Mechanical Vibrations

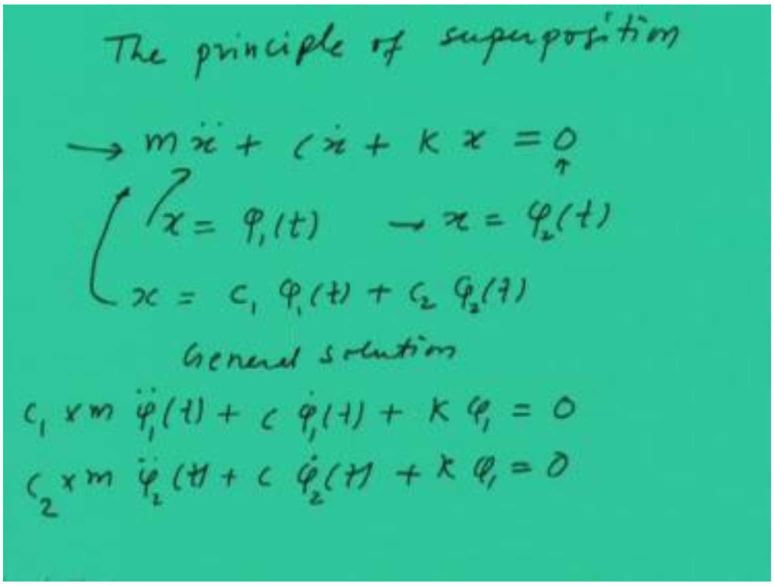
Prof. Rajiv Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 2
Single DOF Undamped Free Vibrations
Lecture - 1
Vibration Model, Equation of Motion - Natural Frequency

Previous lectures we already introduced vibration, its application also. Little bit about various terminologies gradually we introduced regarding the vibrations like free vibration, forced vibration. We have already seen that, the main purpose of the free vibration is to obtain the natural frequency of the system. And that helps us in… when we analyze the forced vibration, so that one of the excitation frequency, is present in the system should not coincide with the natural frequency, so that resonance condition can be avoided. We already mentioned that, damping in the system. They have a little effect on to the natural frequency. Gradually, as we will be going into more deep into the course, we will see… With proper mathematical foundation, we will see how these statements are correct. And especially damping – they help at the resonance condition that, the amplitude… Or they remain in the finite value; otherwise, theoretically, when there is a resonance condition, the amplitude of vibration becomes infinite.

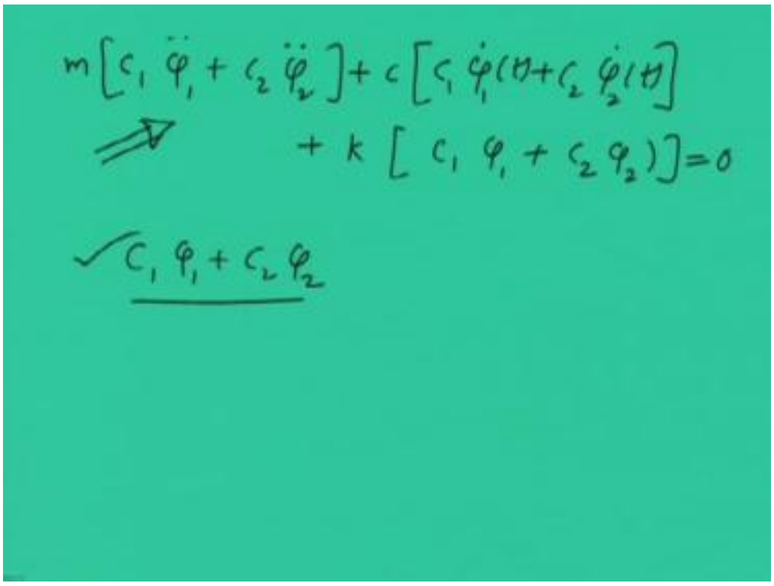
So, today, we will see more deep into the free vibration is we have already explained earlier. We will be mainly concentrating on the linear system. And the advantage of the linear system is we can able to apply the superposition theorem. And about this theorem, let us see specifically for our case how this superposition theorem can be useful. So, first, let us see this theorem how it is not applicable for the non-linear system and how it is applicable for the linear dynamic systems.

(Refer Slide Time: 03:10)



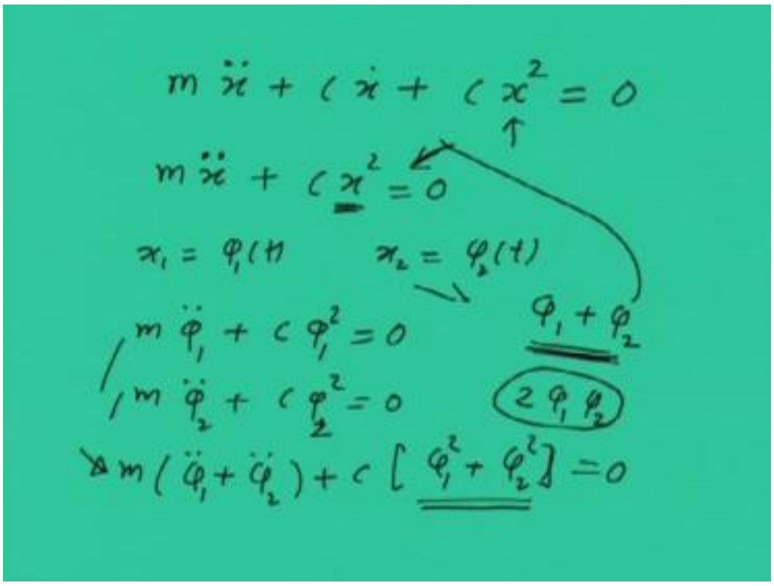
The principle of superposition – let us say we have a dynamic system, a differential equation like this; and because it is having 0 right-hand side. So, let us say we have particular solution of this, because it is a differential equation of second order. So, these are the particular solution – two particular solutions, which of this solution. And we know that, even if we multiply this solution with a constant and if we add them, that is also a solution of the differential equation. And this we call it general solution. Now, we will see how this is a general solution of this. So, let us first satisfy the first solution on to the differential equation. So, we will get this; then we have second term. And so this is the… We have satisfied the first solution. Similarly, we can able to satisfy the second particular solution. So, we will get this. Here dot represent the time derivative. So, these two equations we got.

(Refer Slide Time: 05:34)



If we multiply the first equation with C 1 and second equation with C 2; and if we add them, we will get an equation like this. This is the first two terms; then we have these second terms with first derivatives. And this is the second term with first derivative and… So, in this case, we have just multiplied the first equation with C 1 and second equation with C 2 and we added. So, you can see that, the solution – the general solution, which we assumed of this form; if we substitute this directly in this equation… means if we substitute this directly on this, we will get exactly the same equation as this; that means this is also a solution of the original differential equation. This is for the linear system, because you can see that, all the terms of x – they are having no other power than 1.

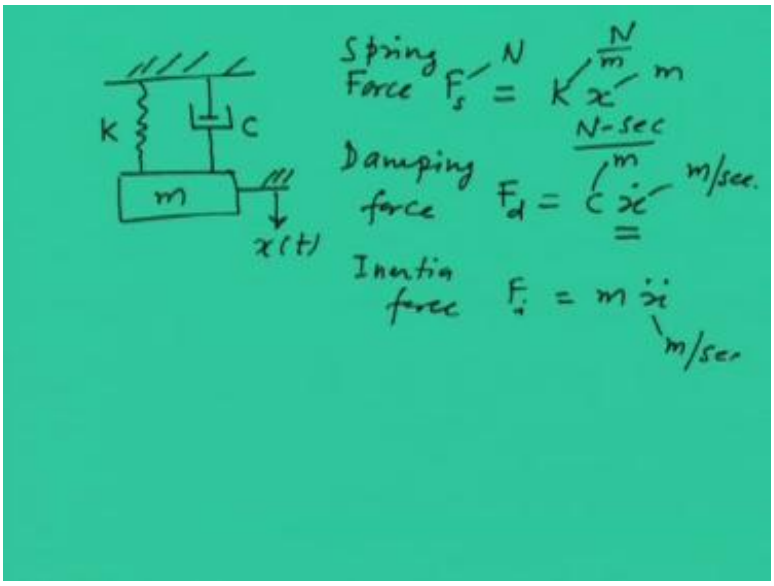
(Refer Slide Time: 07:01)



So, if we have some differential equation in which we have some other power of x like x square; then because of this, we are introducing nonlinearity in the system. Nonlinearity can be there in other terms also like velocity or acceleration. But, in this particular case, we are considering the nonlinearity in this x. Or, to simplify our analysis, let us take only two terms – these two, because this is a second order differential equation. So, let us say we have these, are the two particular solutions of this equation. And if they are solution of these equations or this equation, we can say satisfy them. This is the first solution we have satisfied, second solution. And if we add them – these two as we did previously; let us say those constants we are taking unity here – C 1 and C 2.

So, you can able to see that, the general solution, which we get on these two: phi 1 and phi 2; if we want to substitute this directly here or in this equation, then this term will give us the square of this. But, here you can see that is not square of that; one term is missing; that is, 2 phi 1 phi 2. If we are there here, then we could have got phi 1 plus phi 2 whole square. So, this particular equation, which we are getting from these two, we are not getting from the general solution; that means we are not able to use the superposition theorem here; and because of the nonlinearity, this problem is coming. And we need an extra term to satisfy this condition. So, you can see that, how the non-linear system create problem of using this superposition theorem here.

(Refer Slide Time: 10:17)

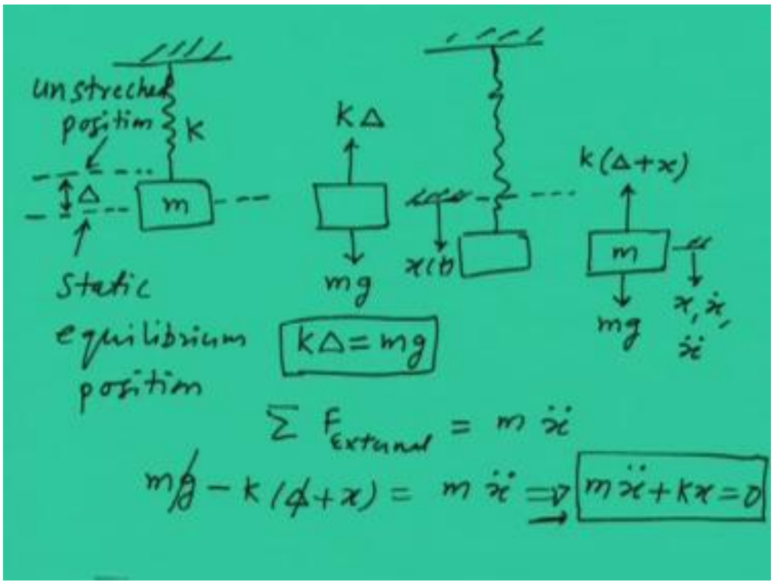


Now, let us consider a most simply vibration mathematical model in which we will be considering a mass, which is suspended by a spring and a damper. I will talk a little bit more about dampers little bit after sometime. But, at present, let us first see the representation of this particular model in which we have one mass is there, which is suspended by a spring; the stiffness of that is K. And there is a damper, which is used to dissipate energy from the system or especially the kinetic energy dissipation take place with the help of this damper. And generally these springs – they give spring force that is F s, which is given as the stiffness of the spring and the displacement, which we are giving to this particular mass. So, whatever the extension or compression of the spring is taking place, the force will be changing proportional to the displacement, because this stiffness is a constant for a particular spring.

Similarly… This is the spring force. Similarly, the damping force, which is very difficult to model mathematically; here we will be taking a very simple model in which the damping force is proportional to the displacement. For the present analysis, we will not consider damping; in the subsequent lectures, we will be considering the damping effect. And the third thing is the mass, which gives the inertia force. So, that inertia force is proportional to the acceleration. From here you can able to make out the… because unit of the force is Newton; displacement is having unit meter. So, this stiffness will be having unit meter or Newton per meter. And this velocity is having meter per second. So, damping will be having unit Newton second per meter. And mass – these are the known quantities – known units for mass and acceleration. Acceleration we know – meters per second square.

Now, we will be obtaining simple equation of motion of a spring mass system in which we are not considering the damping for time being, so that our analysis remains simple. And what is the equation of motion? Equation of motion is nothing, but, the mathematical expression of how the displacement and velocity and acceleration; or, in the other words, how the elastic force, damping force and inertia force are related in a particular dynamic system.

(Refer Slide Time: 13:34)



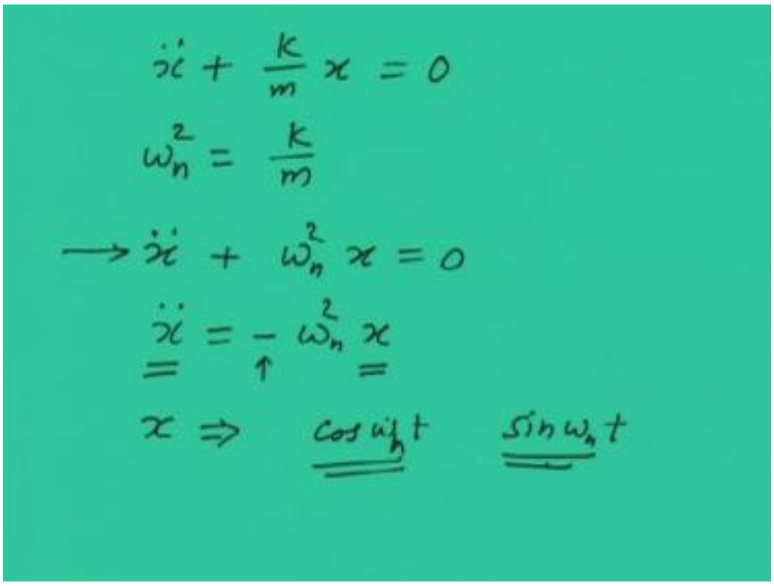
So, in this particular case, let us take a very simple example of a spring and mass system. So, if we hang a mass through a spring like this; obviously, because of the weight of the mass, the spring will get stretched. And let us say this is the position, which is we call it as static equilibrium position. At present, this mass is not oscillating; it is static; but, because of its weight and the spring has some extension, let us say this is the unstretched position of the spring. So, this much extension of the spring has taken place because of the weight of the mass. And this particular position we call it as unstretched position. So, if we take the free body diagram of this mass, which is at present not oscillating; it just in static equilibrium; we will be having mass into g. So, weight of the mass acting downward. And because of the extension of the spring, we will be having K into x as spring force, which will be acting upward. So, from here – from force balance, we can able to see that, K delta is equal to mg.

Now, what we are doing? We are giving a disturbance to this particular mass, so that it gets displaced from its static equilibrium position by some displacement x. So, this is the static equilibrium position; we are giving a displacement to the mass of x t; and this is the spring. So, now, mass is having up and down motion. And for this particular time, it has occupied this position. Now, if we want to obtain the free body diagram of this mass, we have the weight acting downward and because now total extension of the spring is delta plus x, because up to this, there was static extension of the spring. And then x displacement we are giving to the spring. So, total spring force will be K delta plus x. And because now, this particular mass is having a motion; so it will be having velocity and acceleration component also.

So, now, we can apply the Newton's law to obtain the equilibrium equation of this. Newton's second law – it states that, sum of all the external force, which is acting on the body, will be equal to mass into acceleration of the body. So, what are the forces acting? You can see that, we have taken downward direction as positive direction for displacement. So, same convention we will be using it. So, external force mg is acting, which is in the direction of x. This force – spring force is acting upward, that is, negative direction of the x. So, we will put the negative sign here. No other external force is acting on this should be equal to mass of this object into acceleration.

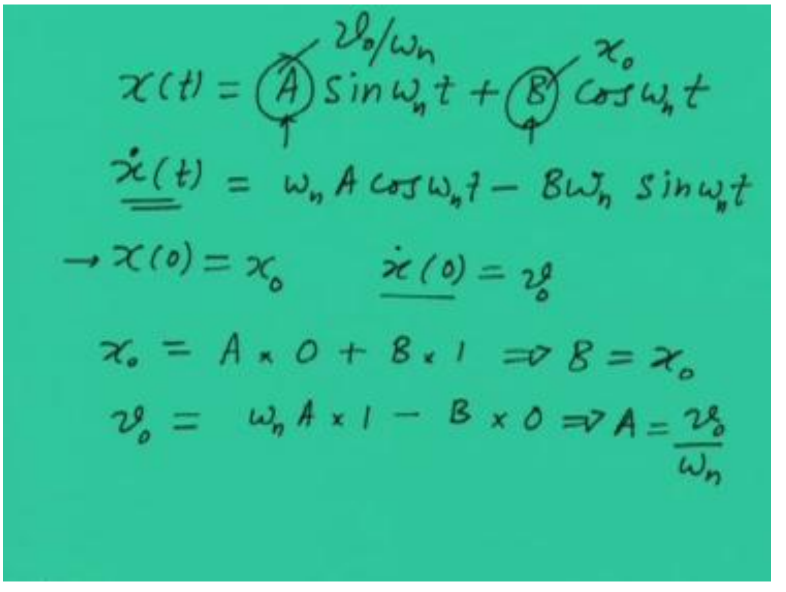
Now, you can see that, we had earlier this expression K delta is equal to mg. So, here K into delta; mg will get cancelled. So, from this, we will get mx double dot. And if we take this term other side, is equal to 0. So, you can see that, this equation we are getting, which is equation of motion of this mass. And in this if you see carefully, the weight of the mass – it is not appearing at all. You can see it is because we have taken the reference position for the displacement from the static equilibrium position – this one. And because of this static equilibrium position as a reference for our displacement x, the gravity effect is not coming in this. This particular equation we can able to write in another form.

(Refer Slide Time: 19:22)



If we divide by mass all sides, we will get K by m x is equal to 0. Generally, we express K by m as omega n square. So, with this, we will be having equation as this. And you can see that, we can able to express this as… So, now, acceleration is proportional to the displacement. We have already seen the definition of the harmonic motion that, acceleration is proportional to the displacement and it acts in the opposite direction. Then, we have the simple harmonic motion. So, in this particular spring mass system also, the displacement will be having simple harmonic motion. And as we know that, we have this simple harmonic functions, that is, cos omega n t and sin omega n t. These are the functions, which will satisfy this particular equation. And because this is a second order differential equation and we have two such functions, we can able to form a general solution out of these.

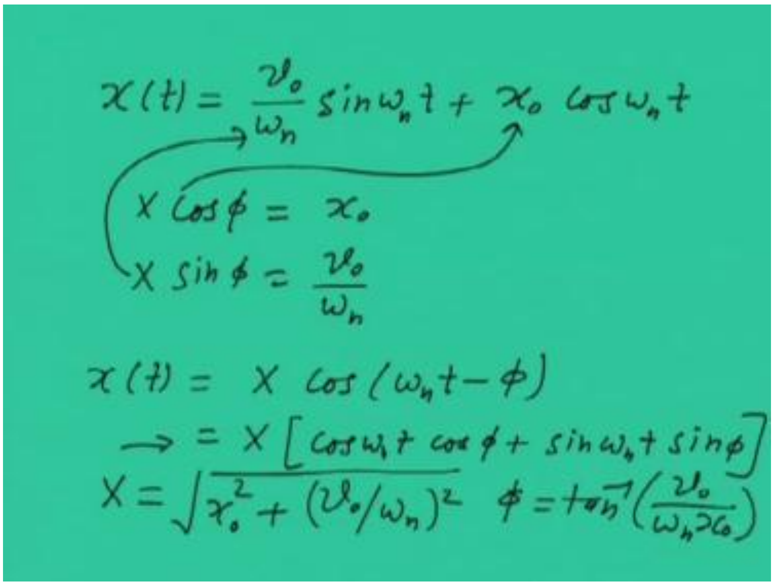
(Refer Slide Time: 20:58)



So, a general solution of this will be A sin omega n t plus B cos omega n t. And omega n as we have seen earlier, that is, the frequency of oscillation of the mass. And here two constants are there: A and B; that we need to obtain through the initial condition. So, initial conditions – we can have displacement at time t is equal to 0 and velocity at time t is equal to 0. Or, if we want to have general solution; if nonzero terms… So, we can express this as x naught and this as v naught. So, if you substitute… For this maybe you can first derive, take the derivative of this once, so that we can get the velocity component also. This is the velocity term.

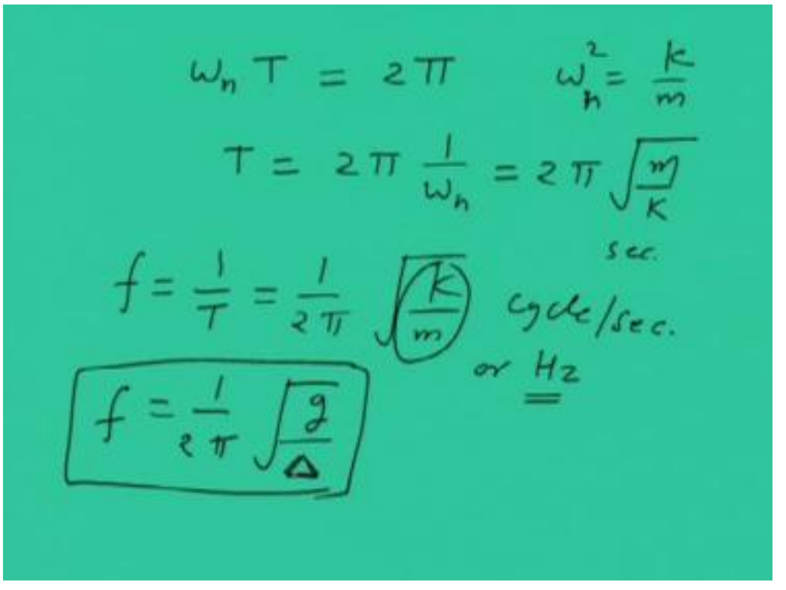
Now, we will substitute this in the first equation. So, we will get 0 is equal to A… This is x naught equal to A… – sin becomes 0 plus B – cos 0 becomes 1. So, here… From here we are getting B is equal to x naught. Then, this if we substitute in the second equation, we will get v naught is equal to omega n A. This becomes 1 minus B; and sin is 0. So, 0. So, from here we can get the A as v naught by omega n. So, we got these two constants; we can able to substitute this in the original equation here. This quantity is nu naught by omega n and this is x naught.

(Refer Slide Time: 23:44)



So, that means we can able to write the solution as nu naught by omega n sin omega n t plus x naught cos omega n t. So, this particular solution we obtained. There is a general solution. This we can able to obtain the motion of the mass. Any initial disturbance we are giving to the mass or in the form of velocity and displacement; how that particular mass execute the motion can be obtained using this expression. And another form of this particular equation we can able to express like this. So, if we write X cos phi as x naught and X sin phi as nu naught by omega n; then we can able to express x t as capital X cos omega n t minus phi, because we know that, if we expand this, this gives us X cos omega n t cos phi plus sin omega n t sin phi. So, if you substitute these quantities, this is here and this is here; that will be same as this expression. And here X is given as… There is x naught square plus v naught divided by omega n square and phi as tan inverse of the velocity – initial velocity – omega n – initial displacement. As we know that, sine and cosine functions – they repeat after every 2 pi radiance; so from this, we can able to get the frequency of oscillation of the mass.

(Refer Slide Time: 26:37)

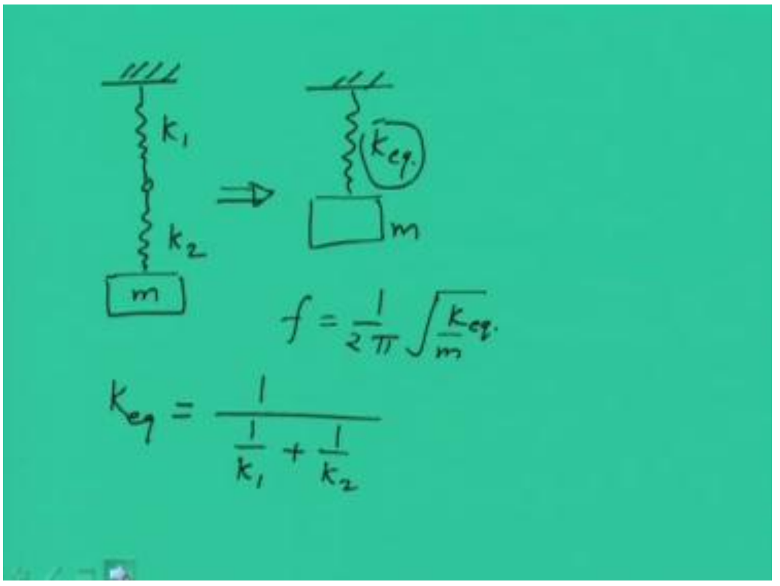


So, we have… There is circular frequency into the time period is equal to 2 pi for the sine and cosine functions, which we have the response. x t we have expressed in the previous case in terms of the sine and cosine functions. So, from here you can see that, T can be written as 1 by omega n. And previously we have defined omega n square as K by m. So, here if we substitute, you will get 2 pi root K by… This will be m by K – m by K. And this is in the second; unit will be second. Similarly, frequency can be given as inverse of this. So, that will be 1 by 2 pi root K by m. And that will be in the cycle per second or hertz. So, it is the unit of the frequency.

And, you can see that, K by m – we can able to replace in other terms; if we go back to the previous slides, we had the relation K delta is equal to mg. So, this can be expressed as K by m. That will be equal to g by delta. So, this expression we will be writing here. That will be 1 by 2 pi g by delta. So, frequency is given like this. So, you can see that, the frequency depends upon the static deflection; delta is the static deflection of the mass. So, we have seen in the expression that, frequency of the mass depends upon the system property, the stiffness and the mass itself. And if the stiffness changes with a system, it is not a property of the material as such directly. But, if we have different systems, if we have beam or we have other kind of springs; so natural frequency will be changing with the… depending upon what kind of system we are considering. Whatever the analysis we have done for a simple spring mass system for single degree of freedom system, this is valid for a large class of dynamic system having single degree of freedom.

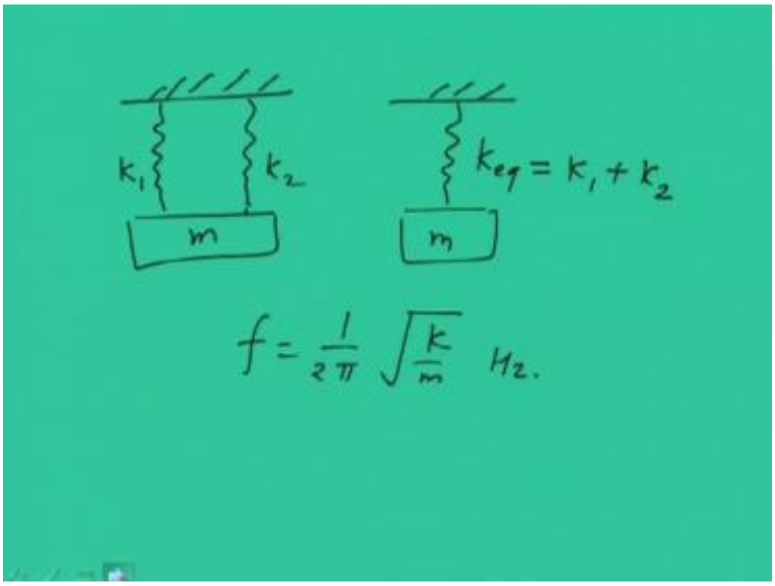
We will see some of the examples how this particular relations, which we have generated can be used for different systems.

(Refer Slide Time: 29:58)



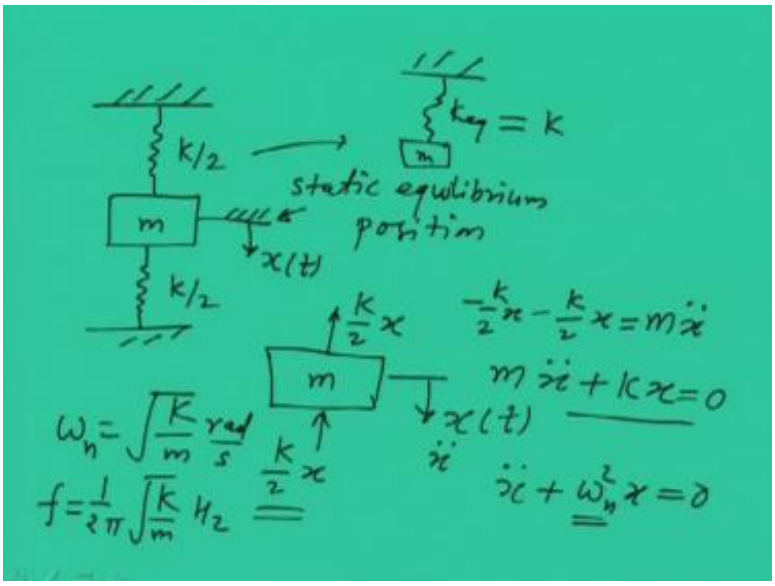
So, let us take one system in which we have a spring K 1. And this is another spring; K 2 is attached below that. And mass m is attached here. So, this particular system if you want to obtain the natural frequency of the system, what we can do; we can find an equivalent stiffness and mass. And if we can obtain this equivalent stiffness, this particular relation, which we generated – we derived earlier can be used. In place of K, now it will be K equivalent. And we know that, these two springs are connected in series. So, their stiffness – equivalent stiffness will be given as…

(Refer Slide Time: 31:13)



And, if we have another system in which springs are connected like this – K 1 and K 2' mass is here. So, in this particular case, also, we can have an equivalent system with K equivalent; where K equivalent now will be given as summation of these two, because they are connected parallel. So, same equation is valid here also, that is, in hertz.

(Refer Slide Time: 31:59)

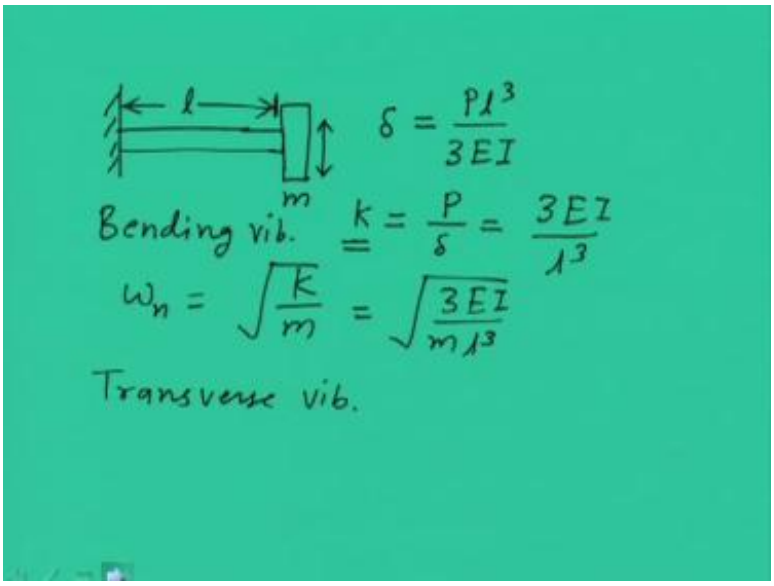


If we have… Let us say there is another system in which mass is there here and it is connected by a spring having stiffness let us say K by 2, and another spring K by 2. So, in this particular case, how we will be obtaining the equation of motion or how we will be obtaining the frequency; for this, maybe what we can do – we can displace this mass from its static equilibrium position. We are assuming that, this particular position is the static equilibrium position. So, if we displace that or we can have the free body diagram of the mass; So, this is the displacement, which we are giving. So, you can see that, the lower spring will get compressed. So, it will give a force upward and the upward spring will get stretched. So, it will give a force again upward direction. And because this mass is now in motion, it is having velocity and acceleration also. So, using Newton's second law, we can able to obtain the equation of motion for this.

So, the equation of motion is summation of all the external forces. You can see these are the external forces, which are acting opposite to the direction of motion. Motion is downward; forces are acting upward. So, we can write minus K by 2 x minus K by 2 x. And these are the two external forces acting; should be equal to mass and the acceleration. So, this will give us equation of motion is, which is exactly same as the previous one. And here if we divide by m, again we can able to express this equation as… And we have already seen that, this is the frequency – circular frequency of the motion, which is we can able to express here as omega n root K by m; the unit of this is radiance per second. And if you want in the hertz, then we have to divide by 2 pi. So, it is the hertz frequency. So, you can see that, how we can able to obtain the equation of motion of this, which is identical to the single spring mass system. So, here if we want to represent this particular system as simple spring mass equivalent system; so equivalent stiffness is equal to K itself. So, even it is looking like they are connected as series; but, they are not connected as such in series, because mass is there in between the two springs.

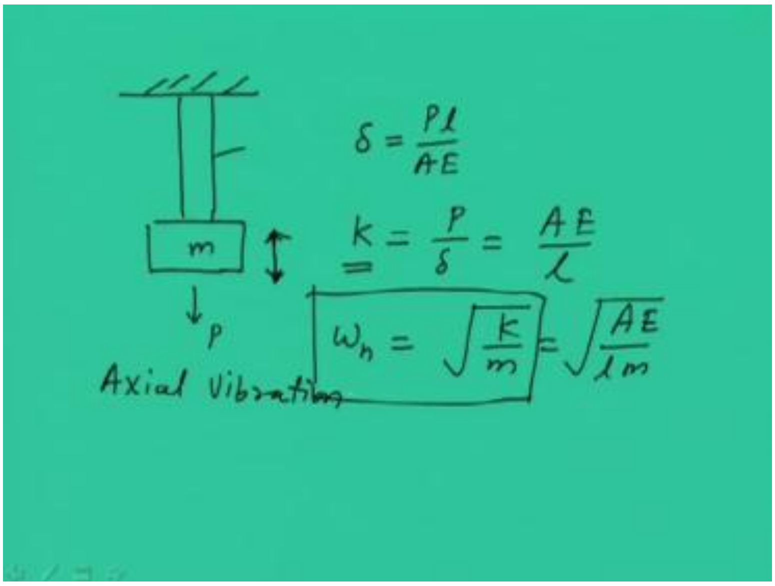
Let us take some more examples especially when we are talking about beams in which bending is taking place or axial rod, when the axial force we are exerting or some other kind of springs like coil springs; how we can able to use this formula for obtaining the natural frequency of the system.

(Refer Slide Time: 35:57)



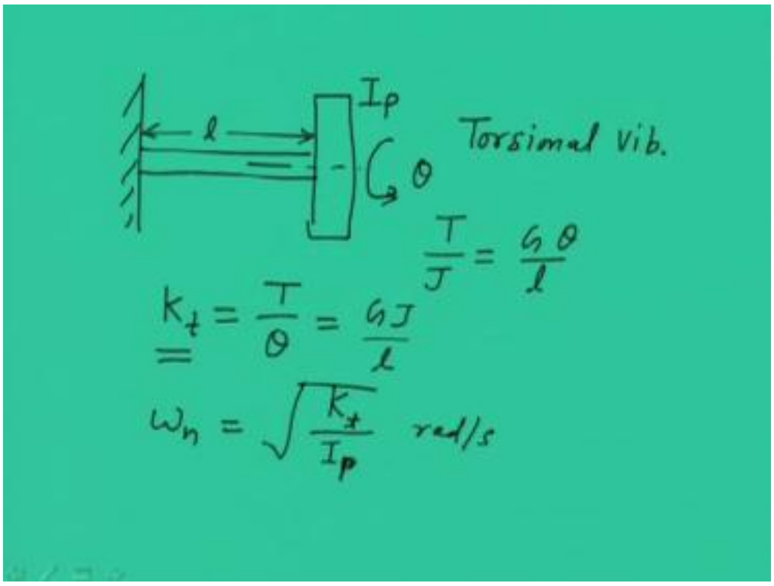
So, let us take one simple cantilever beam and a mass is attached at the free end. So, here we are assuming that, mass of the shaft or the mass of the beam is negligible; only it has flexibility. And this disc, which is having mass m is having no flexibility. And from strength of material concept, we know that, if we apply force P at the free end of the cantilever beam, the deflection at free end can be written like this; where, P is the load; l is the length of the beam; E is Young's modulus; I is the second moment of area of the cross section of the beam. And the stiffness as we know is defined as load divided by deflection. So, using this formula, we can able to write this as 3 EI by l cube. So, once we know this stiffness, which is the equivalent stiffness. So, the natural frequency of the system can be written as… So, this stiffness can be substituted here. So, we will get the natural frequency of this system in which this mass is having up and down motion. This is the… We call it bending vibration or transverse vibration; sometimes we call it transverse vibration.

(Refer Slide Time: 37:57)



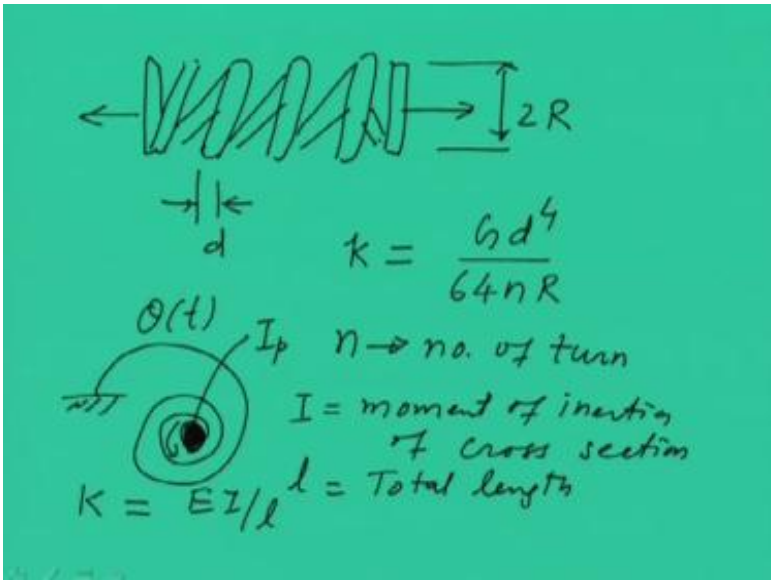
Now, we will see some more examples on this. Let us say there is a rod. And at the tip of the rod, one mass is attached. This rod is having no mass; it is only having flexibility. We are giving this particular mass a disturbance such that it oscillates up and down. For this particular case, we know from strength of material that, if we apply axial force to this particular rod, the deflection of the rod is given like this. And from here we can get the axial stiffness of the rod; that is, P by delta; load by deflection – that will be given as AE by l. So, once we got this stiffness, the natural frequency of oscillation of this mass in this axial direction can be obtained by the same formula. That will be AE – lm. So, you can see that, this particular formula which we derived for simple spring mass system is varied for… So, various class of problems we can have… This is also called axial vibration.

(Refer Slide Time: 39:41)



Now, we will see some other class of problems. Let us say there is a shaft and there is a heavy disc. And we are giving a twist to this disc theta. So, in this particular case, we are giving twist to that disc, so that it oscillates about its axis. And this particular vibration is called torsional vibration. And this again from strength of material formulae, we have relation that, torque – J – G theta by l; G is the torque; J is the polar moment of area; G is the modulus of rigidity; theta is the angular twist of the disc about this axis; l is the length of the rod. And from here now we can define the torsional stiffness of the shaft as torque divided by angular twist, is similar to the linear stiffness in which we have force divided by linear displacement. Here we have got torsional stiffness. So, torque divided by the angular displacement; that is can be given as GJ by l. So, once we have that torsional stiffness, the natural frequency of the system will be torsional stiffness divided by a polar mass moment of inertia of the mass. So, polar mass moment of inertia of the mass is I p. So, here you can see instead of linear stiffness, torsional stiffness is coming – K t, that is, subscript; should clarify this; K t; t is the subscript. It is representing the torsional stiffness. I p – I subscript p is the polar mass moment inertia of this disc about this axis. So, here instead of mass, now, polar mass point of inertia is coming; but, unit remains the same – radiance per second.

(Refer Slide Time: 42:24)

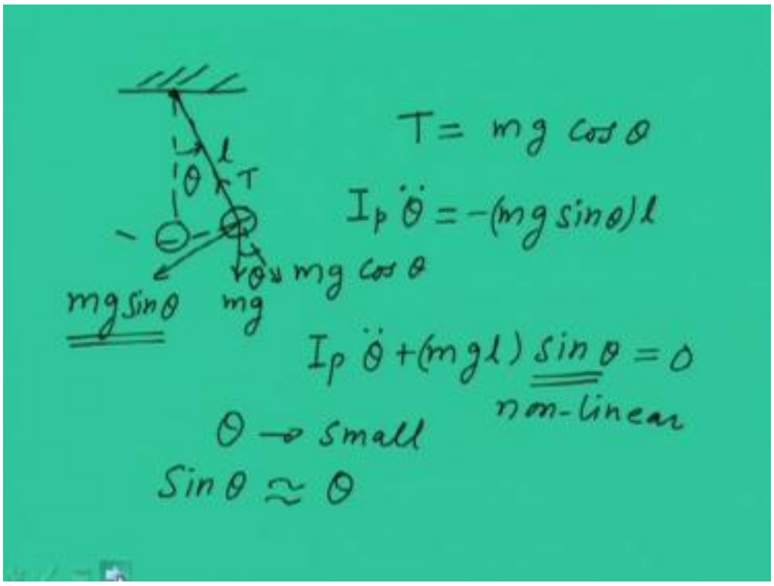


Now, we will take few more examples specifically for various kind of springs. Let us see one kind of spring, which is having various number of turns. And this spring – if we are stretching it; so we have various dimensions like coiled by which this spring has been made; the diameter of that is small d; the mean radius of the spring is 2R. And for this, we know that, the stiffness is given in terms of this geometry and the n; n is the number of turns in the spring. So, using this relation, we can get the stiffness. And from there we can obtain the natural frequency of any particular mass. Another kind of spring, which we deal with coil spring; which is something like this; which generally we used earlier days in watches.

So, here we tighten up the spring from here and we store the energy in the spring and the particular dimensions like whatever the I of this particular strip cross section; there is a moment of inertia of cross-section area; and l is the total length of this strip. So, with this dimension, we can able to obtain the stiffness of this as EI by l. And once we know the stiffness, we can get the natural frequency of the system. If we have some mass here having polar mass moment of inertia, we can able to know how much torsional natural frequency it will be having if it is having oscillation about this point. So, oscillation about this point will be… that is, we will be expressing as a theta.

Now, let us say we will consider a most simplest form of the mechanical system, which has oscillation, that is, a pendulum. We will obtain the equation of motion of a pendulum. Actually this equation of motion of the pendulum in general form is a non-linear equation. We will see how these non-linear terms come; and how we can able to linearize those equations, because we know that solution of the non-linear equations are difficult. So, we will be linearizing those equations and we will be obtaining the natural frequency of the linear system.

(Refer Slide Time: 46:13)

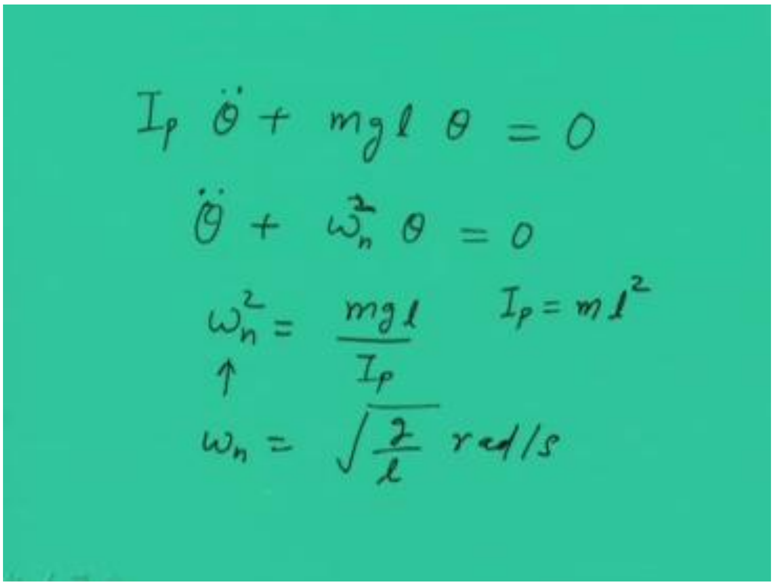


So, let us take a very simple example of a pendulum, is the general position of the pendulum. The weight of the pendulum – the bob is acting here downward. This is the length of the cord from here to here. This is the angle – angular position of the pendulum. So, if we take the component of this mass in the direction of the cord, because we know this particular angle is theta; so this will be mg cos theta. And perpendicular to this, there is a tangential to the path will be mg sin theta

Now, to get the equation of motion, what we will be doing it here? We will be… Let us first balance the… This particular mass – if you are taking the mass free body diagram, what will happen? We will be having a tension in the cord and the tension in the cord, because this particular bob is having no motion in the direction of radial direction. So, if we take the force balance, T will be equal to mg cos theta. Directly in the radial direction, T is equal to mg cos theta. But, because it is having a motion in the tangential direction; so there we have to apply the Newton's second law of motion. So, that is I p – polar mass moment of inertia of the bob about its rotation into the angular acceleration should be equal to external moments. So, external moment is coming from this force. And you can see that, it produces a moment opposite to the theta direction. So, that moment will be negative mg sin theta into – l is the length of the cord. So, that will be the moment. So, this gives us equation of motion of the pendulum.

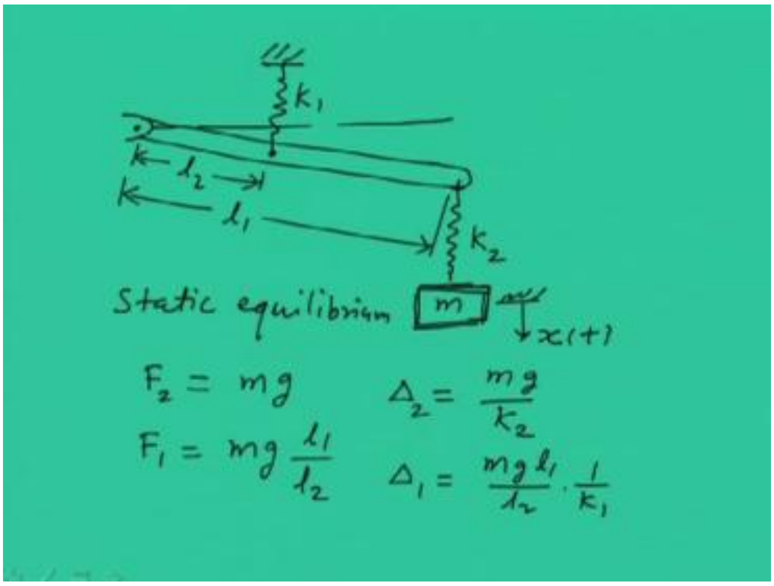
And, you can see that this particular term, which is sin theta, is a non-linear term, because the sin theta can be expressed in terms of theta in polynomial form. So, obviously, it will be having higher degree terms of theta. So, this is a non-linear term. And the solution of this is just impossible. So, what we assume that, theta – whatever the displacement we are giving to this pendulum is small. And if theta is small, then sin theta can be approximated as theta. So, if we do this, this equation reduces to… We will go to the next slide.

(Refer Slide Time: 49:57)



I p theta double dot plus mgl theta is equal to 0. So, we can see this equation is having similar form as we had earlier for spring mass system. And now, omega n is omega n square is this. So, this is the natural frequency of the pendulum. Or, we can able to simplify because I p we know; that will be m into l square, because radius of ((Refer Slide Time: 50:38)) will be l; m is this. So, if we substitute this here, we will get omega n as g by l, which is in radiance per second. If we want in cycle per second, we have to divide by 2 pi.

(Refer Slide Time: 51:16)

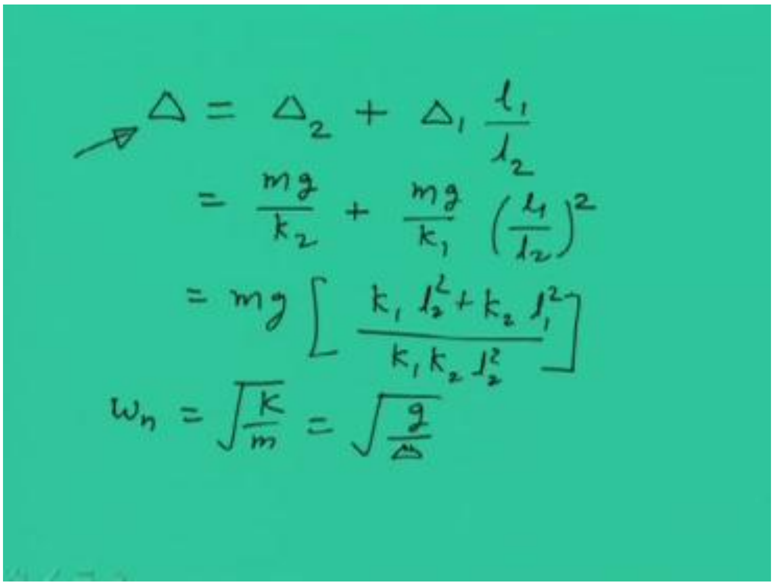


To have better understanding of how to obtain the natural frequency of a particular system, especially for single degree of freedom system; now, I will be taking another example. The example is in which we have one rigid member like this. At the end of that, one spring is attached and mass is suspended. The stiffness of this is K 2. There is another spring here having stiffness K 1. The length of the rod from here is l 2. And total length is l 1.

Now, we are interested in finding the natural frequency of this particular mass if we give some disturbance to this. We are neglecting the mass of this particular member. We will be neglecting the mass of this particular member. And this is rigid; there is no deformation of this particular member. This particular deflection, which we have taken is from the static equilibrium position of this system; that means we are allowing to deform these springs to the extent possible due to the gravity. And then whatever the position of the mass is there; that is our reference point for measurement of the displacement x, which is time dependent. So, this mass will be osculating about its mean position. This static equilibrium position is basically mean position of oscillation of this mass. So, only mass is there on this object and rest of the other object is having no mass. And only two springs are connected. So, now, our aim is to obtain the natural frequency of the system. So, for this static equilibrium position, what will be the force acting on the spring 2? On spring 2, we will be having force is equal to weight of the mass. And what will be the force acting on to the spring 1? That will be when we are applying mg force here, how much force it is getting at K 1 we can able to calculate, because we know the our dimensions l 1 and l 2. So, that will be mg l 1 by l 2.

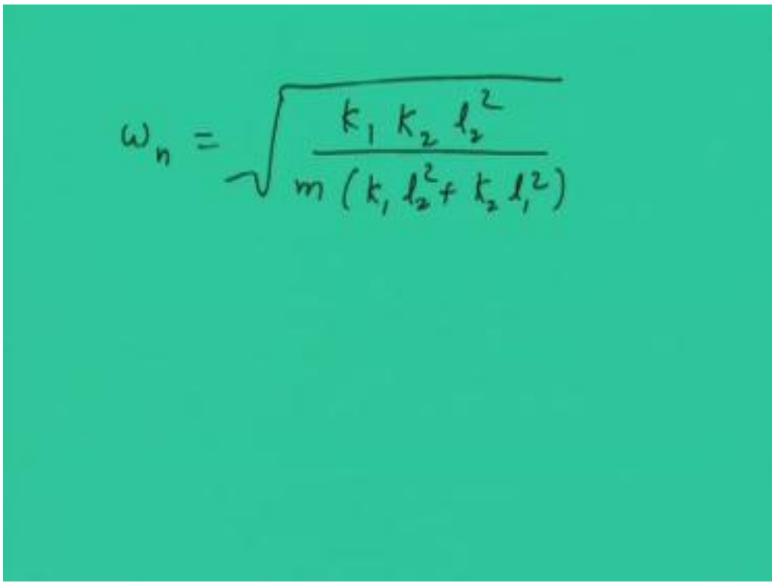
And, now, once we know the load, the extension of the spring K 2 can be obtained. We know that, this spring is getting mg load; we know the stiffness of that. So, this much extension the spring 2 is getting. Similarly, we can get the extension of the spring 1. That will be load acting divided by the stiffness K 1. And now, we want how much this particular mass is getting displaced due to delta 2 and delta 1 displacement. You can see that, whatever the delta 2 displacement is taking place, mass will be having same displacement. But, when we are talking about delta 1 displacement here, that is, the stretch of the spring K, equivalent displacement at this end we need to obtain. At the free end, we need to obtain. So, the total extension…

(Refer Slide Time: 55:42)



The total displacement of the mass due to two extensions will be delta 2 plus delta 1 and l 2 by l 1, because it will get magnified at the free end because of these lengths. Now, we can substitute the values of these here and we can get the total static deflection of the mass m. And this can be simplified as this. Here we are interested in this static deflection directly, because that is another way of obtaining the natural frequency of the system, because we know the natural frequency of the system is given by K by m or g by delta. So, this particular delta we have already obtained. Now, if we substitute this g here, we can get the natural frequency of the system directly.

(Refer Slide Time: 56:59)



So, natural frequency of the system will be this. So, you can see that, here we have not obtained the free body diagram of this member. But, directly we obtained how much static deflection of this particular mass is taking place. And based on that, we could able to get the natural frequency of this system.

So, today we started with simple free vibration analysis of a spring mass system. And we have seen that, even the simple analysis of a simple spring mass system is valid for a large class of vibrations problems. And including the transverse vibration, torsional vibration, axial vibration and even the concept of the equivalent stiffness, which is applicable to the single degree of freedom system covers a large class of problems. And we have already seen that, instead of obtaining the equivalent stiffness, if we can get the static deflection of the particular mass; that will also help us in finding the natural frequency of the system. In the subsequent class, we will see some other methods of obtaining the natural frequency for similar kind of problems especially based on the energy methods.