Mechanical Vibrations

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Module - 3
Single DOF damped free vibrations
Lecture - 1
Viscously damped free vibration,
Special cases oscillatory, non-oscillatory and critically damped motions

Today we will be studying another kind of free vibration in which we will be considering the dissipation of the energy when this particular dynamic system is vibrating. So, this is called damped free vibration in which case we will be having decay of the vibration if we are talking about the free vibration or the vibration will decay after sometime or due to the damping effect. This damping may be in the system; it can come from various sources. One is the material damping itself in which may be the friction at the entire molecular level can take place, and because of that the energy is dissipated from the system in the form of heat or there can be some fluid and solid interaction is there.

And because of that the heat is getting dissipated from solid to the fluid like a solid member is vibrating in a fluid, and the fluid is damping, or it is taking the energy of the solid. And after some time it is trying to decay the vibration of the solid structure or machine. Another form of the damping which is there in the system is may be aero dynamical forces which is especially in the turbines you will find that we are injecting the steam and to the turbine blades, and because of these interactions there can be damping in the system or sometime we have two solids which are in contact.

So, some kind of friction damping between two solids can also lead to the dissipation of the energy. So, how to model this particular damping is your question, and it is very difficult to model damping as compared to the stiffness or mass, because whatever the models are available that leads to very complicated analysis of the damped free vibration. So, that is why in most of the analysis when we obtain the natural frequency of the system, we neglect the damping effect. And we will see from the present study especially on the natural frequency, the effect of damping is very less.

But especially they are very important when the system is going under resonance condition. Some simple model of the damping people have developed especially when the force is proportional to the velocity of the vibrating body. Like in stiffness, the stiffness in the spring force is in that we know it is proportional to the displacement. Similarly, in this particular model the damping force is proportional to velocity. If we want to model a damping force, which is proportional to the displacement, then what will be the effect that we will try to see whether we will get some damping effect or not, through some illustration I will try to explain that.

But let us see what is the effect of damping if you are talking about a force versus displacement relationship. So, obviously, if there is a damping in the system, this particular plot in a particular cycle, they should enclose some area, and that is the energy dissipation per cycle of the system. And let us see through figure how the linear damping force which is proportional to velocity or a damping force which is proportional to the displacement they are different and what is the characteristic of that.

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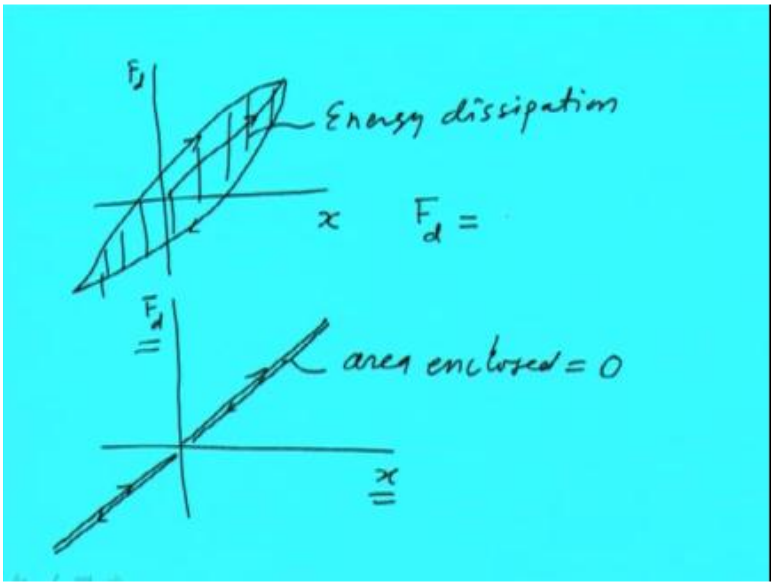

So, let us first take a damping force model which is proportional to the velocity. So, in this we expect that when we start the system, it will go like this and during the oscillation it will come take this. And during this process it encloses one area, and this area is the energy dissipation over one cycle of the motion. And if we have one model in which damping force is proportional to displacement; this particular diagram how it will look like, because it is linear. So, it has to go in the straight line. Then while it is returning from here, again it will follow the exactly same path, and it will go other side.

And then while returning it will follow exactly same path like this, and it will keep oscillating along this line. And if you see carefully what is the area enclosed by this? That will be zero; area enclosed by this curves will be zero; that means per cycle there is no dissipation of the energy. So, this particular model in which the damping is proportional to displacement, it is not dissipating any energy. So, that is why we have to have some other relationship between the damping force and the displacement. So, the most simple model which gives simpler mathematical expression when analyzing the most simple model of the damping is this which gives simpler mathematical expressions.

In practical application in two wheelers there is a shock absorbers are present in the front wheel and the back wheel. You must have seen near all the wheels, we have two rods, and they have coils; that is the springs. And inside there is a piston and a cylinder is there in which liquid is filled. So, during motion of the vehicle when duration of the route is there, the vehicle starts oscillating it, but after some time that oscillation dies out. Why it happens? Because of the damping effect, whatever the dampers are there in this system it dies out the vibrations. So, let us see what is the principle how the working principle behind the dampers.

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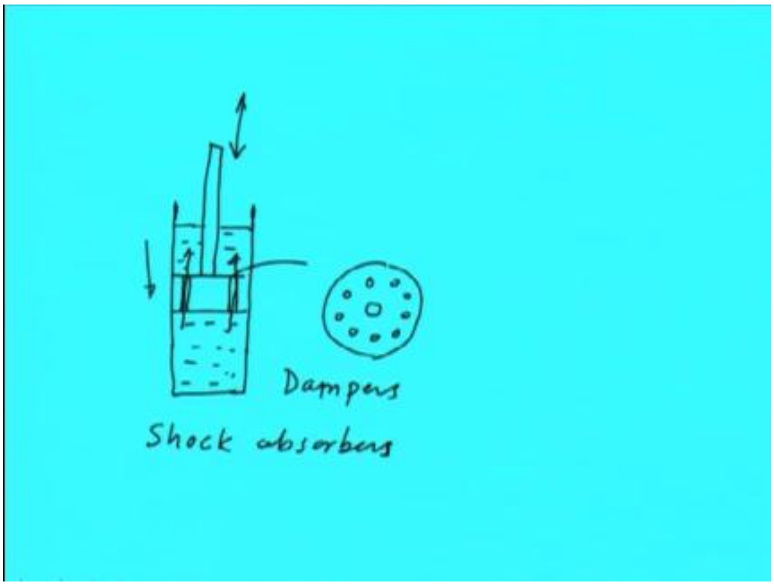

So, in those dampers there will be one cylinder, and there is piston which is attached through a rod. Now during motion of the vehicle which is having to and fro motion up and down motion, this particular piston will also go up and down. This particular cylinder is filled with fluid, and in this piston there are small holes or if you see the top view of this there will be small holes on this piston. So, yes, this piston is going up and down. These fluids if it is going downward, these fluids, which are getting compressed here, they will try to go through these holes towards the upper cylinder.

And during this process they will be having friction, because it will be injecting with very high velocity through these holes there will be dissipation of the energy. And when this piston is going upward, then these fluids will again come back from these holes to the lower portion of the cylinder. So, by that way again they will be losing the energy of the vibrating system. So, this is the basic principle of the shock absorbers. So, here what I have shown in which the dampers are there which is dissipating the energy. Apart from this around this there are coils also, springs are also there; they give the to and fro motion.

So, that if there is sudden jerk, then we do not feel those jerks directly, but that will be taken care by the springs. So, now with this background of the dampers, let us try to model mathematically this system, and try to analyze them analytically that how this motion take place, how the damping effects the motion of the particular system and the natural frequency of the system.

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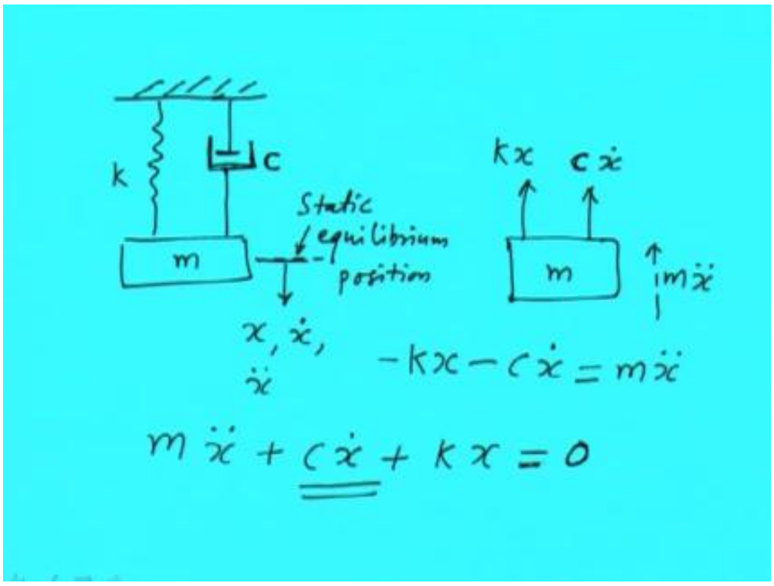

So, for this let us take a simple model in which we have one spring and mass, and apart from this. Now we are attaching one damper also. So, damper we represent like this in which this represent some kind of piston. This is cylinder and ceases the damping coefficient. K is the stiffness of the spring and mass m, and here let us say this is again static equilibrium position of the mass. So, when we are taking the reference as the static equilibrium position, the gravity effect will not come.

Now when we disturb the system through x, then we will be having the free body diagram of this particular mass in which we will be showing all the external forces. So, stiffness force will be upward, because the spring is getting extended. So, it will apply force upward on to the mass, because it is moving; it will be having velocity also, acceleration also. So, in this particular case we are considering the viscous damping that is proportional to the velocity of the motion of the body and cease the damping coefficient.

Apart from this, we can have some external forces but since we are dealing with free vibration. So, let us not consider any other external force, and apart from this there will be inertia force. So, this is the free body diagram of the mass. Now we can apply the Newton’s second law of motion. So, that says that some of the external forces. So, k x is one of the external force minus, because this is acting opposite to the displacement direction which is downward, but this force is upward. Similarly, the damping force these are the two external forces should be equal to acceleration of the body.

So, these equations we can able to write in more standard form like this. So, you can see that this equation is similar to the previous one; only extra term is upper damping is coming .This particular equation, because it is homogenous because the right hand side is zero. So, we can have the general solution of this.

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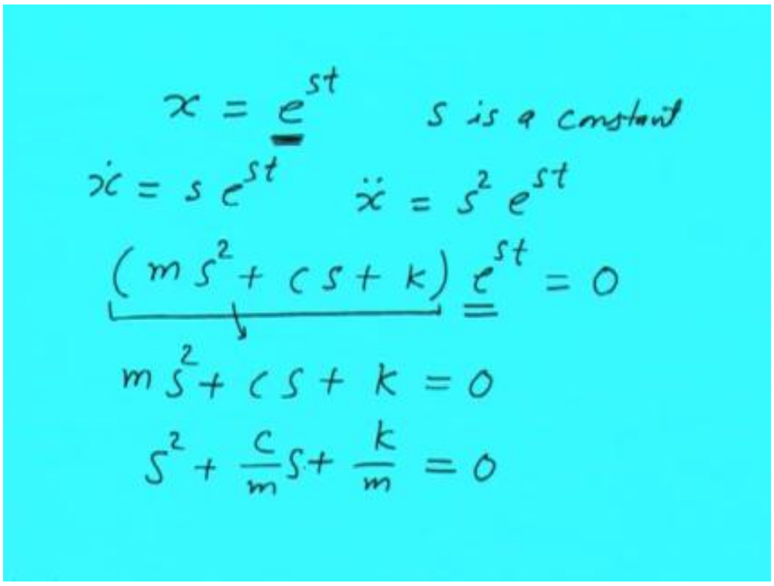

So, let us take the general solution of this as or in the previous case when we did not consider the damping, then we saw that this particular equation of motion in the acceleration was proportional to displacement. And because of that we assure that the motion was simple harmonic, but now another term damping has come into the equation. So, we will not be having the simple harmonic motion of the motion. So, that is why now we are looking for the general solution of this particular equation. So, once we assume this solution where S is a constant, because the equation of motion contain derivative of this displacement and velocity. So, the velocity can be written as like this, and acceleration can be obtained as S square e s t.

Now if we substitute this in the equation of motion, we will get m S square plus c S plus k, and this term is common is equal to 0. So, this quantity cannot be 0. So obviously, the terms within the bracket has to be zero. So, that whatever the assumed solution is the solution of the differential equation. So, we are getting a condition if we are able to satisfy this condition that this quantity is 0, then this particular expression is the solution of the original equation. So, let us equate this to zero. So, there is a quadratic equation in terms of S. So, we can able to solve this equation. Better we can write this as in this form; the solution of this because this is a quadratic in S.

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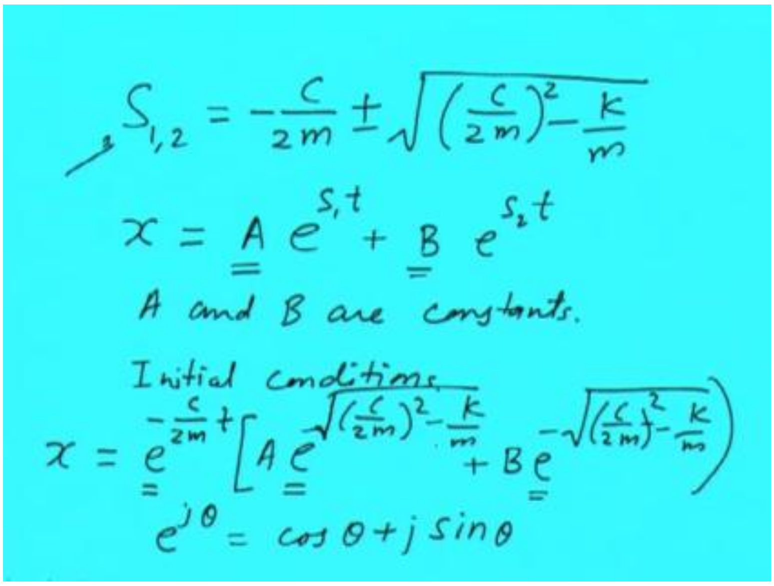

It can be written as it will be having two roots S 1 and 2. So, it can be written as this. So, plus and minus gives two roots; so we have S 1 and S 2 two constants we have got. So, we have two solutions, and those two solutions can be combined to get the general solution. So, S 1 t is one of the solution, A is a constant and B S 2 t. So, this is a general solution where A and B are constants, and these constants we have to obtain using the initial condition of the system. Initial conditions how much displacements we are giving to the mass or how much velocity we are giving to the mass from there we can able to obtain these.

So, we need two initial conditions to get the two unknown constants, and if we substitute the S 1 and S 2 from here, we can able to write this as x. I am taking some of the common terms. So, this particular term I am taking common after substituting here A, then square root of c by 2 m square minus k by m, then second term which is negative of this square root. So, it is the general solution of the differential equation for damped vibration.

Now here you can see that this particular quantity within the square bracket, if within the bracket whatever the quantity is there if this is positive, then we will be having these terms and these terms as non-oscillatory terms. Because all are decaying or exponential terms will be there, but if this quantity within the square bracket is negative, then we will get the complex quantity in the exponent of e, and that quantity we can able to express in terms of the sin and cosine function. Like if we have j theta, then it can be written as in terms of the sin and cosine. Then will be having some harmonic component in the response. So, oscillations will be possible, and there is another condition in which this term is Zero. There is a very critical case. So, we will see these three cases separately, and we will analyze them.

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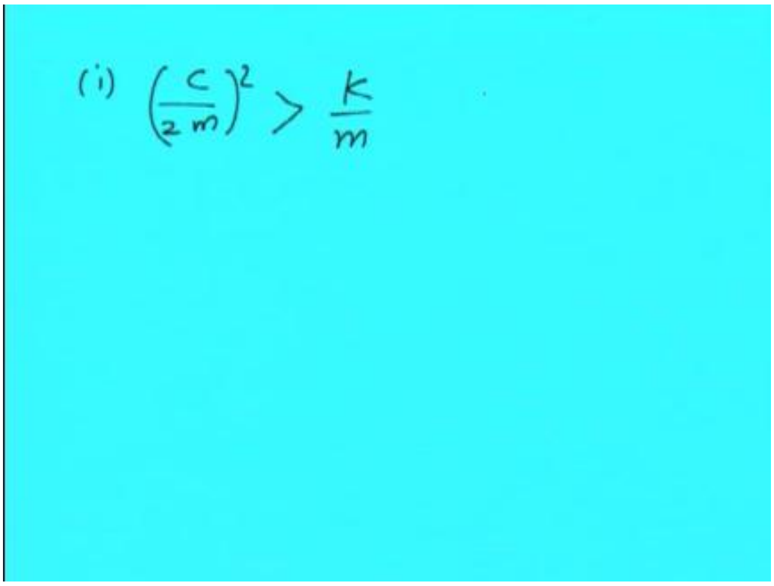

So, first case when within this square bracket this term, or I should say this quantity is greater than k by m. So, that means within the square root we will be having positive quantity.

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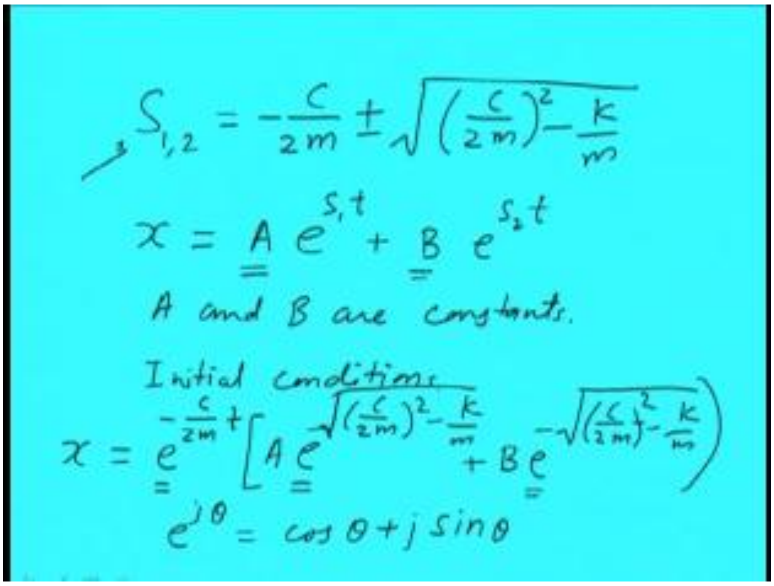

So, all the equations here, they will not be have any oscillatory terms.

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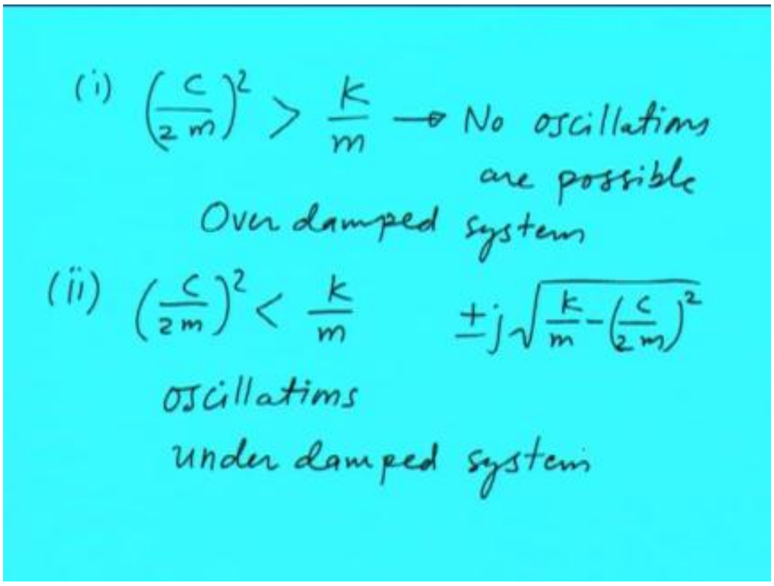

So, no oscillation will be possible, and this is called over damped system. What is happening in this system? We are disturbing the system, and it is gradually coming to the steady position without any oscillation about the mean position. Second case is c by 2 m it is less than k by m. So, in this case within the square root whatever the term is there that will be negative. So, we will be having this kind of terms; imaginary quantity will be coming outside because negative is inside. So, I have written in the form of imaginary terms. So, you can see because of this we will be having cos and sin terms, and we will be having oscillations in the motion, and this is called under damped system.

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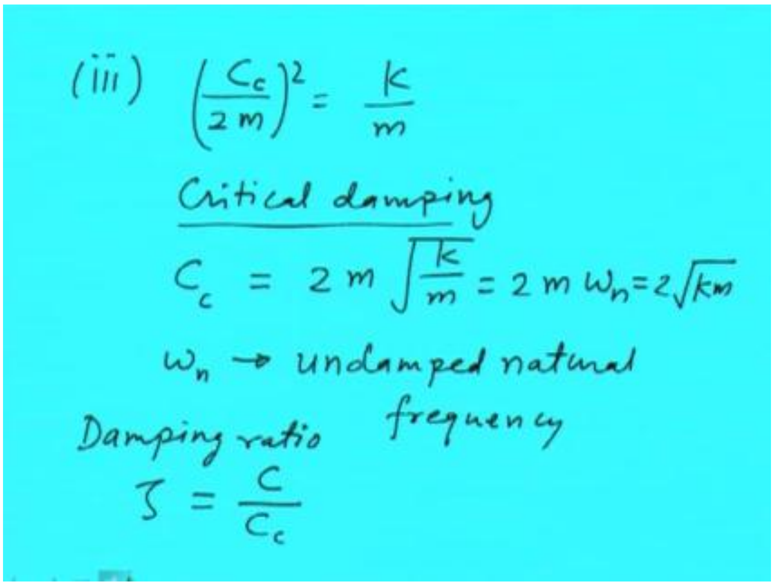

Third case when these two quantities are equal; that means k by m, then we call this damping as critical damping, and we represent Cc is critical damping. So, critical damping is defined like this. So, we can able to see that it will be k by m, or it can be simplified as because this we know we write as omega n; that is omega n I am calling now as undamped natural frequency, because root k by m is the natural frequency of the system when there is no damping. So, that is why this undamped natural frequency term is coming. This even we can able to simplify as k by m.

And this is the definition of the critical damping, and we define another term that is a non-dimensional term that is called damping ratio which is zeta, and it is defined as the damping coefficient divided by the critical damping, and this is non-dimensional quantity. So, it is having lot of advantage you. We will see how the expressions get simplified because of this particular term.

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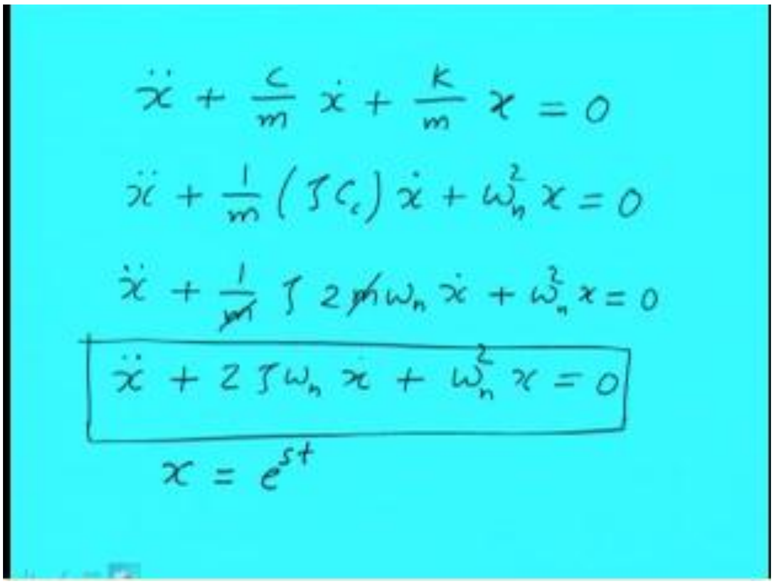

So, let us write the equation of motion again and we will try to simplify the equation of motion in terms of these non-dimensional terms. So, this was the equation of motion for damped system, and now we are gradually introducing the non-dimensional terms. So, c we are writing as zeta Cc; zeta is the damping ratio, this is a critical damping x dot plus this we can write as natural frequency. There is undamped natural frequency. And then we can simplify this as, because this critical damping can be written as plus omega n square into x and again it can able to simplify this further that will be 2 zeta omega n, because m will get canceled x dot plus omega n equal to 0.

So, this is a very important form of the equation of motion. It is useful for especially when we will be dealing with the multi degree of freedom system and when we will be trying to find the damping and the natural frequency of the system through experiment. So, this equation of motions will be very useful in that case. Now we will obtain the similar quadratic equation as we obtained earlier for one assumed solution general form of solution to this.

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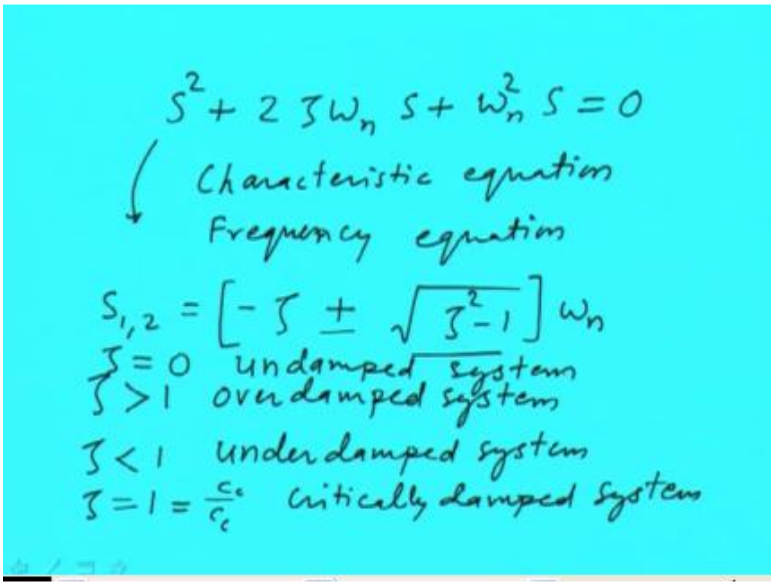

From that we will be getting S square plus 2 zeta omega n S plus omega square n S. Basically, this equation is in mathematics is called characteristic equation, and in vibration we call it as frequency equation. And because this is quadratic we can get the roots of these two. We will be getting two roots of this, and this will be of this form; zeta is nothing but damping ratio. So, this is the two roots of this characteristic equation. Now from here you can see that when zeta is greater than 1, then this quantity becomes positive always.

So, both the roots of these equations are positive. So, will not be having any oscillatory terms in the solution because exponential terms are all real; so there is no oscillation. So, this is called over damped system as we defined earlier. If this zeta is less than 1, this quantity within this square bracket will be negative and we will get an imaginary quantity here. And when we will substitute this root in the solution, we will get some oscillatory terms. So, that is why it is called under damped system in which we will be having some oscillations may be decaying kind of thing. Then another case is zeta is equal to 1; in this case this quantity will be 0.

So, you can see that we will be having; this we already defined, because zeta was C by Cc, and the damping itself is Cc. So, that is why it is coming 1. So, that is why it is called critical damped system. And there is another case in which zeta is 0, and this is very obvious that when zeta is 0, then it is undamped system means that is without damping, and there is no damping in the system.

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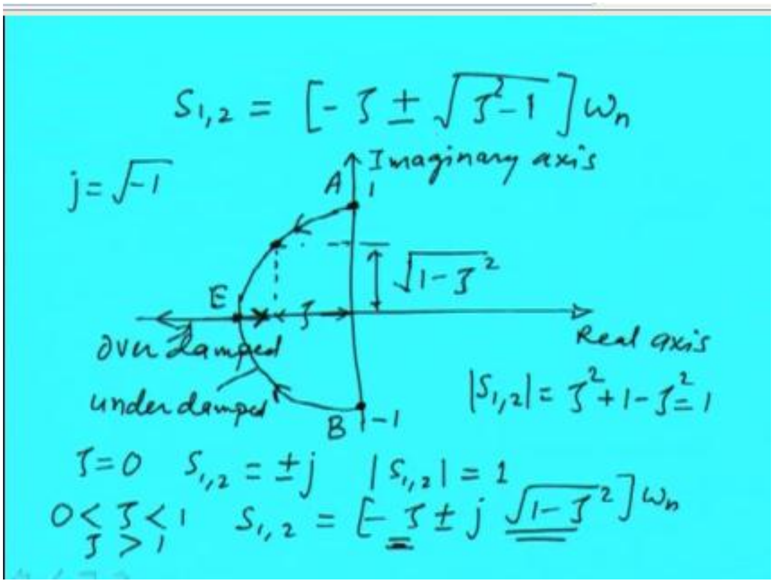

Let us again write the roots of these two equations here, because we want to plot these roots in a complex plane to have better understanding of these roots. So, let us say we have a complex plane in which it is a real axis, and this is the imaginary axis. Now for zeta is equal to 0, what we get from this expression when zeta is 0? We get S 1 2 as plus minus j. J is nothing but minus of the under root. So, these two quantity both are same, because in this case magnitude of S 1 2 is nothing but 1, and this is complex; this a imaginary number.

So, on this plane we can see that this two points which is corresponding to 1 and minus 1 on imaginary axis, they represent these two points. So, the undamped system is represented by these two points. Let us say they are A and B, and when we have that zeta between 0 and 1, then we will be having this roots S minus zeta plus minus j 1 minus zeta square. And this can be represented by circle. So, at any position on the circle, or this represent zeta; that is real part, this is a negative direction, okay.

And this quantity represent 1 minus zeta square which is a complex quantity, this quantity, and basically this represent the equation of a circle, and the magnitude of this is again 1, because if you take the magnitude of this two quantity that is one as 1 2 will be zeta. This will be zeta square this first term and then plus 1 minus zeta square. So, that gives us 1. So, from A to let us say this point is E; from A to E and from B to E is representing the under damped system.

Now if we have the zeta greater than 1, then you will see that here two roots will be there. One will be always increasing toward negative direction this direction, and another will be going toward the positive direction this direction; at the limit it will become 0. So, these are the over damped system over damped case. So, now you can see that how we have drawn the roots of this equation in the complex domain and various cases of the damping we have illustrated. So, once we have obtained the roots of the characteristic equation and we have interpreted them also for various kinds of motions like under damped system, over damped system, critical damped system or undamped system.

Now let us obtain the explicit expressions of this kind of response, and through illustration we will show how these response look like. Once we give different kind of damping in the system. So, first let us take the under damped system in which the zeta is less than 1, so how the response takes the mathematical form.

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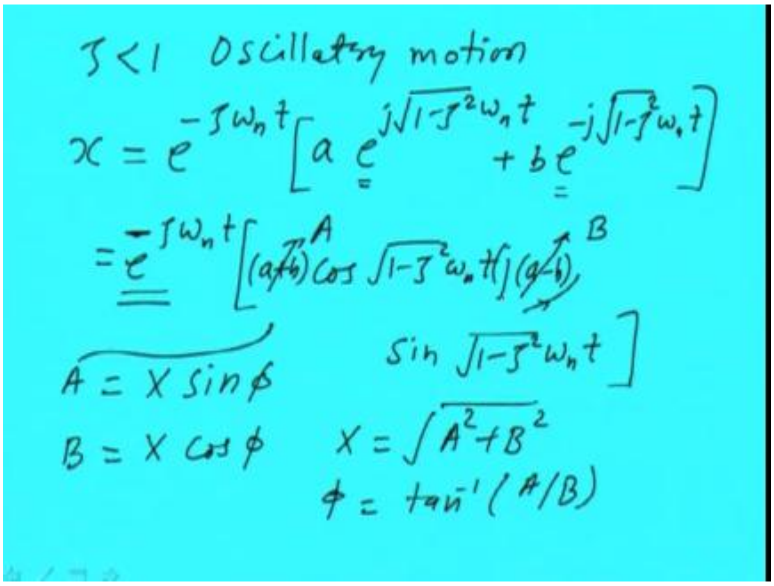

So, in this case the zeta is 1; that is we will be calling as oscillatory motion, and if you substitute those two roots in the equation of motion, they can be written in this form. Small a is a constant, then this is negative of this, and this particular terms we can able to write in the cos and sin terms using a regular relations. And if we do this we can able to express this a plus b cos of 1 minus zeta square omega n t, and similarly, j a minus b here you can write sin 1 minus zeta square omega n t.

Now you can see that the response it contains harmonic terms also this term which is decaying in nature. So, you can see that it or it is having negative here. So, once the motion will start it will try to decay the motion because of this term and which is due to the damping in the system. This particular equation we can able to simplify in another form. If we write this particular quantity let us say is A, this is including the imaginary part is B. If we say A is equal to X sin phi and B is equal to X cos phi, then we have the relationship between the X and A and B like this or this phi as tan inverse A by B.

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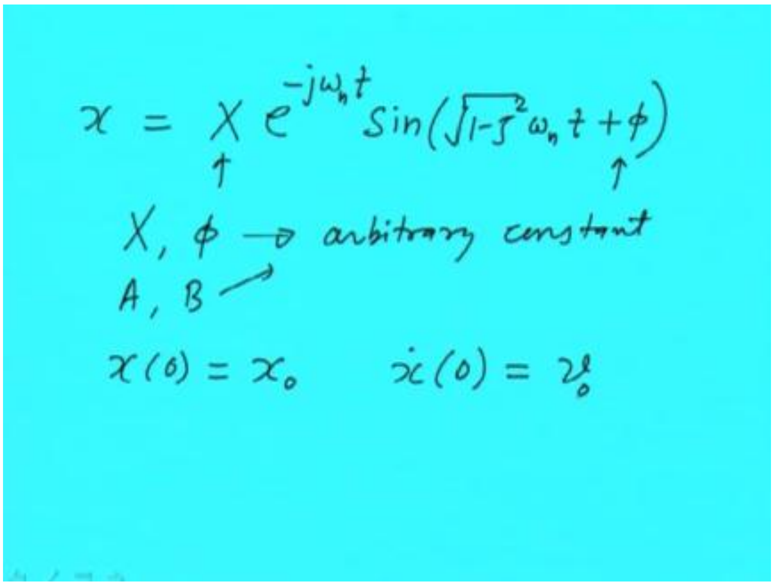

Now if you substitute this quantity here and here we can able to write another form of the expression that is amplitude and this is the decaying function and sin term with some phase; phi is the phase, and x is the amplitude; this is the phase. Here either amplitude or phase their arbitrary constant and these we have to obtain through the initial condition as we know that the differential equation of the motion is of the second order. So, we need to obtain these using the initial condition this or the terms like A or B; they are all arbitrary constant. So, these constants how we can able to obtain; let us say we have displacement at time t is equal to x naught and velocity at time t is equal to v naught.

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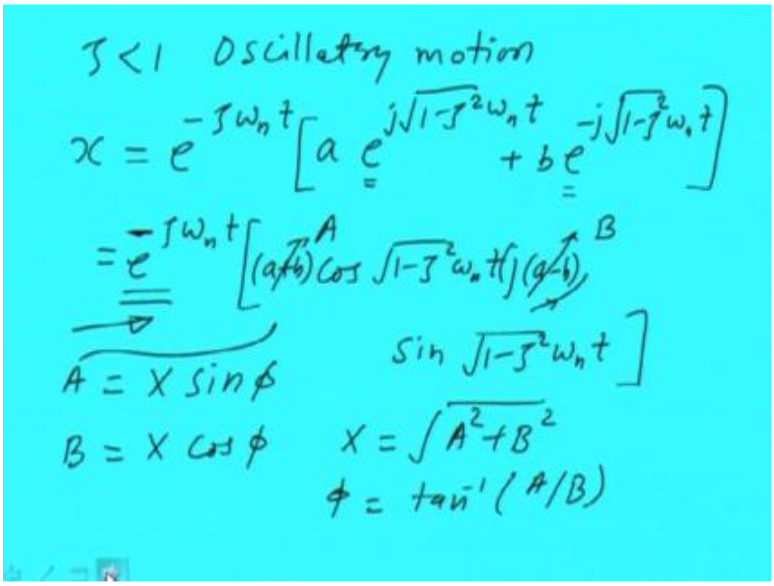

Then we can apply these boundary conditions in the let us say this particular form of the equation. So, will we need to differentiate this once to get the x dot term.

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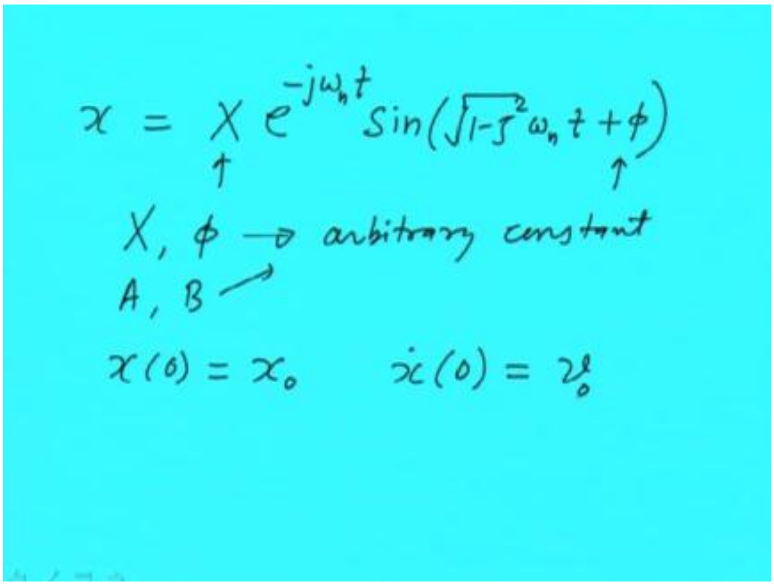

So, that x dot terms will be of this form.

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$$\dot{x} = (-\zeta\omega_n)e^{-\zeta\omega_n t}\left[A\cos\sqrt{1-\zeta^2}\,\omega_n t + B\sin\sqrt{1-\zeta^2}\,\omega_n t\right] + e^{-\zeta\omega_n t}\Big[A\{-\sin\sqrt{1-\zeta^2}\,\omega_n t\}.\sqrt{1-\zeta^2}\,\omega_n + B\{\cos\sqrt{1-\zeta^2}\,\omega_n t\}\sqrt{1-\zeta^2}\,\omega_n\Big]$$

Velocity term will be minus zeta omega n zeta omega n t, then the rest of the terms as it is plus B sin 1 minus zeta square omega n t, and then there will be another term. So, we are differentiating with respect to time, and we are getting these expressions 1 minus zeta square omega n t 1 minus zeta square omega n; this is within this bracket. And then another term is there. It is a big equation omega n t 1 minus zeta square omega n. So, this is the velocity term.

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$$x = X e^{-j\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right)$$

$X, \phi \rightarrow$ arbitrary constant

A, B

$x(0) = x_0 \qquad \dot{x}(0) = v_0$

So, both the initial condition these two initial conditions we need to substitute in first.

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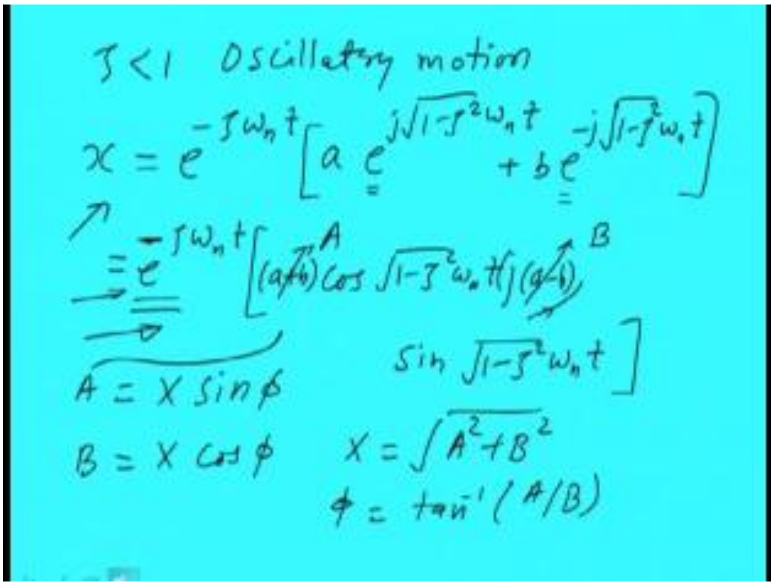

In this expression which is given as here.

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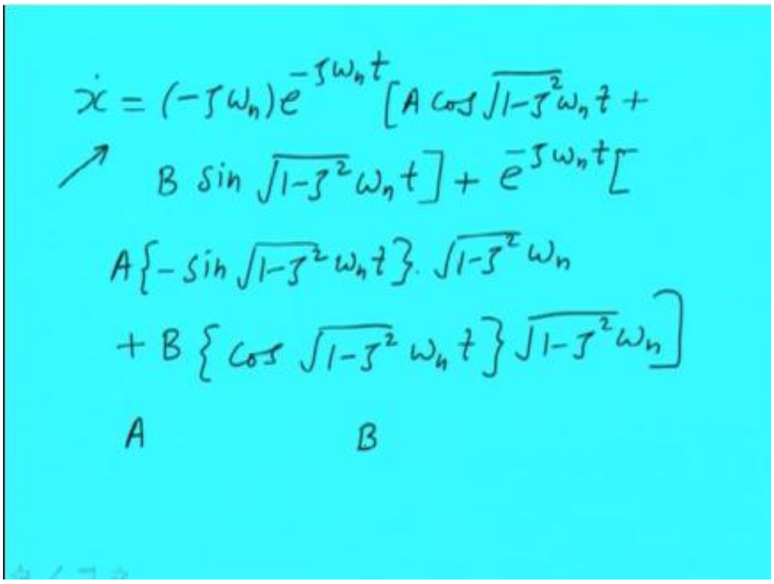

And in the velocity equation which is given as here, and we will be having two linear equations to solve for A and B, and you can see that A and B will be given as…

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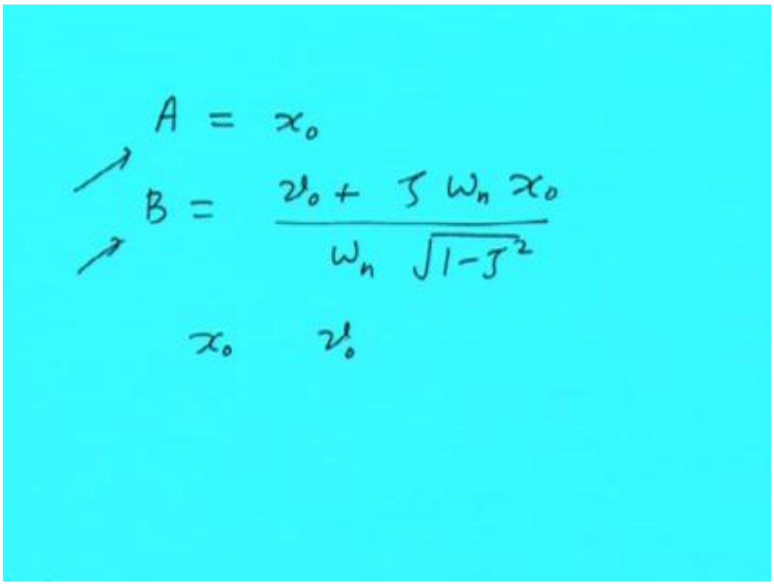

X naught and B will be given as v naught plus zeta omega n x naught divided by omega n 1 minus zeta square. So, these are the integral constants for a particular initial condition of displacement and velocity.

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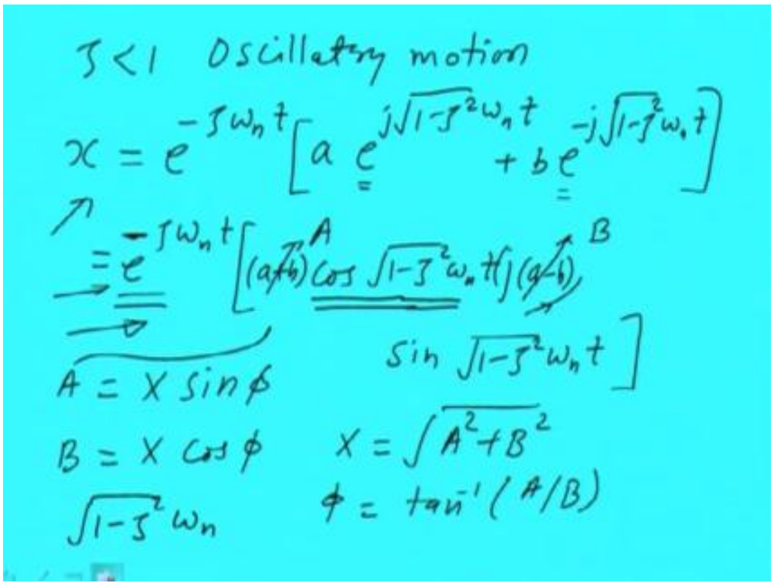

And if you see carefully the previous equation especially for that displacement or velocity, we can see that there are harmonic terms and in which terms like 1 minus zeta square omega n is there.

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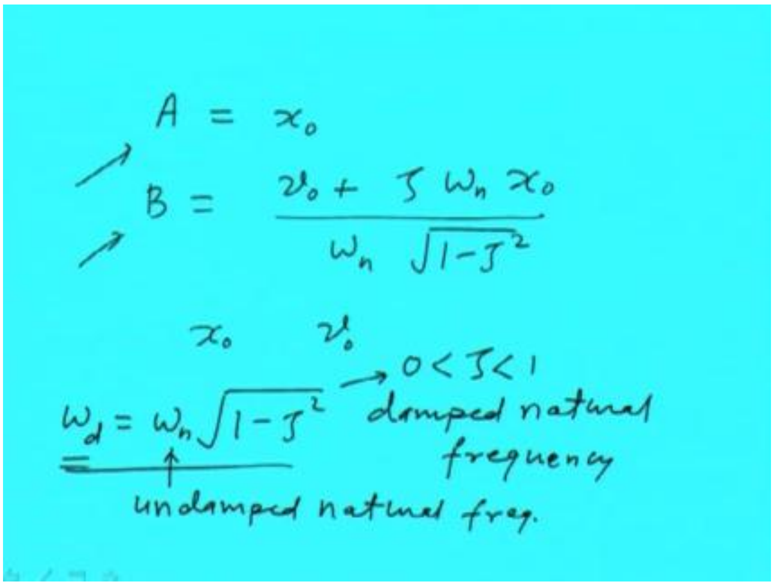

Let us write them in a separately omega n. So, this term we call it as omega d; that is the damped natural frequency. So, you can see that when zeta is 0, the damped and undamped natural frequencies are same, because this is the undamped natural frequency. And here you can able to see the effect of damping on to the damped natural frequency when damping is there because this is a square quantity. So, either positive or negative value of the zeta, always it will be decreasing the damping, because this term within the square bracket will be less than 1. So, this particular case as we are doing for the zeta 0 to 1, so in this range you can see that always omega d will be less than the omega n.

So, damped natural frequency is always less than the omega n, but if you will see through illustration when damping value. If you put especially in the structural members, the zeta value will be around 0.1 for that the effect of the change in the natural frequency to the damping will be very less.

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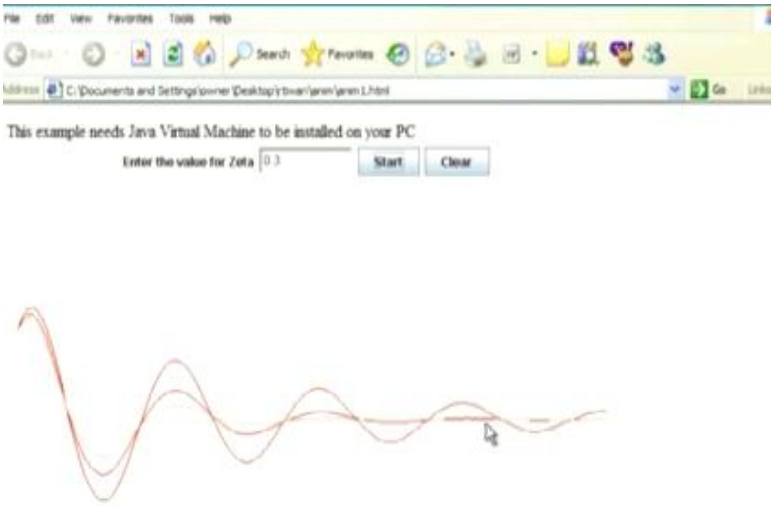

Now I will be showing how this particular vibration or the signal or the response changes with the various kind of damping like here I will be showing through simulation. So, at present I am giving zeta value as 0.1 as you can see here, and now I am giving the start thing. So, you can see that how the signal is decaying when we give the zeta is equal to 0.1. Now I am giving another value of the zeta and I am drawing this plot on the same graph, so that you can compare how it changes. We can see that when we increase the damping, the vibration is getting dampened quiet quickly

But if you see carefully always both the graphs they are having peak at the same time also the minimum, the mean at the same time or the minimum also the same time at every place. So, that represent that their frequency is not changing as they are going along this, but only the amplitude is getting changed. This will be more clear, if we take another set of damping. You can see that this is very fronted graph, but at this point all three are meeting at this position; all three are maximum, similarly, as they are going along this direction.

So, this is the example of the under damped system in which we have given the zeta value from 0.1 up to 0.3. The 0.3 is quiet high value; from here itself we can able to see that how quickly this vibration decays, but most of the structures they have lot of oscillations before they get into stationary position. So, we are seeing through animation for different damping ratio how the cause gets decay with respect to time. Let us see that plot more carefully and in this particular plot, which we cross through animation we have shown.

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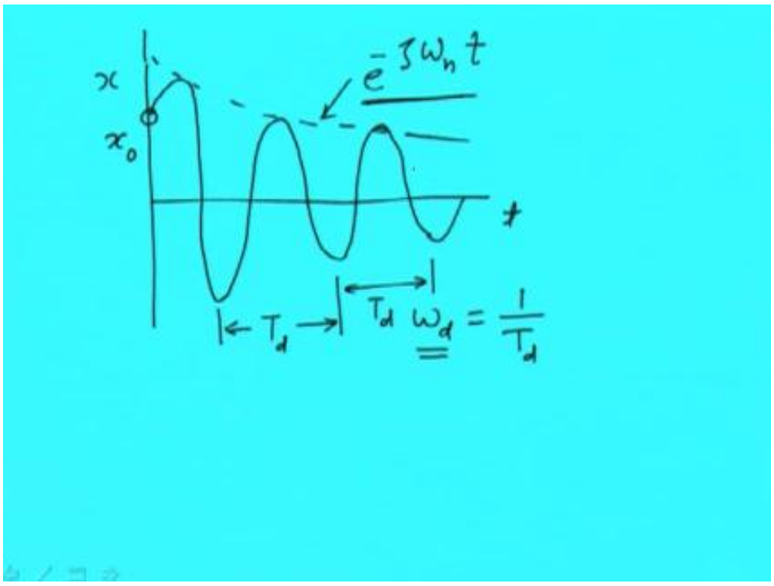

This is the displacement; this is the time. Then this is the particular damped vibration signal. And this particular displacement is the initial condition which we are giving to the particular dynamic system that is spring that mass this much displacement initially we are giving it, and we are allowing it to oscillate like this. So, it is decaying gradually and this decay if we join these amplitudes. So, this is nothing but the exponential decay which is there in our response term at outside the bracket, and because of this only it is decaying.

Within the bracket we had the oscillatory terms, and because of that it is oscillating like this. And in this case from here to here this is the damped time period, which is related with the damped natural frequency like this. So, time period remains same always as you progress here always it will be same, because damped frequency is same, but amplitude is decreasing gradually. Now we will consider non-oscillatory motion in which that is over damped system and in this particular case we already observed that the two roots of the characteristic equation, the one becomes negative, and another becomes positive, but both are real.

One becomes toward the zero; that is I will repeat this one. The two roots both decreases and we have both of them as real. And let us see how the expression of the response take place in this particular case how they have the exact form.

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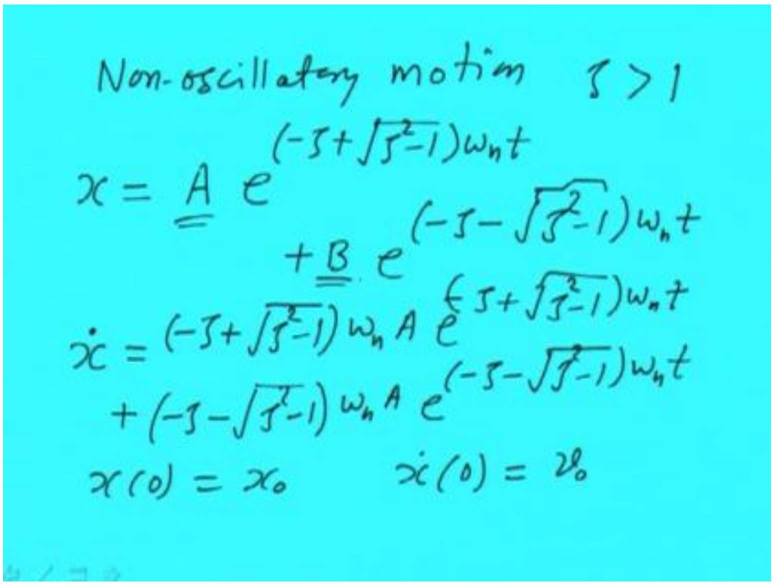

So, for non-oscillatory motion in which we are considering zeta greater than 1, the response we will be writing as exponential minus zeta plus 1 minus zeta square, or this will be because zeta is greater than 1. So, it is better to write in this form, then the second term; for the second root that is negative. So, this is the displacement expression for the oscillatory motion. As we did in the previous case, we can get the velocity by differentiating this, and this expression will be again lot of terms will be there; just for completeness I am writing them here.

So, this is the differentiation of the first term and this term as it is within bracket omega n t. This is coming from the first term, then from the second term we have this quantity. And so these are the displacement and velocity. Again we have two initial condition at x is equal to 0 x naught and at this velocity at time t is equal to 0 v naught. We will substitute in this two equation, and we will solve for A and B which are integral constants.

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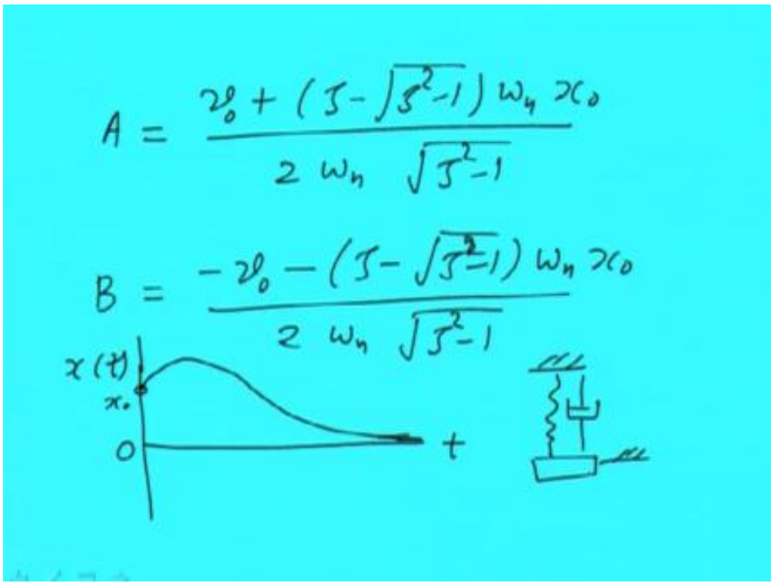

And they will take this form that is A naught; A will be nu naught plus zeta square minus 1 omega n x naught by 2 omega n zeta square minus 1. And similarly B will be minus nu v naught that is the initial velocity of the mass square omega n x naught divided by same denominator. So, these are the integral constant in the explicit form. In this particular case the response will be because we set this non-oscillatory there is no oscillatory term because all roots are same. So, we may have the oscillation like this.

So, it will not cross this particular line below other side; that means if you are giving some initial condition here x naught and v naught, it will come to the static equilibrium position with respect to after some time, but it will not go other side of the static equilibrium position. So, if we have the spring mass system with damper if this is the static equilibrium position, if we disturb this to the downward direction, gradually it will come to its original position, but it will not go up. So, that is the over damped system.

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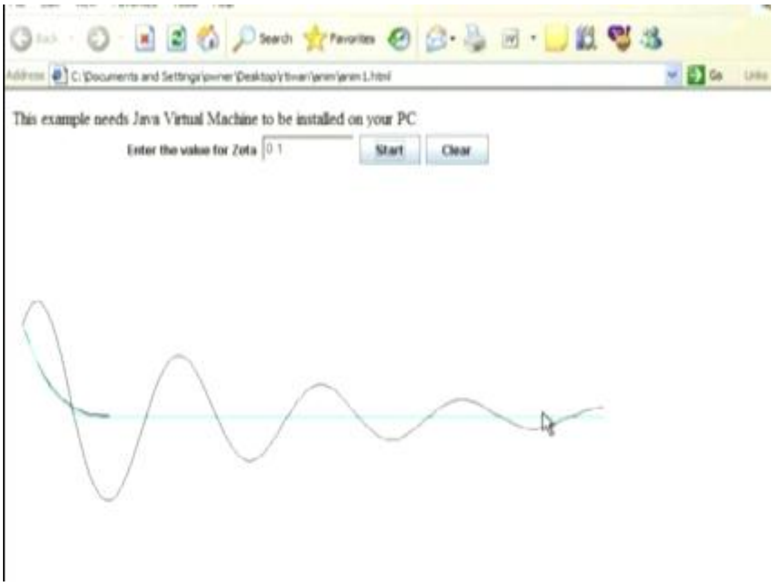

So, through simulation I would like to show how the over damped system response can be simulated. So, I am giving zeta 1.1 which is more than 1. So, you can see that how it decays quickly. So, this is no oscillation, directly it decays and in previous case now I would like to show with under damped system how it was. So, you can see that with over damped system how quickly the system undergoes to its static equilibrium position.

So, today we have seen the damped free vibration analysis. We initially saw that how the damping comes into the dynamic system and then how we can able to mathematically model them. For the present case, we have taken the viscous damping in which the force of the damping is proportional to the velocity. And it gives mathematical expression for analyzing the damped vibration that is linear, and they can be solved easily. And we have seen that various kind of damping, the various level of damping may give different kind of response like we can have under damped system or we can have over damped system or critical damped system or even the negative damping is there, the system may go into instability zone.

In the next lecture, we will extend this method especially for finding this damping ratio and natural frequency, how we can able to obtain experimentally, and we will be taking some examples related to that. Also we will explore other form of the damping which is there in the mathematical models of the various kind of damping.