if we disturb the system, it decays after sometime. But in force response especially at the resonance when we are talking about, then whatever the forcing we are giving to the system that is balanced by the damping dissipation. The energy in the dynamic system is generally dissipated in a form of heat or it is radiated away like if we have a piece of metal and if you do the bending of that to and fro, you will feel the heat in that metallic wire. And that is the form of dissipation of the energy; similar mechanism is there in the dynamic system.

Or if there is a buoy or a float which is going up and down in a water surface, you will see that the waves will be radiated out and away from that float, and whatever the energy you are giving to the float is radiated away in the form of waves. We have already seen various form of damping present in the system; that is right from the viscous damping to the intermolecular interaction between these molecules of the metal. And even the interaction between these two solids that is the dry friction or viscous is nothing but interaction between the fluid and the solid.

And this dissipation of the energy generally we calculate over a cycle, what is cycle means when we have the oscillation of the particular object; what is the time period of that over which the time period, whatever the energy is getting dissipated; based on that we formulate the damping model. So, it is very important to get the loop or the hysteresis loop of the damping in the system. Hysteretic loop is nothing but when we have some oscillation per cycle that particular if we have the plot of the damping force versus the displacement.

So, that particular curve will in enclose one area; that is called hysteretic loop. And this particular area if we can able to find out, then we can have some quantification of the damping in the system. And in subsequent lectures, we will see that how these hysteresis loop area can be calculated and that is nothing but the dissipation of energy.

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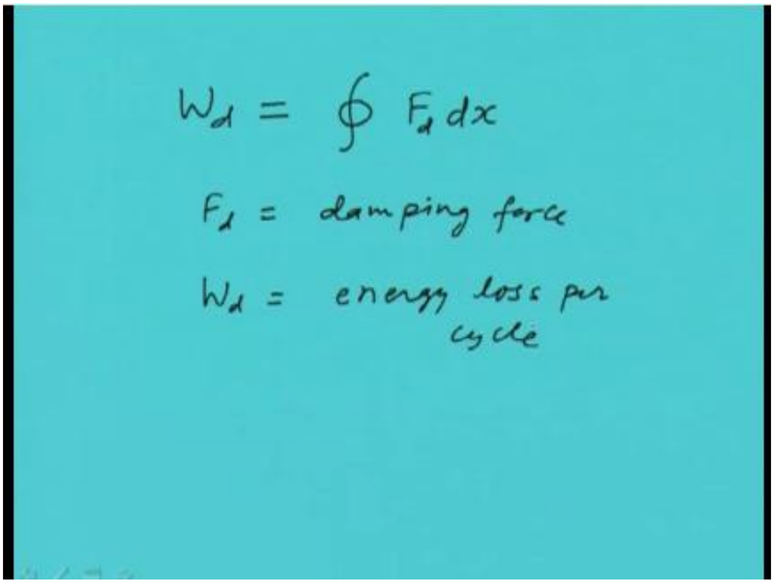

So, this particular dissipation of the energy can be expressed as integration over a cycle of damping force or work done by the damping force, where F d is the damping force and x is the displacement, and this is the energy lost per cycle. This particular energy loss depends upon various practices including temperature, amplitude or frequency of oscillation. Now let us take the viscous damping model, and we will try to find out the energy which is getting dissipated because of that model.

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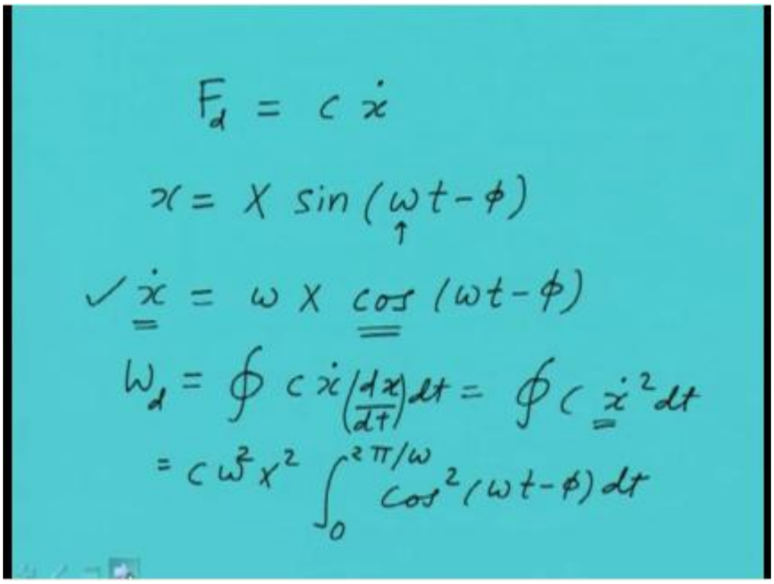

So, here we have the damping force in the viscous damping model as this, and as I mentioned for the forced response, the displacement can be expressed as some amplitude and sinusoidal function with frequency same as the excitation frequency. And because of damping, there will be some phase also, and from this we can able to get the velocity if we differentiate. So, this is the velocity, and the energy loss can be calculated that is damping force is this into displacement, and this displacement can be written as d t like this. So, this itself is a velocity. So, c x dot square d t.

Now here we can substitute this expression, and if we substitute we will get c omega square x square integration, because the period of this particular harmonic functions are 0 to 2 pi. So, now I am putting the limit 0 to 2 pi by omega. That is in seconds and cos square omega t minus phi d t. So, this particular expression we have substituted here. And if we integrate this we will get omega d as I will give some more steps of this, so that things are clear 2 pi by omega and here this pervious this term can be expanded.

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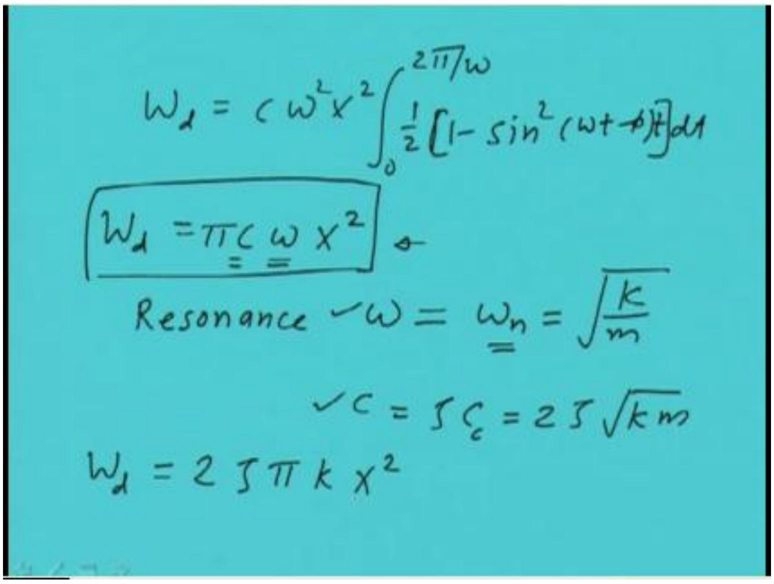

And we can able to write like, this and if we simplify after integration we will get. So, this is the energy dissipation in the force response, especially when we are talking about the resonance at the beginning lectures I have already explained what is the resonance. When the frequency of excitation is equal to the natural frequency, we have a resonance. And if you substitute this in this expression which itself is root k by m that is undamped natural frequency. So, these we will be substituting here, also the c, this damping we can able to write in more convenient form like this. So, if we substitute these two in the energy loss, we will get after simplification this term. Now we will be plotting the damping force with respect to the amplitude or the displacement how the plot looks like.

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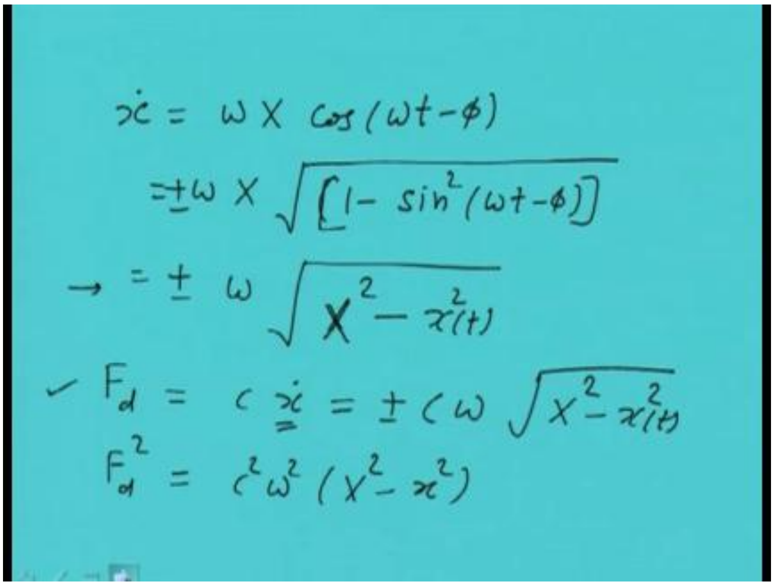

We already know that velocity is given as this, and this can be rearranged like this plus minus under root 1 minus sin square omega t minus phi. And if we take the X inside, this can be written as X square minus X square this term becomes the small x, because that is a displacement with respect to time. This is the amplitude, and this particular term can be used to obtain the damping force which is given as c x dot. And if you substitute x dot from here, we will get damping force like this. This is the time dependent; this is the amplitude. And if we square this term, we will get F d square is equal to c square omega square X square minus small x square.

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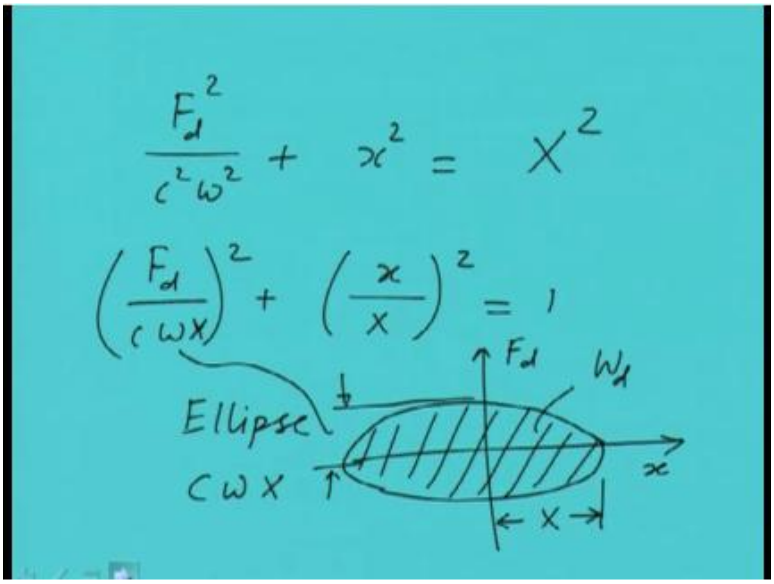

And this we can able to rearrange like this, F d square by c square omega square plus x square is equal to capital X square. Now again slight rearrangement can be done of this; x will also come capital X here plus small x capital X whole square is equal to 1. Now this we know that this is the equation of an ellipse. And if we want to plot this, this will take this form. This is x, this is F d. And you can see that this is the amplitude maximum displacement and the height of this is given as that is C W X. So, it is the equation of ellipse. So, this is nothing but the hysteresis loop you can see that how it is enclosing area. And area under this curve is nothing but energy loss.

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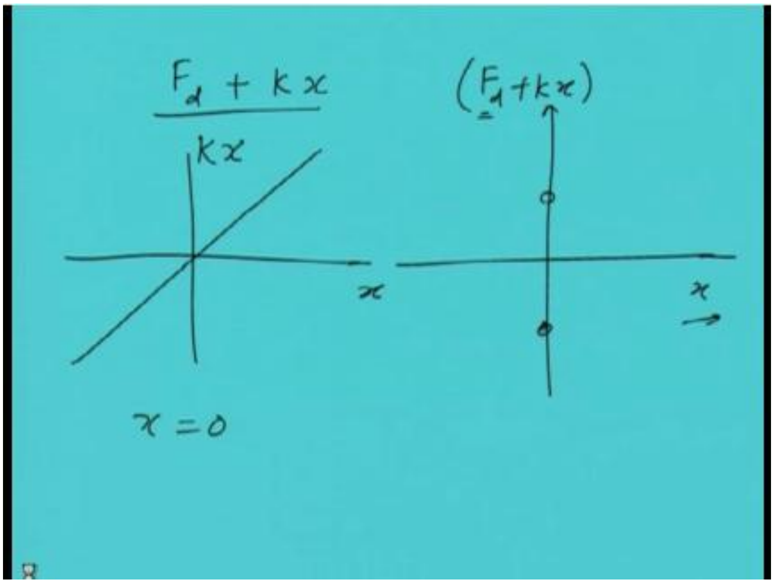

And if we want to add in this damping force the stiffness force also which is k x because k x in that particular plot the k x I will write again k x versus x is a linear like this. And it is not enclosing any area, but if you want to plot this; that means here it is F d plus k x and here x. Then you will see that we are adding k x term in F d at various location when we are talking about at x equal to 0. This particular term remain the same. It is not changing; that means it remains as the previous plot this one. So, these two points will remain same in this graph.

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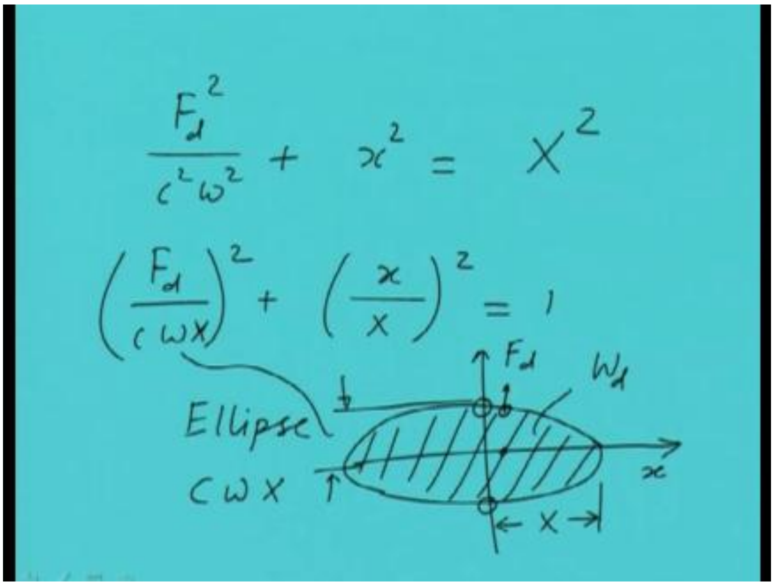

But as we will increase X further, then will be adding the k x, because now the X is having some finite value. So, you will see that this point will move upward.

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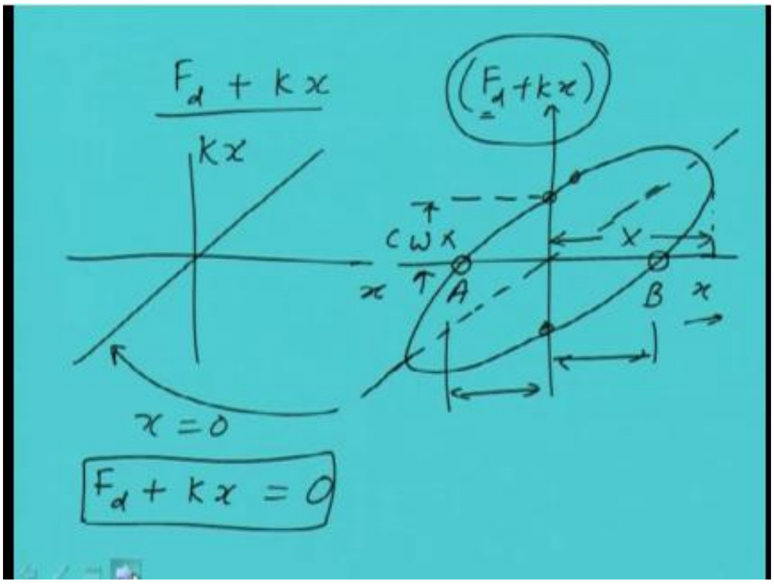

So, somewhere here; similarly, if we plot it all these points you will see that the ellipse will take a particular shape like this, where the major axis will be same as this axis. And various points can be because if we see this will remain the maximum amplitude and these points this is the same as the previous one C W X, okay. Even we can able to locate these points. So, the loop closer curves which are intersecting the x axis; that means when this quantity becomes 0, then these two intersections are taking place. So, we can have this condition equal to 0 to locate. Let us say this is A point, and this is B point. So, now we need the coordinate of these two. So, from this condition we can get it; let us see how can obtain this.

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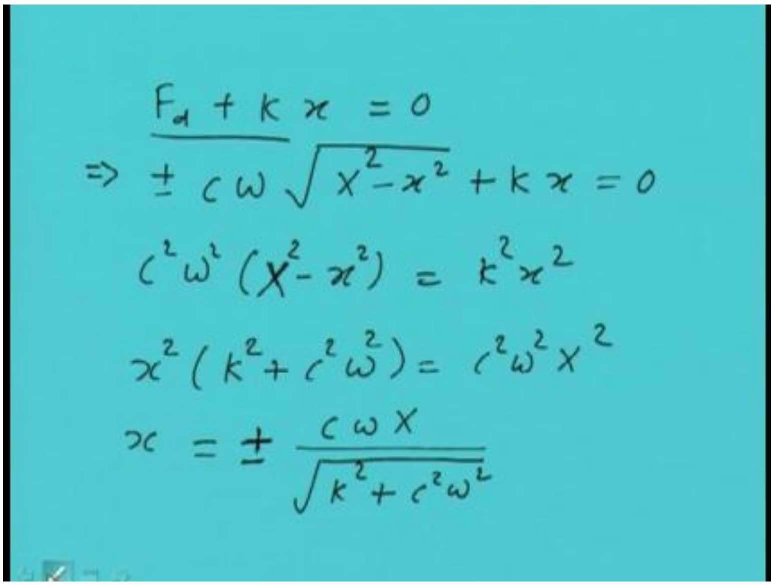

So, F d plus k x which can be written as c omega capital X square small x square plus k x is equal to 0. This is 0, and if you square it, we will get this. And if we rearrange this, we can rearrange this, k square plus c square omega square is equal to c square omega square capital X square. Now you can see that you got the location where this x is 0 or this quantity is 0 c w capital X divided by k square plus c square omega square. So, plus minus two locations we got.

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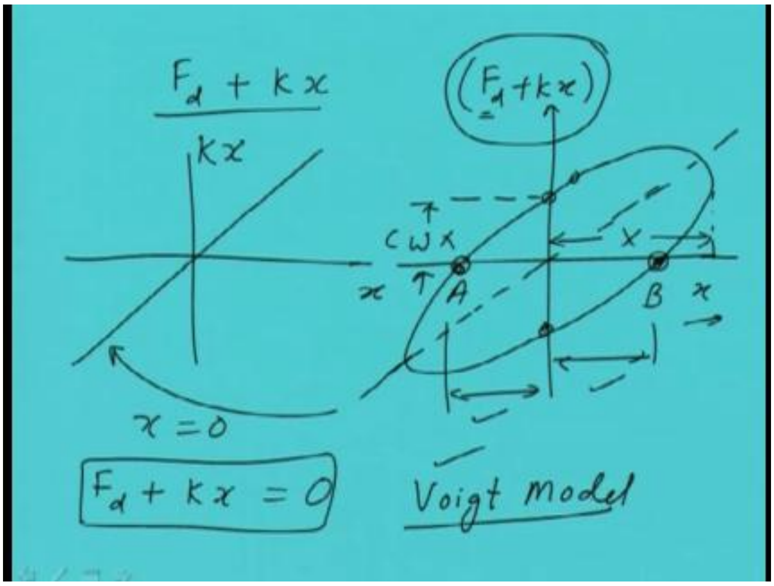

And these are corresponding to the A and B point; one is positive, and another is negative. Both are having same magnitude; at the same location they are becoming 0. So, if you see this particular curve that is conforming this well known Voigt model in which a despot is connected with the spring in parallel. So, it is conforming this particular model. Now we will define some of the term which is related with the damping energy.

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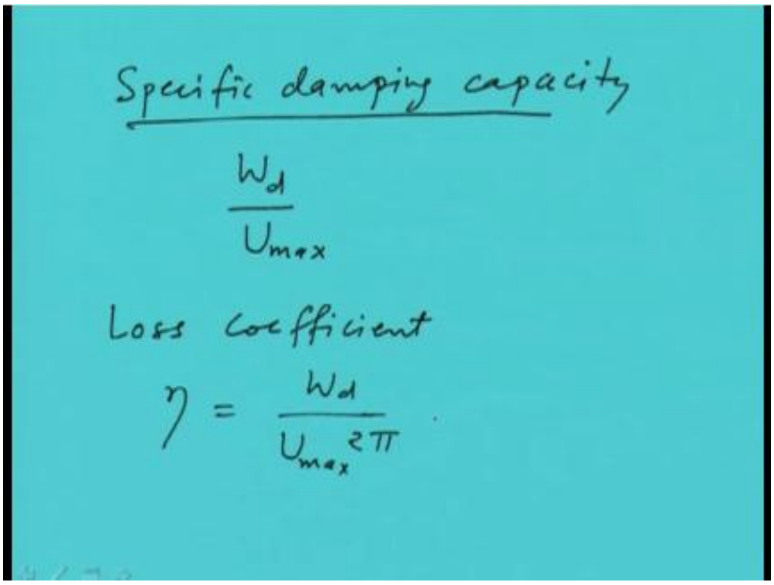

And first one is the specific damping capacity, and this is defined as the energy dissipation divided by the maximum potential energy in the system. And there is another of term which is called loss coefficient which is similar to the specific damping capacity, but there is slight variation in this that is omega max by pi. So, this is defined as the ratio of damping energy loss per radian divided by the maximum potential energy in the system.

Now once we have obtained the expression for the loss of energy due to the damping if we have another form of damping, then on the similar times we can able to obtain the energy loss. And once we obtain the energy loss, then we can simplify our analysis by some concept which is called equivalent damping system. Let us see how this works.

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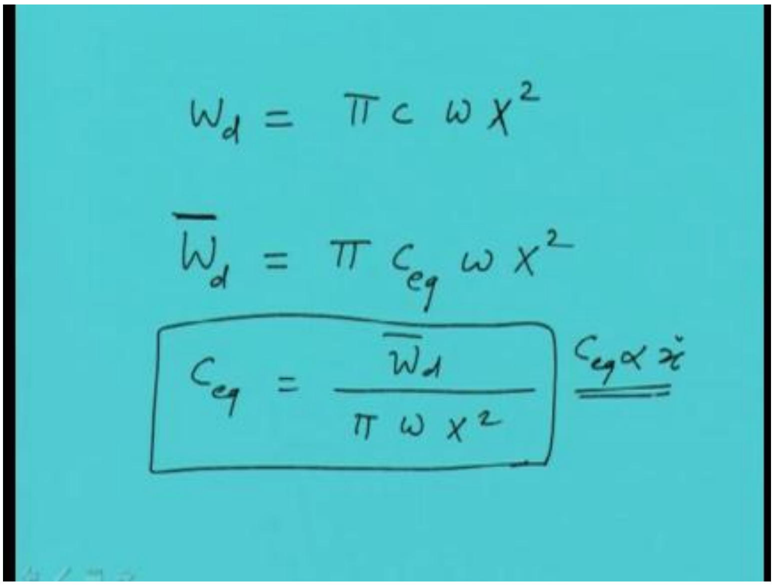

So, we have the expression for the energy loss with the viscous damping model; this was the expression we derived. Now let us say we have obtained the damping energy loss by some other means, and W d bar is that energy. So, we can able to get the equivalent viscous damping by equating this. So, we can see that we can able to obtain the equivalent viscous damping from this expression and which you can able to use in the standard linear model of the damping which is proportional to the velocity. So, our expressions will be simpler.

So, today we have seen the expressions for the logarithmic decrement, also how to obtain the energy loss per cycle of a particular viscous damping model or any other kind of damping model how we can able to get it. Also we saw the hysteresis loop especially for the viscous damping model, and finally, we saw that if we can get the energy loss for a particular damping model. Then that particular model can be converted to an equivalent viscous damping model in which the analyses are simpler. And in the future lecture, we will see the coulomb damping and there is material damping, and we will try to find out the equivalents of those models in the viscous damping model tops.