Mechanical Vibrations
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Module - 11
Finite Element Analysis
Lecture - 01
Finite Element of Formulation for Rods, Gear Train and Branched System

So, in the last class, we already introduced what is the torsional vibration. We considered some simple cases of rotors like single degree of freedom system, 2 degree of freedom system. And also we extended the method for multi degree of freedom system using transfers matrix method. And then we also indicated in the last lecture that there is another method which is called finite element method which is very popular and powerful. Specially it can be applied to any general rotor system that today we will be studying. And as compare to the transfer metrics method this particular method is having one small disadvantage in the sense that in the system degree of freedoms are more. Then the matrices, which we have to handle or they become larger and larger, but with the present computer computing power I think that should not be concerned to us.

So, today I will be explaining finite element method for torsional vibration. And we giving some basic steps also so that the formulation part of the specially developing elemental matrices is clear to us. In the previous lectures we have already covered how to obtain the equation of motion for the continuous system to torsional case. And we ended up with partial differential equation for such systems. So, I will not be covering that part here. So, directly we will be going for the solution of those equations through finite element method. Apart from that we will be covering another kind of element that is the gear element and once we will develop the gear element. Then through examples will demonstrate the finite element method. So, that the basic concepts are clear.

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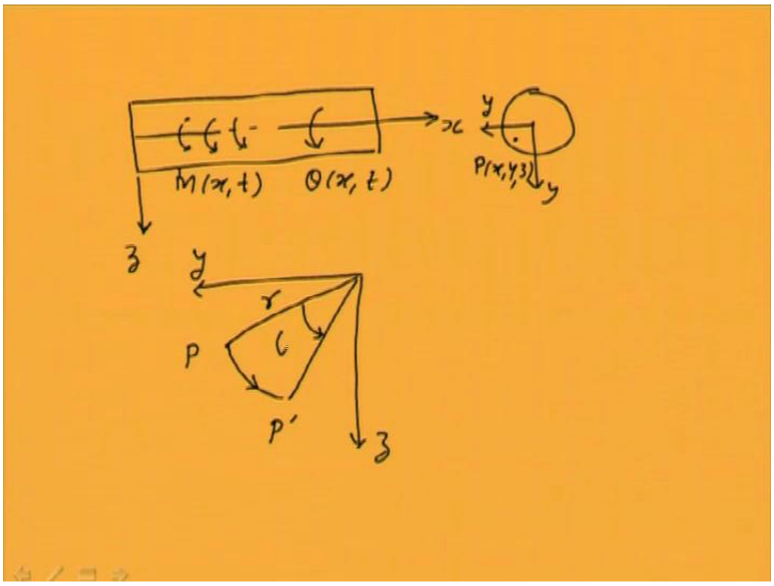

I am explaining with the help of simple diagram what is the continuous system and what is the equation of motion which we already developed earlier. So, this is the x axis of the rod and this is the side axis and it is loaded with a distributed moment which is function of special coordinate also the time. And at any general location of the rod the angular displacement is taking place that is xt. And if you see the side view of this rod we can able to point out 1 1 point that is let us say point p the location of that is let us say x, y, z.

And if we want to see the motion of that particular point p in a large view point p is here on to the shaft. The radial position of that is r and once it is getting displaced it is reaching the point p prime and this angle we will call it theta. So, here we are assuming that the point p is moving in a same plane it is not going toward the axial direction. Now, the equation of motion for this kind of continuous system in which the stiffness and mass are distributed can be written like this.

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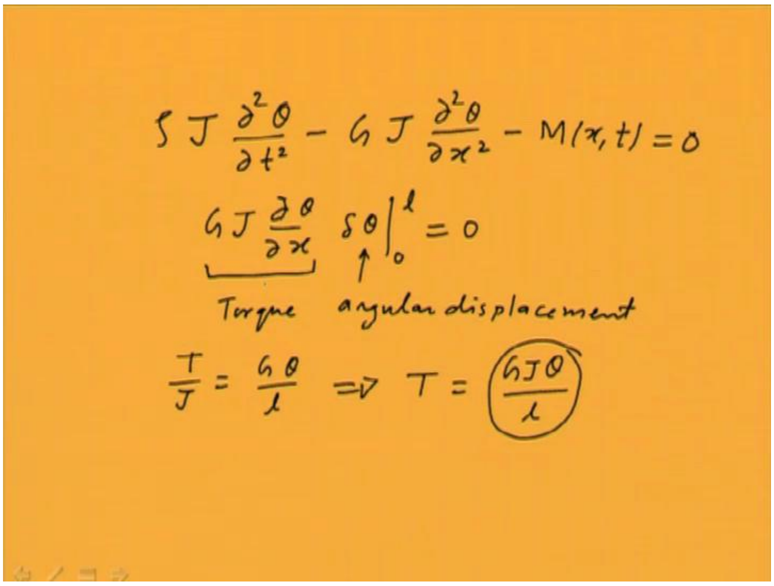

This is the density this is a polar moment of area and this theta is the angular displacement t is a time G is the modulus of rigidity and this is the special partial derivative. So, this is the equation of motion for partial differential equation which we already derived earlier. Uh apart from this boundary conditions are there that is at ends of the shaft. So, in this you can see this particular quantity is nothing but torque. And this is the angular displacement through strength of material formula we know T by J is equal to G theta by l. So, this we can able to see that T is equal to G theta Jl. So, this is identical to the, this particular term and that is why we are calling this as a torque. Now, we will be solving this partial differential equation a with the help of finite element method.

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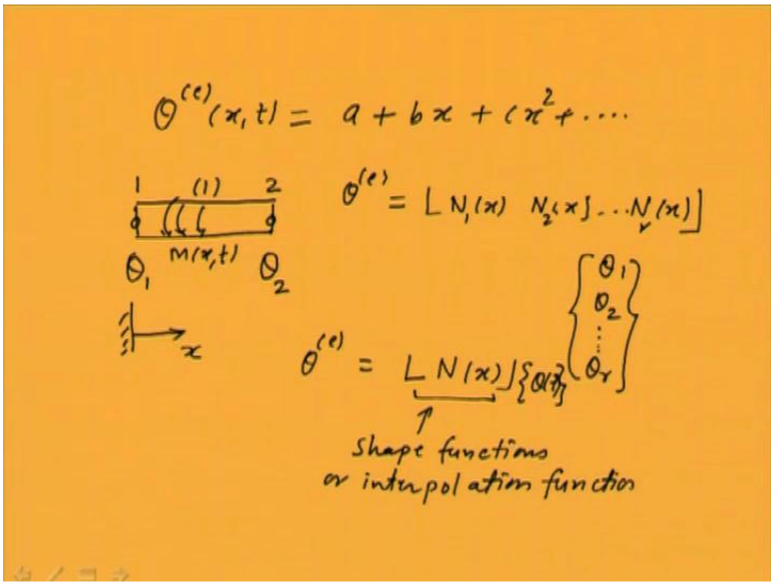

So, in the finite element method we assume the solution in a polynomial form. So, this is the solution of what 1 particular element which we assume in a polynomial of form. Let us consider 1 element of the beam in which we have node 1 and 2 and this let us say element 1. So, corresponding to this node 1, we can have displacement theta 1 and corresponding to node 2. We can have displacement theta 2 and the reference axis that is a local coordinate system axis can be chosen from left hand side of the element. In this particular element maybe we can have a moments also that is also distributed these are the external loads on the beam element.

This particular solution which we have assumed is can be written in a more general form like this. That is vector contain angular displacements that various nodes of the element which again we can able to write in more compact form like this. Theta here is time dependent and is special different special coordinate different and this is called shape functions or interpolation function. Once we have assumed the solution in a polynomial form we will substitute this solution in the equation of motion. So; obviously, because this solution is approximate in nature we will not able to satisfy the equation completely. So, we will get some residue.

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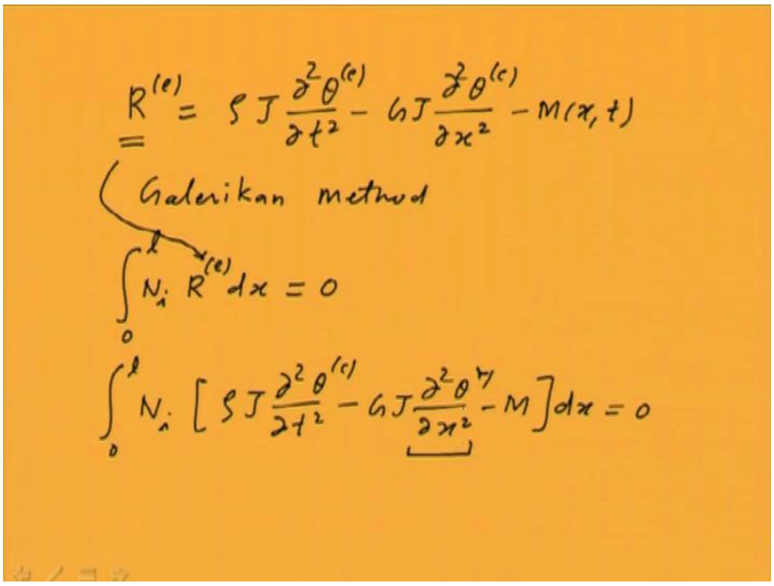

So, let us write the residue. Here theta we are writing with subscript or superscript this is the superscript e that revision for the element. And this is the external torque which is acting on the element. So, this is the residual and here we will be using Galerkin method of the to minimize this particular residue. And we will be using shape functions as a weight function to minimize this residual over the whole length of the beam l is the length of the beam. So, we will be substituting this here I am showing this step. So, that the concepts are not clear. Here you can see that specially the derivative with respect to x of the theta that is angular displacement is 2. So, we would like to reduce this because by reducing this we can able to choose lesser degree polynomial is our solution. So, now we will be reducing this differentiation with by integration by parts.

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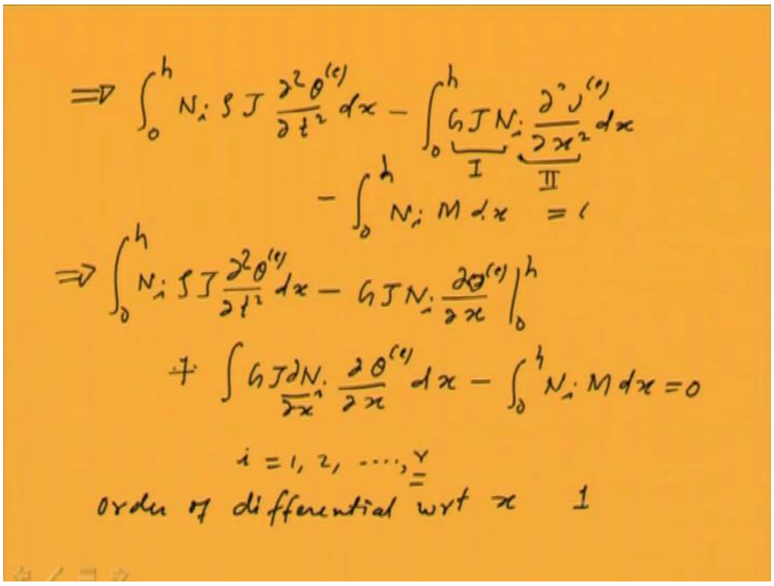

Here the first term remains the same there is no change in the first term second term I am writing as it is present. In next step, we will doing the integration by parts and the third term it remains as it is. So, here we will be doing the integration by part this will be taking as first term this will be taking as second term. So, let us do that integration. So, here first term will remain as it is then integration of the second term 0 to h h is the element length plus GJ here we are doing the differentiation of the first term. And then this is the integration of the first term and this and the last term remains the same here you can see that I we have attached to the, this shape function.

And they can have values up to r or that we will be deciding what should be the r r can be a integer number. In this particular equation in which we could able to reduce the order to differentiation of x you can see that now the maximum order of differentiation in this whole integral and other terms are order of differentiation with respect to x is 1. So; that means, according to finite element method we require completeness of theta and its derivative with respect to x. So; that means, we will be requiring completeness requirement of theta and del theta by del x.

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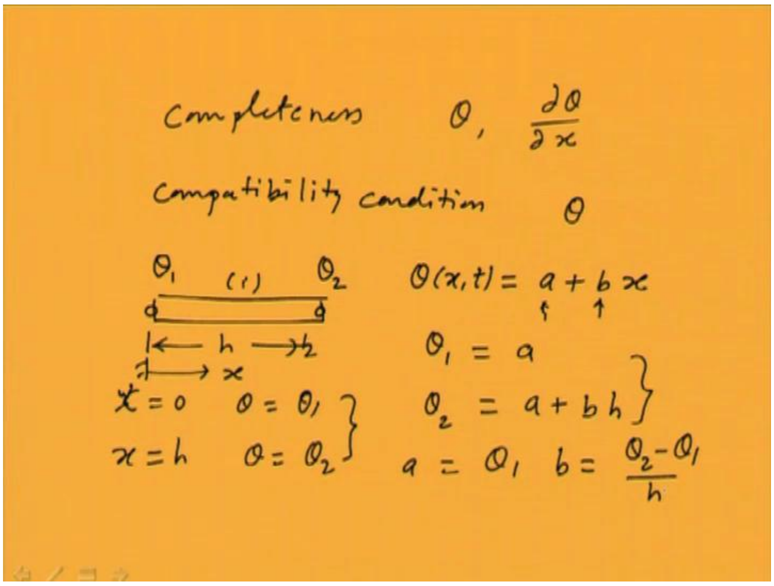

And if we see go back to the previous integral equation inside the integral the derivative of x is also x here. So, we require the compatibility condition of what order less; that means, up to theta will be requiring the compatibility condition. So, now we have seen the compatibility condition. So, what we can do we can come back to the element here and because we require compatibility up to first derivative. So, at each node we can assign 1 or 1 variable that is theta 1 and theta 2 this for element 1. So, a if we choose the solution that is xt is a linear function that is a plus b x. So, this particular solution will satisfy the, our requirement of completeness and compatibility condition both.

And will be if we see the boundary conditions of the element here we have 2 boundary condition 1 is at x is equal to 0; that means, because xx is here. Ah theta is theta 1 and x is equal to h h is the length of the element theta is theta 2. So, here we have 2 boundary conditions so; that means, we can able to obtain 2 constants in the polynomial. So, that is also satisfying our requirement. So, if we substitute these 2 boundary conditions we can get these relations. Second boundary condition a plus bh now this equation can be solved for a and b a is directly we can get theta 1 and b is nothing but theta 2 minus theta 1 by h.

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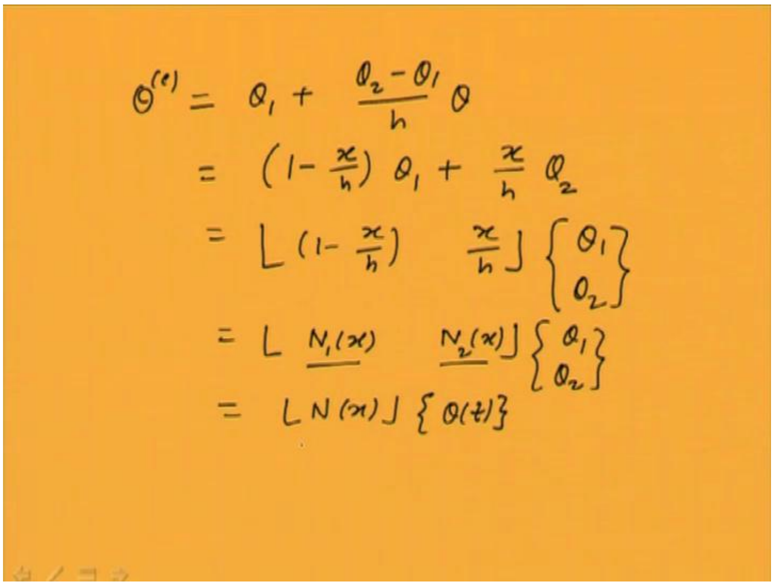

Now, we can substitute this in the previous equation, so theta 1 plus theta 2 minus theta 1 by h into theta. So, this can be rearrange as 1 minus x by h theta 1 plus x by h theta 2. And this we can able to write in a matrix vector form like this is a row vector and this is column vector. And this is the, our standard form of the solution which we wrote earlier also like this. So, these N x N 1 x Nx N 2 x are nothing but shape functions. So, this we can in more compact form like this. So, here you can see that shape function is function of x and the nodal displacements are function of time. So, this solution which we have obtained we will be substituting in this particular equation which we obtained earlier. We can have the r as 2 here. So, we can have 2 such equations here. So, we can able to combine them.

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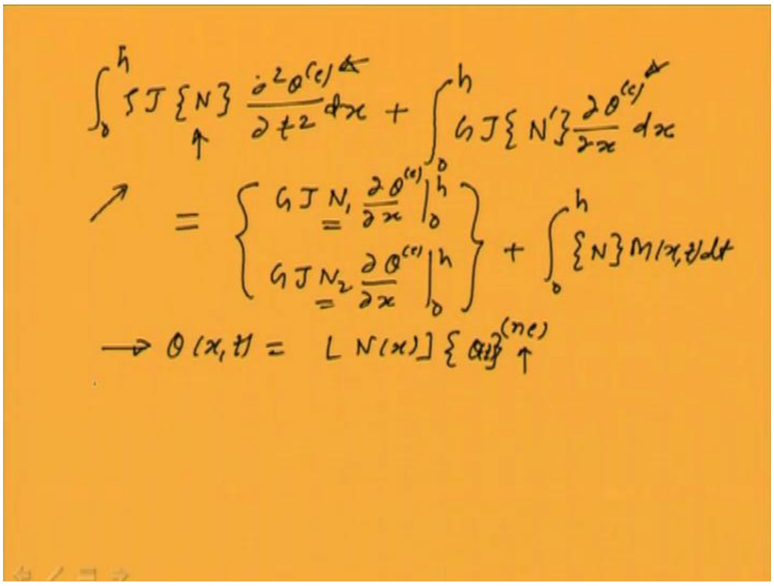

So, let us combine them. So, combine form on the previous equation is this. So, we have combined here 2 equations that is because this is a vector N 1 and N 2 is the. Then second term here we are writing N prime prime represent the derivative with respect to x. And this N other term which are not integral terms we are keeping in the right hand side. So, I am expanding that term second term is. So, here you can see that N 1 and N 2, 2 equations we have combined here in 1 plus the external momentum ah. Now, we will be substituting the solution theta xt is equal to N x theta that is time dependent. We will call this as nodal a displacement and represent the nodal displacement of the element. This particular solution we will be substituting in above equation. So, we will get here we have taken out the theta double dot n which independent of x.

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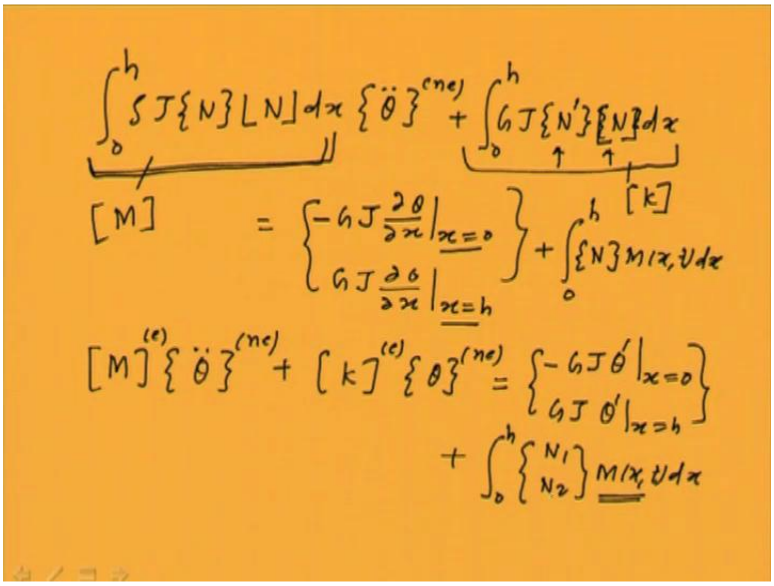

So, we have taken off from the this integral then we have N prime represent derivative with respect to x. This is row and this is a row vector; this is a column vector is equal to here we are substituting the value of x is equal to 0. So, you can see that the x value of x is equal to 0 of N 1 becomes 1 and N 2 becomes 0. So, we will be left with this this particular term. Then here here minus will be coming in the next term we will be left with this particular term because here h at x is equal to h N 1 is 0 and N 2 is 1. So, we will be left with this particular term plus we will be having the terms corresponding to the external moment all integrals are from 0 to h. This is the element length now what will be assigning this particular term we will call it as a m matrix that is mass matrix.

This particular term we will be calling as a k matrix that is stiffness matrix. So, equation can be written in a more compact form like this these for the element. So, let us call it with a superscript e is equal to now we can see that we can able to write this in more compact form by putting theta prime which represent the derivative with respect to x. And the forcing term forcing term is something like this in the expanded form. So, if we know the distribution of m we can able to integrate this over the element. Next we will be obtaining the exact form of the mass matrix and the stiffness matrix. So, mass matrix has been defined earlier is 0 to h row J N vector and row vector into x.

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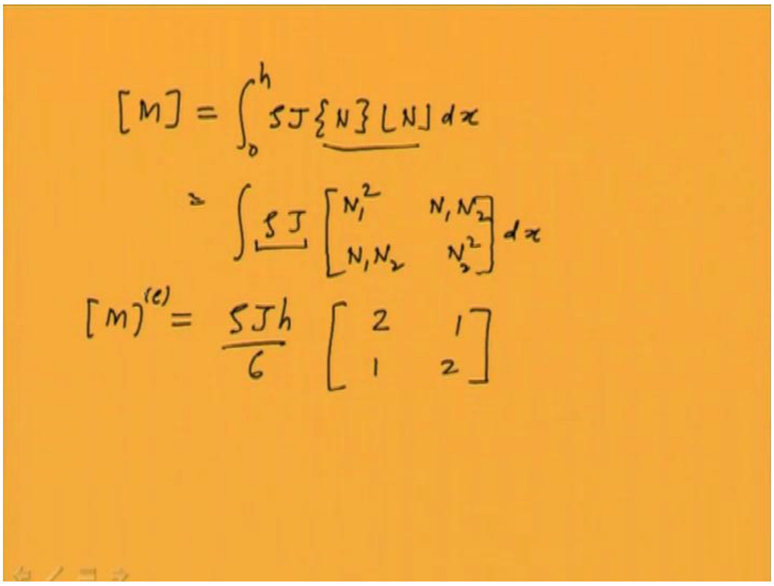

So, this will be row J if we multiply these 2 we will get N 1 square N 1 N 2 N 1 N 2 N 2 square dx, because element is uniform. So, row and J can be considered as constant. So, they can come out. So, if we integrate this we will get as a mass matrix of a 1 element.

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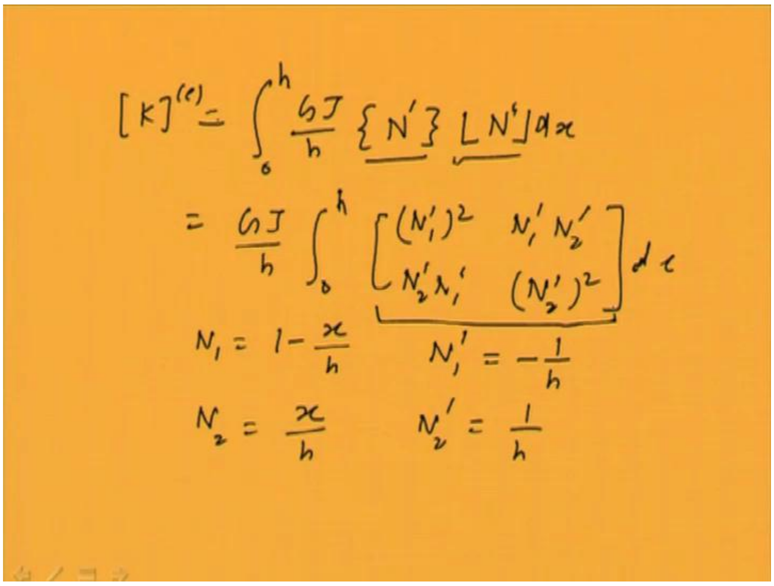

Or similarly we can able to obtain the stiffness matrix. Here N prime vector N these 2 vectors are there. So, if we multiply them we will get N 1 prime whole square N N 1 prime N 2 prime N 2 prime N 1 prime N 2 prime whole square dx. We have N 1 as 1 by x by h. So, N 1 prime will be minus 1 by h and similarly N 2 we have x by h. So, N 2 prime will be 1 by h. So, you can see that this particular matrix is a constant.

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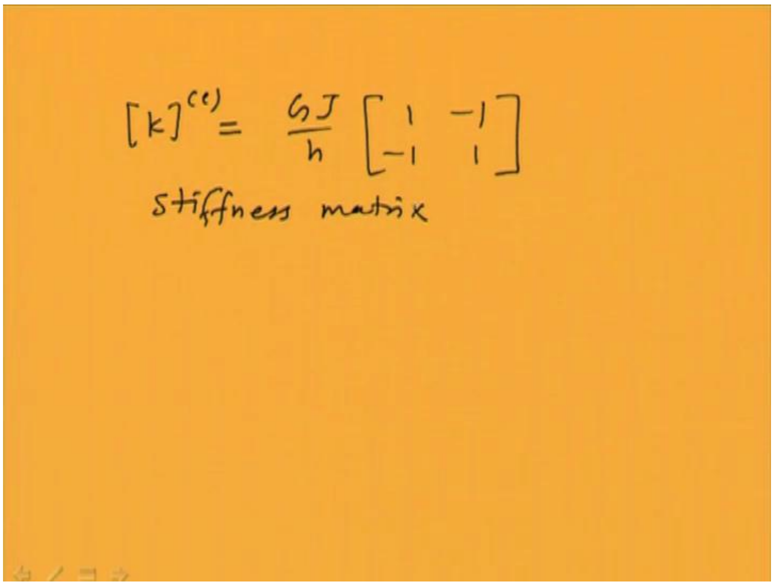

So, that stiffness matrix becomes this. So, this is the stiffness matrix for the torsional vibration. So, now we have already obtained the equation of motion for 1 particular element gone through 1 example will demonstrate the the finite element procedure. So, for this we will be taking 1 simple example of 2 disc rotor system which we earlier also we have considered. So, this form demonstrating the procedure.

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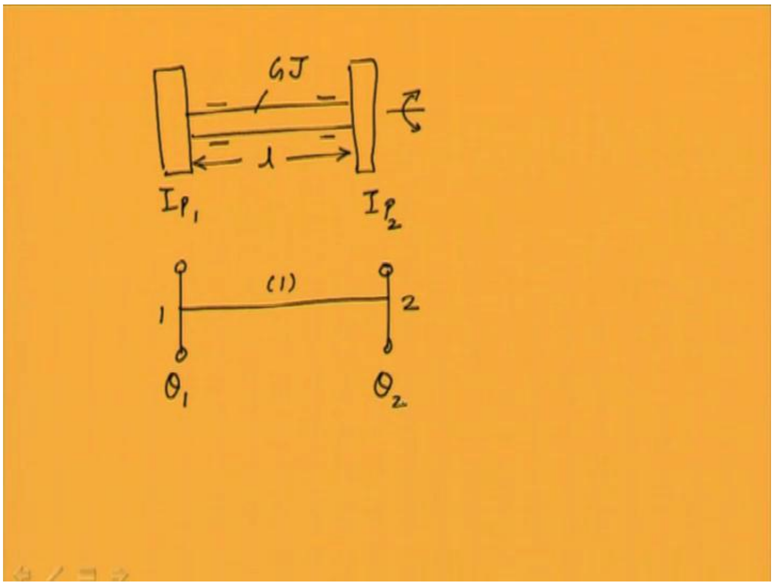

So, we have 1 rotor system like this let us say these are polar mass moment of inertia of the rotor is supported on friction as bearing. So, it can able to rotate freely on this, this is the length of the shaft which is massless the other property of the rotor this, this modulus of rigidity and polar moment of area. This particular problem we can able to solve by assuming a single element like this. A single element we have node 1 and node 2 and corresponding to node 1 we have theta 1 and theta 2 displacement. Now, for this particular element we can write the equation of motion. So, the finite element elemental equation will be given by here we are not considering the mass of the shaft.

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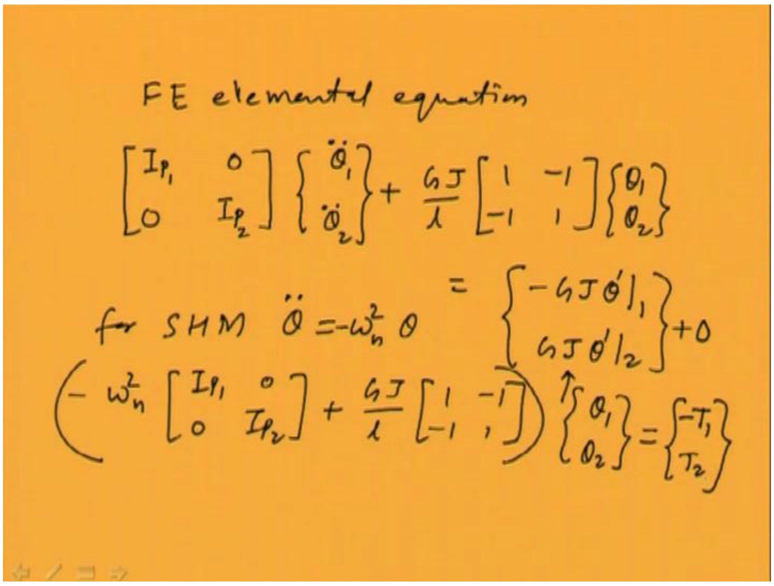

So, the mass matrix will be diagonal because the contribution will be coming from the discs only. So, here the mass of the shaft is has not been considered then we have the stiffness matrix and equal to the internal torque. This is at node 1 and this is at node 2 you can see the other term are nothing but torque as we have already defined earlier. So, this is their internal torques and there is no external moment because we are solving the free vibration. So, there is no external momentum in this for simply harmonic motion we can able to write theta double dot dot s this with minus. So, if we substitute in this equation we will get this form of the equation uh. Then is equal to these torques let us say T 1 and T 2. Now, we can apply the boundary conditions here the shaft is mounted freely on the 2 bearings. So, both the ends of the bearings the shaft is free. So, there is no torque at the ends. So, when there is no torque these both the torques will be 0.

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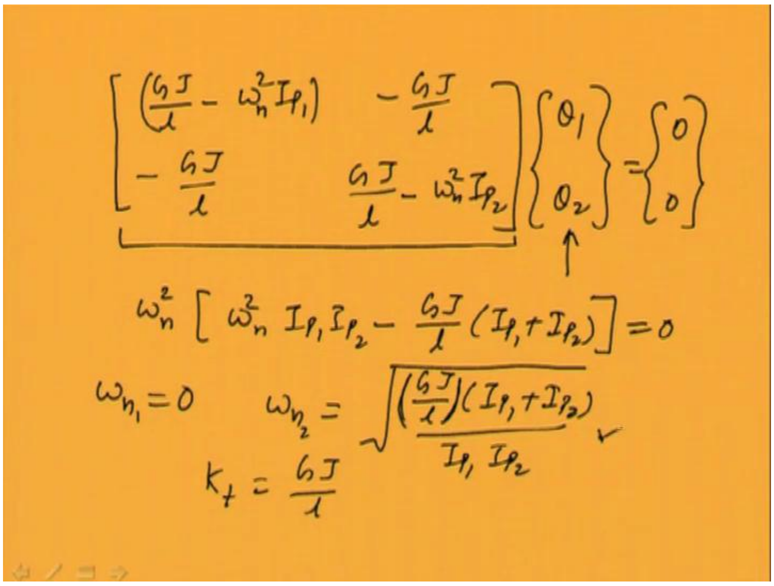

So, the equation after applying the boundary condition takes this particular form and there is a homogeneous equation now because right hand side it is 0. Now, because this quantity cannot be 0, so we have to have this determinant is equal to 0. So, see if we put the this determinant as 0 this is a N we will get a polynomial in terms of omega N and that if we solve it. We will get this form Ip 1 plus Ip 2 and from here you can see that the first natural frequency of the system is 0 and second natural frequency of the system is this. And if we recall if we substitute this particular quantity as kt kt is Gj l.

Then this particular equation is exactly same as we earlier obtained and there using finite difference method or using the close form solution. So, in this particular case even when we consider the the approximate solution in the form of polynomial it gives exact solution. This is very special case in other cases like beams we may not get the similar trend. So, let us take another example in which we will be demonstrating the assembly procedure of the finite element method for that we will be taking 1 small problem. In this 1 cantilever beam will be considered.

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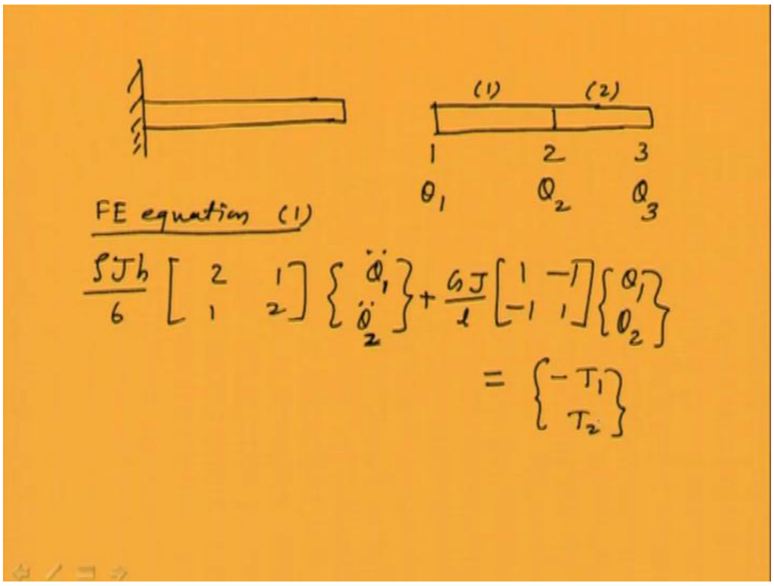

So, this is a cantilever beam having fix and we want to obtain the natural frequency of this particular system. So, for finite element purpose what we will do we will divide this particular beam into 2 elements 1 and 2. So, we will be having node 1 2 and 3 and then corresponding displacements are these. Now, we can able to write the finite element equation for element 1 like this is the, this is the mass matrix and you can see for element 1 there are theta 1 and theta 2 as displacements. Similarly, if we have stiffness matrix and these are the angular displacements at node 1 and 2 and they should be equal to the internal torque like this.

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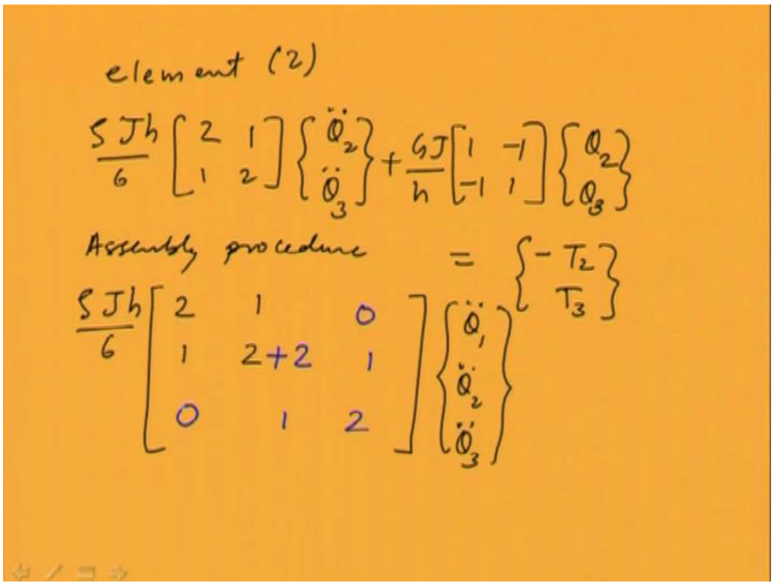

So, we will be writing equation of motion for the element 2 on this we have most of the terms for the mass matrix exactly same as the previous element. Only difference will be coming in this vector where theta 2 and theta 3 are the displacements of the element 2. Then stiffness matrix remain the same only difference will be coming in these 2 and theta 3 vector is equal to the internal torque. Now, we can combine the element 1 and element 2 I am showing the procedure for this assembly procedure. So, here first time entering the mass matrix for both the elements. So, first element is contributing here for second element will be coming here and rest of the elements are 0. So, this is the assembly this blue color represent that element 2 and black color represent element 1 contribution.

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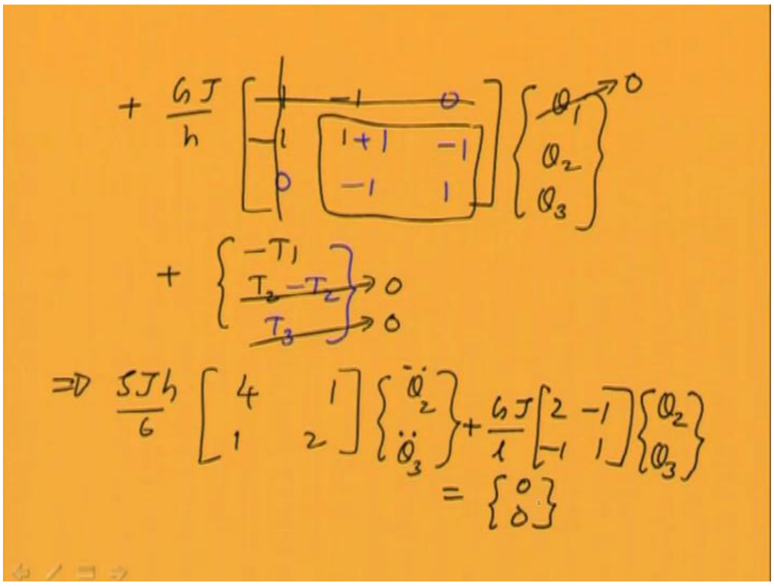

Similarly we can able to assemble the, this stiffness matrix. Like this theta 1 theta 2 theta 3 and contribution from the first element will be this and for the second element will be having contribution here. Rest of the terms are 0 for force factor we have minus T 1 T 2 this is coming from this second element. So, here you can see that this term is getting 0 and because here 1 end of the beam is fixed; that means, at node 1 it is fixed. So, there the displacement with be 0 and other end of the beam which free end that is a cantilever. So, there torque will be 0. So, those boundary conditions will be applying in this, so this become 0. Then T T 3 is 0, because that is free end and in the previous slide a theta 1 is 0 that is a mass matrix.

So, you can see that in this particular case finally, we will be left with 2 by 2 matrix because we need to eliminate the first row and first column because they are getting multiplied by 0. So, we will be left with this particular terms and in the similarly in the stiffness matrix we will be left with particular term. So, after applying the boundary condition our equation of motion is reduce to 2 by 2 size of this form. This is coming from the mass matrix let us go back to the previous slide this 2 into 2 4 and here we have only 2 vectors this 1 plus stiffness term theta 2 theta 3. And right hand side both are 0 now for simple harmonic motion we know that the we can able to write the theta in terms of a theta double dot in terms of a omega square and theta.

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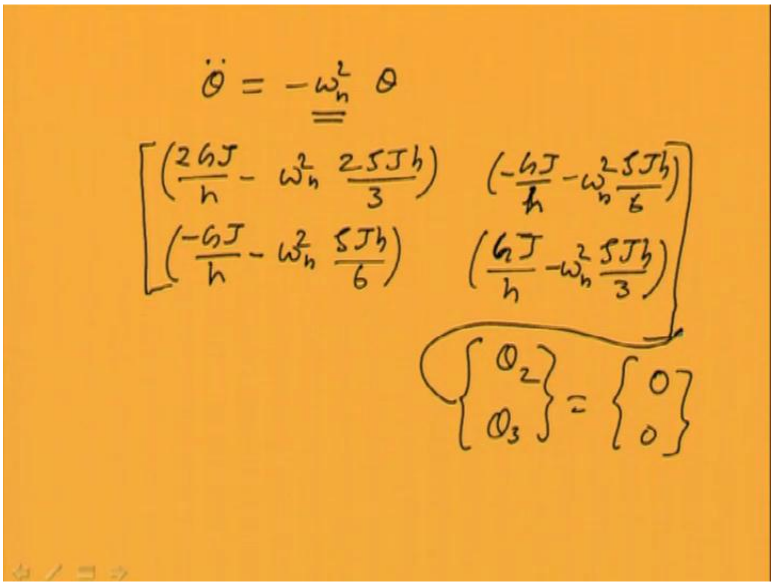

So, we will be having for simple harmonic motion because we are analyzing the free vibration where this is the natural frequency. So, previous equation reduce to we can able to club in a matrix form. This is first term, then second term in this both the mass matrix and stiffness matrix has been clubbed in 1 place and the last term is this. And it is getting multiplied by theta 2 and theta 3 and that is equal to 0. So that is homogeneous equation, because this will go here this is a homogeneous equation and for this, because we cannot have the trivial solution. That means both the displacement theta 2 and theta 3 cannot be 0. So, the determinant of this particular matrix has to be 0 and from there we will get a frequency equation. In terms of omega and I am writing the final form of the vectors.

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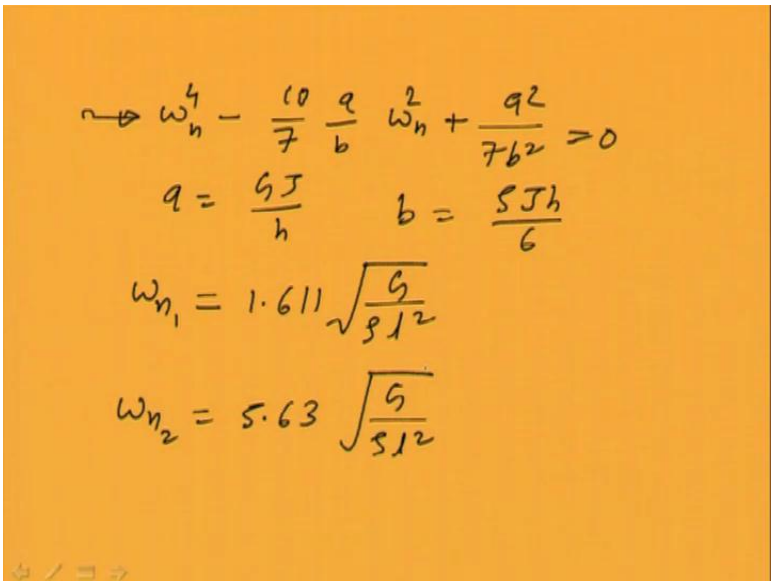

A and b are defined some parameter I will defining this where a is GJ by h and b is row J h by 6. So, we can able to solve this for omega n and this will give us 2 natural frequency of the system. First one is this second is and because earlier we have already obtained the closed form solution for the cantilever case. So, let us see those expressions directly that is a closed form solution.

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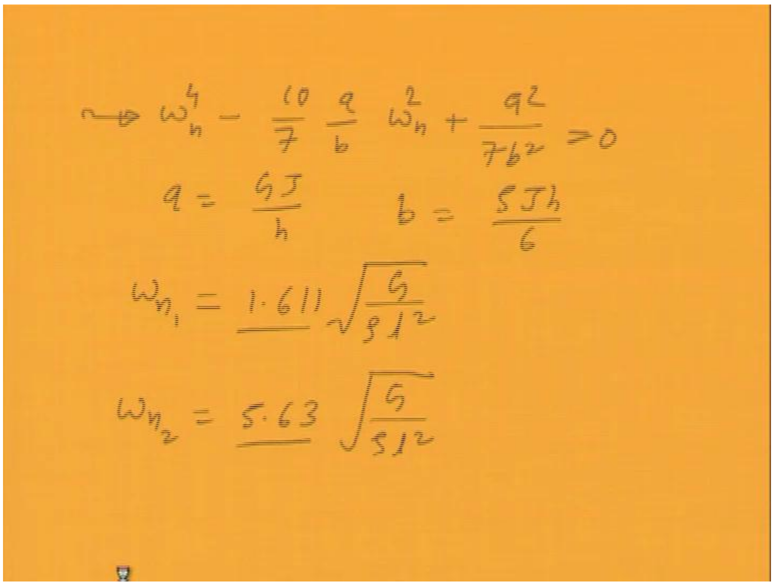

This is the closed form solution for cantilever beam case. So, n can be 1 3 5 various modes they represent. So, for omega 1 exact solution is 1.5 7 0 8 and this quantity and omega n 2 is 4.7 2 G row l square. So, if we compare with our result our result is 1.6

which is close to 1.57 and this is 5.63 which is likely away from the 4.72 from for the closed form solution. Ah here 1 aspect is there we have considered only 2 element for demonstration purpose. But we can able to increase the number of element if we are writing a program computer program for this particular problem in which we can able to increase may be 3 element or 6 element or 10 or maybe we can go up to 100 element.

And we will see that as we will increase number of element our results will be closer to the exact solution. Or that is generally we perform for the finding out the convergence criteria means what should be the number of element we should have. So, that we are close to the exact solution. Now, we will be a developing elemental equation for gear element a we have already analyze the gear a through the previous transfer metrics method also by the direction method. Let us develop the elemental mass matrix and the stiffness matrix for the gear element then through 1 example will demonstrate the procedure of finite element method.

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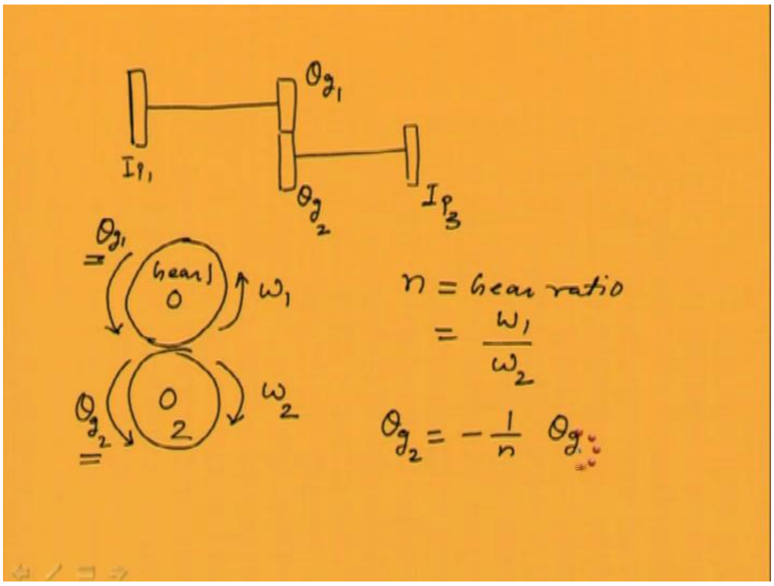

Let us take a gear a like this is a rotor let us say the p 1 this is a gear this is another gear and is connected to shaft through to a another rotor. Let us say this is I Ip 3 now the displacements of the gears are g 1 2 and g theta g 1 and theta g 2. Now, let us draw the gear separately this is gear 1 and this is gear 2 and let us say it is rotating with omega 1 and then the gear 2 will be rotating with omega 2. And the related with the ratio that is omega 1 divided by omega 2. Now, in finite element method generally the convention is to represent the the displacements are in the same directions. So, for 2 gears we are writing the, these are the torsional displacements gear is rotating at some RPM, but it is getting torsional vibration, because it is transmitting power. Now, we have taken these direction same because in finite element method that is the convention. So, the relationship between these 2 displacements will be like this. Minus sign is, because in the in the figure they are in the same direction, but it should be in the opposite direction for actual case. Now, from the now from the now from the previous at the analysis we know that if want to a do the equivalent system for this gear element. We have to divide the specially the polar mass moment of inertia by the gear is, so gear is a square.

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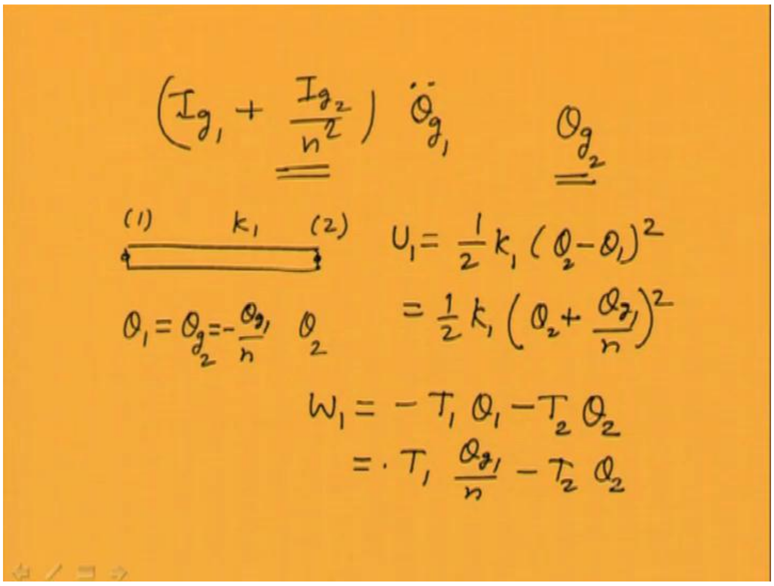

So, that will give us the, what is the inertia force, which we will be getting in terms of the the gear displacement one. So, here we have obtained the equivalents of the polar moment of inertia of gear 2, because theta theta g 1 and theta g 2 related. So, we now want to eliminate the theta g 2 from the equation now let us write the elemental equation for the stiffness. So, for that let us take the shaft having stiffness k 1 and is having element node 1 and node 2 and correspondingly theta 1 and theta 2 are the displacements. And this theta 1 is actually the gear 2 displacement because this particular shaft which we have chosen in previous figure is this particular shaft. So, this end is element node 1 node 2, but this particular displacement is assigned to node 1.

So, that is why we are calling as theta g 2 is equal to we know that is theta g 1 by n. Now, let us write the strain energy due to this that will be k 1 theta 2 minus theta 1 whole square of this is the relative twist between the 2 ends of the shaft. So, this the strain energy we can substitute for theta 1 that is theta g 1 by n whole square. Now, worked on is given as T 1 and T 2 are the reaction stocks at node 1 and 2. Again we can able to replace theta 1. So, we will be getting theta g 1 by n minus T 2 theta 2. Now, now we can able to use Lagrangian equation to obtain the equation of motion.

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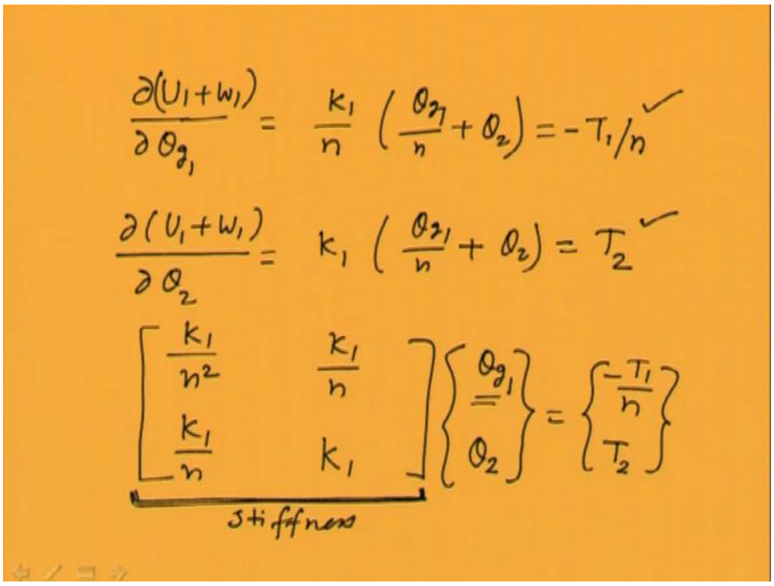

So, for Lagrangian equation we need to differentiate the strain energy plus the work done by the variables which we have that is theta g 1 and theta 2. So, we will get theta g 1 by n plus theta 2 is equal to minus T 1 by n and this second term we will getting theta g 1 by n plus theta 2 is equal to T 2. Now, these 2 equations can be combined in a matrix form like this in which first term is theta g 1 another theta 2. So, here you can see that we have eliminated the actual displacement of gear 2 a gear 2; that means, theta g 2 by theta g 1, because they are related.

So, this equation will form of this form minus T 1 5 second term is T 2. This particular equation is the stiffness matrix which we need to consider for the analysis of the gear element. And we have already seen the change in the mass matrix will be this much. So, now, we will be taking 1 example of the branch system in which 1 shaft is giving power to 2 shafts. And using finite element method which we already developed the elemental equation for a stiffness and mass matrix will be demonstrating the procedure.

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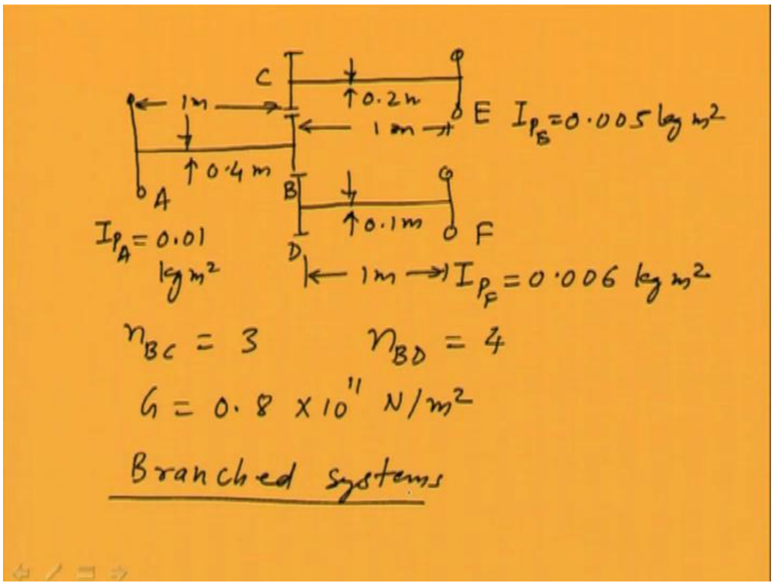

So, in this particular problem we have 1 rotor which is connected through shaft a gear I have suggested rotor is A this is gear is p this gear is connected to gear C. And that itself is connected to another rotor E another disc I should say and this again it is another gear that is D and this is another rotor disc that is F. Various parameter of these are like polar mass moment of inertia of disc A is 0.01 kg meter square of a disc E is 0.005 kg meter square. And of disc F is again 0.006 kg meter square. Diameter of various shafts are which is point 2 meter diameter is having 0.1 meter diameter is having point 4 meter diameter. Lengths for simplicity we have taken as the 1 meter all the lengths are 1 meter of the all 3 shafts.

So, these are the various dimensions of the various shafts gear ratio between B and C is 3. And gear ratio between B and D is 4 and modulus of rigidity again as this much. So, these are the parameters of the branched system this is a branch system in which power is coming from 1 shaft and going to 2 shafts. So, once we have given all the parameter of the shaft now we will be dividing these 3 shaft system to 3 elements and we will be writing the elemental equation for each of this. And then we will be assembling the elemental equation and we will be applying the boundary condition and finally, then we will be getting the natural frequency of the system.

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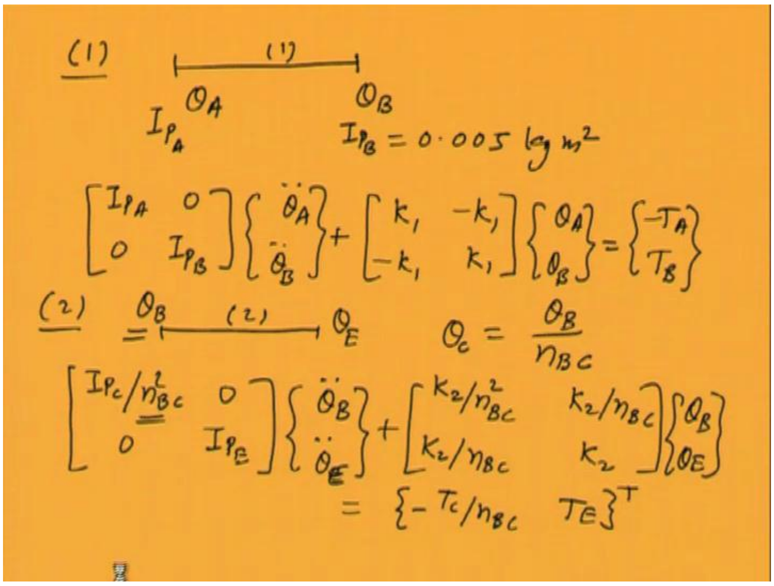

So, let us do it is element 1. A element 1 is in which we have theta A theta B and Ipa Ipb in this I forgot to give the Ipb value that is 0.005 kg meter square. I will be giving other gear of polar mass moment of inertia as I will be going to respective elements. So, the elemental equation will take this form this is the first element k 1 is the stiffness of element 1. And these are the torques internal torques similarly for element 2 we can able to here we have theta B, and here we have theta E or theta C is related with theta B by gear ratio BC. So, we could able to eliminate theta C from this element. So, elemental equation will be Ipc BC square 0. Here theta B double dot is there that is what we have to divide by the gear ratio then the stiffness matrix also will get changed. We have to divide by gear ratio square of first term for second term as it is only n a. And this we have to divide here also we have to divide by nbc. So, remain as it is theta B theta E and right hand side is theta C by nbc and theta E transpose.

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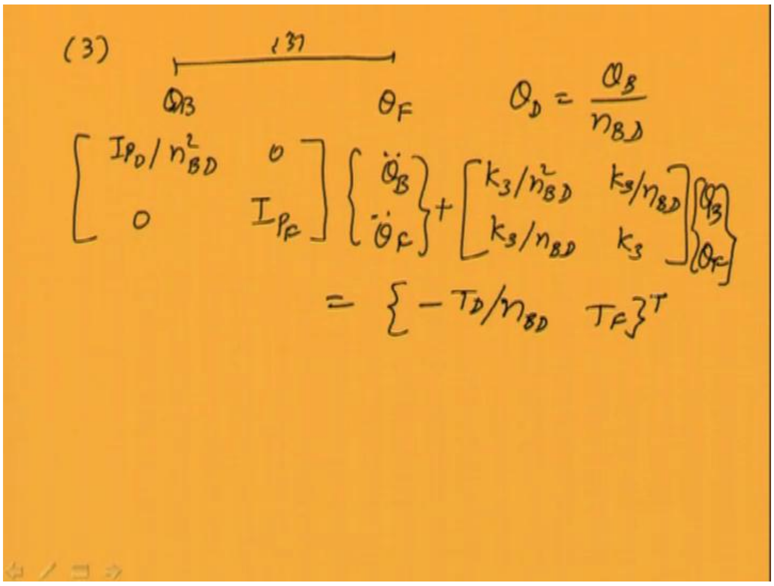

Element 3 is having 1 side theta B again, because we have related the displacement of disc D with B that is D is related with B through BD gear ratio BD here is F. So, the elemental equation for this is now we obtained all the elemental equations. Now, we can able to assemble them. So, this equation also I will be showing in the final thing.

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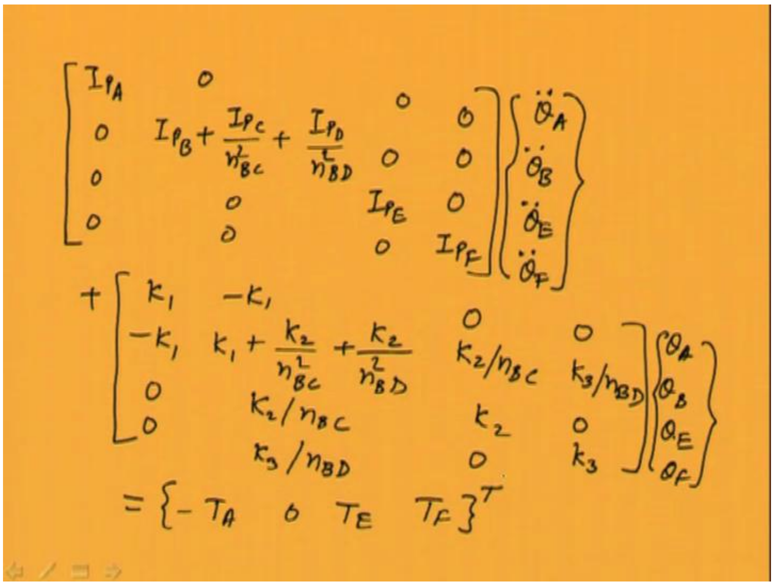

This equation will be showing the final there is no need to show the writing part. Now, after assembly, the equation take this following form now here you can see that we have 4 displacements that is at node ABED only. Now, we can apply the boundary condition here and ((Refer Time 1:04:08)) especially we can see that particular term of the torque get canceled totally. Because when we assemble the matrices those stocks they cancel each other now if we apply the boundary condition we have boundary condition that the end A is free. So, this torque will be 0 and E is also free, so this 0 and F is also free. So, all these torques are 0 we will go back to the here you can see this end E is free end F is free also end A is free. So, torques will be 0 at these ends now we can able to write the previous equation in this particular standard form.

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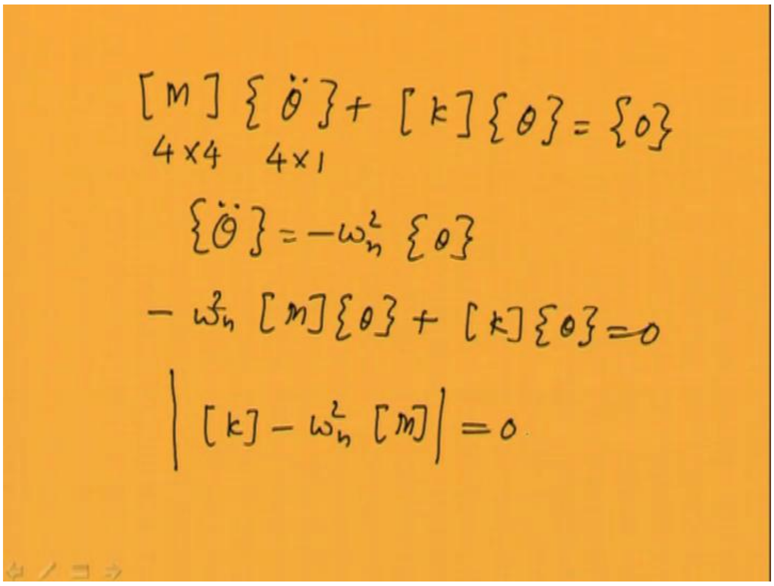

And in this right hand side terms are 0 the size of the matrix is 4 by 4 into 4 into 4. Now, for simple harmonic motion we can write this as... So, the equation becomes and from here if we take the determinant of this quantity 0 will get a polynomial in terms of omega n. And if we solve the these polynomial we will get 3 roots of the omega n.

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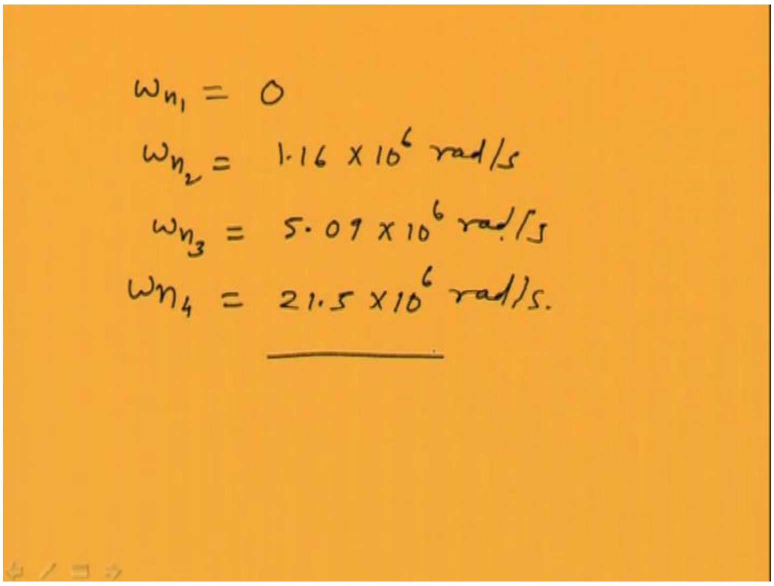

First will be 0; first is 0, because this free end second will be 1.16 10 is to 6 radians per second. Third natural frequency will be 5.09 10 is to 6 radians per second. And fourth natural frequency would be 21.5 10 is to 6 radians per second, because in this particular case we had 4 discs. So, we worked 4 natural frequency at the system for this particular example we had 4 discs. So, because of that we got 4 natural frequency at the system and because ends of the shafts were free. So, we have got 1 natural frequency is 0 which corresponding to the, a rigid body mode of the vibration. So, today we saw the finite element of the analysis of the torsional vibration of a rods we develop the elemental equation for mass and stiffness matrix for a continuous system. And then we demonstrated the finite element procedure even the assembly procedure using examples. Then we took the gear element stiffness and mass matrix development. And that also we demonstrated through example, with these examples I think all the concepts are clear to you.