

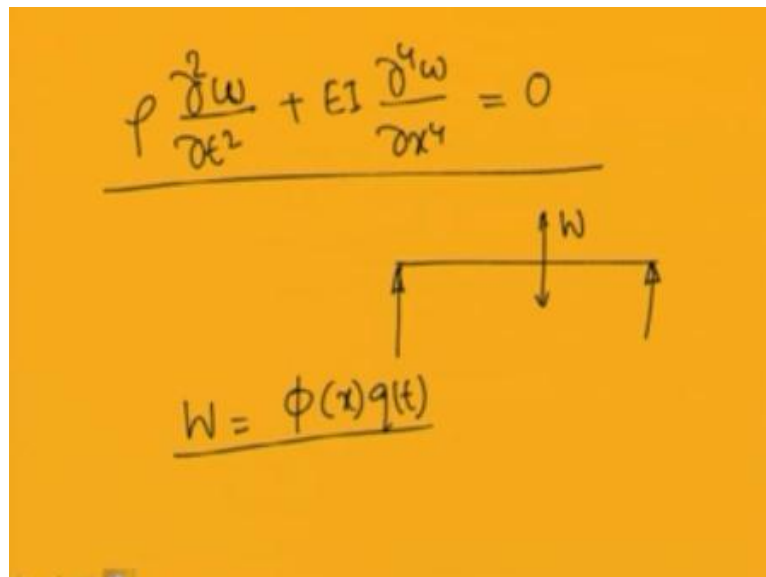
Mechanical Vibrations
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Module - 9
Continuous Systems Closed Form Solutions
Lecture - 5

Transverse Vibration of Beams: Equations of Motion and Boundary Conditions

In last class, we have studied about the vibration of continuous system in which, we have taken the Euler-Bernoulli beam. And we have found the mode shapes for fixed-fixed beam, simply supported beam and cantilever beams. So, in those cases, we have seen the vibration of the beam can be written in terms of this Euler-Bernoulli equation.

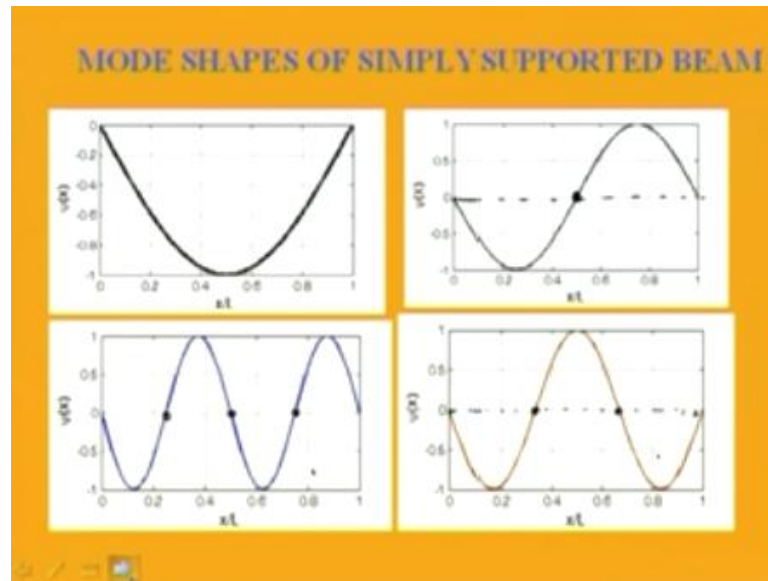
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The image shows a handwritten slide with a yellow background. At the top, the Euler-Bernoulli equation is written as $\rho \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$. Below the equation is a diagram of a beam of length L supported at both ends by upward arrows. A downward arrow labeled w is shown in the middle of the beam. At the bottom, the deflection is given as $w = \phi(x)q(t)$.

That is $\rho \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$. So, in this case this is the governing equation, that is the Euler-Bernoulli equation, here ρ is the mass for unit length EI is the flexural rigidity of the system w is the deflection, so if this is a cantilever beam. So, in this cantilever beam, so the vibration will take place in the transverse direction. So, this transverse direction deflection I have taken it equal to w . So, this w is a function of x and t and, so you have found this w , we have written it in terms of $\phi(x)$ and $q(t)$. So, $\phi(x)$ is the mode shape and $q(t)$ is the time modulation of the system. So, for simply supported case we have seen the, mode shape in this form.

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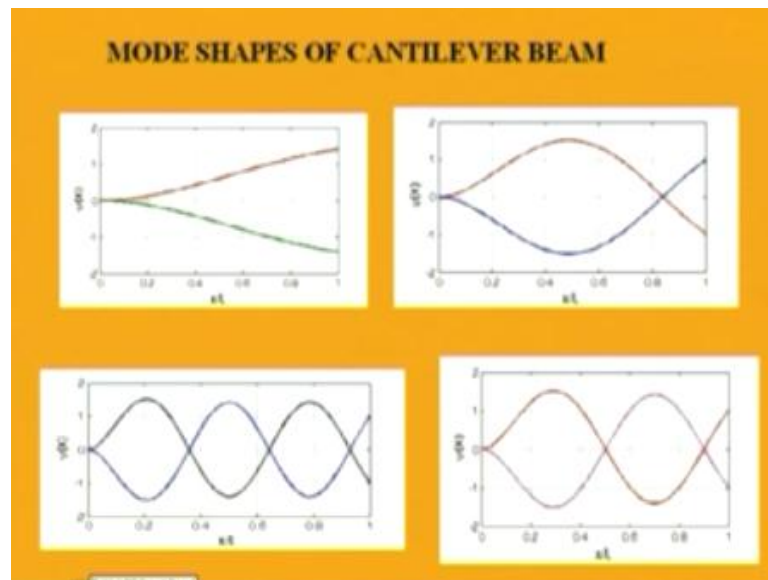
So, I have shown the mode shape, this is the mode shape for the first mode, this is for the second mode, this is for the third mode, and this one is for the fourth mode. So, you can observe that there is a node at $l/2$ or at the middle of this in case of a simply supported beam, when it is vibrating with first mode. So, in case of first mode, there is no node and in case of second mode, there is a single node from here. So, this is the beam, so in the first mode, this is the vibration of the system and the second mode this is the vibration of the system. So, this is along the length of the beam and here you can find at this point, there is no vibration. So, this point node, this point is the nodal point, so initially for the second third modal. So, this is the line and you have this and this point will have the 0 vibration. Similarly, in case of fourth mode, these are the points, where there will be 0 vibrations. So, you can see the simulation of this by using, you can write a simple program in matlab to find these things. So, in the program you can write the expression for this ϕx .

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$$\begin{aligned}\phi(x) &= A \sin \frac{n\pi x}{L} \\ \omega &= \beta^2 l^2 \sqrt{\frac{EI}{\rho L^4}} \quad \beta l = \frac{n\pi}{L} \\ W(x,t) &= \phi(x) q(t) \quad q(t) = C_1 \sin \omega t + C_2 \cos \omega t \\ W_n(x,t) &= \left(A_n \sin \frac{n\pi x}{L} \right) (C_1 \sin \omega t + C_2 \cos \omega t)\end{aligned}$$

So, already you have derived, this $\phi(x)$ equal to $A \sin n \pi x$ by l and this ω equal to $\beta^2 l^2$ into root over EI by ρL^4 . And in case of simply, supported beam, you have found this βl value equal to $n \pi$ by L . So, using this expression, you can write a simple code in Math lab to simulate the vibrations of the simply, supported beam. So, in this case you can write this vibration $W(x,t)$ will be equal to $\phi(x)$ into $q(t)$. So, $q(t)$ is nothing but this is the $C_1 \sin \omega t$ plus $C_2 \cos \omega t$, so where C_1 and C_2 depends on the initial conditions. So, this $W(x,t)$ you can write equal to A , so for n th mode, you can write this $W_n(x,t)$ equal to $A_n \sin n \pi x$ by l into $C_1 \sin \omega t$ plus $C_2 \cos \omega t$, so this A in term is arbitrary. So, you can normalize, this term to get the actual value of $A = 1$. So, let me show the simulation for this, so you can write this expression and you can simulate this value.

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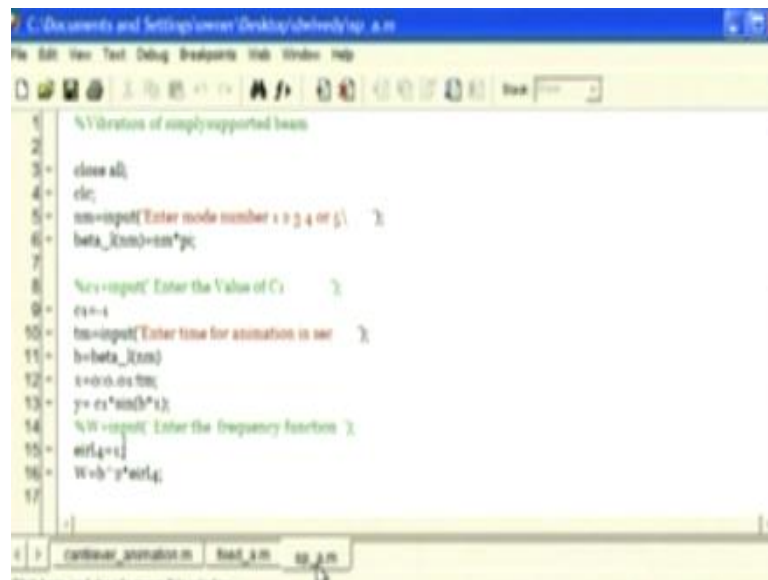
So, let us, do the simulation for this case.

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```
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2  
3  
4 %Vibration of Cantilever beam  
5  
6 close all;  
7 clear;  
8 nn=input('Enter mode number (1,2,3,4 or 5) ');  
9 beta_1(1)=1.875;  
10 beta_1(2)=4.694;  
11 beta_1(3)=7.853;  
12 beta_1(4)=10.996;  
13 beta_1(5)=14.137;  
14 %c1=input('Enter the Value of C ');  
15 c1=1;  
16 tti=input('Enter time for animation in sec ');  
17 b=beta_1(nn);  
18  
19
```

So, in this case of a simply, supported beam, so this is for...

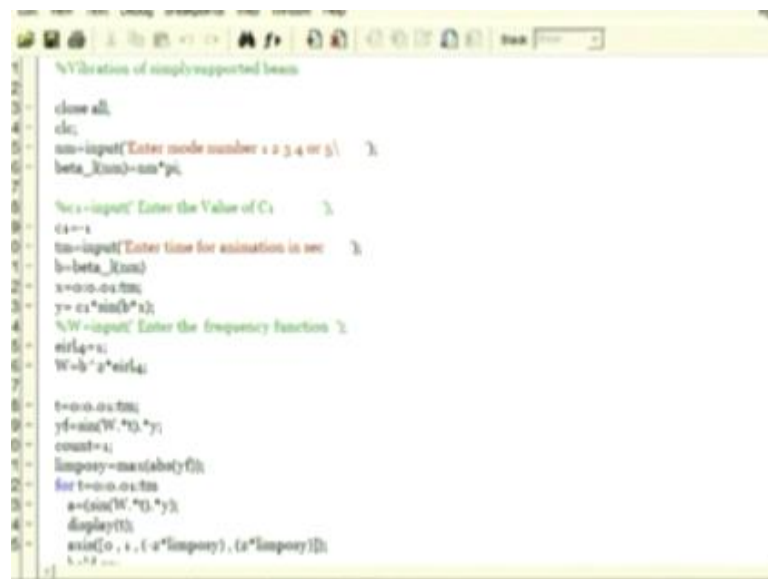
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```
1 %Vibration of simply supported beam
2
3 close all;
4 clc;
5 nm=input('Enter mode number : 1 2 3 4 or 5) ');
6 beta_l(nm)=nm*pi;
7
8 %C1=input(' Enter the Value of C1 ');
9 C1=1;
10 tm=input('Enter time for animation in sec ');
11 b=beta_l(nm);
12 t=0:0.01:tm;
13 y=C1*sin(b*t);
14 %W=input(' Enter the frequency function ');
15 w1=1;
16 W=b^2*w1;
17
```

A simply, supported beam the program is written for a simply supported beam.

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```
1 %Vibration of simply supported beam
2
3 close all;
4 clc;
5 nm=input('Enter mode number : 1 2 3 4 or 5) ');
6 beta_l(nm)=nm*pi;
7
8 %C1=input(' Enter the Value of C1 ');
9 C1=1;
10 tm=input('Enter time for animation in sec ');
11 b=beta_l(nm);
12 t=0:0.01:tm;
13 y=C1*sin(b*t);
14 %W=input(' Enter the frequency function ');
15 w1=1;
16 W=b^2*w1;
17
18 t=0:0.01:tm;
19 yf=sin(W.*t.*y);
20 count=1;
21 limposy=max(abs(yf));
22 for t=0:0.01:tm
23     a=(sin(W.*t).*y);
24     display(t);
25     axis([0 , t , (-4*limposy) , (4*limposy)]);
26     t=t+0.01;
27 end
```

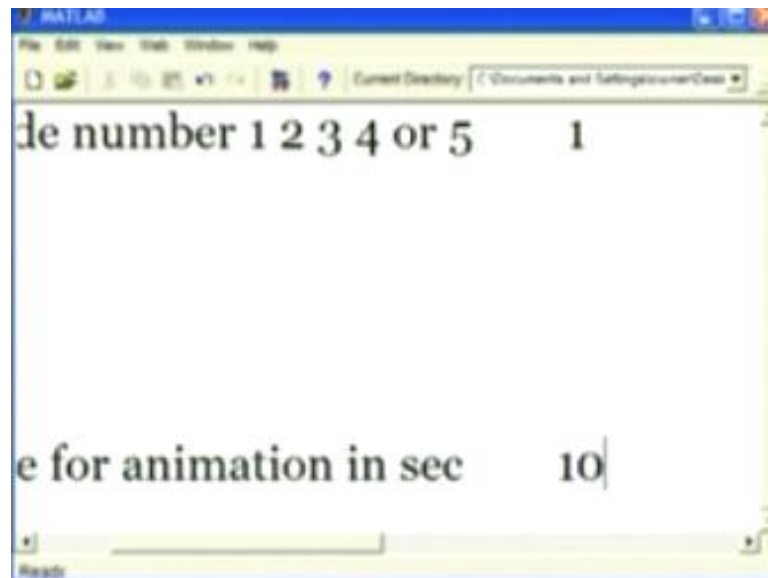
So, in this case, you can see here in this program, I asked for the number of mode. So, whether it is first second or third fourth and, so the beta l value for nth mode is known to us. So, for beta l equal to n pi, so this beta l for nth mode will be n pi and we can write taking this EI by rho l fourth equal to 1, you can find this W function.

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```
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11- b=beta_k(x);
12- z=0.05;
13- y=c1*sin(b*x);
14- %W=input('Enter the frequency function ');
15- w1g=x;
16- W=b*z*w1g;
17-
18- t=0.0:0.05:1;
19- yf=sin(W.*t).*y;
20- count=1;
21- limposy=max(abs(yf));
22- for t=0.0:0.05:1
23-     a=sin(W.*t).*y;
24-     display(t);
25-     axis([0, 1, (-1*limposy), (1*limposy)]);
26-     hold on;
27-     % plot(x,a);
28-     grid on;
29-     plot(x,a,'LineWidth',3);
30-     pause(0.0001);
31-     %M=getframe(gcf);
32-     plot(x,a,'LineWidth',3);
33-     clf
34-     count=count+1;
35- end
```

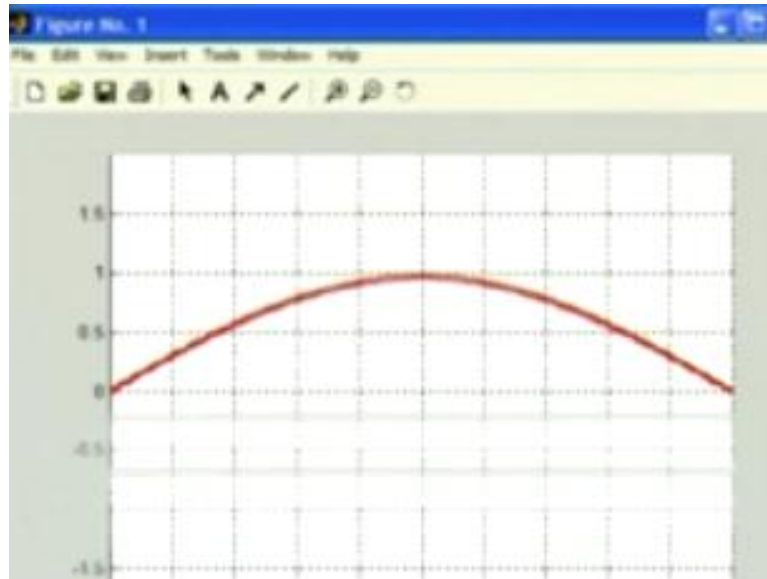
This W is written in terms of yf this is $\sin \omega t$ and this is the mode shape $C_1 \sin \beta x$. So, to simulate this thing, so you just run this and check.

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So, for different modes, you can find for different mode, let us find it for the first mode. So, for the first mode, so if you give the time let us simulate it for 10 second.

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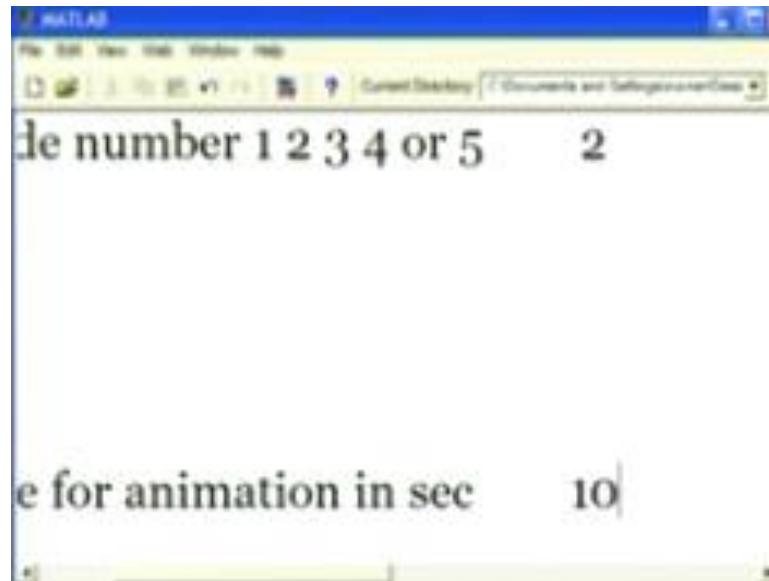
And you can see for the simply, supported case, so the vibration takes place like this. So, at these 2 points the vibrations are 0. So, at these 2 points it is supported, so similarly, we can find for the second mode. So, for the second mode...

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```
11- b=beta_x(x);
12- x=0:0.01:1;
13- y=c1*sin(b*x);
14- %W=input('Enter the frequency function \');
15- w14=1;
16- W=b^2*w14;
17-
18- t=0:0.01:1;
19- yf=sin(W.*t.*y);
20- count=1;
21- limpor1=max(abs(yf));
22- for t=0:0.01:1
23- a=sin(W.*t.*y);
24- display(t);
25- axis([0,1,(-1*limpor1),(1*limpor1)]);
26- hold on;
27- % plot(x,a);
28- grid on;
```

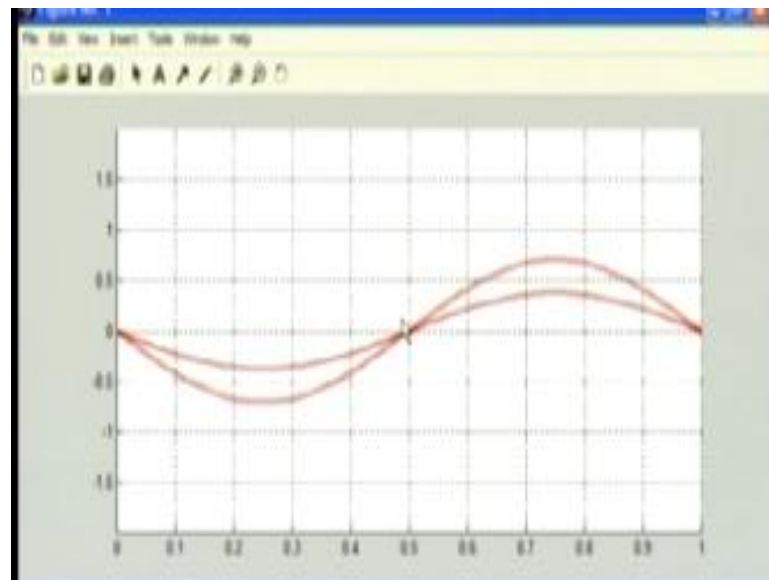
So, for the second mode you can find it.

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So, let us, take for the second mode and you can observe that. So, for the second mode we can find, so let us run it for this.

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So, in case of second mode, you can observe that at this point at 0.5 there is no vibration of the beam. So, there is a no vibration of the beam at $x = 0.5$ in case of the second mode and similarly, you can find for the third mode. So, for the third mode case.

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```
11- b=beta_x(tau)
12- t=0:0.001:0.01
13- y=c1*sin(b*t)
14- %W=input('Enter the frequency function:')
15- w1=q=c;
16- W=b^2*w1*dq;
17-
18- t=0:0.001:0.01
19- y1=sin(W*t)^2;
20- count=1;
21- impory=ones(size(y));
22- for t=0:0.001:0.01
23- a=(sin(W*t)^2);
24- display(t);
25- axis([0, 1, 1, 1, 1, 1]);
26- hold on;
27- % plot(x, y);
28- end on;
```

For this case, let us see for the third mode. So, in this case...

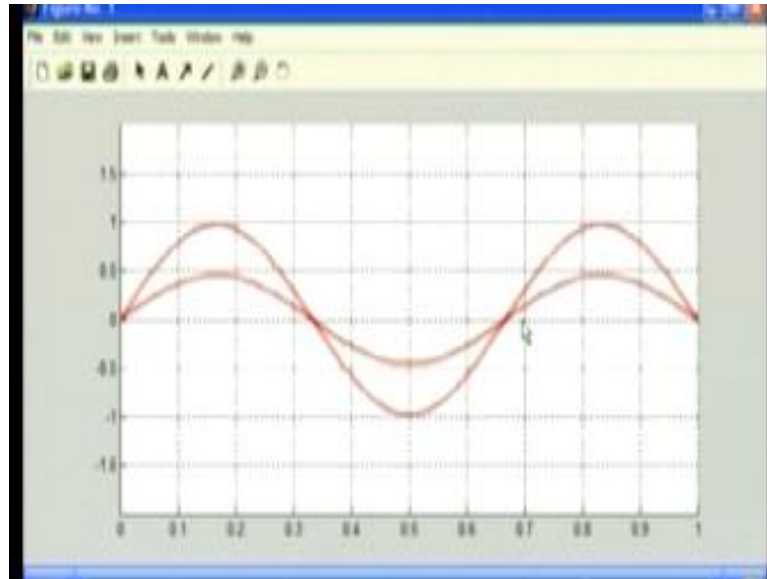
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```
the number 1 2 3 4 or 5      3

for animation in sec      10
```

I am writing it for third, so if you simulate it for...

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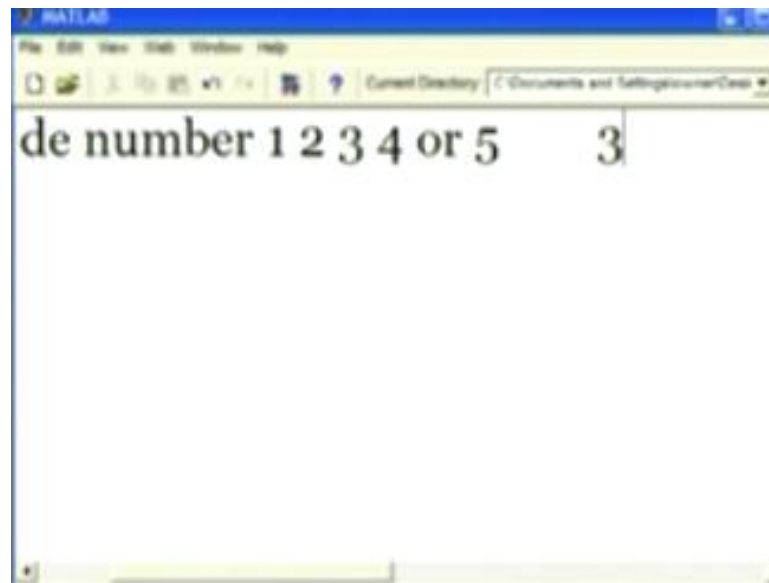
You can see clearly, you can visualize that there are 2 nodes present in this case. So, at this 2 nodes that is no vibration and the beam vibrates in $\sin n \pi x$ by 1. Similarly, you can simulate it, for forth mode and other modes also. So, let us see the simulation for the case of a cantilever beam.

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```
1) Vibration of Cantilever beam
close all;
clc;
n=input('Enter mode number (1,2,3,4) ');
beta_1(1)=0.873;
beta_1(2)=4.694;
beta_1(3)=7.853;
beta_1(4)=10.996;
beta_1(5)=14.137;
t=input('Enter the Value of t ');
t=1;
t=input('Enter time for simulation in sec ');
l=beta_1(n);
z=0:l:l;
t=0:0.01:t;
```

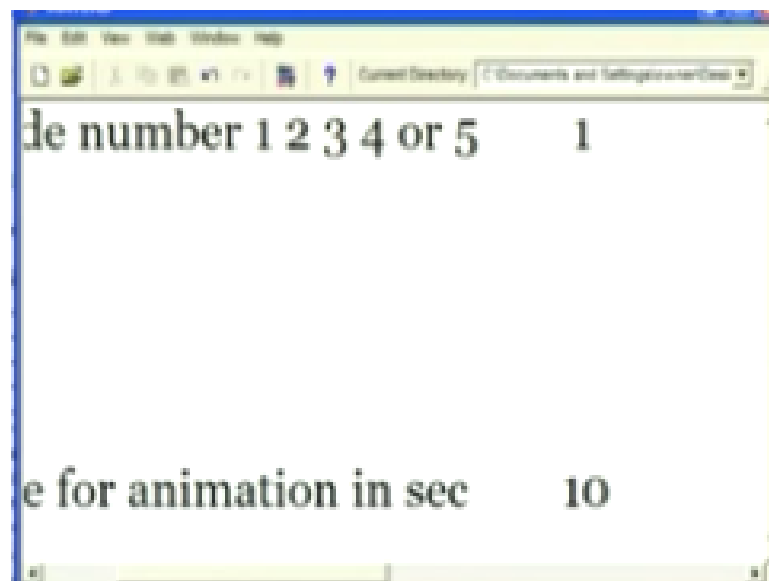
So, in case of a cantilever beam, you can find the vibration.

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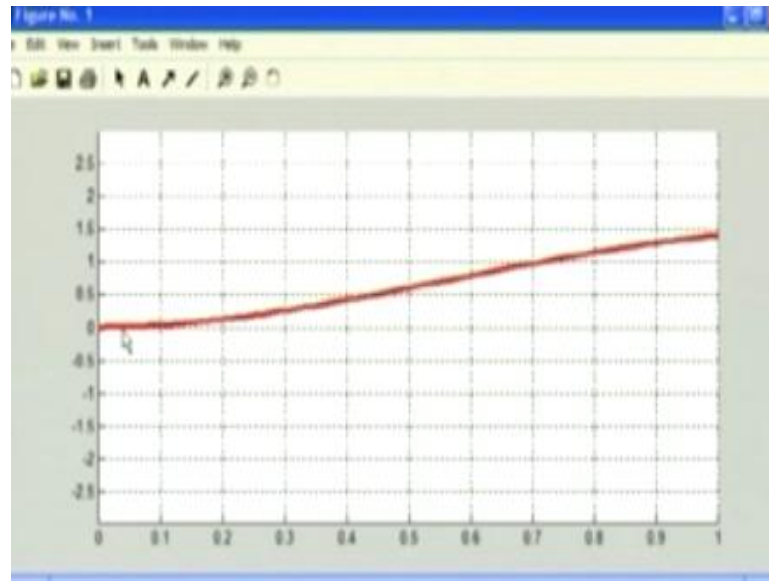


So, for the first mode, let us see say for the first mode. So, in case of the first mode...

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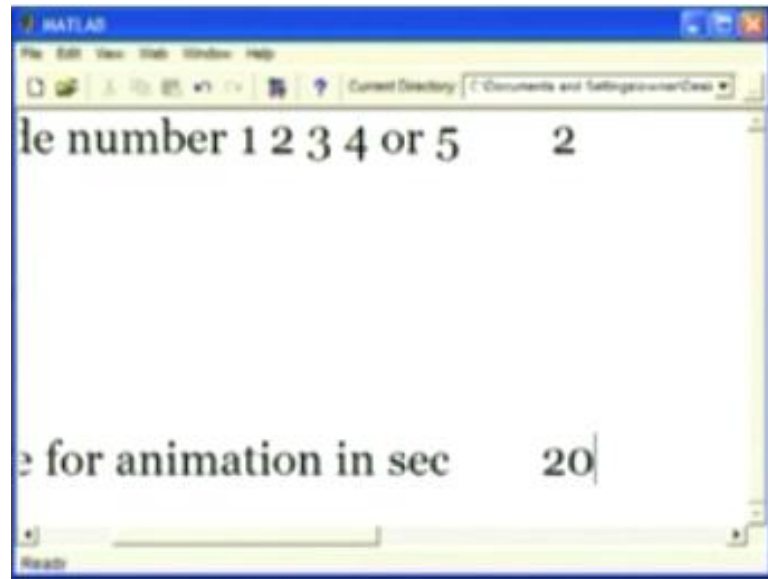
So, you can see that the free end is moving up and down. And there is maximum deflection in the free end and in the fixed end you can observe up to certain extent, there is no vibration of the free end. So, up to certain extent the vibration in case of free fixed end, its slope will be equal to 0. Similarly, for the second mode.

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```
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4  
5 %Vibration of Cantilever beam  
6  
7 close all;  
8 clc;  
9 m=input('Enter mode number (1,2,3,4 or 5) ');  
10 beta_x1=1.875;  
11 beta_x2=4.694;  
12 beta_x3=7.855;  
13 beta_x4=10.996;  
14 beta_x5=14.137;  
15 %c=input('Enter the Value of C ');  
16 c=1;  
17 t=input('Enter time for animation in sec ');  
18 h=beta_x(m);  
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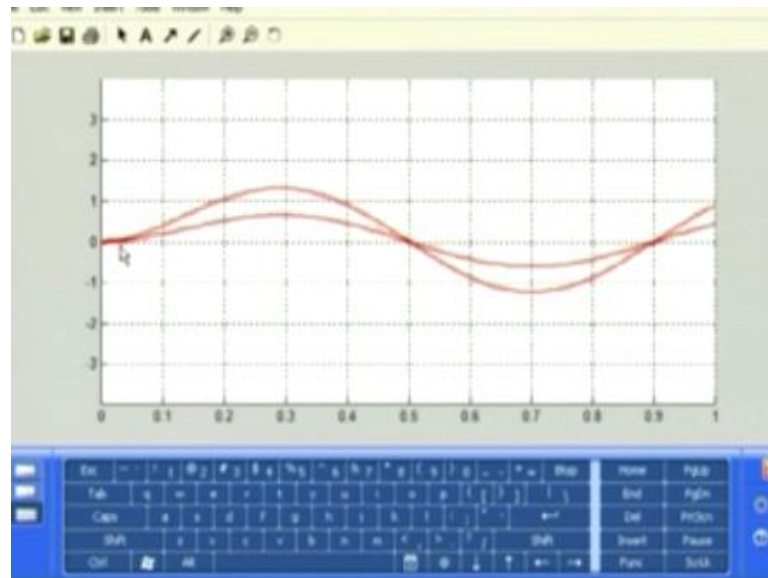
So, if you want to find for the second mode in case of a simply, supported beam. So, let us find.

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For the second mode, so in this case you can. So, in this case, you can observe that the beam is vibrating. So, that is a node formation here and you can similarly, simulate for the third and fourth mode in this case. So, in case of third mode, you can observe the error 2 nodes present in case of third node third mode of vibration, so in that case.

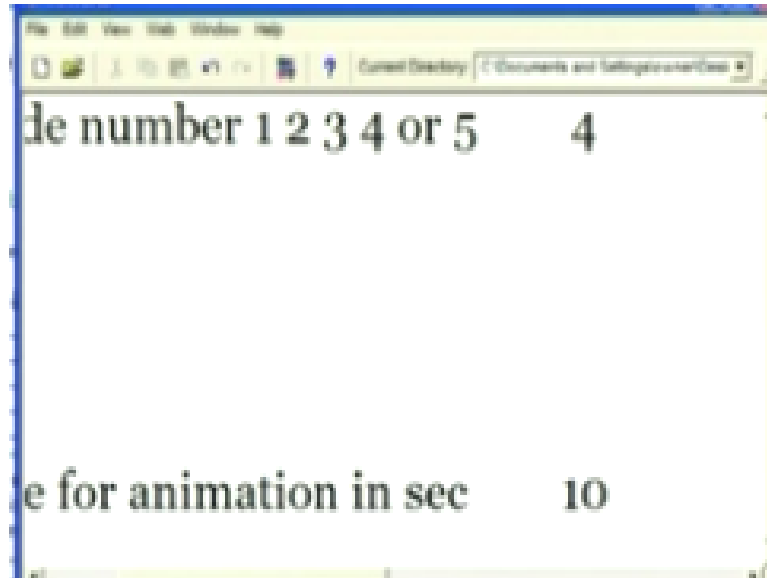
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So, you can clearly, observe there are 2 nodes 1 node is here, and other node is here in case of third mode of vibration in case of a cantilever beam and in the fixed end, you can

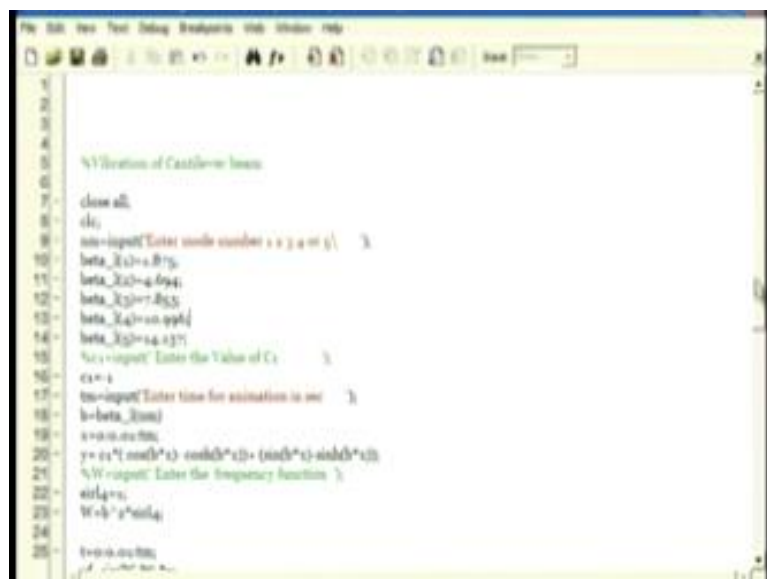
clearly observe the slope is 0 and displacements are also 0. So, to see the forth mode, so in that case also you can find.

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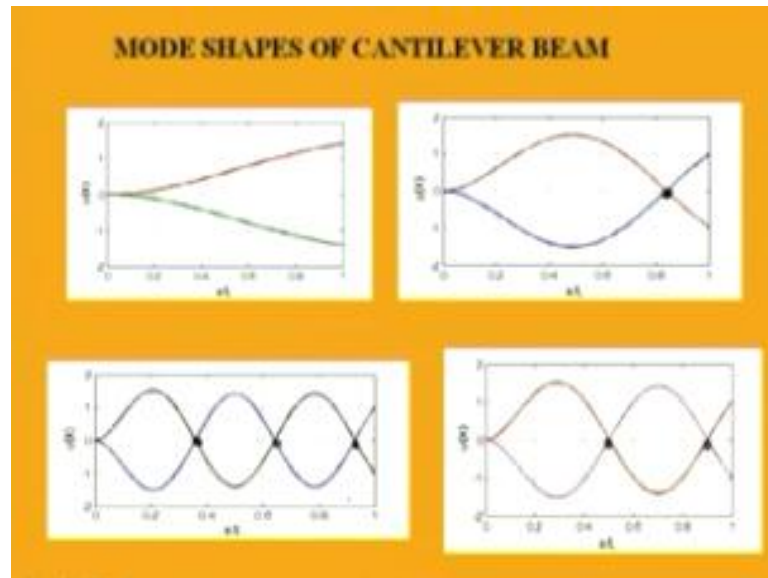
So, in case of forth mode, so clearly you can observe 3 nodes are found in this beam in the cantilever beam. Similarly, for higher nodes n minus 1 number of nodes, you can find and the free end will vibrate with a maximum and the fixed end that is 0 deflection and the slope is 0 also in this case.

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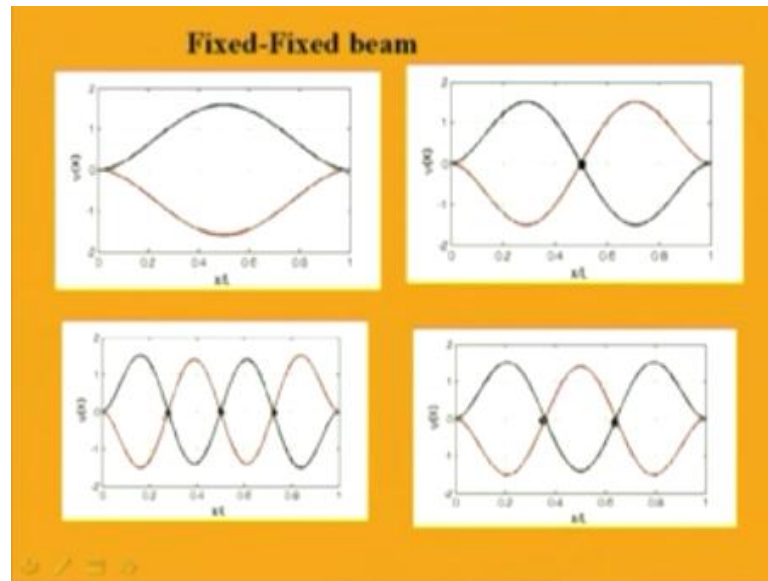
So, in case of a cantilever beam, you can observe that first frequency correspond to beta 1 that is equal to 1.75 the second modal characteristic function. That is beta 1 2 that equal to 4.694. And for the third mode, it is equal to 7.853 and fourth mode, 10.996 and fifth mode, it is equal to 14.137. So, taking those value, so you can simulate for different modes and you can find.

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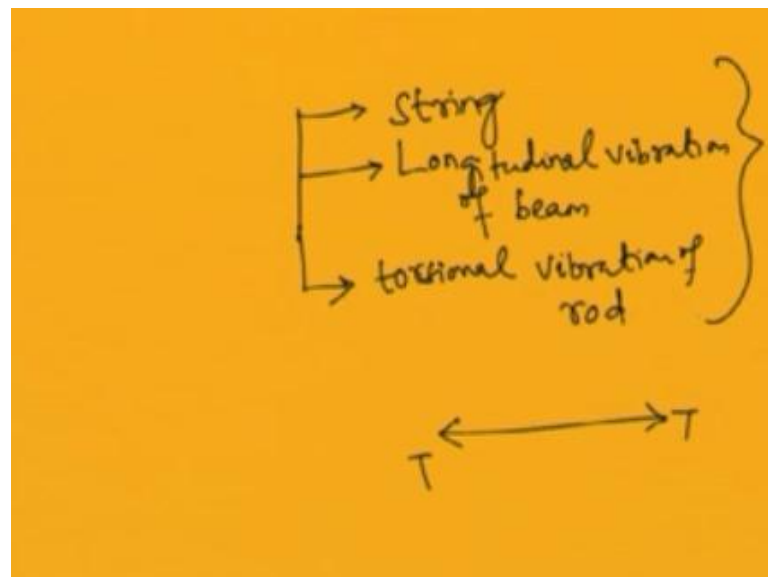
The mode shapes and the response of the system. So, in case of cantilever beam already, you have seen that the vibration. These are the mode shapes this is for the first mode and this is for the second mode and in the second mode a mode is found at this position. In case of third mode, there are 2 nodes and in case of fourth mode, there are 3 nodes present in this beam. So, in these cases, you have found that for nth mode there will be n minus nodes, that is at n minus node n minus 1 points on the beam there will be no vibration.

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In case of fixed-fixed beam also if you find, so this is at the mode shapes. So, this is this is a Euler-Bernoulli beam. So, in this case, both the ends are fixed and slopes are 0 here and in case of second mode, you can see there is 1 mode in case of third mode that is 2 nodes. And in this case you have 3 nodes present, so the systems represented by wave equations are.

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So, we have studied about 3 systems in the first system. So, you have studied, about the string. So, the torque string, so in this case the strings in tension, so the vibration of the

string in tension and the longitudinal vibration of beam, longitudinal vibration of beam and the torsional vibration of rod.

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$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$c = \sqrt{\frac{T}{\rho}} \quad \rho = \text{mass/length}$$

$$c = \sqrt{\frac{E}{\rho}}$$

$$c = \sqrt{\frac{G}{J}}$$


So, these 3 systems, you can write the equation motion for 3 systems by using this wave equation that is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, so in these cases, so the C. So, in case of a torque string C equal to or equal to $\sqrt{\frac{T}{\rho}}$, where rho is the mass per unit length. So, mass per unit length and in case of this longitudinal vibration of rod C equal to $\sqrt{\frac{E}{\rho}}$ and in case of torsional vibration of rod it is equal to $\sqrt{\frac{G}{J}}$. So, here in case of, so in all these cases, you can check that the C represent the represent a velocity term and already, you have solved this wave equation and you have found.

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$$u(x,t) = \phi(x) \cdot q(t)$$
$$\left. \begin{aligned} \phi(x) &= A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x \\ q(t) &= C_1 \sin \omega t + C_2 \cos \omega t \end{aligned} \right\}$$

The response $u(x,t)$ can be written in this form $u(x,t) = \phi(x) \cdot q(t)$, where this $\phi(x)$ you have written as $A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x$. And $q(t)$, you have written in this form $q(t) = C_1 \sin \omega t + C_2 \cos \omega t$. So, for different boundary conditions, today will find the mode shapes for the systems. So, let us first take the system of a string for this case for the fixed-fixed string.

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The diagram shows a horizontal line representing a string of length L , fixed at both ends. A vertical double-headed arrow indicates the displacement $u(x,t)$ perpendicular to the string.

$$u(0,t) = 0$$
$$u(L,t) = 0$$
$$\phi(0) = 0$$
$$\phi(L) = 0 \quad \phi(x) = A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x$$
$$\phi(0) = A + B \cdot 0 \Rightarrow A = 0$$
$$\phi(L) = B \sin \frac{\omega}{c} L = 0$$
$$\sin \frac{\omega}{c} L = 0$$

We have already, found the mode shapes and in this case, when it is fixed as both ends are fixed in this string. So, the displacements are 0, so $\phi(0) = 0$ and $u(0,t) = 0$ and $u(L,t) = 0$.

also equal to 0. So, as $u(0, t) = 0$ and $u(L, t) = 0$ for all time, so you can write this $\phi(0) = 0$ and $\phi(L) = 0$. So, when $\phi(0) = 0$ in that case $A \cos(\omega c x) + B \sin(\omega c x) = 0$. So, at $x = 0$, this will be equal to 0, so this is expression for $\phi(x)$. So, I can write this $\phi(0) = 0$, so this is equal to $A \cos(0) + B \sin(0) = 0$ implies this $A = 0$.

Similarly, $\phi(L)$ will be as $A = 0$ $\phi(x)$ becomes, $B \sin(\omega c L) = 0$. So, you can write this is equal to $B \sin(\omega c L) = 0$. So, as $\omega c L \neq 0$, so this equal to \sin , so B will not be equal to 0 as $B = 0$ when $B = 0$ $\phi(x)$ will become, 0. So, the case will be a trivial solution of the system. So, as we are interested for the non trivial solution of the system is, when the beam when the string is vibrating as we are interested to see the vibration of the string. So, in that case, we should $B \neq 0$. So, as $B \neq 0$. So, our frequency equation will be $\sin(\omega c L) = 0$.

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$$\frac{\omega c L}{c} = n\pi$$

$$\omega = \frac{n\pi}{L} c$$

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{T}{\rho}}$$

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

So, as $\sin(\omega c L) = 0$, so this $\omega c L$ will be equal to $n\pi$ or ω will be equal to $n\pi$ by L into C . So, for this case of string you found this as $\sqrt{\frac{T}{\rho}}$. So, ρ is the mass per unit length and T is the tension in the string. So, if this is the string and this is the tension tension in the string, so this ω , so as you have seen this is n . So, it depends on the number of modes, so as you will go on increasing this n , so you will have different values of ω . So, this can be written as

the n th modal frequency of the system. So, the n th modal frequency of the system can be written as $n\pi$ by l into root over T by ρ . So, the modal or the mode shape of the n th frequency. So, $\phi_n(x)$ can be written as.

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$$\begin{aligned}
 & \text{Diagram of a string of length } l \text{ fixed at both ends.} \\
 & u(0,t) = 0 \\
 & u(L,t) = 0 \\
 & \phi(0) = 0 \\
 & \phi(L) = 0 \quad \phi(x) = A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x \\
 & \phi(0) = A + B \cdot 0 \Rightarrow A = 0 \\
 & \phi(L) = B \sin \frac{\omega}{c} L = 0 \\
 & \sin \frac{\omega}{c} L = 0
 \end{aligned}$$

So, $\phi(x)$ equal to $A \cos \omega$ by C , so already you have seen this A equal to 0, so $\phi(x)$ equal to $B \sin \omega$ by C x .

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$$\begin{aligned}
 & \frac{\omega l}{c} = n\pi \\
 & \omega = \frac{n\pi}{l} c \\
 & \boxed{\omega_n = \frac{n\pi}{l} \sqrt{\frac{T}{\rho}}} \\
 & \underline{\phi_n(x) = B_n \sin \frac{n\pi x}{l}}
 \end{aligned}$$

So, for n th mode, I can write this equal to $B_n \sin n\pi x$ by l , so ϕ_n equal to $B_n \sin n\pi x$ by l , so for different modes, as this ϕ_n equal to $B_n \sin n\pi x$ by L .

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$$\phi_n(x) = B_n \sin \frac{n\pi x}{L} \quad \text{--- ①}$$
$$\int_0^L \phi_n(x) \rho(x) \phi_n(x) dx = 1 \quad \text{--- ②}$$
$$\rho B_n^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx = 1$$
$$\tilde{P}' M \tilde{P} = I$$

So, for different mode this B_n is different, so for first mode this is B_1 , second mode B_2 and n th mode it is B_n . So, this is similar to the mode shapes, you are you obtain in case of multi degrees of freedom systems. So, in that case you have normalized that mode shapes and you have found the normalized mode shape of that system. You can write this n th mode amplitude equal to 0 or you can write this $\phi_n(x) \rho(x) \phi_n(x) dx$, so 0 to 1, so this should be equal to 1. So, ϕ_n , so you can normalize this mode shape by using this formula, similar to the things. You have done in case of the multi degree of freedom system here; you have taken the weighted modal matrix and found this. So, let P is the weighted modal matrix. So, you have found this thing P weighted modal matrix transpose $M P$.

So, you have seen this equal to or you have found this equal to I , so in this case, you have got the diagonal term is 1. So, similarly, by making this normalization, you are making the mode shapes. So, you can normalize the mode shapes, of the n th mode, so to normalize this thing. So, you can write this $\phi_n(x) \rho(x) \phi_n(x) dx$ equal to 1. So, for the string this mass per unit length, I can take it constant. So, this is equal to 1, so this is equal to ρ . So, this mass per unit length equal to ρ I can take it out, similarly, substituting this equation this equation in the second equation, this first equation in the second equation. I can write ρ into $B_n^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx = 1$. So, when I am normalizing this thing, so I can write ρB_n^2

integration 0 to l sin square n pi x by l dx equal to 1. Already, you know the sin square theta.

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$$\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)$$

You can write in this form half 1 minus cos 2 theta, so this integration, previous integration.

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$$\begin{aligned} \phi_n(x) &= B_n \sin \frac{n\pi x}{L} \quad \text{--- (1)} \\ \int_0^L \phi_n(x) \rho(x) \phi_n(x) dx &= 1 \quad \text{--- (2)} \\ \rho B_n^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx &= 1 \quad \text{--- } \tilde{P}'M\tilde{P} = I \end{aligned}$$

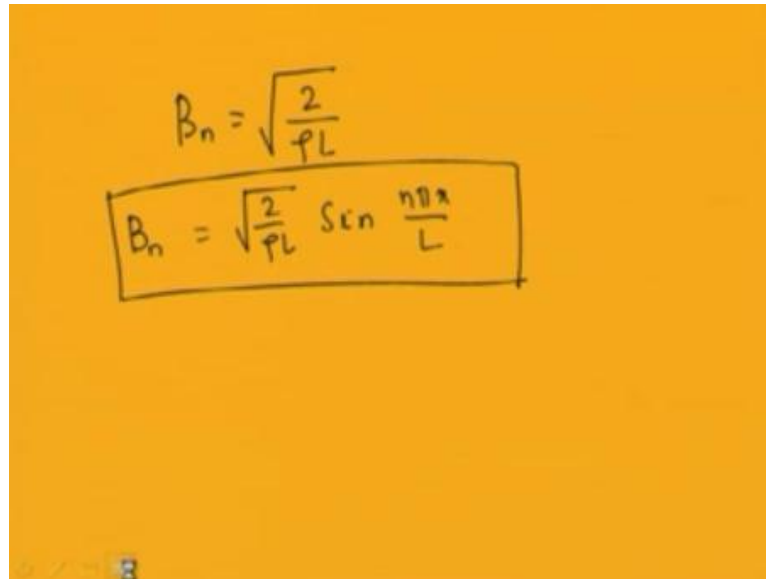
So, this integration you can find, by writing this way.

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$$\begin{aligned}\sin^2 \theta &= \frac{1}{2} (1 - \cos 2\theta) \\ \frac{1}{2} \rho B_n^2 \int_0^l (1 - \cos \frac{2n\pi x}{l}) dx & \\ &= \frac{1}{2} \rho B_n^2 \left[x - \frac{\sin \frac{2n\pi x}{l}}{2n\pi/l} \right]_0^l \\ &= \frac{1}{2} \rho B_n^2 l = 1 \quad B_n^2 = \frac{2}{\rho L}\end{aligned}$$

So, rho beta n rho B n square, so integration the sin square n pi x, you can write this is equal to 0 to 1 1 minus cos 2 n pi x by l dx. So, this integration will become rho B n square, so this is x minus this cos integration will be sin. So, this is sin 2 n pi x by l divided by, so you can divide this by 2 n pi by l. So, this is from 0 to l. So, this integration, you can find that this is equal to. So, you can substitute x equal to l, so there is a half term here, as we have sin square theta equal to half 1 minus cos 2 theta. So, I can have this half term, so this is half rho B n square. So, the sin 2 pi nx for value of sin 2 pi n x by l for x equal to l, this is equal to sin 2 n pi. So, this sin 2 n pi equal to 0 similarly, when x equal to 0 this is sin 0. So, sin 0 equal to 0, so you will have this term only, equal to half rho B n square l. So, this is equal to, so you can substitute this, and you can write this equal to 1. So, you can write this B n square, so B n square will becomes, equal to 2. So, this is equal to 2 by rho L or B n you can write B n.

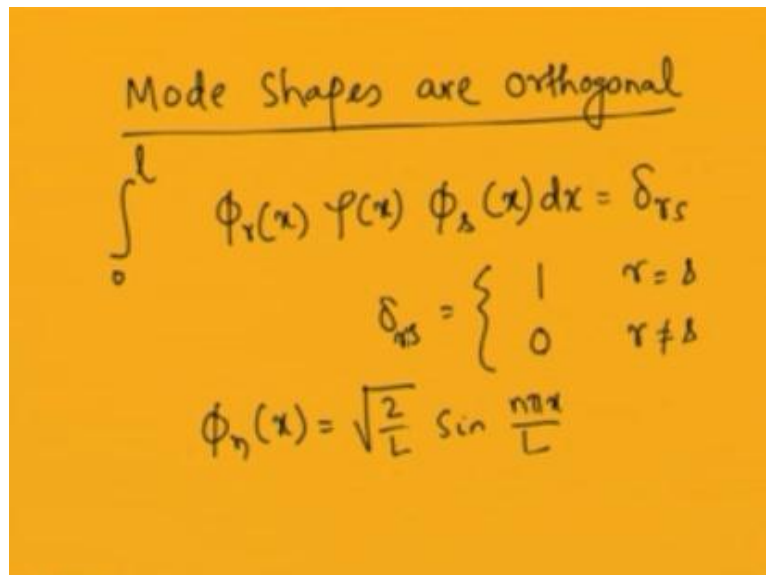
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The image shows two handwritten equations on a yellow background. The first equation is $B_n = \sqrt{\frac{2}{\rho L}}$. The second equation, enclosed in a rectangular box, is $B_n = \sqrt{\frac{2}{\rho L}} \sin \frac{n\pi x}{L}$.

You can write equal to root over 2 by rho L. So, the normalized mode shape, you can write as B_n equal to root over 2 by rho L sin $n\pi x$ by L. So, this is a normalized mode shape or Eigen function of the string. So, now, you can show that this Eigen functions or this mode shapes are orthogonal. To show mode shapes are orthogonal.

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The image shows handwritten text and equations on a yellow background. At the top, it says "Mode Shapes are orthogonal" with a horizontal line underneath. Below this, the integral equation is written: $\int_0^L \phi_r(x) \rho(x) \phi_s(x) dx = \delta_{rs}$. Below the integral, the Kronecker delta is defined as $\delta_{rs} = \begin{cases} 1 & r = s \\ 0 & r \neq s \end{cases}$. At the bottom, the mode shape is given as $\phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$.

So, we can, so this mode shapes are orthogonal with respect to this mass matrix in case of multi degree of freedom system, you have seen. So, in this case also, you can show that the mode shapes are orthogonal by writing these things. So, 0 to l rho phi r, so let us

take the rth mode and this phi r into rho x into phi s mode, let us take r and s mode. So, we should show that, if the mode shapes are orthogonal. So, this integration should be equal to delta rs, where delta rs is the Kronecker delta term. So, delta rs equal to, so this will be equal to 1 when r equal to s and it will be equal to 0 when r not equal to s. So, if this satisfy this equation, then we can show that or we can tell that systems are this mode shapes are orthogonal, so to prove these things. So, already, we have found the mode shape for the nth mode. So, phi n x equal to root over 2 by 1 sin n pi x by l, so by substituting this expression in this expression, so we can show that.

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$$\int_0^l B_r \sin \frac{r\pi x}{L} \cdot \rho(x) B_s \sin \frac{s\pi x}{L} dx$$

$$\rho \int_0^l$$

So, when r not equal to s, so this becomes, 0 to l for phi r I can write this is equal to Br sin n pi or r pi x sin r pi x r pi x by l into rho x into for s mode. I can write in this form sin s pi x by l dx, so I can take this rho x constant by taking rho x constant equal to rho. So, this becomes, 0 to l and for this Br I can write this equal to...

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Mode Shapes are orthogonal

$$\int_0^L \phi_r(x) \phi_s(x) dx = \delta_{rs}$$
$$\delta_{rs} = \begin{cases} 1 & r = s \\ 0 & r \neq s \end{cases}$$
$$\phi_n(x) = \sqrt{\frac{2}{PL}} \sin \frac{n\pi x}{L}$$

Already, we have found this B_n , it is equal to root 2 by rho L. So, 2 by rho 1, I can put it here. So, this becomes.

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$$B_n = \sqrt{\frac{2}{PL}}$$
$$B_n = \sqrt{\frac{2}{PL}} \sin \frac{n\pi x}{L}$$

So, you obtain this B_n equal to 2 by rho L. So, you can substitute that root over 2 by rho L and you can find.

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Mode Shapes are orthogonal

$$\int_0^L \phi_r(x) \psi(x) \phi_s(x) dx = \delta_{rs}$$

$$\delta_{rs} = \begin{cases} 1 & r=s \\ 0 & r \neq s \end{cases}$$

$$\phi_n(x) = \sqrt{\frac{2}{PL}} \sin \frac{n\pi x}{L}$$

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$$\int_0^L B_r \sin \frac{r\pi x}{L} \cdot \psi(x) B_s \sin \frac{s\pi x}{L} dx$$

$$P \int_0^L \sqrt{\frac{2}{PL}} \cdot \sqrt{\frac{2}{PL}} \sin \frac{r\pi x}{L} \cdot \sin \frac{s\pi x}{L} dx$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\therefore \frac{2}{PL} \int_0^L \left\{ \cos \frac{(r-s)\pi x}{L} - \cos \frac{(r+s)\pi x}{L} \right\} dx$$

So, this will become 2 by ρ 1 root over into again another 2 by ρ 1 for this B_s . So, you will have $\sin r \pi x$ by l into $\sin s \pi x$ by l dx . So, you know the integration or you know this term $\sin r \pi x$ by l into $\sin s \pi x$ by l can be written. Or you know the term $\sin \alpha$ into $\sin \beta$ $\sin \alpha$ into $\sin \beta$ can be written as half $\cos \alpha$ minus β minus $\cos \alpha$ plus β . So, if you substitute this thing \sin for the $\sin \alpha$ into $\sin \beta$. So, you can find for the from this equation ρ into 2 by ρ 1 integration 0 2 l and another half will come here, so this is half. So, this is $\cos \alpha$ minus β , so for this α and β , you can write this as r minus s into π by l to x minus $\cos r$. So, this is r

plus s into pi by l into x into dx. So, you have reduce this product of sin terms in terms of cos terms. So, this becomes, so you can divide this 2 2 and rho rho, but canceled. So, this becomes.

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$$\begin{aligned}
 &= \frac{1}{L} \int_0^L \left[\cos\left(\frac{r-s}{L}\pi x\right) - \cos\left(\frac{r+s}{L}\pi x\right) \right] dx \\
 &= \frac{1}{L} \left[\frac{\sin\left(\frac{r-s}{L}\pi x\right)}{\left(\frac{r-s}{L}\pi\right)} - \frac{\sin\left(\frac{r+s}{L}\pi x\right)}{\left(\frac{r+s}{L}\pi\right)} \right]_0^L \\
 &= 0
 \end{aligned}$$

This is equal to 1 by L integration 0 to L s cos r minus s by l into pi x minus cos r plus s by l into pi x dx. So, this integration becomes, 1 by L, so for this cos, you can write this is integration is sin. So, this is sin r minus s into pi by l x by, this is r minus s pi by l minus similarly, this becomes, sin r plus s by l pi x by r plus s by l into pi. So, this is from 0 to l. So, you can see that these terms at x equal to l and x equal to 0 it vanishes and this becomes 0. So, when r not equal to s, so you have checked that this integral will be equal to 0.

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$$\int_0^L B_r \sin \frac{r\pi x}{L} \cdot \underline{\phi(x)} B_s \sin \frac{s\pi x}{L} dx$$

$$P \int_0^L \sqrt{\frac{2}{rL}} \cdot \sqrt{\frac{2}{sL}} \sin \frac{r\pi x}{L} \cdot \sin \frac{s\pi x}{L} dx$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$P \cdot \frac{2}{rL} \int_0^L \left\{ \cos \left(\frac{r-s}{L} \pi x \right) - \cos \left(\frac{r+s}{L} \pi x \right) \right\} dx$$

And when r equal to s, so this term, you can write this as Br square sin square. So, when r equal to s.

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$$P \int_0^L B_r^2 \sin^2 \frac{r\pi x}{L} dx = \frac{1}{2} \quad r=s$$

$$\left. \begin{aligned} \int_0^L \phi_r(x) \phi_s(x) dx &= 0 \\ \int_0^L \phi_r^2(x) dx &= 1 \end{aligned} \right\} r=s$$

Your integration term becomes, Br square sin square, so rho into sin square r pi x by l dx. So, already you have integrated this thing, in case of r equal to n and that integration becomes, equal to. So, this integration equal to this integration equal to 1 by 2, you have seen and the total integration will be equal to 1. So, you have observed that, this is equal to 1 and, so you can prove that, when r equal to s, so when r equal to s. So, this

integration become 1 and when r not equal to s this becomes 0. So, this proves the orthogonality principle of this mode shapes. So, the mode shapes are orthogonal to each other. So, the mode shapes are orthogonal to each other; that means, so when r not equal to s. So, the mode shapes, this ϕ_r into ϕ_s x ϕ_r x into ϕ_s x equal to 0 when r not equal to s and when r equal to s 0 to 1. So, that case it becomes, ϕ^2 x dx. So, this will become 1. So, this proves the orthogonality property of the mode shapes similarly, if you recall.

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$$\underline{P}' \underline{K} \underline{P} = \lambda \rightarrow \text{MDOF}$$

↳ Eigenvalue

$$\int_0^l T \frac{d\phi_r(z)}{dz} \cdot \frac{d\phi_s(z)}{dz} dz = \omega_r^2 \delta_{rs}$$

This P weighted modal matrix KP, you have checked that, this will become the Eigen function Eigen values of the system. So, using this modal weighted modal matrix, this is the stiffness matrix and this is the weighted modal matrix. So, P weighted modal matrix transpose, this is stiffness matrix and this is P matrix for a multi degree of freedom system. So, this becomes, equal to lambda that is the Eigen value of the system. Similarly, in this case also you can prove that or you can show that, this integration 0 to l rho, so integration 0 to l. So, this T d phi r x by dx into d phi s x by dx, so this will be equal to omega r square delta rs, where omega r is the rth mode frequency. And omega r square is equivalent to the Eigen, value of the system. So, in this case will prove this thing, so already we got the expression for phi x. So, phi r x is the d phi r by dx is the derivative of phi x. So, you can write this thing this equation in this form, so this will become T into.

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$$\tilde{P}' \underline{K} \underline{P} = \lambda \rightarrow \underline{MDOF}$$

↳ Eigenvalue

$$\int_0^L T \frac{d\phi_r(z)}{dz} \cdot \frac{d\phi_s(z)}{dz} dz = \omega_r^2 \delta_{rs}$$

So, by using the Normalized mode I can write this as.

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$$T \frac{2}{PL} \frac{\pi r}{L} \frac{\Delta \pi}{L} \int_0^L \cos \frac{r\pi x}{L} \cdot \cos \frac{s\pi x}{L} dx$$

$$\frac{L}{2} \delta_{rs}$$

$$T \frac{2 \pi^2 r s}{PL^3} \cdot \frac{L}{2} = \frac{\pi^2 r^2}{PL^2} T = \omega_r^2 \delta_{rs}$$

$\omega_n = \frac{n\pi}{L} \sqrt{\frac{T}{\rho}}$

T into 2 by rho L pi r by L into s pi by l integration 0 to L cos r pi x by L into cos s pi x by L dx, so this thing can be written equal to. So, this is, so this integration you can find this equal to 1 by 2 delta rs. o, this is equal to L by 2 delta rs; that means, when r equal to s this value equal to 0 and when r when r equal to 0 this value becomes, delta rs equal to L and when r not equal to s delta rs equal to 0. So, from this when r equal to s, you can find this value equal to 2 by rho L into pi square rs by l square. So, this is pi square r l

by, so $2\pi^2 rs$ by ρl^3 . So, this thing can be written as ρl^3 into 1 by 2 , so this becomes, π^2 square, so this as r equal to s . So, it can be written as π^2 square r square, so this r into s equal to $r^2 \pi^2$ square r^2 by ρL^3 into T .

So, as ω are already, you know this is equal to ωr equal to $n\pi$ by l into root over T by m . So, this ωn , so this is ωn , so ωr will become, $r\pi$ by l into root over T by ρ . So, I am using ρ , so this ωr^2 term, you can write equal to $n^2 \pi^2$ $n^2 r^2$ by l^2 into T by ρ . So, already we have T by ρ and this is $\pi^2 r^2$ by l^2 . So, this thing can be written as ωr^2 square ωr^2 square. So, I can write this integration equal to $\omega r^2 \Delta rs$, so when r equal to s this integral will reduce to $\omega r^2 \Delta \omega r^2$ and when r not equal to s this value will become 0 . So, we have seen mode shapes are orthogonal and from this also.

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$$\int_0^l \rho \phi_r(x) \phi_s(x) dx = \delta_{rs} \quad r, s = 1, 2, \dots$$

$\delta_{rs} \rightarrow$ Kronecker delta

$$\int_0^l T \left(\frac{d\phi_r(x)}{dx} \right) \cdot \left(\frac{d\phi_s(x)}{dx} \right) dx = \underline{\omega_r^2 \delta_{rs}}$$

We have checked that this integration 0 to l $\rho \phi_r \times \phi_s$, where ϕ_r and ϕ_s are r th mode shape and s mode shape. So, this is equal to Δrs , when $r = s$ equal to $1, 2$, so you can take any number of modes, so this Δrs is the Kronecker delta. And also we have proved that 0 to l , this ω or $T d\phi_r \times dx$ into $d\phi_s \times dx$ equal to $\omega r^2 \Delta rs$. So, in this way, you can find or you can show that different modes are orthogonal to each other. And, so you can use this property of orthogonality,

you can find the free vibration response of the system, so a general free vibration response of the system.

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$$\begin{aligned}
 u(x,t) &= \sum_{n=1}^{\infty} \phi_n(x) q_n(t) \\
 &= \sum_{n=1}^{\infty} \left(B_n \sin \frac{n\pi x}{L} \right) \left(C_{1n} \cos \omega_n t + C_{2n} \sin \omega_n t \right) \\
 &= \sum_{n=1}^{\infty} \left(D_{1n} \cos \omega_n t + D_{2n} \sin \omega_n t \right) \sin \frac{n\pi x}{L}
 \end{aligned}$$

That is $u(x,t)$ will be the summation of all the modal frequencies all or you can find this $u(x,t)$ from the contribution of different modes. And you can write this $u(x,t)$ as summation $\sum_{n=1}^{\infty} \phi_n(x) q_n(t)$. So, this equation can also be written like this, so already you have seen for a string this is equal to $\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} (C_{1n} \cos \omega_n t + C_{2n} \sin \omega_n t)$. So, this is equal to $\sum_{n=1}^{\infty} (D_{1n} \cos \omega_n t + D_{2n} \sin \omega_n t) \sin \frac{n\pi x}{L}$. So, for $\phi_n(x)$ I can write $B_n \sin \frac{n\pi x}{L}$ and for $q_n(t)$ I can write this is equal to $C_{1n} \cos \omega_n t + C_{2n} \sin \omega_n t$. So, I can multiply this B with the C and these 3 constants can be reduce to 2 constants.

And So, I can write this expression for the free vibration response of the system like this. So, this will be equal to $\sum_{n=1}^{\infty} (D_{1n} \cos \omega_n t + D_{2n} \sin \omega_n t) \sin \frac{n\pi x}{L}$. I can write this as $\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cos \omega_n t + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \omega_n t$. So, some other C term or D term I can use, so this becomes $D_{1n} \cos \omega_n t + D_{2n} \sin \omega_n t$. So, where this ω_n are the natural frequencies of the system. So, this into, so this into this mode shape term also will be there. So, this into $\sin \frac{n\pi x}{L}$, so already you know the relation between this ω_n and ω for different mode shapes different modes. So, for different modes you have seen this.

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$$\frac{\omega l}{c} = n\pi$$

$$\omega = \frac{n\pi}{l} c$$

$$\omega_n = \frac{n\pi}{l} \sqrt{\frac{T}{\rho}}$$

$$\phi_n(x) = B_n \sin \frac{n\pi x}{l}$$

Omega equal to already, you have found the expression for omega for this transverse vibration of the string and this can be written in this form. So, omega n equal to n pi by l root over T by rho. So, this is for the nth mode and the final expression for this for the case of the string you can write in this form.

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$$u(x,t) = \sum_{i=1}^{\infty} \phi_i(x) q_i(t)$$

$$= \sum_{i=1}^{\infty} \left(B_{1i} \sin \frac{n\pi x}{L} \right) \left(C_{1i} \cos \omega_i t + C_{2i} \sin \omega_i t \right)$$

$$= \sum_{i=1}^{\infty} \left(D_{1i} \cos \omega_i t + D_{2i} \sin \omega_i t \right) \sin \frac{n\pi x}{L}$$

So, u x t equal to D 1 i cos omega i t plus D 2 i sin omega i t into sin n pi x by l now, let us find the mode shapes for the case of the longitudinal vibration of rod.

(Refer Slide Time: 43:22)

Longitudinal vibration of rod

$\sigma = E \frac{\partial u}{\partial x} = 0$

$u = \phi(x) \underline{q(t)}$

$\frac{\partial u}{\partial x} = 0$

$\frac{\partial \phi(x)}{\partial x} \Big|_{x=0} = 0$

$\frac{\partial \phi(x)}{\partial x} \Big|_{x=l} = 0$

So, in this case will take different boundary conditions and will find the frequency equation and the mode shapes of the system. So, let us take the case of a free-free beam. So, in case of the free-free rod, so this is the free-free rod. So, in this case both the ends are free. So, when both ends are free this ends are not stress, so there is no stress in this in this ends as there is no stress in the ends. So, I can write that the stressed or stress equal to $E \frac{\partial u}{\partial x}$ equal to 0 as $\frac{\partial u}{\partial x}$ equal to 0. So, I can write $E \frac{\partial u}{\partial x}$ that is the stress σ equal to 0.

So, I can write this $\frac{\partial u}{\partial x}$ that is the strain equal to 0 at both the ends. So, this u you can write in this form u equal to $\phi(x) q(t)$. So, as u equal to $\phi(x) q(t)$ and this expression is valid for all the time this boundary conditions. That is $\frac{\partial u}{\partial x}$ equal to 0 at the free end and at the at both the free end are 0 or the slopes this $\frac{\partial u}{\partial x}$ equal to 0 at both the ends for all time. So, you can write this $\phi(x)$ at x equal to 0 or $\frac{\partial \phi(x)}{\partial x}$ at x equal to 0 equal to 0 similarly $\frac{\partial \phi(x)}{\partial x}$ at x equal to 1 also equal to 0.

(Refer Slide Time: 45:37)

$$\begin{aligned}\phi(x) &= A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x \\ \frac{d\phi(x)}{dx} &= -A \frac{\omega}{c} \sin \frac{\omega}{c} x + B \frac{\omega}{c} \cos \frac{\omega}{c} x \\ \underline{x=0} \quad \frac{d\phi(x)}{dx} &= 0 \Rightarrow B \frac{\omega}{c} - \frac{A\omega}{c} \cdot 0 = 0 \\ &\quad \underline{B = 0} \\ \phi(x) &= A \cos \frac{\omega}{c} x\end{aligned}$$

So, as $\phi(x)$ you can already get the expression for $\phi(x)$ that is equal to $A \cos \frac{\omega}{c} x$ plus $B \sin \frac{\omega}{c} x$. So, you can find this $\frac{d\phi(x)}{dx}$, so that is equal to $A \frac{\omega}{c} \sin \frac{\omega}{c} x$ minus plus $B \frac{\omega}{c} \cos \frac{\omega}{c} x$. So, when x equal to 0, so for x equal to 0 $\frac{d\phi(x)}{dx}$ equal to 0 $\frac{d\phi(x)}{dx}$ equal to 0. So, this implies that this B into this \cos term $B \frac{\omega}{c} \cos \frac{\omega}{c} x$ minus $A \frac{\omega}{c} \sin \frac{\omega}{c} x$ equal to 0. So, this becomes equal to 0. So, this implies this B term equal to 0. So, from this boundary conditions you have seen that this B equal to 0. So, as B equal to 0. So, here $\phi(x)$ now becomes $\phi(x)$ equal to $A \cos \frac{\omega}{c} x$, so now $\frac{d\phi(x)}{dx}$ at the right hand.

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$$\left. \frac{d\phi(x)}{dx} \right|_{x=l} =$$

D phi x by dx at the right hand that is x equal to l can be written as...

(Refer Slide Time: 47:26)

$$\begin{aligned}\phi(x) &= A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x \\ \frac{d\phi(x)}{dx} &= -A \frac{\omega}{c} \sin \frac{\omega}{c} x + B \frac{\omega}{c} \cos \frac{\omega}{c} x \\ \underline{x=0} \quad \frac{d\phi(x)}{dx} &= 0 \Rightarrow B \frac{\omega}{c} - \frac{A\omega}{c} \cdot 0 = 0 \\ &\quad \underline{B = 0} \\ \phi(x) &= A \cos \frac{\omega}{c} x\end{aligned}$$

So, this becomes A omega by c into.

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$$\left. \frac{d\phi(x)}{dx} \right|_{x=l} = A \frac{\omega}{c} \sin$$

A omega by c sin.

(Refer Slide Time: 47:36)

$$\begin{aligned} \phi(x) &= A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x \\ \frac{d\phi(x)}{dx} &= -A \frac{\omega}{c} \sin \frac{\omega}{c} x + B \frac{\omega}{c} \cos \frac{\omega}{c} x \\ \underline{x=0} \quad \frac{d\phi(x)}{dx} = 0 &\Rightarrow B \frac{\omega}{c} - \frac{A\omega}{c} \cdot 0 = 0 \\ &\quad \underline{B = 0} \\ \phi(x) &= A \cos \frac{\omega}{c} x \end{aligned}$$

Sin omega by c x.

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$$\frac{d\phi(x)}{dx} \Big|_{x=l} = A \frac{\omega}{c} \sin \frac{\omega}{c} x \Big|_{x=l} = 0$$
$$A \frac{\omega}{c} \sin \frac{\omega}{c} l = 0$$
$$\propto \underline{A \sin \frac{\omega}{c} l} = 0$$

So, this is equal to at x equal to l equal to 0. So, I can write this as A into ω by c sin ω by c l equal to 0 or I can write this as A sin ω by c l equal to 0. So, from this as A cannot be 0, because when A equal to 0 you will have a trivial state solution; that means, there will be no vibration of this rod this may be rod. So, there will be no vibration of this rod. So, you can take this as a free-free beam. So, in this case you can take this as a, this as a free-free rod.

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So, in this both the ends are free, so it is not stressed. So, as this is not stressed in this case you can find that the stress are 0. So, E into strain will be equal to 0 or strain will be equal to 0 that is $\frac{du}{dx}$ will be equal to 0 and as you know this u can be written in terms of $\sin kx$ and $\cos kt$. So, for all times to come strain at this 2 ends will be equal to 0. So, for that reason, so you can find this expression of the mode shape like this $A \sin \frac{\omega}{c} x$ if I am making this A equal to 0 then there will be no vibration of this rod. So, for this free-free rod there will be no vibration when A will be equal to 0, but as the rod is vibrating we are or we are studying the vibration of this rod. So, A cannot be equal to 0. So, as A will not be equal to 0.

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$$\left. \frac{d\phi(x)}{dx} \right|_{x=l} = A \frac{\omega}{c} \sin \frac{\omega}{c} x \Big|_{x=l} = 0$$

$$A \frac{\omega}{c} \sin \frac{\omega}{c} l = 0$$

$$\propto \underline{A \sin \frac{\omega}{c} l = 0}$$

$$\sin \frac{\omega}{c} l = 0 = \sin \frac{n\pi}{l}$$


So, in that case I can write this $\sin \frac{\omega}{c} l$. So, this is $\omega \sin \frac{\omega}{c} l$ will be equal to 0 equal to $\sin n\pi$ by l . So, as this is equal to $\sin n\pi$ by l .

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$$\frac{\omega l}{c}$$

So, I can write this omega L by. So, omega l by c.

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$$\begin{aligned} \frac{d\phi(x)}{dx} \Big|_{x=l} &= A \frac{\omega}{c} \sin \frac{\omega}{c} x \Big|_{x=l} = 0 \\ A \frac{\omega}{c} \sin \frac{\omega}{c} l &= 0 \\ \propto A \sin \frac{\omega}{c} l &= 0 \\ \sin \frac{\omega}{c} l = 0 &= \sin \frac{n\pi}{l} \end{aligned}$$


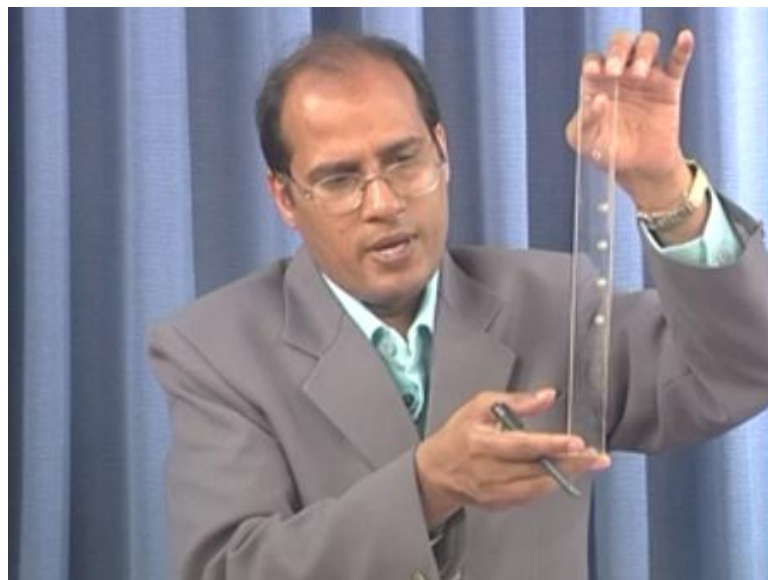
Omega l by c equal to n pi.

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$$\frac{\omega l}{c} = n\pi$$
$$\text{or } \omega = \frac{n\pi}{l} c = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}$$

Or you can write this omega equal to n pi by l into n pi by l into c. So, this is equal to. So, in case of this rod this is can be written this c equal to E by rho. So, where E is the Young's modulus and rho is the mass per unit length of this rod.

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So, in this rod the mass per unit length I am taking equal to rho E is the young's modulus. So, in this case when it is vibrating that time the frequency the nth mode you can write in this form.

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$$\frac{\omega l}{c} = n\pi$$
$$\text{or } \omega = \frac{n\pi}{l} c = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}$$
$$\boxed{\omega_n = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}}$$

So, this is equal to $n\pi$ by l into E by ρ and the mode shape you can write or Eigen function you can write in this form.

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$$\left. \frac{d\phi(x)}{dx} \right|_{x=l} = A \frac{\omega}{c} \sin \frac{\omega}{c} x \Big|_{x=l} = 0$$
$$A \frac{\omega}{c} \sin \frac{\omega}{c} l = 0$$
$$\text{or } \underline{A \sin \frac{\omega}{c} l = 0}$$
$$\sin \frac{\omega}{c} l = 0 = \sin \frac{n\pi}{l}$$


So, the Eigen function will become $A \sin$.

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$$\frac{\omega l}{c} = n\pi$$
$$\text{or } \omega = \frac{n\pi}{l} c = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}$$
$$\boxed{\omega_n = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}}$$

ϕ

So, phi becomes.

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$$\phi(x) = A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x$$
$$\frac{d\phi(x)}{dx} = -A \frac{\omega}{c} \sin \frac{\omega}{c} x + B \frac{\omega}{c} \cos \frac{\omega}{c} x$$
$$\underline{x=0} \quad \frac{d\phi(x)}{dx} = 0 \Rightarrow B \frac{\omega}{c} - \frac{A\omega}{c} \cdot 0 = 0$$
$$\underline{B = 0}$$
$$\phi(x) = A \cos \frac{\omega}{c} x$$

So, as A not equal to 0. So, this phi x B equal to 0 you can write phi x equal to A cos omega by cx.

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$$\frac{\omega l}{c} = n\pi$$
$$\text{or } \omega = \frac{n\pi}{l} c = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}$$
$$\omega_n = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}$$
$$\phi = A \cos \frac{\omega}{c} x$$

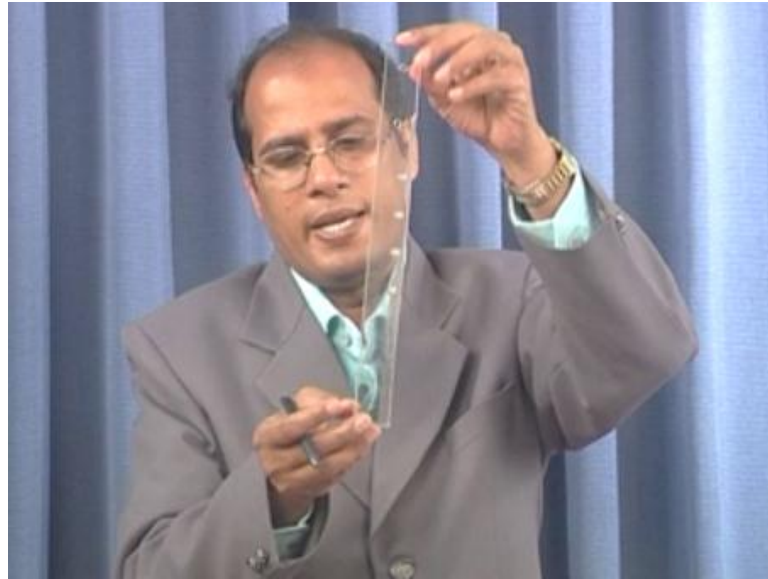
So, phi equal to A cos omega by c x and for omega I can substitute this value equal to n pi by l into root over E by rho. So, this omega by c can be written or this omega by c equal to n pi by l.

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$$\phi_n = A_n \cos \frac{n\pi}{l} x$$

So, this expression for phi equal to A_n for phi n can be written A_n cos n pi by l x. Cos n pi by l x. So, you can see that this free-free rod vibrates in a cosine as a cosine form.

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So, in this rod the vibration will take place. So, in this case the vibration will take place in this longitudinal direction. So, in this longitudinal direction the wave propagation will take place in a cosine wave, so this ϕ_n is equal to $A_n \cos \frac{n\pi}{L} x$.

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$$\phi_n = A_n \cos \frac{n\pi}{L} x$$
$$\omega_1 = \frac{\pi}{L} \sqrt{\frac{E}{\rho}}$$
$$\omega_2 = \frac{2\pi}{L} \sqrt{\frac{E}{\rho}}$$

So, you can write this first modal frequency equal to π by L into \sqrt{E} by ρ and second modal frequency will be equal to 2π by L into \sqrt{E} by ρ . And similarly the n th modal frequency will be $n\pi$ by L into \sqrt{E} by ρ ; that means, the n th modal

frequency is n times the first modal frequency of the system. So, the general vibration or the free vibration response of the system you can write.

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$$u(x,t) = \sum_{i=1}^{\infty} (D_{1i} \cos \omega_i t + D_{2i} \sin \omega_i t)$$

This $u(x,t)$ will be equal to summation. So, this will be equal to summation like in the previous case of string I can write this as D_{1i} equal to 1 to infinity. So, $D_{1i} \cos \omega_i t$ plus $D_{2i} \sin \omega_i t$ into.

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$$\phi_n = A_n \cos \frac{n\pi}{L} x$$
$$\omega_1 = \frac{\pi}{L} \sqrt{\frac{E}{\rho}}$$
$$\omega_2 = \frac{2\pi}{L} \sqrt{\frac{E}{\rho}}$$

So, in this case, it becomes cos...


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$$u(x,t) = \sum_{i=1}^{\infty} \left(D_{1i} \cos \omega_i t + D_{2i} \sin \omega_i t \right) \cos \frac{i \pi x}{l}$$

So, it will be $\cos n \pi x$ by $\cos n \pi x$ by l or. So, this is equal to ω_n by x by l is your. So, for n I can substitute this equal to l . So, $l \pi x$ by l . So, $u(x,t)$ that is the free vibration response of the system becomes l equal to 1 to infinity $D_{1i} \cos \omega_i t$ plus $D_{2i} \sin \omega_i t$ into $\cos i \pi x$ by l . So, in this D_{1i} and D_{2i} are the modal participation of the i th mode. So, these 2 this D_{1i} and D_{2i} you can obtain it from the initial conditions of the system. So, let us find the mode shapes for the case of a clamped free beam. So, let one in this clamped and other end is free.

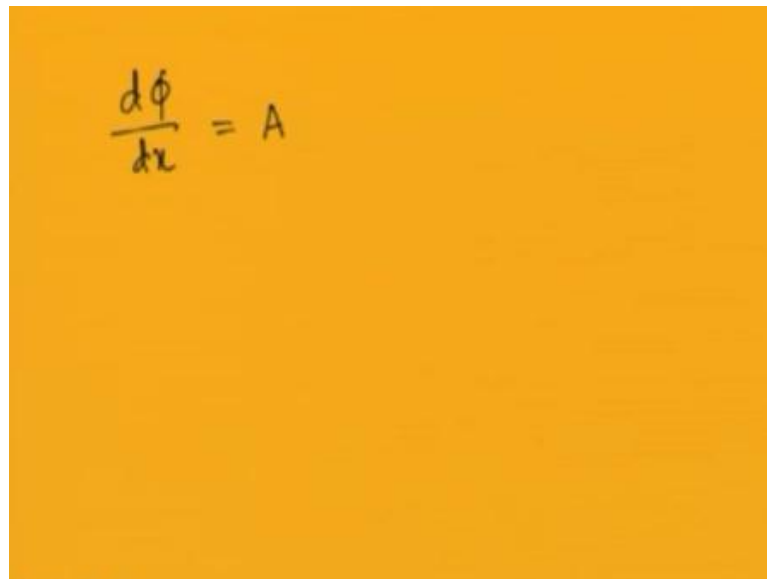
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Clamped-Free rod

$$\phi(0) = 0$$
$$\left. \frac{d\phi}{dx} \right|_{x=l} = 0$$
$$\phi(x) = A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x$$
$$\phi(0) = 0 \Rightarrow A = 0$$


So, this end is clamped and other end is free. So, in this case you have to find the mode shapes of the longitudinal vibration of the rod. So, in case of a clamped free beam clamped free. So, this is a clamped free rod also, we are interested to find the longitudinal vibration of this rod. So, in this case the boundary conditions are. So, at the fixed end displacement will be equal to 0 and at the free end the stress will be equal to 0. So, are the stress equal to 0 already we have seen that it can be written at the free-free free end $\frac{d\phi}{dx}$ equal to 0 and at the fixed end $\phi(x=0)$ equal to 0. So, our boundary condition will be $\phi(0) = 0$ and $\frac{d\phi}{dx}$ at $x = l$ will be equal to 0. So, we can write this $\phi(x)$ in this form. So, $\phi(x)$ will be equal to $A \cos(\omega c x) + B \sin(\omega c x)$. So, in this case I can substitute this $\phi(x=0) = 0$. So, that will give $B = 0$ plus $A = 0$ or $A = 0$. So, that will give $A = 0$ $\phi(0) = 0$ implies $A = 0$. So, now differentiating it once, so I can write this.

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$$\frac{d\phi}{dx} = A$$

$\frac{d\phi}{dx}$ $\frac{d\phi}{dx}$ becomes $A \omega c$. So, now the...

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Clamped-Free rod

$$\phi(0) = 0$$
$$\left. \frac{d\phi}{dx} \right|_{x=l} = 0$$
$$\phi(x) = A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x$$
$$\phi(0) = 0 \Rightarrow A = 0$$

Remaining terms as A equal to 0. So, will have $\phi(x)$ equal to $B \sin \omega c x$.

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$$\phi(x) = B \sin \frac{\omega}{c} x$$
$$\left. \frac{d\phi(x)}{dx} \right|_{x=l} = B \frac{\omega}{c} \cos \frac{\omega}{c} x \Big|_{x=l} = 0$$
$$\Rightarrow B \cos \frac{\omega l}{c} = 0$$

So, we have $\phi(x)$ equal to. So, I can write this $\phi(x)$ equal to, so $\phi(x)$ equal to $B \sin \omega c x$. So, $d\phi(x)$ by dx , so $d\phi(x)$ by dx will be equal to $B \omega c \cos \omega c x$. So, now, at x equal to l equal to l this equal to 0. So, this implies that this $B \cos \omega c l$ equal to 0. So, today class we have studied the free vibration response of string and also for the longitudinal vibration of the rod. Or the case of string we have shown that the Eigen functions or orthogonal and for the case of fixed-fixed

string. We have found the frequency equation and the mode shapes and also for this longitudinal vibration of rod. We have found the frequency for the case of the free-free rod. Next class, we are going to study about the mode shapes of torsional vibration of rods and will summarize the free vibration of continuous systems.