Mechanical Vibrations

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Module - 8
Torsional Vibration
Lecture - 1
Simple systems with one, Two or three discs, Geared system

So, today we will be studying one type of vibration which is called torsional vibration, which is very pertinent in the machinery and various kind of application we can find this torsional vibration in machinery. Like if we have inertia force variation in the machine like in the IC engines. And because of the inertia change the shaft of the IC engine may get periodic variation in the torque. And this periodic variation in the torque may leads to torsional vibration of the crank shaft. Another application in which we can have a torsional vibration is this, in which we have the impulse load like in a punching machine.

There sudden releases of the energy are take place when there punching operation is going on. And this leads to sudden change in the torque in the fly wheel shaft of the punching machine, which leads to torsional vibration. Another kind of vibrations of torsional vibration we can have in the general test when there is a sudden fault in the power line. And again when the sudden closure of the fault takes place, it can have torsional vibration. And there are other kinds of machine elements like gears which transmit high power they may leads to torsional vibration in the shafts. And some other kind of elements like, when we have the turbine blades, where injecting the steam on to the blade that may leads to some kind of torsional vibration in the steam engine shafts.

Now, will be studying these type vibrations and we will try to see what are the, what is the motive behind studying these kind of torsional vibration. First thing which we would like to find out what is the natural frequency of the torsional vibration. And this will be helpful to us in finding out to at least avoid the resonance condition in the machinery, because if machine is having some excitation frequency or the forcing frequency it should not coincide with the natural frequency of the shaft. So, that the resonance can be avoided. That is the main motive toward analyzing the natural torsional natural frequency of the shaft.

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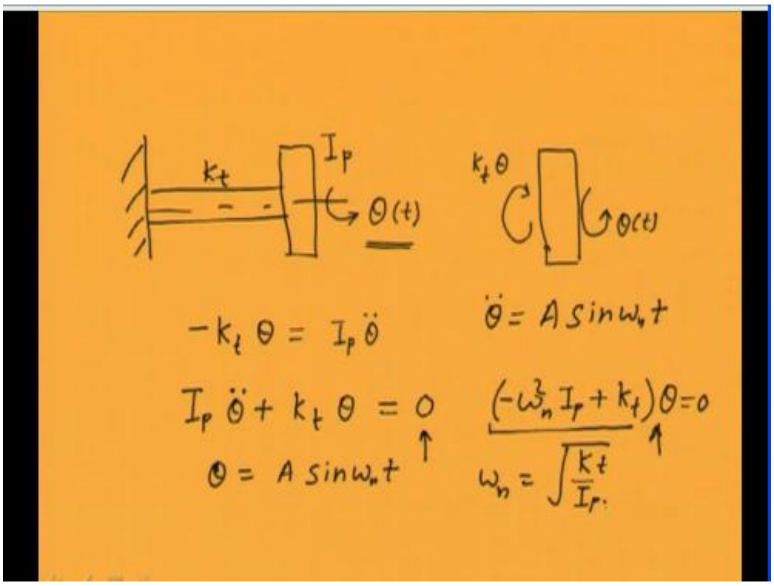

Let us consider a single degree of freedom system. This is a cantilever kind of shaft in which we have polar mass moment of inertia of the disc, and torsional stiffness of the shaft. Here, we are assuming the shaft as mass less, and is having only flexibility. If we want to see; what are the torsional vibrations in this particular shaft then the motion of the disc about its own axis is called the torsional displacement. So, this particular displacement will be time dependent and it will, the disc will be oscillating about its mean position with this displacement. So, to analyze the dynamics of this particular disc; we need to obtain the free body diagram of this particular disc.

Let us see, this is the direction of the displacement we have given to the disc. So, opposite to this will be getting a force which will be given by the shaft to the disc. And from Newton second law we can able to obtain the equation of motion of this particular disc. So, some of the external force which is acting on this particular disc can be written as, this is the external force which is acting on to the disc should be equal to the inertia force. Inertia moment of the disc, this is the inertia moment of the disc. So, equation of motion can be written as this is more standard form of the equation of motion. This particular equation of motion is homogeneous. Since, we have no forcing in the system, and the solution of this we expect to be harmonic motion, because the system is going under free vibration.

So, the solution can be assumed as amplitude and some harmonic term like this. And here we can substitute the solution in the equation of motion for that we need the double derivative of the solution. So, both the quantity we will be substituting in the equation of motion. So, we will get in the simplified form now, since the theta cannot be 0. So, for non trivial solution the quantity within the bracket should be 0, and that will give us the natural frequency of the system. This where K t is the natural this is the torsional stiffness of the system, and I p is the polar mass moment of inertia of the disc. Let us consider another system in which we have 2 disc which is connected by a single shaft.

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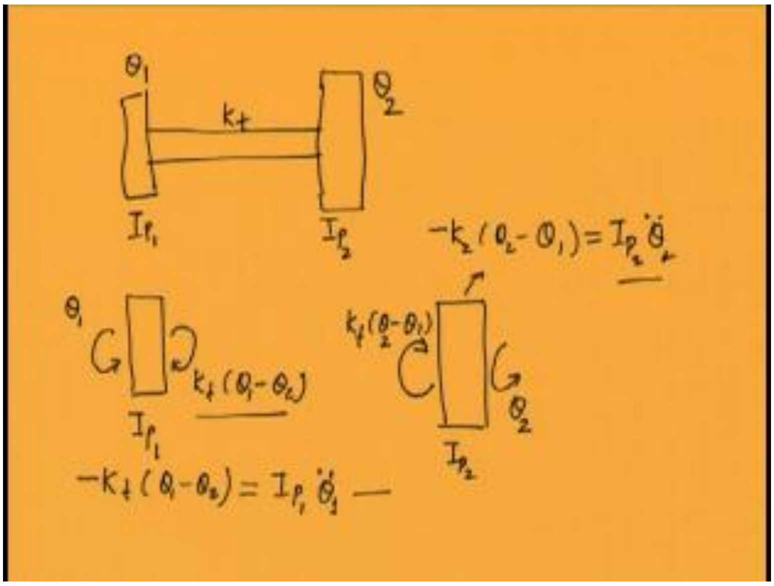

So, this is a disc one, having polar mass moment of inertia I p 1, and this is the another disc having all polar mass moment of inertia I p 2. The stiffness of the shaft is; let us say K t. Now, to analyze this system we need to draw the free body diagram of both the disc first, and then will be writing the equation of motion. So, this is the equation, this is the free body diagram of the disc 1, in which the first disc is having the displacement theta 1. Let us say and the second disc is having displacement theta 2. Let us say this particular disc is having this is the positive direction of the theta, and there will be torque from the shaft will be coming which is given as this.

Here, because the 2 disc are having different displacement. So, the twist actual twist of the shaft will be the relative displacement between the 2 disc into the torsional stiffness this. And another free body diagram which is of this disc 2 can be drawn here, let us say is having displacement theta 2 here. And will be having torque which will be opposite to the displacement direction is given like this. Now, using this 2 disc free body diagram; we can able to obtain the equation of motion. So, let us consider the first disc, here itself we can able to write minus K t is this is the torque external torque on the disc 1 should be equal to the inertia term of the disc 1. Here, this is a I p 1 for this particular disc we can able to write on the single lines like equation of motion this.

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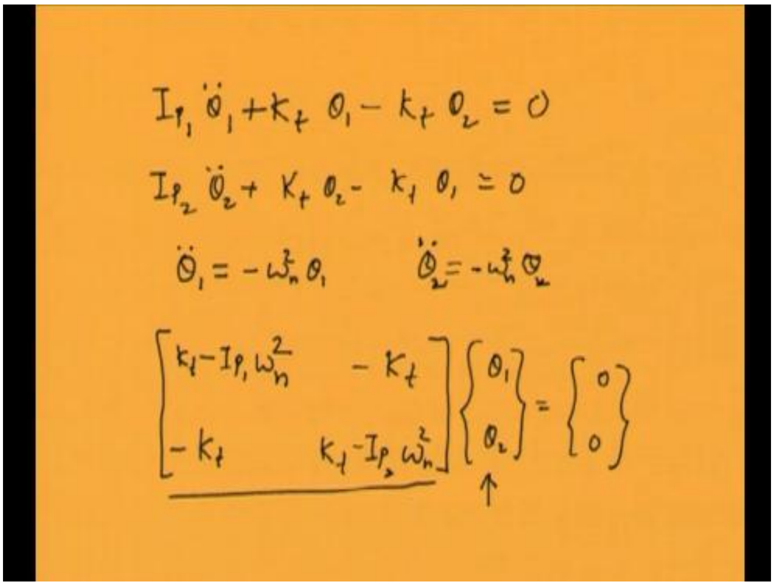

And, these 2 equations can be written on a standard form like this. And for circle disc on these 2 equations are the equations of motion of the 2 disc solution for that we can able to assume as harmonic motion is taking place, because we are looking into the free vibration of these 2 discs. So, this is the solution for the free vibration for both the disc as we did for the single degree of freedom system also. These two solutions can be substituted in these two equations of motion, and that will give us solution in this form. Put these 2 equations in a matrix form, and the entries in the matrix will be something, these are the natural frequencies. Now, for non trivial solution of these particular equations, because this theta 1 and theta 2 cannot be 0, so the determinant of this particular matrix should be 0. And that will give us a frequency equation. The frequency equation will be a quadratic in omega n square, and it is having this form.

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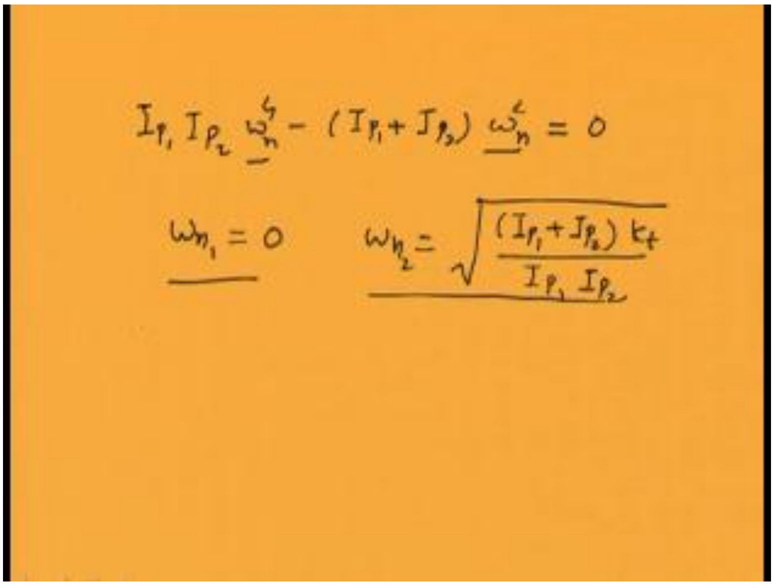

So, here you can able to see that the 2 solutions will be like this, because omega n can be taken common. So, that can be equated to first equal to 0 and then second. So, these are the 2 natural frequency of the system. We can able to see that the first natural frequency is having 0 value, and will try to see the physical meaning of these 2 natural frequency. So, for that may be we have to go for the finding the mode shapes of corresponding to these natural frequencies. So, let us go for the calculation of mode shape; for that we have go back to the previous equation.

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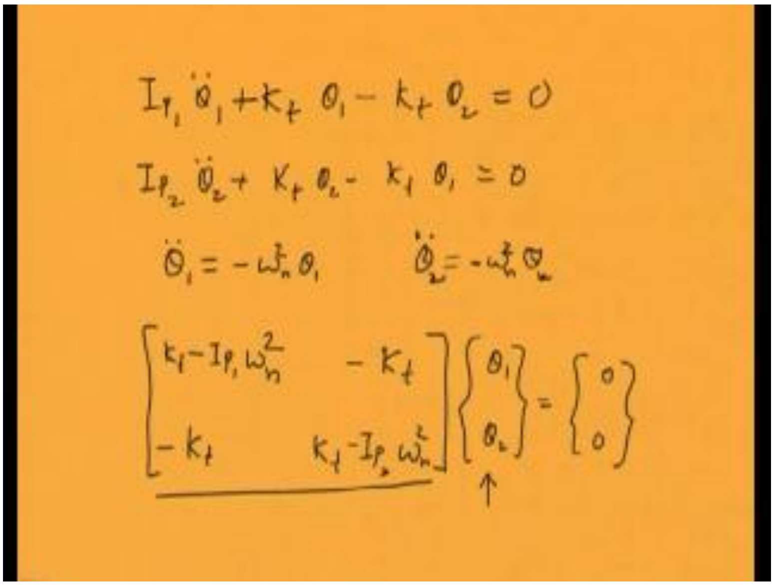

This particular equation in which will be substituting omega n in this term, and then will be solving for theta1 and theta 2.

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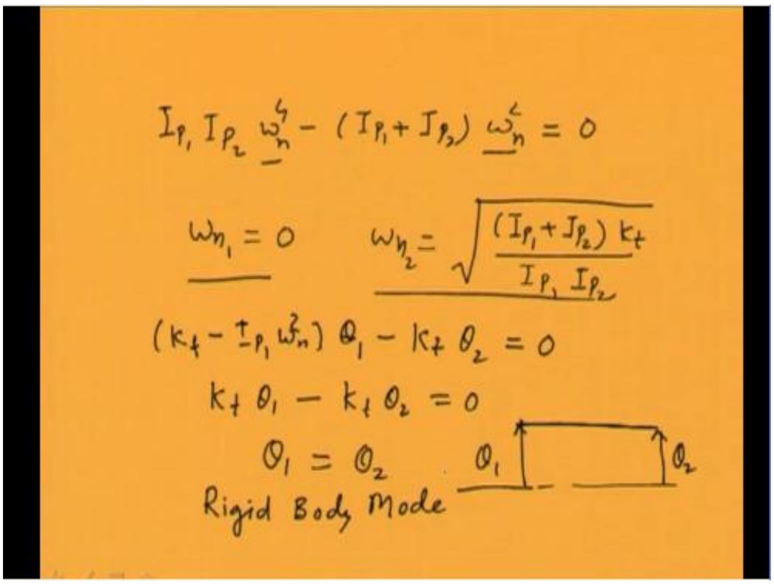

So, let us take the first equation. So, when we substitute omega n is equal to 0, we will get K t theta 1 minus K t theta 2 is equal to 0. You can see that; we got theta 1 is equal to theta 2 for this. And if we want to draw the mode shape of this, this corresponding to theta 1 this corresponding to theta 2 they are equal. So, this is the mode shape for the first natural frequency. You can this is corresponding to the rigid body mode in which we do not have the twist of the shaft both the disc are having same displacement. So, there is no twist of the shaft and is having no practical value because no stresses are generating into the shaft. Let us see the second mode shape that is having some practical value. So, we will again, we will be writing the same equation.

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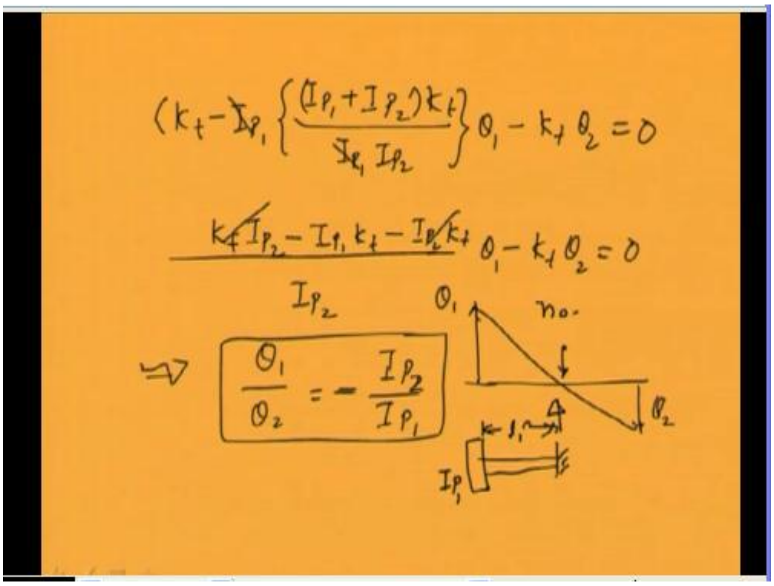

And, here instead of omega n square we are substituting the actual value of the natural frequency. Here, if we solve this equation; let us solve at least one more step on this, we can see that this term get canceled. So, equation is getting simplified. So, K t I p 2 minus I p 1 K t minus I p 2 K t theta 1 minus K t theta 2 is equal to 0. So, here this particular term is again getting canceled. So, finally you will get theta 1 by theta 2 is equal to minus of I p 1 sorry, I p 2 by I p 1. This is the relative displacement of the 2 disc corresponding to the second mode which is you can see in is inversely proportional to the mass moment of inertia. That means; if disc is having large mass moment of inertia it will be having less displacement. And this negative sign represent that 2 disc will be having different phase.

So, let us draw the mode shape for this. So, if you are considering theta 1 as positive. So, theta 2 will be negative, and their amplitude will be inversely proportional to the mass moment of inertia, polar mass moment of inertia of the disc. In this second natural frequency we have seen that the mode shapes there is a point; if we go back to the previous slide here. There is no displacement of the shaft that means; we can able to considered that left half portion of the shaft on this point is a cantilever kind of shaft. And similarly in the right hand side also we can able to consider that shaft as a cantilever.

So, we can consider this as like this in which this end is fixed completely. Similarly, we can able to draw another cantilever within the right hand side. And for this polar mass moment of inertia is let us I p 1 length of this is let us see l 1. So, it will be interesting to find out the location of this particular position, where displacement is 0. This is called a node here, there is no angular displacement is taking place. So, we have seen from the previous mode shape that this particular 2 rotor system can be considered as 2 single degree of freedom systems.

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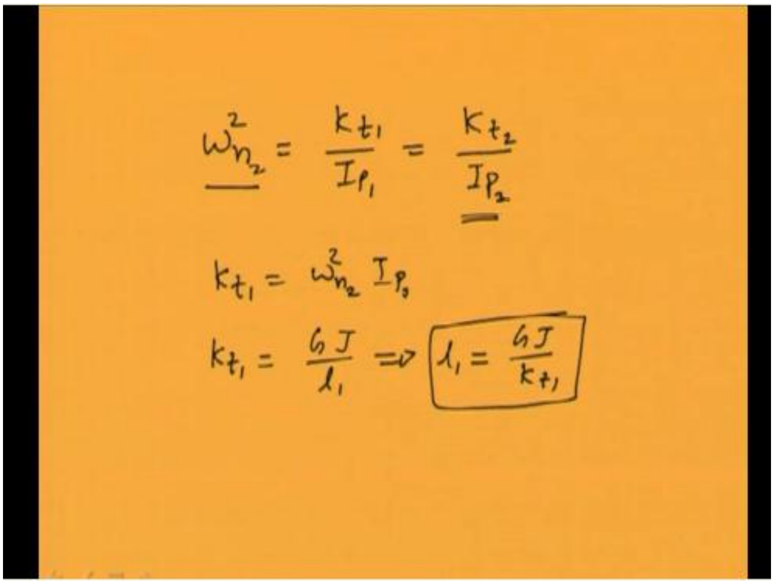

And, the natural frequency of that particular system can be written like this, which will be equal to the natural frequency of the second single degree of freedom system for the cantilever case. And this particular equation because we know the natural frequency of the system it will remain the same for both the cases. And I p 1and I p 2 we know. So, from here; we can able to get the K t 1 which will be this and K for proportional vibration we know that; this is given by l 1. And this particular equation will give us the position of the node. So, in the same level we can able to obtain the position of the node from this also, but that will this is a redundant equation. Now, you consider another kind of system in which two discs are there, but shaft is having a different dimensions in between. So, that particular type of shaft is called stub shaft, because in some application we may find these kinds of shafts.

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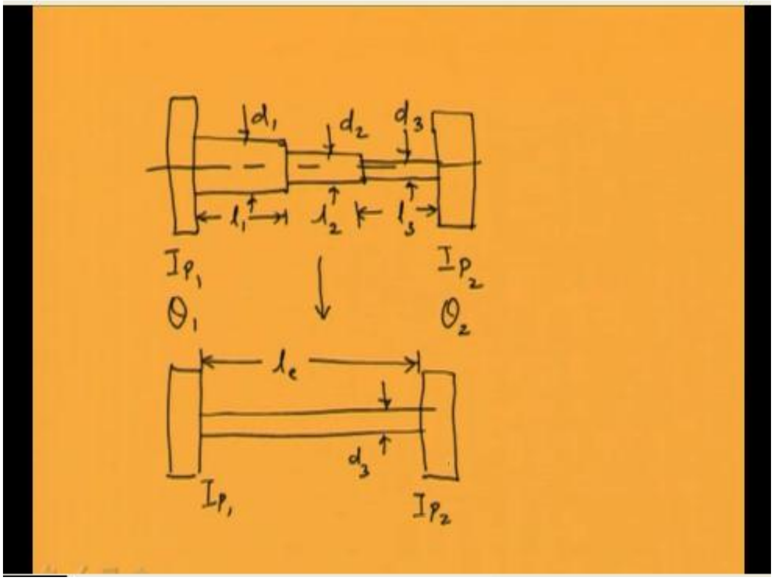

So, let us first see the system this is the axis of the shaft, this is the one segment of the shaft then there is a step like this then another step is there, and then there is a disc here. So, disc I p 1 I p 2 and corresponding displacements are theta 1 and theta 2. In this particular case; let us say the lengths of these are given as this. And they have different diameters also. So, this particular system is exactly similar to the previous case. Only difference is, here we have different shaft diameters of the shaft. So, it will be very good if we can convert this particular system to equivalent uniform shaft system and; that means, if this particular system we can able to convert to a system like this.

Let us see this equivalent shaft diameter is l 3, and this is the equivalent length of the shaft. So, if we can convert this system to this then the analysis which we perform in the previous section is valid. So, let us see how we can able to do it, to convert the stub shaft into the uniform shaft the main aim is because the shaft is giving the only the stiffness in into the torsional vibration. So, if we can have some kind of equivalent stiffness then we can able to obtain the equivalent system.

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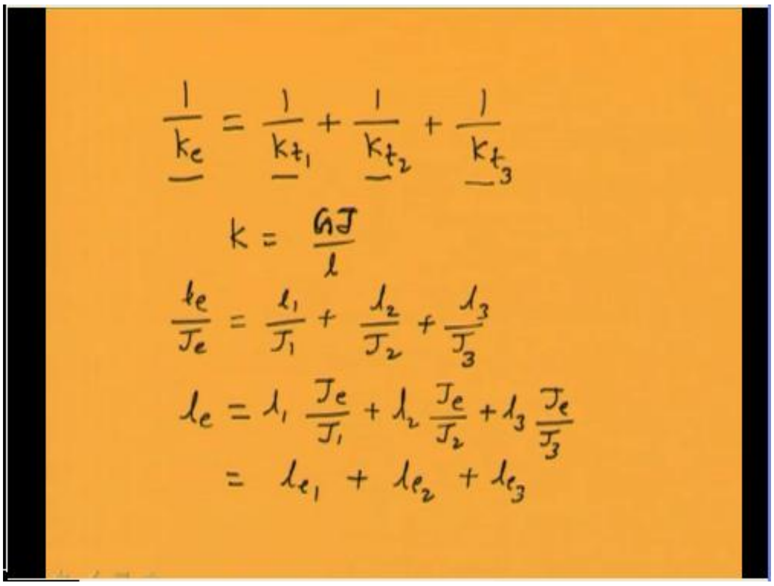

So, because these stub shafts are connected by series. So, we can able to write the equivalent stiffness of the uniform shaft like this, because we have 3 stubbed shafts. So, these are the torsional stiffness of the 3 stub shaft, and this is the equivalent stiffness of the overall system. This can be, because we know K is given by G J by l where this is the modulus of rigidity J is the polar moment of area, l is the length of the shaft segment. So, using this equation can be converted into or because G is common, so I have eliminated from both the sides which can be written as like this.

Once we obtained the equivalent length of the uniform shaft then the analysis in the previous section is valid here, we can obtain the mode shape of the system, we can obtain the natural frequency of the system. Now, the question comes because those mode shapes are corresponding to the equivalent system, how we can able to convert that into the real system would like to see now.

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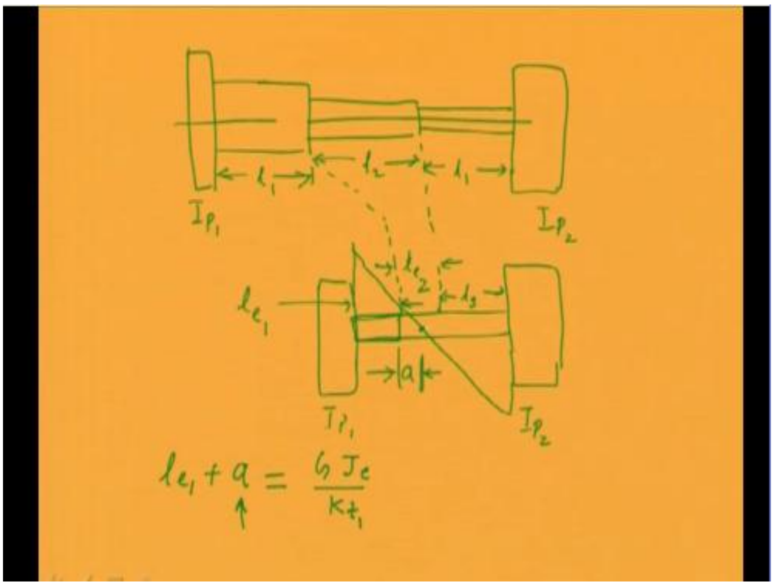

So, let us draw the actual system first, this is the actual system. We converted this into equivalent system like this, because we took smaller diameter for the equivalence, so will be getting lesser length of the shaft. The mode shape for this particular system is let us say this one, where this is the node nodal position. Now, if we want to find out the various positions of the shaft on to the equivalent system, if this is the l 3 in equivalent system that will remain l 3. And corresponding to this second segment; let us say this is this is l 2, and this is l equivalent 2. And the third one is that is corresponding to l 1equivalent l 1, and the actual system is l 1.

Now, let us say the position of the node this is the node position. On the equivalent system is a from the this position l l a 1 position. So, in the equivalent system we can able to locate the node position as we did in the previous case. Where, K t 1 is the, this stiffness of the first part of the system this one after this. And from here we can able to obtain a, because other quantities are known. Now, we need to find out the equivalence of this in the actual system where, is the load in the actual system. Find out the position of the node in the actual system.

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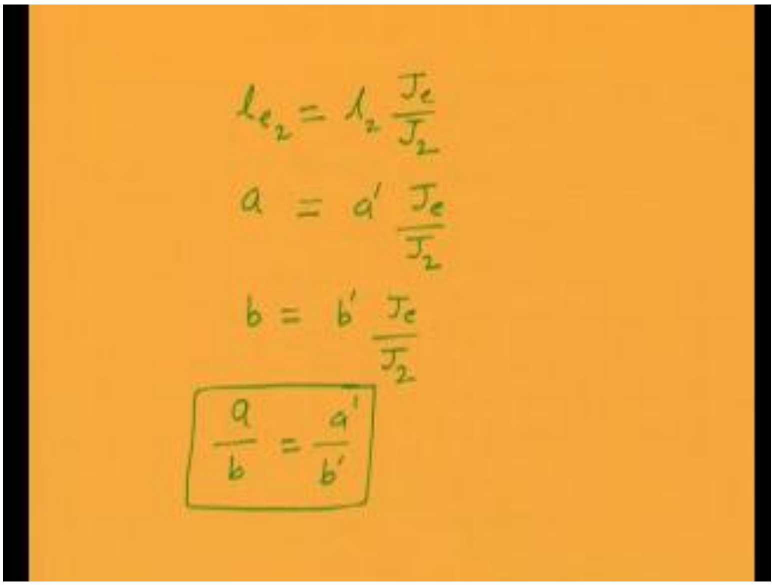

Let us see the previous relations which we had for the equivalent system for the segment 2, this. So, if a is the equivalent node position for the segment 2 then let us say a prime is in the actual system. Or similarly if we have from other side; that is from the right side of the element 2, the node position is given by b in equivalent system. And b prime is in the actual system then we can have this relation. And if we take the ratio of these 2 equations we will get a by b is equal to a prime by b prime. So, you can see that once we are located node in the actual in the equivalent system, in the actual system also the node positions will be there in this in the same proportion.

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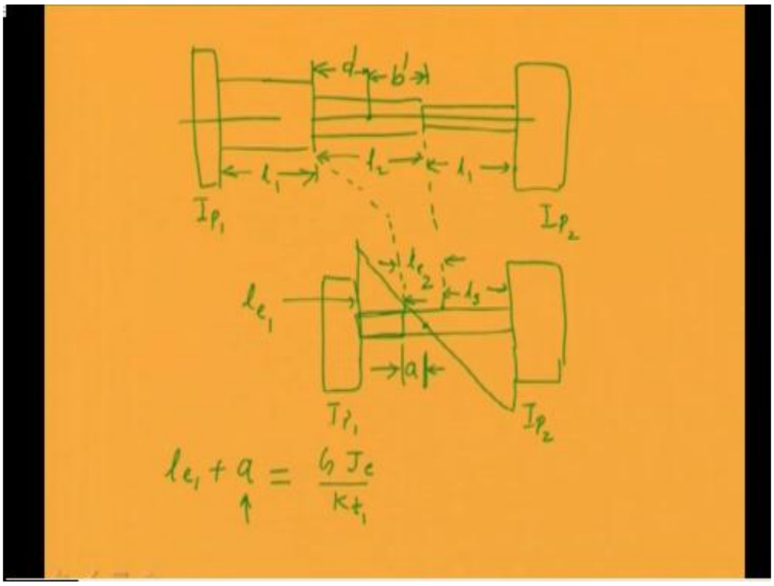

Where this a prime and b prime are like if node is here. So, a prime will be this, and b prime will be this. Now, we will consider this 3 rotor system and in this particular 3 rotor system the basic principle of the 2 rotor system is equally valid here. Here, 3 rotors are collected by 2 shafts. So, we can have if we want to see the motion especially the how they will be vibrating in free or normal modes, either they can have one mode in between the shaft, or they can have 2 nodes in the shaft. If they have one node in the shaft then; obviously, the 2 extreme discs will be having opposite phase. That this will be seen in more detail, and if there are 2 nodes in between the shaft then the 2 disc at the extremes will be having same phase.

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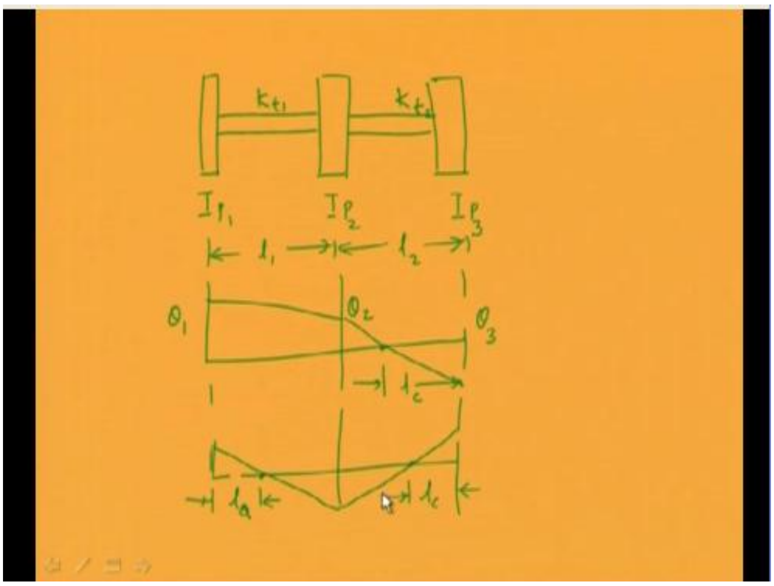

So, let us see this particular analysis. So, this is a one rotor is connected by a shaft is another rotor. For simplicity we are considering the uniform shaft, but we can able to take non uniform shaft also. But then we have to go for the equivalent system in that case. So, let us say they are mass moment of inertias are like this. And this is these are the torsional stiffness of the shaft, lengths of the shaft are let us say; know if we directly want to go for the mode shape analysis first, and then from there we will try to find out the natural frequency of the system.

Let us say there is only one node in this particular system. Then, how the mode shape look like. So, let us say node is here. So, this is the node shape which will be. So, here we have theta 1 theta 2 theta 3, theta 3 is negative. As I said earlier and there is one node and then theta 1 and theta 3 will be having opposite phase. In another case; when we have 2 nodes, one node is here another node is here. Let us say the position of those nodes are; l a from left hand side, and l c from right hand side. On the same lines we can able to write this as l c. This is the position of the node from the right hand side.

This regarding the boat shapes we have seen that either it can have one node in between the shaft, or it can have 2 nodes in there in between the shaft. So, let us consider the general case when it is having 2 nodes in the shaft. So, how we can able to analyze; left part of the shaft will be a cantilever, because there is no displacement at this point. So, we can consider this as a cantilever beam, another is the right extreme right position there also we can able to consider that as a cantilever beam. And in between this the middle portion will be having a fixed fixed beam fixed fixed rod and the disc is in between those fixed ends. So, there are 3 systems now, but because they are vibrating in natural mode all the natural frequency of these systems will be same. So, let us right those equations of motions now one by one.

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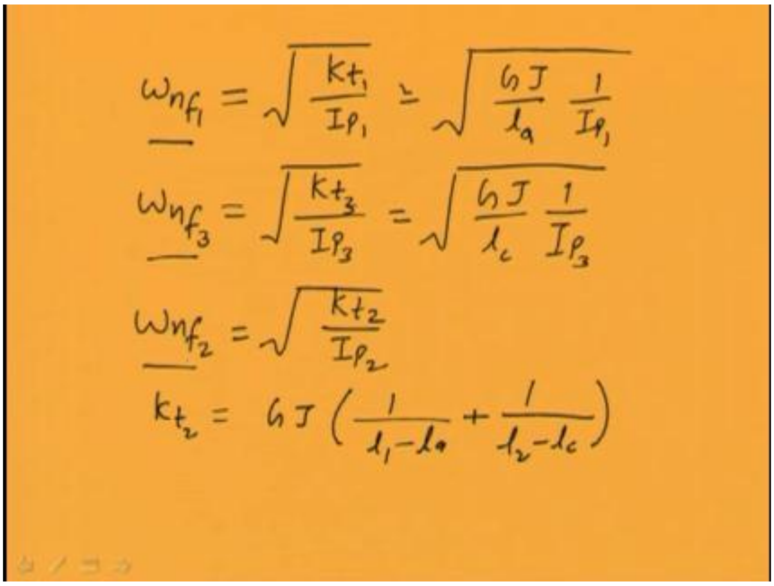

Disc at extreme left we can able to right the frequency equation as, and if you substitute for the torsional stiffness this value, l is the length of that left hand side a cantilever beam. On the same line we can write for the third disc, this is natural frequency for the third disc. And for the middle rotor or will be having natural frequency this K t 2 is given as now these 3 frequencies 1 2 and 3 for 3 disc because they are belonging to the same system the natural frequency should be same. So, we can able to equate these 3 equations and our aim would be to obtain either l a or l b. Eventually, these equations when you equate these 3 equations, we will get a quadratic equation either in terms of l a or l b. So, let us see the steps.

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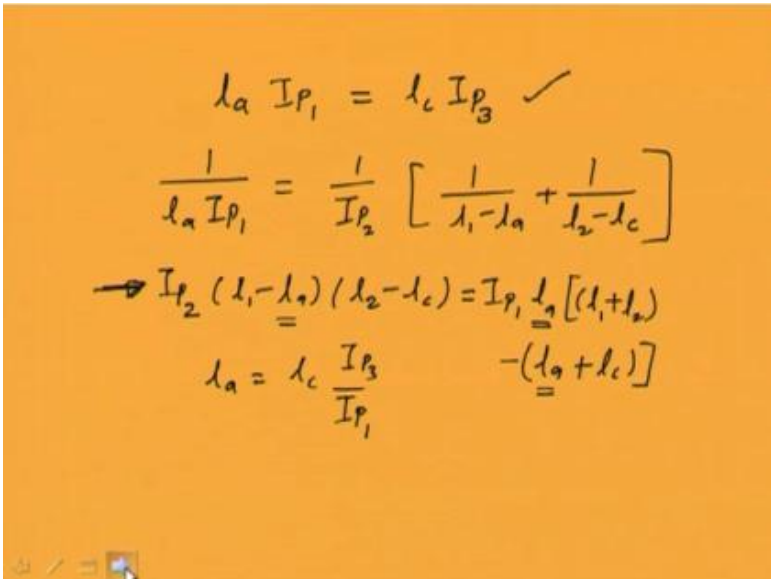

So, if equate a natural frequency of the first disc and the third disc. We will get this equation, and if we are equate the natural frequency of the first and the second disc; we will get equation which can be written as, in this particular equation. Now, this particular equation contain both l a and l c. So, from this equation we can able to eliminate l a from this particular equation. So, if we do it we will what we have to do we have to substitute; let us say having l a is equal to l c I p 3 by I p 1, and we if you substitute this here and here.

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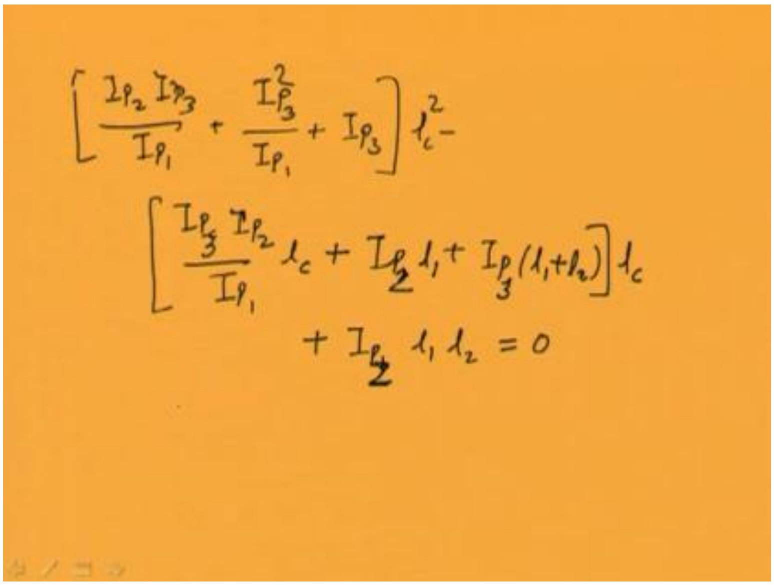

And, if you solve it we will get this following equation; l c square minus is a big equation l c and then plus I p b l 1 l 2 is equal to this is 2. So, this is a quadratic equation in l c we expect 2 solutions from this. And those will be corresponding to the one correspond to the 2 node system. And another corresponding to the single node system, and let us go back to the mode shapes.

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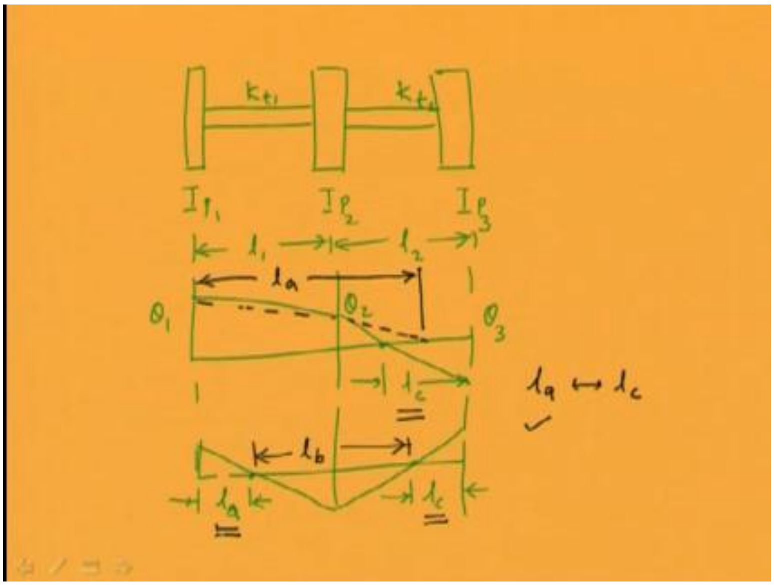

So, we will be from that quadratic equation we will be getting 2 solutions for l c. So, if you see in this mode shape on will corresponds to this, and another will be corresponding to this. And because we have relationship between the l a and l c. So, we will be getting 2 values of the l a also. So, one value will be this one, another value will be if we extend this particular line up to here. So, l a will be second value of value l a will be corresponding to this having no meaning, but because our solution gives to 2 value of l a. So, that is the interpretation of those 2 solutions. Now, let us consider some examples. So, that our concepts you are more clear, first we will consider the 2 degree of freedom system.

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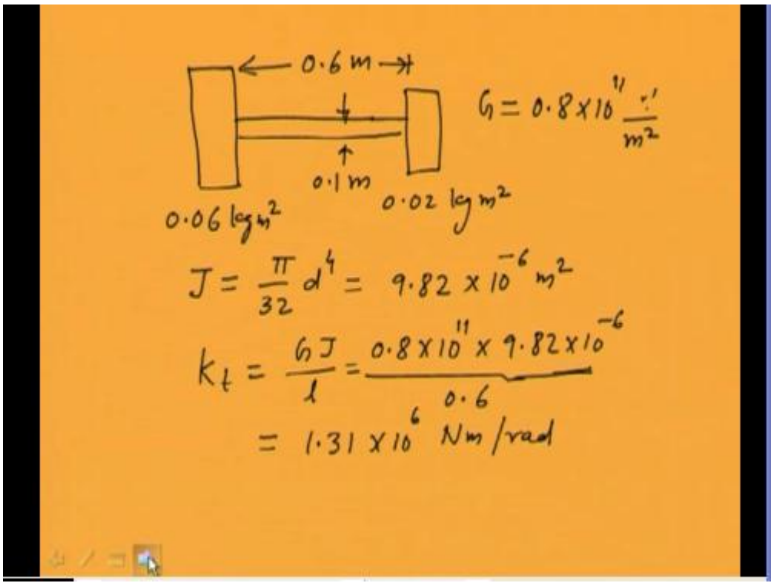

So, first shaft is having 0.06 kg meter square mass moment of inertia, second shaft is having 0.02 kg meter square, and diameter of the shaft is let us say 0.1 meter, length of the shaft is 0.6 meter. We need to obtain the natural frequency of the system. Let us say the modulus of rigidity is given as 0.8 10 rise to 11 Newton per meter square. Now, with this information we can able to calculate the polar moment of area of the shaft that will be this. So, we know the diameter. So, it comes out to be this much. Now, because the torsional stiffness is given by G this expression, so we can able to substitute these quantity here. Length of the shaft is 0.6. So, this gives us the stiffness of the shaft as Newton meter per radian. Now, we will be substituting these values in the natural frequency equation.

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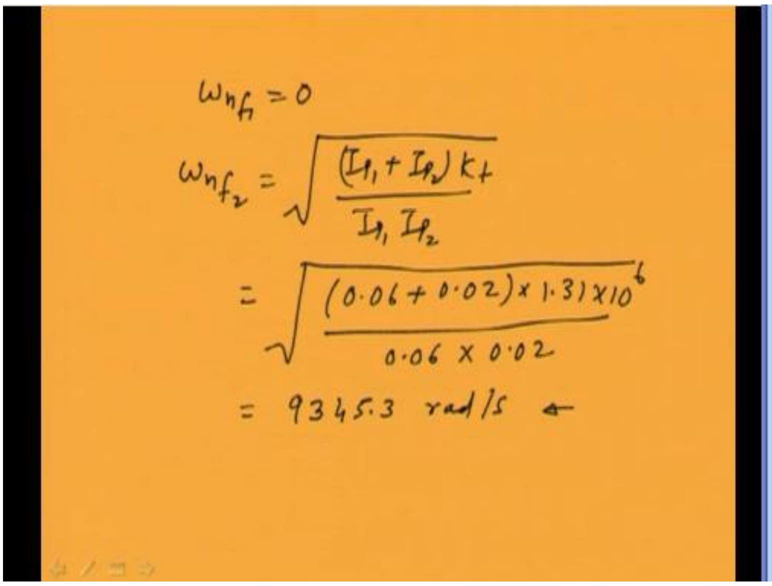

First natural frequency remains 0 rigid body mode, second natural frequency we obtained is I p 1 plus I p 2 K t divided by I p 1 and I p 2. So, if we substitute these values here; 0.06 plus 0.02 into 1.31 10 raise to minus or 10 raise to 6 divided by 0.06 into 0.02. So, this gives us 9345.3 radians per second as the second natural frequency. You can see this natural frequency is quite high in a range of 9000 radians per second. As compared to that transverse vibration generally these natural frequencies are very high.

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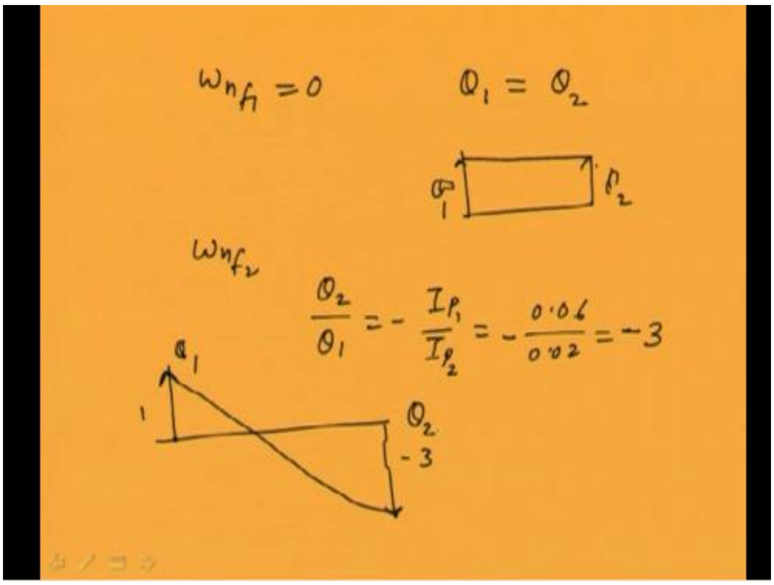

Now, let us see the modes shapes corresponding to the first natural frequency will be having both the displacement same. So, mode shapes remains like this, both are same theta 1 theta 2. Corresponding to the second natural frequency; will be having mode shape which is 0.06 by 0.02 minus. So, it comes out to be minus 3. So, the mode shape will be something, if we have 1 here, second will be having minus 3 displacements. So, this is corresponding to the second natural frequency mode shapes.

Or, now we will be considering another kind of system in which gears are there, and in such systems also if is a problem of torsional vibration. And here our aim would be to obtain the natural frequency of the system, so that we can avoid any resonance condition from this. So, generally in this particular kind of systems; because of the gear will be having different speeds of the shafts may be because there will be gear ratio after the gears. So, there will be either speed reduction or speed increase in the shaft will take place. So, let us see how we can able to analyze for torsional vibration case. Such systems here our main aim would be; if we can able to convert this kind of systems to the some kind of equivalent single shaft system. Then, we can able to analyze using the analysis which we already covered previously.

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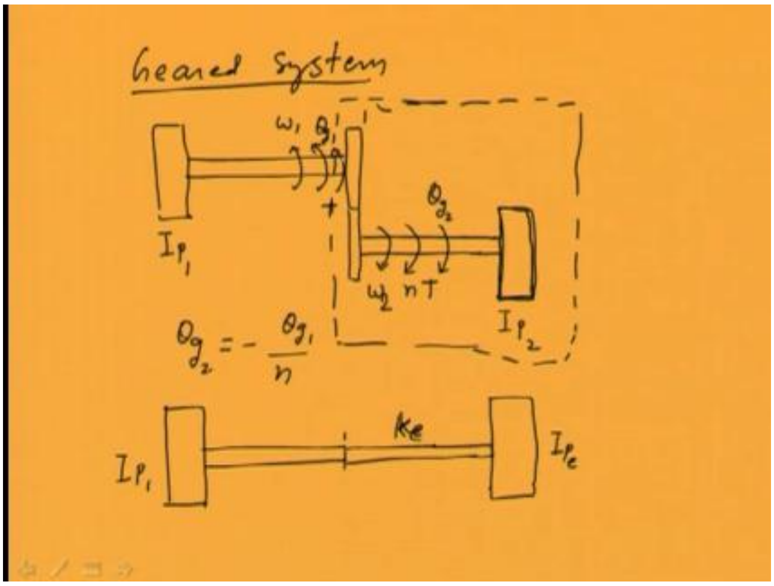

So, let us see the geared system; so let us say we have one shaft which is connected by gear. And then with another gear we are transporting torque to the another shaft. Let us say the mass per moment of inertia of this particular rotor is I p 2. And here, there is another rotor or disc which is having mass moment of inertia I p 1. Here, we are assuming that there is mass moment of inertia the polar mass moment of inertia, if gears are negligible, but later on we can able to add that also. For simplicity in this particular analysis; we are not including the polar mass moment of inertia of the gears.

So, this is the gear system, in which we have speed of this omega 1 the twist of the gear 1, torsional displacement of the gear 1 is theta 1. And the torque which is coming on to the shaft at gear 1 is let us say T. Now, because of this gear pair there will be change in the angular velocity of the shaft 2, let us say that is omega 2. And there is change in the torque that is n T; n is the gear ratio, and there will be change in the angular twist of the shaft. That is let us say g 2 that g 2 theta g 2 is related with theta g 1 by the gear ratio.

Actually this is theta g 1 let us say, distinguish between this rotor and the gear displacements. Now, we want to analyze this particular gear system. So, if we consider this particular system is enclosed in a black box. Let this is a black box, and we want to find out an equivalent system in which let us say this is the I p 1, this is the shaft. And after this corresponding to this particular box, let us say we are having equivalent stiffness K e and equivalent polar mass moment of inertia this one. So, if we can convert the geared system into this particular system what we will find that we already analyze these 2 disc system.

So, our analysis will be valid whatever the, we have covered previously for gear system also, if we can find out the equivalence of that. So, our aim would be to obtain first to obtain the equivalent system of the gear system. And the here because for finding the equivalence we will be having 2 quantities which will be of our concern; one is the strain energy in the system. So, whatever the strain energy is there in the original system should be equal to the stain energy in the equivalent system. Similarly, another quantity that is the kinetic energy of the actual system, and of the equivalent system should be same. And based on this 2 energy will be obtaining what should be the equivalent stiffness of the equivalent system and what should be the polar mass moment of the inertia of the equivalent system.

So, let us first see the equivalence of the strain energy. So, to obtain to equate the strain energy in the system of the actual system and the equivalent system what we will be doing, first we will be fixing let us say the, this particular disc will fix it. And will give to gear 1 theta g 1 rotation. So, what we are doing we are fixing the I p 2 the disc 2 by some means and we are giving gear 1 theta g 1 displacement. And we will try to find out how much strain energy stored in the gear this shaft 2.

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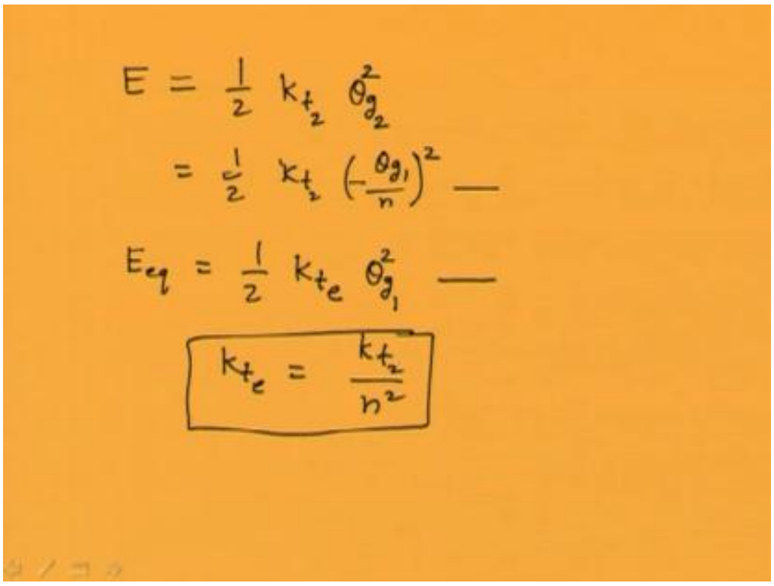

And, will be is given by this is the strain energy in shaft 2 that will be half stiffness of the shaft. And the angular twist of that is theta g 2, and we have a relationship between the theta g 1 and theta g 2 like this, this is the square of that. Here, also square minus sign will be having no meaning here because this square of the quantity. So, here will be this is a strain energy stored in the actual system. Now, the strain energy in the equivalent system can be written as; this is the torsional stiffness of the equivalent system. And it is getting twist same as the gear 1 this. So, now this 2 energy if we equate; we can see that we get this relationship, that means; if we divide the torsional stiffness of the actual system of the shaft 2 by the gear ratio square will get the stiffness of the equivalent shaft system. To find out the equivalent polar mass moment of inertia of the equivalent system, now we have to equate the kinetic energy of the actual system and the equivalent system.

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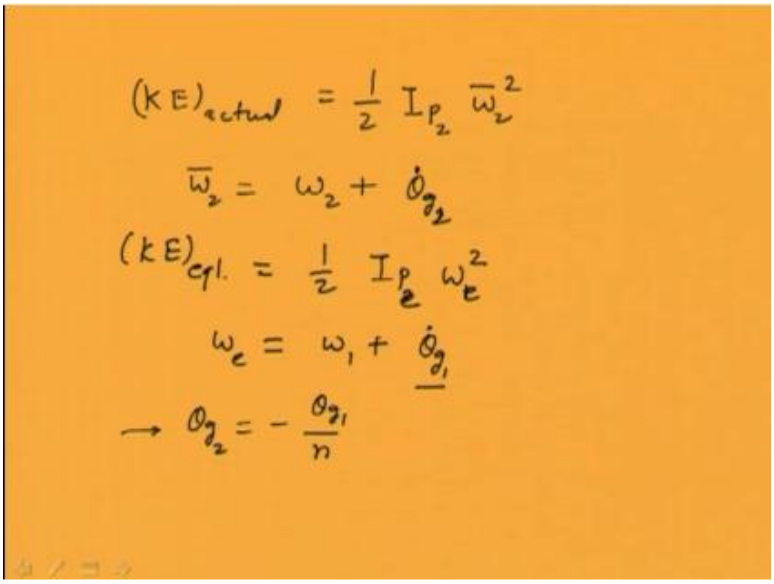

So, first let us see kinetic energy as omega 2 prime square where, omega 2 prime is nothing, but actual speed of the rotor plus its angular twist. And kinetic energy of the equivalent system can be written as omega 1, let us say equivalent square. So, here omega equivalent is omega 1 plus theta g 1 dot, this is the angular displacement of the gear 1. Now, because we have relationship between the gear displacement 1 and gear displacement 2, so we can substitute these equations in the kinetic energy, and we can equate them.

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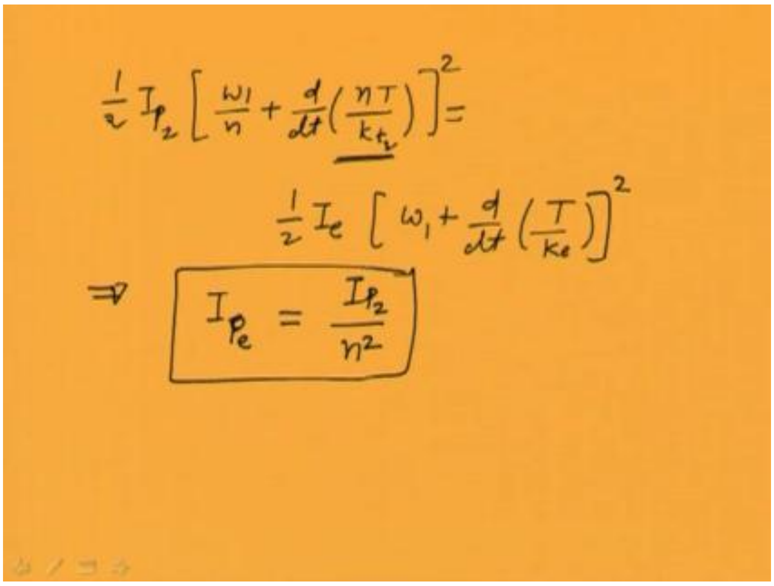

So, let us start equating them. So, this is the kinetic energy of the actual system in which we are placed omega 2 by omega 1 by and plus theta g 2 has been replaced by this. This quantity is nothing, but theta and time derivative with respect to time. And whole square should be equal to kinetic energy of the equivalent system. Now, if we simplify this; we can able to see that the equivalent mass moment of inertia, polar mass moment of inertia comes out to be I p 2 by n square. So, here also we can able to see that the polar mass moment of inertia of the equivalent system can be obtained; if we divide the polar mass moment of inertia of the actual system divided by the square of gear ratio.

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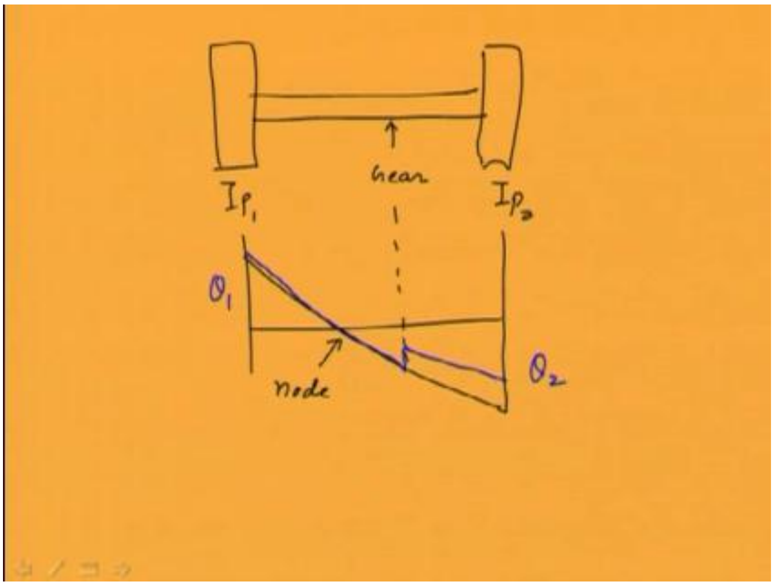

Now, once we obtain the equivalent system which is something like this, in which we have one disc here, another disc here, and gear is somewhere here. So, if we want to obtain because this system you already analyzed previously. So, the mode shape can be obtained from this, this is the mode shape. Now, this particular mode shape is for the equivalent system. Now, how we will be obtaining the mode shape for the equivalent actual system to be seen, because this is the location on the gear. This is the node position where is node displacement is taking place. So, for this particular case the node is coming before the gear here.

So, let us see what is the correction we need to make on the, this mode shape to get the actual mode shape? So, what we need to do up to this point, there is no change because there is no change in the speed of the shaft up to that point, because gear is here. After this, because whatever the displacement is taking place, because these are nothing but theta 1 and theta 2. So, whatever the displacement is taking place in this range it has to divided by n that gear ratio. So, the mode shape will change like this. So, you can see that there is a step change of the mode shape, at the gear ration will take place. And this is the modification we need to incorporate, because we have converted the actual system to equivalent system and we analyze system. Now, to get the mode shape in the actual system this particular modification has to be considered. So, this is one important aspect we need to consider in this particular case.

In this particular lecture we have seen the introduction to the torsional vibration; we consider general degree of freedom system, 2 degree of freedom system, 3 degree of freedom system. And then we have seen how we can analyze the geared system in which the 2 shafts are having different speeds. Now, in future in the next lecture we will be considering more general methods available for analyzing the multi degree of freedom systems.