# Mechanical Vibrations

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## Module - 5
## Two DOF Free Vibrations
## Lecture - 1
## Generalized and Principle Coordinates, Derivation of Equations of Motion

Today we are going to study about two degrees of freedom systems. So, in the previous classes you have studied about the free and force vibration of single degree of freedom systems. In case of free vibration of single degree of freedom system, you have studied the system without damping and with damping. In case of system with damping, you have studied three different cases; one is under damped case, second one is the critical damped case, and third one is the over damped case.

In case of steady state response of the system, you have determined the steady state response using different methods. So, you have used a simple force polygon method to derive the magnification factor or you have derived the steady state response of the system. So, the total response of the system you know equal to the transient response of the system and the steady state response of the system. And in case of a system with damping, so that transient response dies down and you have only the steady state response.

So, in case of single degree of freedom systems, you have studied also about the rotating unbalance, support motion, whirling of shaft and different vibration measuring instruments. So, today class we are going to study about two degree of freedom systems. So, you know that many machine components cannot be modeled as a single degree of freedom system. So, it may be modeled as a two degree of freedom system.

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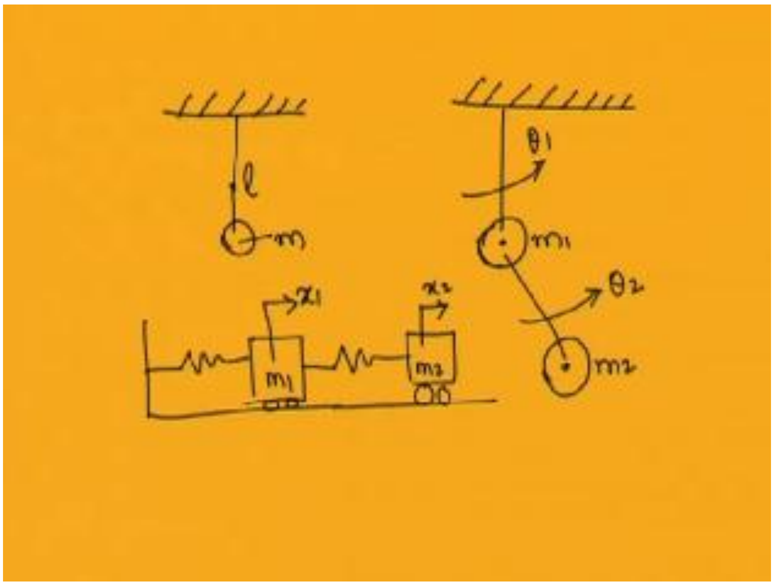


For example, if you take a simple system like a double pendulum. So, I am drawing a double pendulum. So, already you have studied about the case of a single or simple pendulum. So, in case of a simple pendulum, you have a bob. So, with mass m and it is hanged by a mass less string of length l. So, in this case you have derived the equation of motion. Now to derive the equation of motion for this case in which let you have two mass m.

So, this is m 1 and m 2, and they are hanged by two springs. So, you require more number of coordinates to express the motion of this system. So, you require to define the motion of this point and you require to define also the motion of this point. So, define the motion of these two points. So, you require minimum two coordinates to express the motion. So, these two coordinates may be this theta 1 theta 2. So, it may be the motion of these.

So, theta 1 and the motion of this also theta 2; so, you can find the motion of theta 1 and theta 2. So, these are the two minimum coordinates you require to express this motion of this double pendulum, or you just take this another simple spring mass damper system or spring mass system. So, this is one mass and this is the other mass; they are connected by the springs. So, if you want to express the motion of these two mass, here the minimum coordinates you required the motion of these mass that is x one. So, this is m 1 and this is m 2, and the motion of this mass this is x 2.

So, here also you require the minimum coordinates two. So, when you require two coordinates to express the motion of the system, then this will be two degrees of freedom system.

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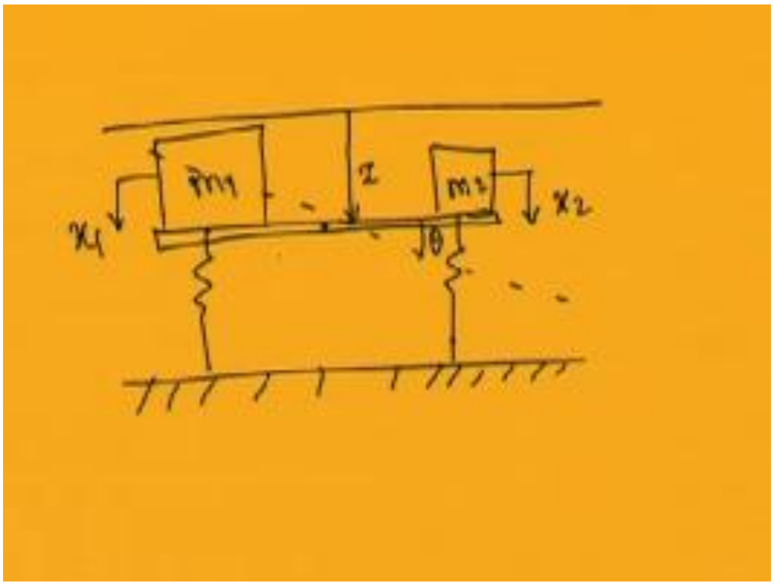

In case of a lathe machine also you can model the lathe machine with its heads stock and tail stock. So, let this part is the head stock, and this part is the tail stock. And it is on the bed and the support can be modeled as a spring. So, you can model these lathe as a two degrees of freedom system with these head stock mass here. So, this is m 1, this is m 2, and this you can concentrate or you can distribute the mass of the bed and other parts in these two parts.

So, in this case the vibration of this you can express in different ways. So, you can write the vibration or motion of this in the transverse direction by this motion of these. Let the motion of this will be x 2, the motion of this may be x 1. So, you can express this motion by x 2 and x 1, or you may express the motion by the angular displacement. So, let this is the equilibrium position, and so you can express the motion by this displacement x and this angle theta.

So, in this case you can express the motion by this translation motion of this mass enter by x and the angle theta. So, there may be different ways to express the motion of the same system.

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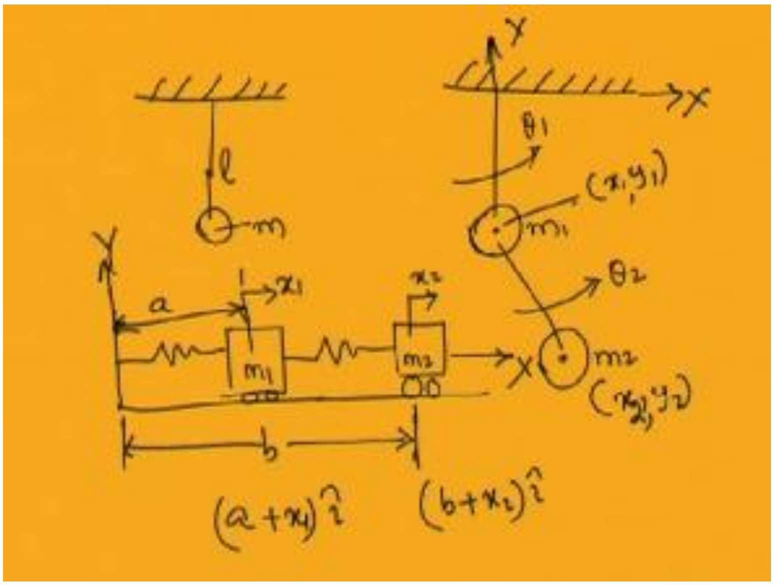


For example, in case of this double pendulum you can express the motion by theta 1 theta 2, or you may express the motion by putting a coordinate system. So, a coordinate system you can put this is X, this is Y. So, you can write this coordinate of this in x 1 y 1. So, you can write it x 1 y 1, and the coordinate this is x 2 y 2. So, either you can express the motion of this double pendulum by using these coordinate x 1 y 1, x 2 y 2 or theta 1 theta 2. So, these coordinates x 1 y 1, x 2 y 2 are the physical coordinate of the system.

So, you are describing a reference point from which you are finding this motion x 1 y 1 and x 2 y 2. So, these are the physical coordinates of the systems. So, in this case also you can have a physical coordinate; you can fix the coordinate system x and y here. So, this will be your Y coordinate, and this is the X coordinate you can fix. So, from this X Y from this coordinate system from the equilibrium point you can take, let this is the equilibrium point position. So, you can take this as a. So, it will be a plus x 1 and similarly, you can take this point.

So, let this is the equilibrium position. So, from this you can take this as b. So, the physical coordinate of this you can write as a plus x 1 i for mass m 1, and for mass m 2 you can write the physical coordinate as b plus x 2 i. So, you can write where i and j are the unite factor in x and y direction. So, in these cases or in case of a multi degree or two degree of freedom system, you can use different coordinate systems. One is your physical coordinate system and other one is the generalized coordinate system.

A generalized coordinate system is a system in which you are expressing the motion of the system by using minimum number of coordinates. So, in case of this double pendulum, this theta 1 theta 2 are the generalized coordinates and x 1 y 1, x 2 y 2 are the physical coordinates of the system. So, by using this generalized coordinates or physical coordinates, you can derive the equation of motion.

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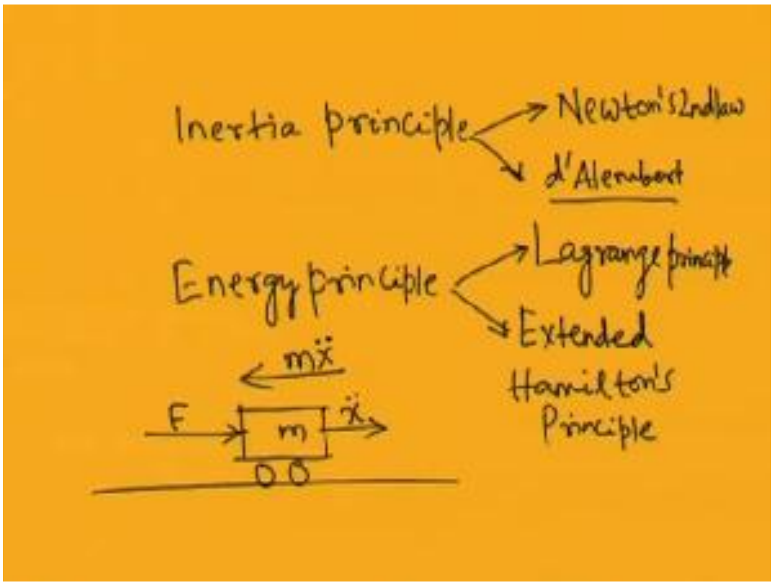


Already you know to derive the equation motion; you may use the inertia principle. So, you may use the inertia principle or you may go for this energy principle. So, today class I will tell you how to derive this equation of motion by using this both inertia principle and energy principle. In case of inertia principle, you know you can use Newton’s method or you may use d’Alembert principle. And in case of energy principle, you can go for Lagrange principle or you may go for extended Hamilton principle.

So, in both the cases in case of Lagrange principle or extended Hamilton principle, you require to find the energy of the system and the work done by the forces of the system. So, energy of the system includes the potential energy and kinetic energy and work done by the non-conservative force also you have to find. In case of inertia principle, you are using Newton’s second law either you are using Newton’s second law or you may go for d’Alembert principle.

So, you know both the methods are similar, and you are converting a dynamic system to an equivalent static system by using the d'Alembert principle. The d'Alembert principle if you recall, the d'Alembert principle is nothing but. So, if you have a mass subjected to a force let it is subjected to a force F; this is a mass m. So, it will be accelerated by an amount a or this acceleration you can write in terms of x double dot. So, according to this d'Alembert principle you can convert this dynamic system to an equivalent static system by incorporating the inertia force of the system.

The inertia force is nothing but mass into acceleration, and it acts in the direction opposite to the direction of acceleration. So, the inertia force direction will be this. So, the inertia force is m x double dot, and it is acting opposite to the direction of acceleration. So, this is the direction of acceleration, and it is acting opposite to the direction of acceleration. So, according to Newton's second law, your equation of motion becomes.

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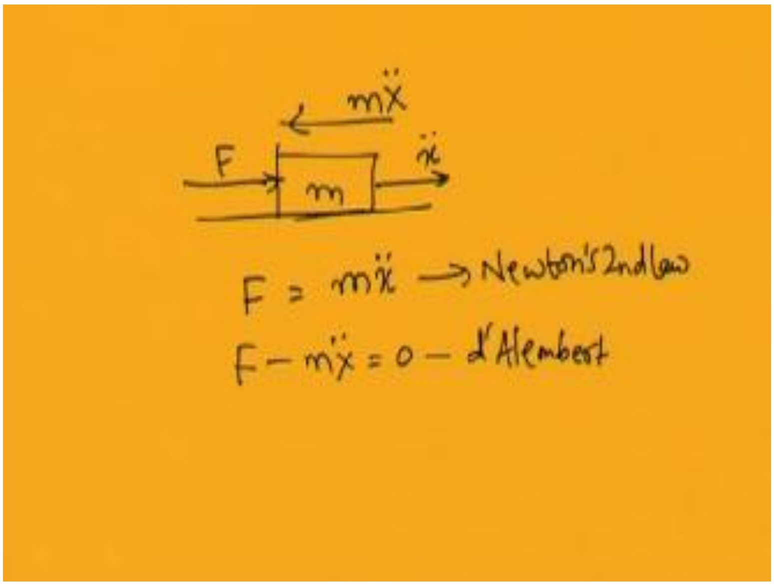


So, according to Newton's second law for this case; so, if you have a mass m, it is subjected to a force F. So, already I told you that inertia force is m x double dot, and this is your acceleration x double dot. So, according to Newton's second law your F equal to mass into acceleration, but according to this d'Alembert principle summation of forces which include the inertia force will be equal to zero. So, F minus m x double dot will be equal to zero. So, this is d'Alembert principle, and this is Newton's second law.

So, already you know by using this Newton's principle Newton's second law or d'Alembert principle, you can derive the equation of motion of a single degree of freedom system. So, today we will derive some of the equation for the two degrees of freedom system by using these Newton's method.

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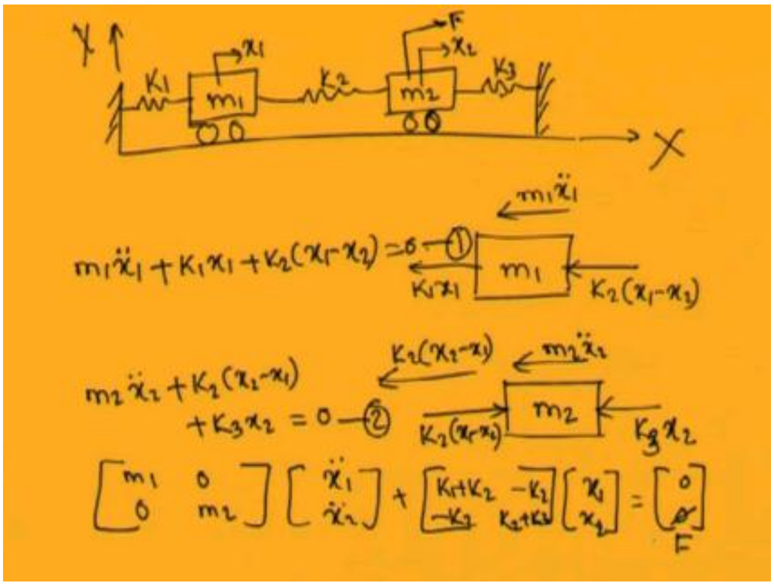


So, let us take a simple two degree of freedom system with two mass and springs. So, let me take two mass. So, this is mass m 1. So, it is connected to another mass m 2. So, they are connected to a rigid constant. So, let me write this spring constant k 1. So, this spring constant k 2 and this spring constant equal to k 3. So, I can write the equation of motion by taking a physical coordinate system here. I can take a physical coordinate system from a reference coordinate system or I can take a generalized coordinate system; let me take a generalized coordinate system. Let x 1 and x 2 are the generalized coordinate system.

So, this x 1 and x 2 are the coordinates I have taken from this equilibrium position of mass m 1 and mass m 2 respectively. For a physical coordinate system, I can take this is the reference axis. So, this is I can take X axis and this I can take Y axis. So, for a physical coordinate system, I can take this X and Y coordinate system and this equilibrium position I can take this equilibrium position from this end I can take it as a, and form this I can take it as b.

So, the physical coordinate if I want to write, then this physical coordinate I can write it equal to a plus x 1 and b plus x 2, but by using these generalized coordinates now I am going derive the equation of motion by using d'Alembert principle or by using Newton's second law. Both are same; already you have seen one in case of d'Alembert principle, you are converting that system to an equivalent static system. So, for finding this equation of motion, I have to draw the free body diagram. So, already for a single degree of freedom system you know how to draw the free body diagram.

So, this is for mass m 1, I can draw the free body diagram and then for mass m 2, I will draw the free body diagram, and then I will write the equation of motion. So, for this mass m 1, the forces acting on this mass m 1 are the spring force k 1 x 1. So, as the mass will move towards right, then the spring will be extended and in turn it will exert a force k 1 x 1 on the mass. So, it will exert a force k 1 x 1 on this mass k 1 x 1. So, the other force acting on this to the right of this, this is spring k 2.

So, due to the motion of this mass m 1 it will be compressed by an amount x 1, and due to the motion of this x 2 it will exert a force on this mass m 2 in this direction. So, the net of this spring or net motion of this spring will be equal to x 1 and x 2 in this direction. So, as the net motion of the sprig in this direction is x 1 minus x 2. So, the force acting on this mass will be equal to k 2 into x 1 minus x 2. So, in addition to these two spring forces, the force acting on the mass is the intertie force also. So, this inertia force already you know that it is opposite to the direction of acceleration.

So, it will have a force m 1 x 1 double dot. So, it is equal to m 1 x 1 double dot, and it acts opposite to the direction of motion; similarly, for mass m 2 I can write. So, I can write the equation of motion for this mass m 1 like this. So, summation of all the forces will be equal to zero. So, the equation of motion will be equal to m 1 x 1 double dot plus all the forces you just note that they are acting in the same direction. So, m 1 x 1 double dot plus k 1 x 1 plus k 2 into x 1 minus x 2 equal to zero.

Now I will draw the free body diagram of this mass m 2. So, for mass m 2, what are the forces acting? Forces are acting by spring k 2 and spring k 3. So, when this mass is moving towards right. So, the spring k 3 will be compressed. So, it will exert a force on the mass in opposite direction. So, that force equal to k 2 k 3 x 2. So, this force is k 3 x 2, and this spring already you know that it has a relative motion of k 2 x 1 minus x 2 in this direction. So, on this mass it will exert a force or it will exert a force.

So, if you are writing in the same direction. So, it will exert a force of k 2 into x 1 minus x 2 in this direction or if you want to write in other way; so, you can write a force of. So, in this way also you write x 2 minus x 1. So, either you can write k 2 into x 1 minus x 2 opposite to this direction. So, you just see that a net force acting on the springs would be equal to zero. So, this force plus this force if you added it, it is equal to zero. So, either you can write it in this form k 2 into x 1 minus x 2 in this direction or k 2 into x 2 minus x 1 in opposite direction.

And in addition to this, you also know that the inertia force m 2 x 2 double dot is acting on this. So, in this case the equation of motion becomes. So, no other force is acting on the system. So, the equation of motion becomes m 2 x 2 double dot plus k 2 into x 2 minus x 1 plus k 3 x 2 equal to zero. So, now you have a set of equation; this is equation number one; this is equation number two. So, you have two equations for the system. So, these equations you can arrange in a matrix form also. So, you can write this as mass matrix into x 1 double dot x 1 double dot x 2 double dot plus stiffness matrix into x 1 x 2 x 1 x 2 equal to 0 0, because there is no force acting on the system, and you are considering a free vibration of the system.

So, if you want to consider the force value vibration of the system if a force F is acting here. So, if a force F is acting on mass m 2. So, your second equation will be converted to m 2 x 2 double dot plus k 2 x 2 minus x 1 plus k 3 x 3 equal to F. So, in this case your mass matrix becomes. So, you just note that it is equal to m 1 0, and in this case it is equal to 0 m 2. So, m 1 x 1 double dot plus 0 m 2 x 2 double dot, and in this case this is k 1 x 1, and this is equal to k 2 x 2.

So, k one plus k two, you can write this is equal to k 1 plus k 2 and then minus k 2. Similarly, you can see for this case it is equal to k 2 x 2 minus k 1 x 1. So, this is equal to minus k 2 x 1, and this is k 2 x 2, and this is k 3 x 2. So, this becomes k 2 plus k 3. So, your stiffness matrix becomes k 1 plus k 2 minus k 2 minus k 2 k 2 plus k 3 x 1 x 2; this is 0 0. So, if a force is acting. So, this zero will be replaced by a force F. So, in this way you can find the equation of motion by using either d'Alembert principle or Newton's principle.

So, already you know about the physical coordinate system you can put a reference frame and from which you can measure the coordinates, and you can write the equation either in terms of the physical coordinates or in terms of the generalized coordinates. In this case in this example, I have taken the generalized coordinates x 1 and x 2, and I have drawn the free body diagram of the system; free body diagram of mass m 1 I have drawn first. So, in this case the forces are acting spring force k 1 x 1 and for this middle spring.

So, as it is shared by these two displacements, this is in this side you have a displacement x 1, in this side you have a displacement x 2. So, the spring will have a net displacement of x 1 and x 1 minus x 2, and you will have a force k 2 into x 1 minus x 2 acting on this mass m 1 in this direction, and in the opposite direction it will act on mass m 2. So, you can see that the net force acting on this mass m 1 and m 2 by the spring k 2 will be equal to 0.

So, in this way after drawing the free body diagram, summation of all the forces will be equal to 0; in case of a rotating system you can take the summation of moments also equal to 0. So, by using the equilibrium principle, you can derive the equation of motion of the system. And you can see in this case the mass matrix is. So, the mass matrix is m 1 and 0 and 0 and m 2, and in this case the stiffness matrix is k 1 plus k 2 minus k 2 minus k 2 k 2 plus k 3 x 1.

So, if you take a coordinate system in such a way that the mass matrix and both the stiffness matrix will be uncoupled. Then you can write the equation of motion like this m 1 x 1 double dot plus k 1 x 1 will be equal to 0 and m 2 x 2 double dot plus k 2 x 2 equal to 0. So, you may have a coordinate system; let a coordinate system q 1 and q 2 for which you can write the equation of motion in this uncoupled form. That is you can write m 1 q 1 double dot plus k 1 q 1 equal to 0 and m 2 q 2 double dot plus k 2 q 2 equal to 0.

So, in this case can see that you can reduce these two degrees of freedom system to an equivalent single degree of freedom system, and already you know the solution for a single degree of freedom system. So, those coordinate systems by using which you can write the equation of motion in an uncoupled form is known as the principle coordinate systems. Those coordinate systems by using which you can express this mass matrix and stiffness matrix in an uncoupled form is known as the principle coordinate systems. So, by this time you know three different coordinate systems.

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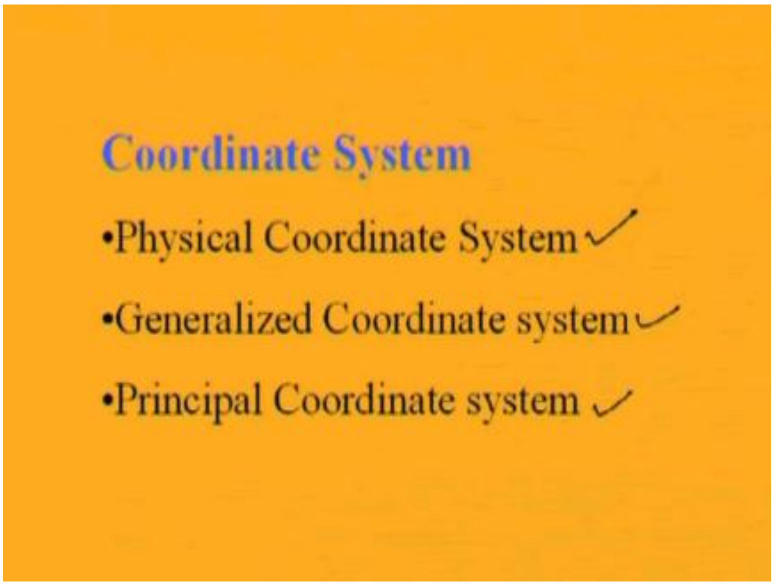


So, one is your principle coordinates, one is the generalized coordinate system, one is the physical coordinates system, and the third one is the principle coordinate system. In case of physical coordinate system, you are describing or you are taking a reference axis and you are describing the position of the mass or the position of the points whose motion is required by these physical coordinates. And in case of a generalized coordinates, you are taking a set of coordinates which are the minimum number of coordinates required to express the motion of the system. And in case of a principle coordinate system, you are taking a coordinate system in such way that the mass matrix and stiffness matrix are uncoupled.

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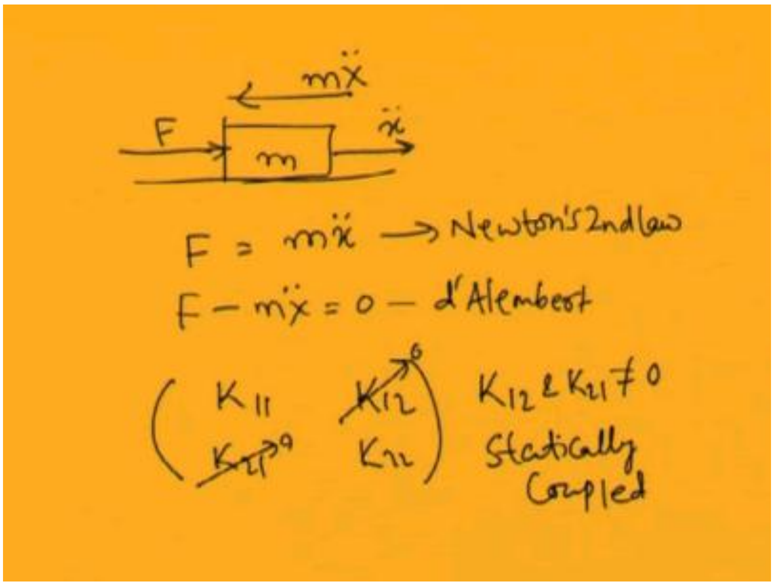


A system in which the mss matrix is uncoupled, you can write a system in which the mass matrix is uncoupled that is in this way you can write. So, the equation will be 0 m or this way. So, in this case the mass matrix is uncoupled. So, the system in which the mass matrix is uncoupled is known as dynamically uncoupled system, or if it is coupled then this is known as dynamically coupled system. So, in case of dynamic coupling, you will have instead of 0 you may have some term.

So, let it is m 1 m 2. So, this is m 1 1 m 1 2. So, this will be m 2 1, this will be m 2 2. So, already m 2 2 you have written. So, in case of a dynamic coupling, you will have this m 1 2 and m 2 1. And in case of a dynamically uncoupled system, you will have this m 1 2 uncoupled; for uncoupled you will have this m 1 2 and m 2 1 equal to 0. Similarly, you can have static coupling of the system. In case of a static coupling, you will have the stiffness matrix uncoupled that is stiffness matrix you can write in this form.

So, you can write k 1 1 k 1 2 k 2 1 k 2 2. So, in this case if k 1 2 and k 2 1 are not equal to zero, the system is statically coupled. So, if this is equal to zero and this term equal to zero, then the system is known as statically uncoupled system. So, for principle coordinates, these are the coordinates of a system in which you have the mass matrix and stiffness matrix uncoupled, or in other words you have a system with both dynamic and static uncoupling terms.

So, in case when you are using the principle coordinates, you can reduce it two degree of freedom system to equivalent single degree of freedom systems, and you can find the solution easily in that way. So, now let us see the other method to find the equation of motion that is the second method the Lagrange principle.

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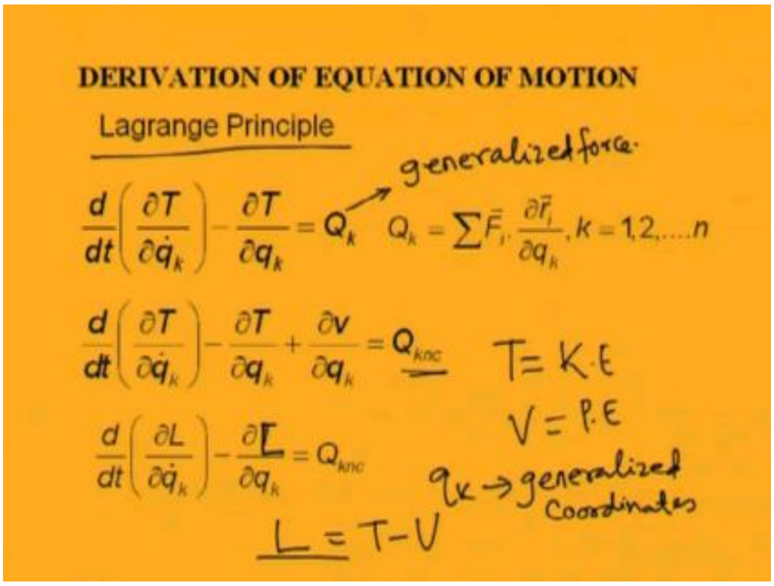


Using the Lagrange principle, we will find the equation of motion of the system. Already I told you in case of Lagrange principle or in case of extended Hamilton principle; we can have the potential energy and the kinetic energy of the system. Let T is the potential energy of the system and u is the kinetic or V is the potential energy of the system. So, T is the kinetic energy of the system, and V is the potential energy of the system. So, according to Lagrange principle d by d t of del T by del q k dot minus del T by del q k equal to this capital Q k, where this capital Q k is known as the generalizes force.

So, this is known as the generalized force, and this small q k is the generalized coordinate. Already you know in case of this double pendulum, theta 1 and theta 2 are the generalize coordinates. So, in case of Lagrange principle you are using a set of generalized coordinates. So, in this case d by d t of del T by del q k dot. So, for two degree of freedom system this k equal to 2. So, k will be equal to 1 and 2. So, in this case you can write the generalized equation of motion using this expression or you may write this like this.

So, it will be equal d by d t of del T by del q k dot minus del T by del q k plus del V by del q k equal to Q k n c; Q k n c is the generalized non-conservative force. So, you know this Q k is the generalized force which contains both conservative force and non-conservative force. So, the first equation is applicable for all the systems whether it is conservative or non-conservative. So, when you are separating the conservative and non-conservative forces, already you know that the non-conservative force or the work done by the non-conservative force equal to negative of the potential energy.

So, by applying that principle you can rewrite this equation in this form. So, it will be equal to d by d t of del T by q k dot minus del T by del q k plus del V del q k equal to Q k n c. Already you know that this Lagrangian of a system in case of a single degree of freedom system was written in this form L equal to T minus V. So, T is the kinetic energy, V the potential energy, and L is the Lagrangian of the system. So, L equal to T minus V. So, you can write the same equation in this form that is d by d t of del L by del q k dot minus del T by del q k equal to this is del L by del q k equal to Q k n c, where L equal to T minus V.

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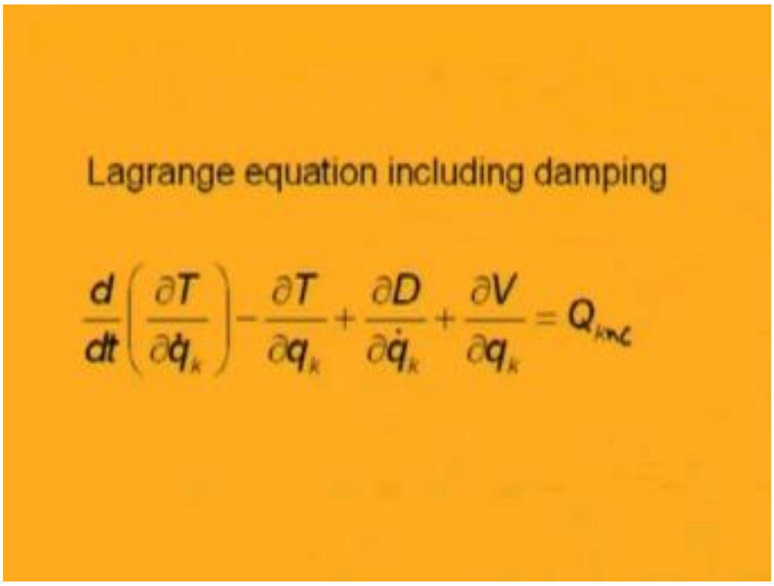


So, if you include the damping of the system, then you can write the equation of motion in this form. So, d by d t of del T by del q k dot minus del T by del q k plus del D by del q k dot plus del V by del q k will be equal to Q k n c. So, in this way also by including damping you can write the equation of motion. So, let us use this Lagrange principle to derive the equation of motion for some systems.

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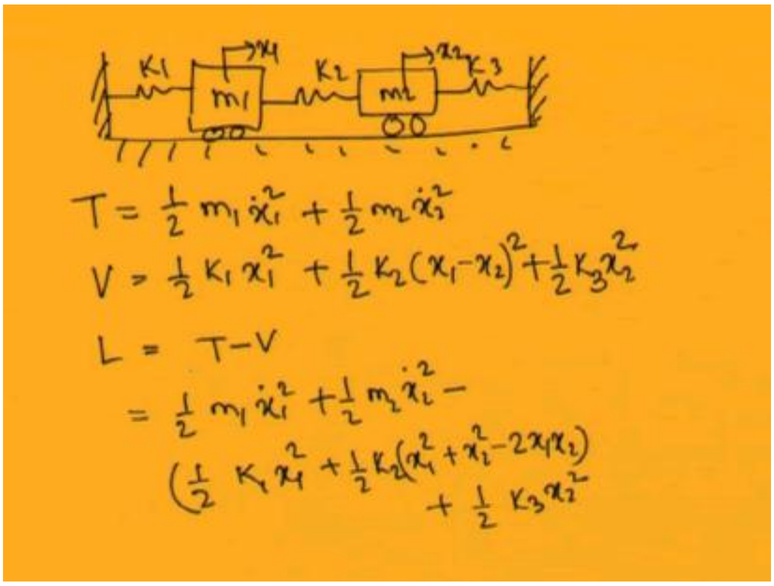


So, for the same system what we have taken before let me draw it again. So, in this case I have a mass m 1. So, it is connected another mass m 2 and I have three springs. So, this is springs k 1, this is k 2, and this is k 3. So, this is the constraint. So, this mass m 1 will have this is x 1 and this is x 2. So, this x 1 and x 2 are generalized coordinates. Now you can write the potential energy of the system and kinetic energy of the system. So, the kinetic energy of the system will be equal to T equal to kinetic energy equal to half mass m 1 into velocity of this. So, m 1 into velocity of this mass equal to x 1 dot, so velocity dot square.

So, already you know that mass into velocity square into half is the kinetic energy of the system. So, for mass m 1, the kinetic energy equal to half m 1 x 1 dot square. Similarly, for mass m 2, the kinetic energy equal to half m 2 x 2 dot square. Now the potential energy of the system is due to only the springs here. So, the potential energy associated with this will be equal to half k 1 into x 1 square plus already you know that the spring will have a relative motion of x 1 minus x 2 or x 2 minus x 1.

So, you can write this is equal to half k 2 into x 1 minus x 2 whole square. You just note that this x 1 minus x 2 whole square equal to x 2 minus x 1 whole square. So, whether you are writing x 1 minus x 2 or x 2 minus x 1, it is immaterial. So, for this third springs, you can write the potential energy equal to half k 3 x 2 square. So, this spring has a displacement of x 1. The second spring will have a relative displacement that is x 1 minus x 2, and the third spring will be displaced by x 2.

So, the potential energy of the system becomes the summation of the potential energy of these three sprigs and the kinetic energy equal to the summation of the kinetic energy of two mass. So, now the Lagrangian of the system can be written as T minus V. So, L equal to T minus V. So, you can write this equal to half m 1 x 1 dot square plus half m 2 x 2 dot square minus half k 1 x 1 square plus half. So, I can expand these things. So, I can write k 2, okay, I can write this is x 1 square plus x 2 square minus 2 x 1 x 2 plus half k 3 x 2 square. So, this is the expression for the Lagrangian of the system.

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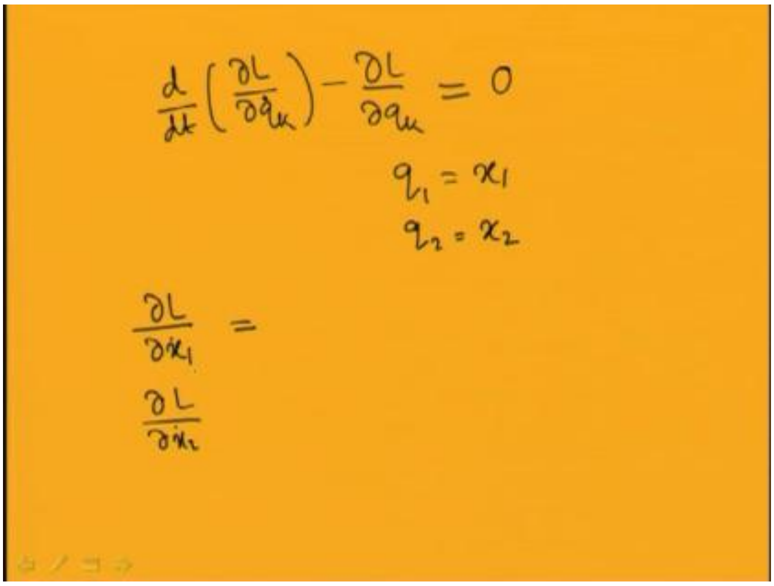


Now by using this Lagrange principle, I have to find d by d t of del L by del q k dot minus del L by del q k. So, there is no damping in the system. So, I can write and there is no force also acting on the system. So, I can write this is equal to zero. So, already you have found the expression for L and you know your q 1 equal to x 1 q 1. So, q 1 equal to x 1 and q 2 equal to x 2. So, first you should find del L by. So, you just find del L by del x 1 dot. Then you just find del L by del x 2 dot and del L by del x 1 and del L by del x 2. So, you can find the equation of motion.

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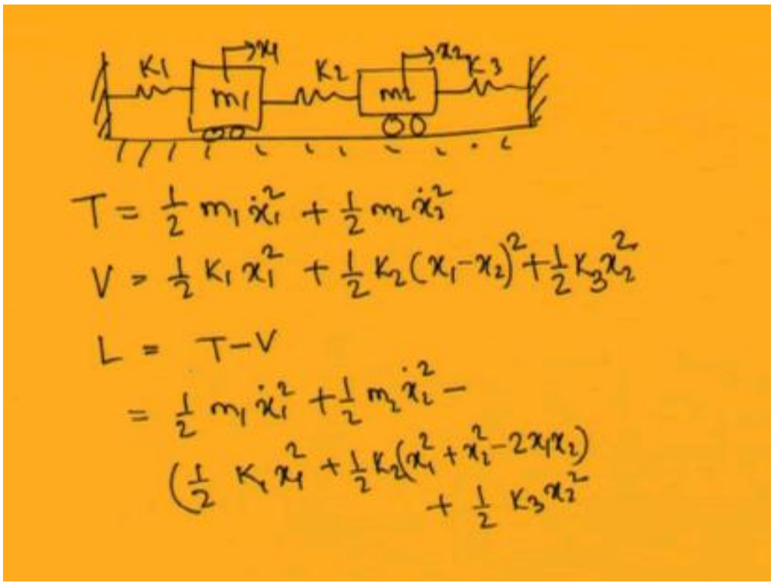


So, now del L by del x 1 dot will be equal to. So, from this equation you can find. So, del L by del x 1 dot will be equal to. So, this the only term associated with x 1 dot. So, it will become half into 2 into m 1 into x 1. So, this two two cancel. So, it will become m 1 x 1 dot. So, this becomes m 1 x 1 dot.

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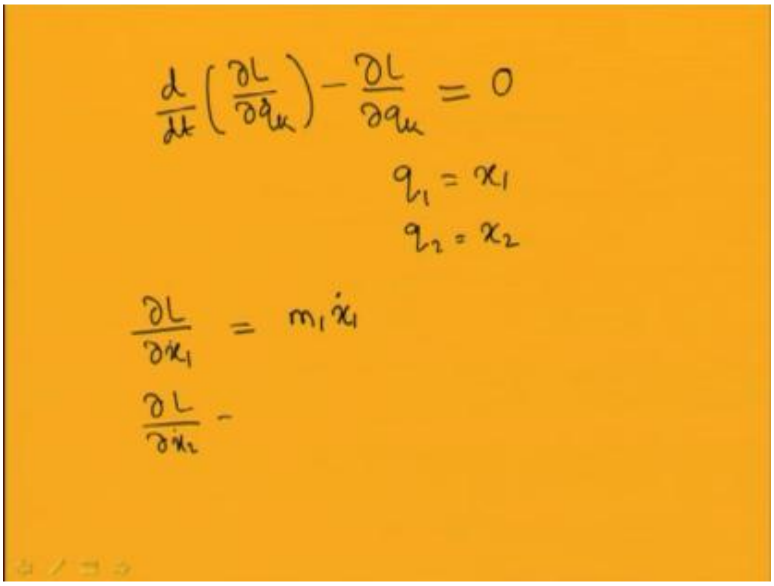


So, I can write this is equal to m 1 x 1 dot.

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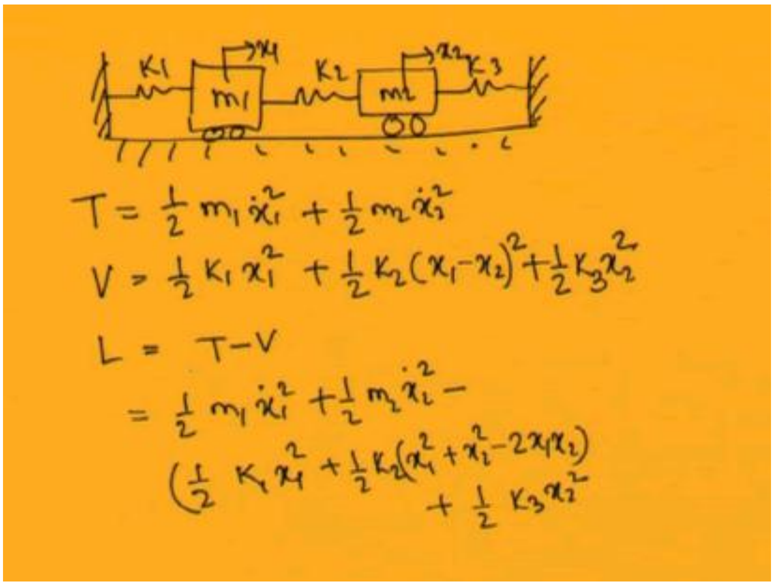


Similarly, I can write this del L by del x 2 will be equal to. So, differentiation of these with respect to x 2 equal to m 2 x 2 dot. So, other terms you just see are not function of x 2 dot.

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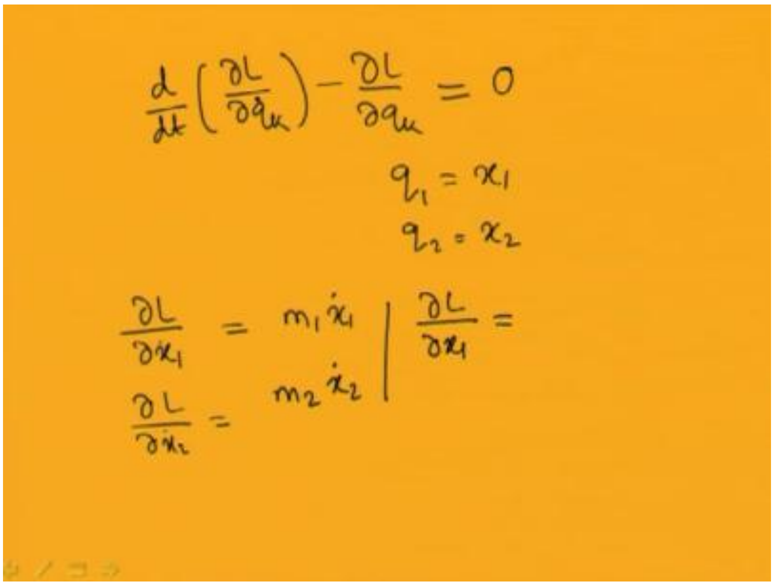


So, those terms derivative will be equal to zero. So, this becomes m 2 x 2 dot. Similarly, you can write del L by del x 1. So, this will become.

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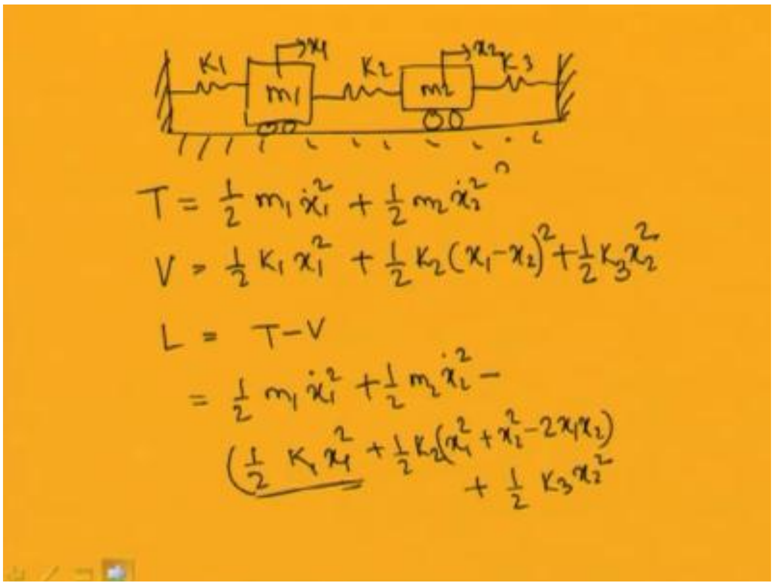

So, from this expression, this x 1 dot term and x 2 dot terms are independent of x 1 and x 2; these are velocity terms. So, velocity and displacements are independent function you can take and so differentiation with respect to x 1 will give. So, this part if you differentiate with respect to x 1, so this part if you differentiate; so, this becomes two two cancels. So, this becomes k 1 into x 1, and here you just see that there are some terms with x 1.

So, k 2 x 1 square terms you can differentiate with respect to x 1. So, two two will cancel. So, this becomes k 2 x 1, then these terms. So, two and two cancels. So, this becomes half k 2 x 1 x 2. So, differentiate with respect to x 1 will become k 2 x 2. So, this will become k 1 x 1 and plus k 2 x 1 and minus k 2 x 2.

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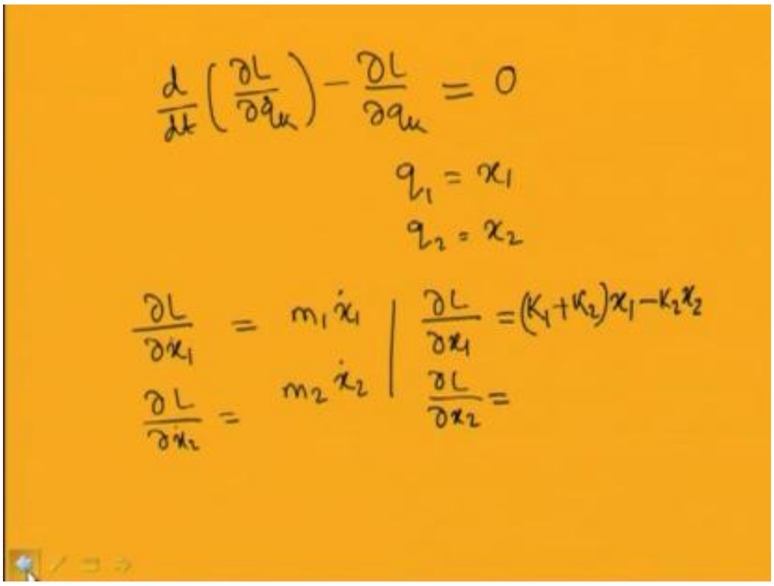

So, i can write this term as. So, this becomes k 1 plus k 2 plus x 1 minus k 2 x 2. Similarly, del L by del x 2 I can find.

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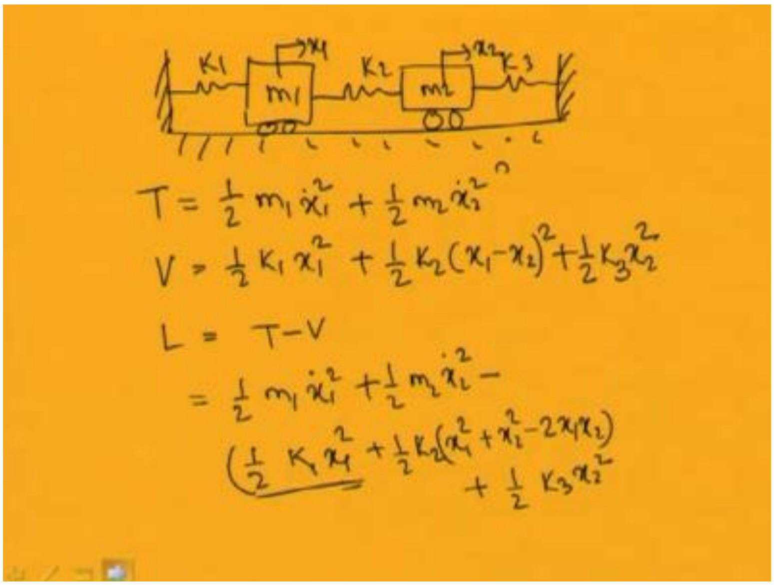

So, you just check this becomes. So, from these terms you can find x 2 square. So, differentiate of this will be equal to 2 x 2. So, this is k 2 x 2, and from this it becomes minus k 2 x 1, and this will give rise to k 3 x 2. So, I can have k 2 plus k 3 x 2 minus k 2 x 1.

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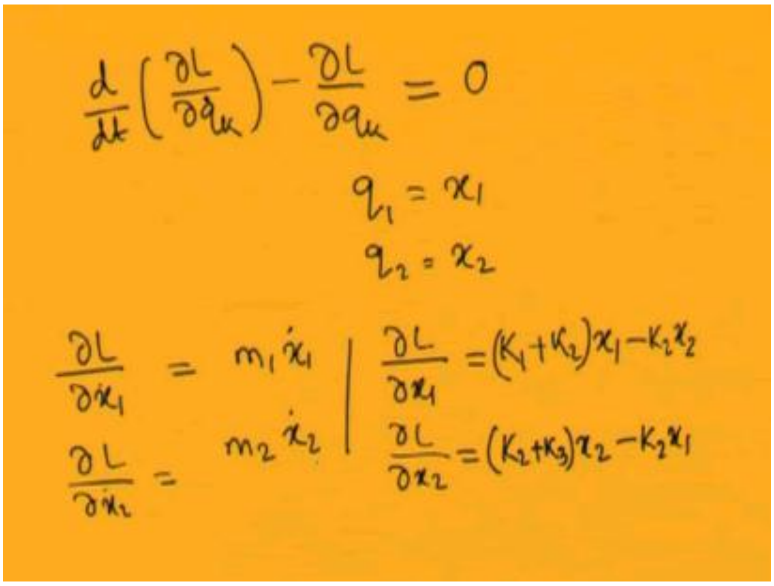


So, this becomes k 2 plus k 3 x 2 minus k 2 x 1. So, now d by d t of del L by del x 1 dot will be equal to m 1 x 1 double dot. And similarly, del L by del x 2 d by d t of del L by del x 2 dot will be equal to m 2 x 2 double dot.

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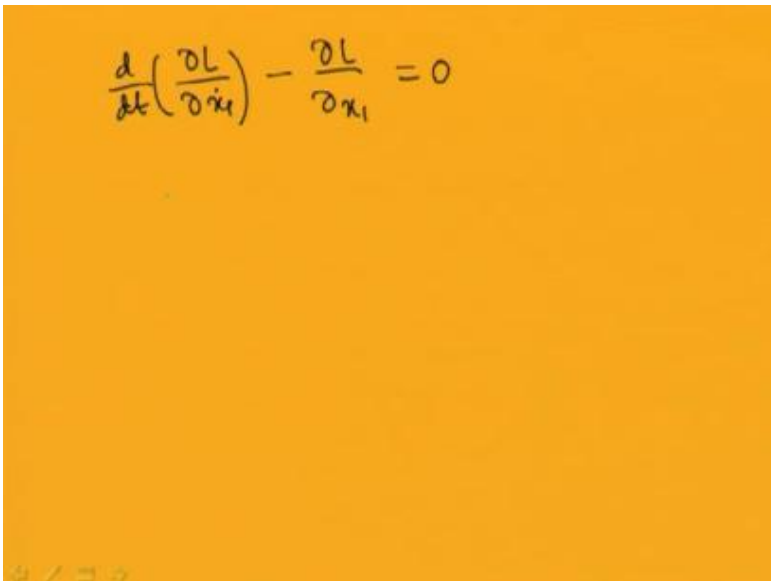


So, my equation of motion will become. So, when I am writing in terms of first equation will become. So, first equation I can write for this. So, del L by del x 1 dot minus del L by del x 1 equal to 0.

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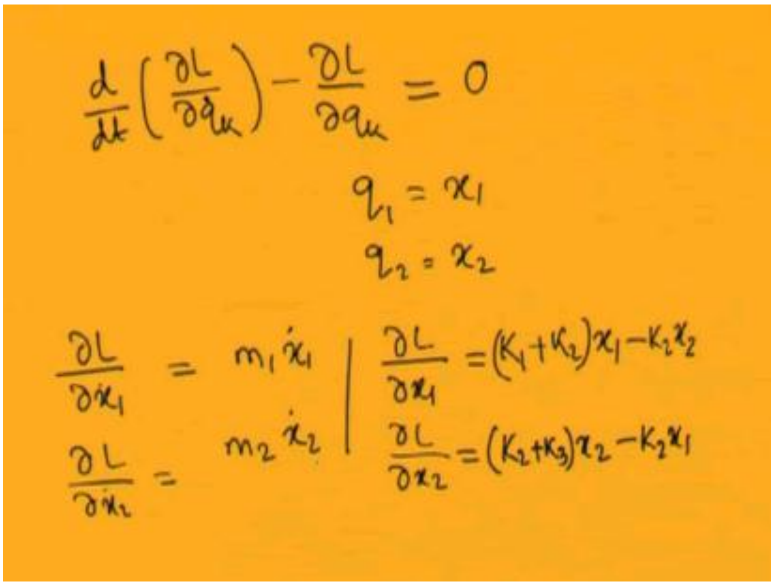


So, from this you are getting m 1 x 1 double dot.

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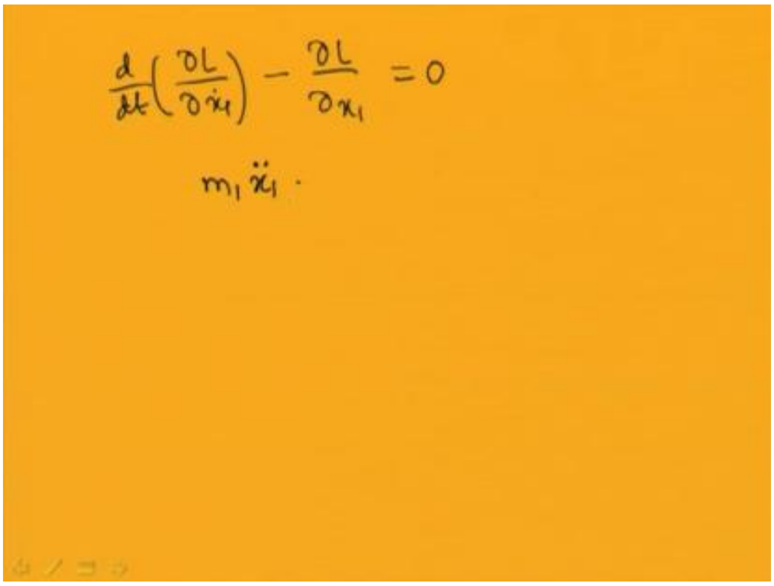


So, this becomes m 1 x 1 double dot.

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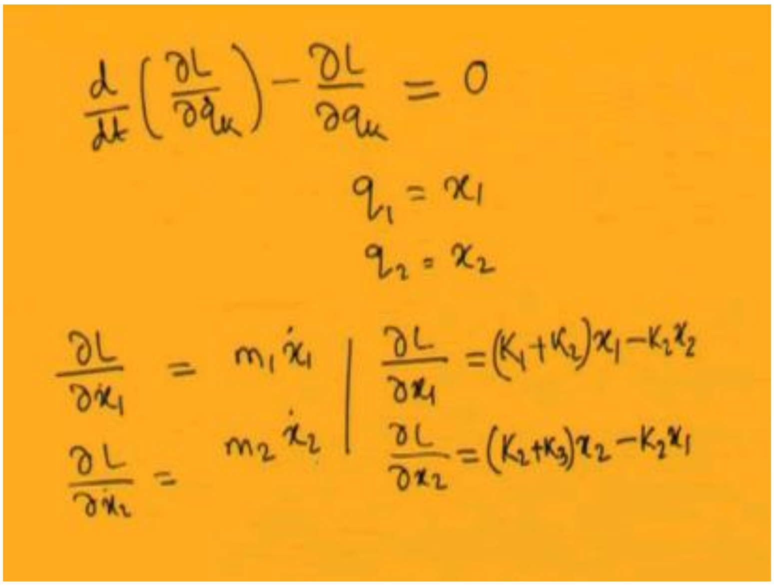


So, for this minus of this; so, you just check this is minus of this.

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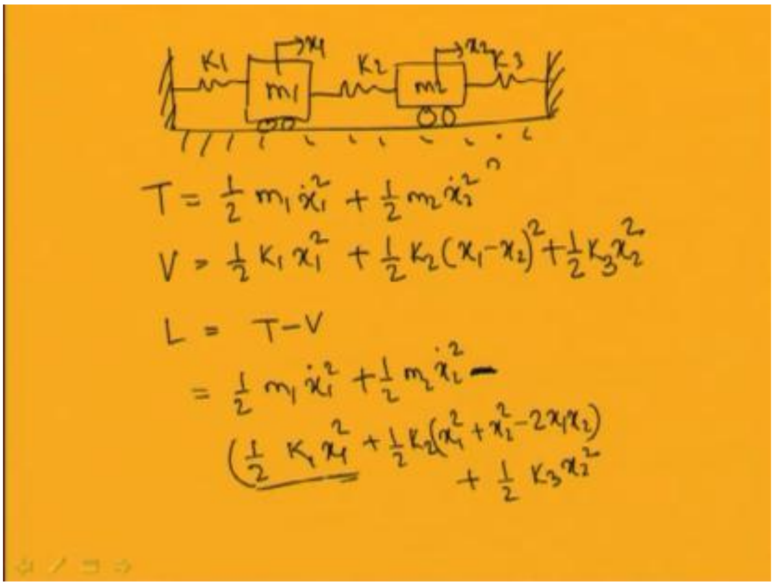


So, before that you just check this is minus. So, you have a minus sign here. So, T minus V. So, L equal to T minus V.

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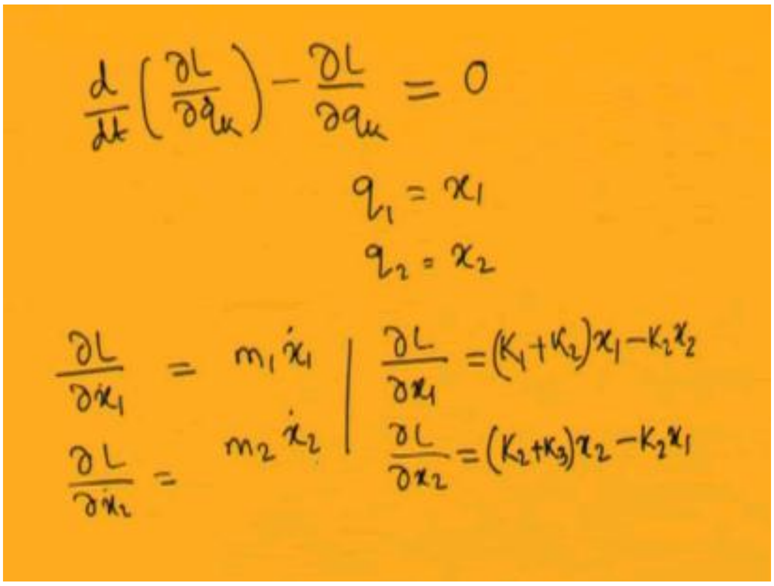


So, when you are differentiating this. So, these terms become negative, and this becomes positive. So, this becomes negative, and this becomes positive. So, for minus del L by del q k. So, you will have minus minus plus.

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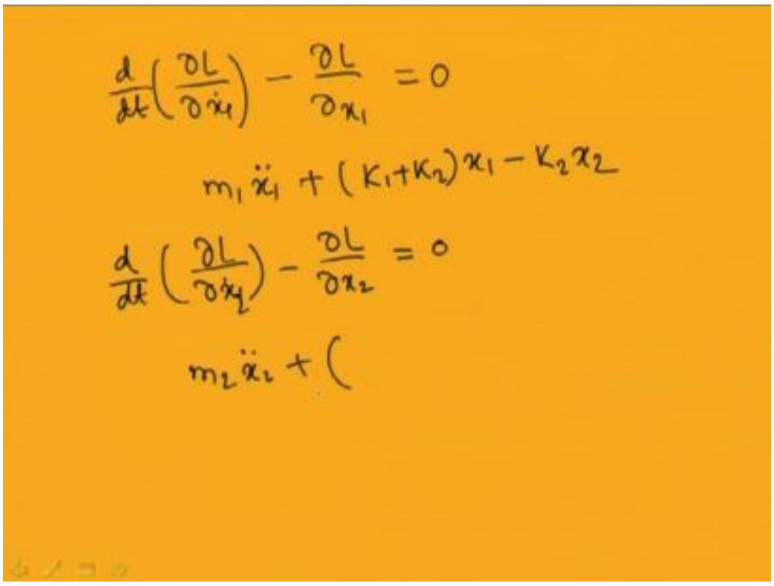


So, this becomes plus k 1 plus k 2 into x 1 minus k 2 x 2, and for the second equation you can have d by d t of del L by del x 2 dot minus del L by del x 2 equal to 0. So, for this case you can write it is equal to m 2 x 2 double dot plus.

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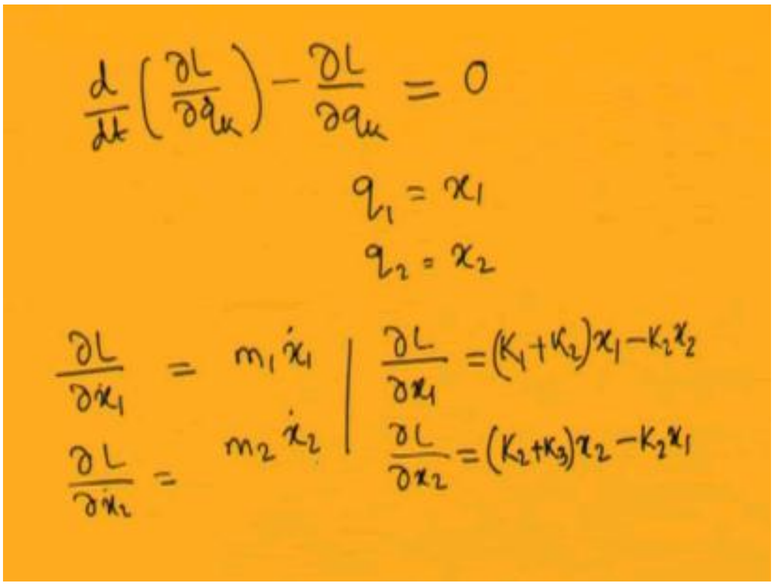


So, you can check it, so plus k 2 plus k 3 x 2.

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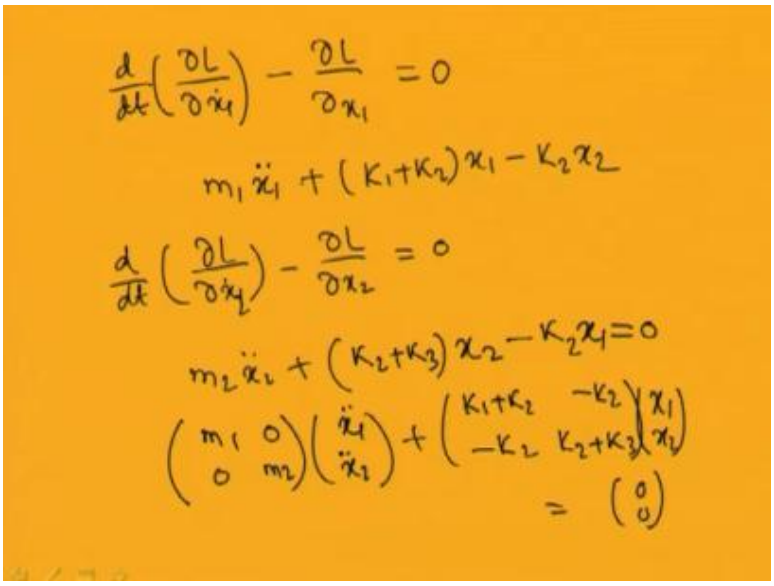


So, this becomes k 2 plus k 3 x 2 and minus k 2 x 1 equal to 0. So, you can write this thing in matrix form. So, matrix form it will become the same equation what you have got before. So, it becomes m 1 0 0 m 2 x 1 double dot x 2 double dot minus or plus. So, this becomes k 1 plus k 2, and this becomes minus k 2, and this becomes minus k 2, and this is k 2 plus k 3 x 1 x 2 equal to 0 0.

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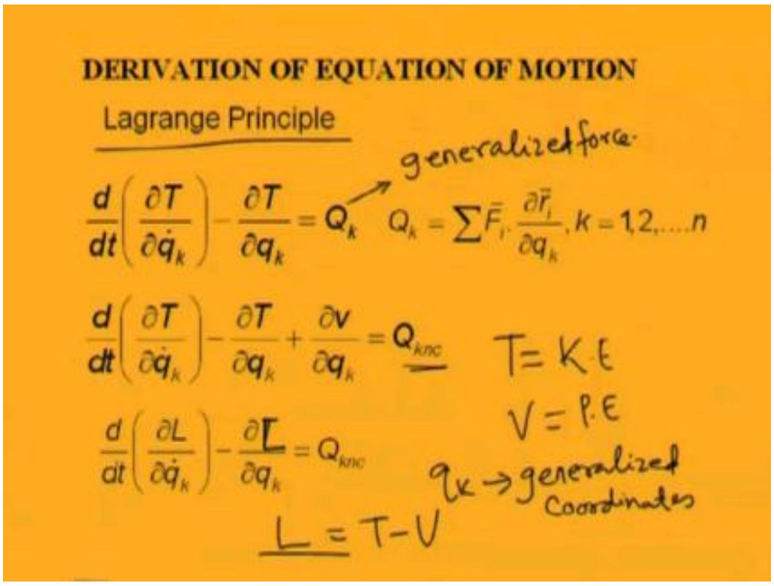


In this equation, this generalized force Q k equal to F i dot del r i by del q k and k equal to 1 to n, where r i is the physical coordinates of the system, F is the force acting on the system. So, if you have n number of force acting on the system by using this formula, you can find the force acting or the generalized force acting on the system for writing the equation of motion.

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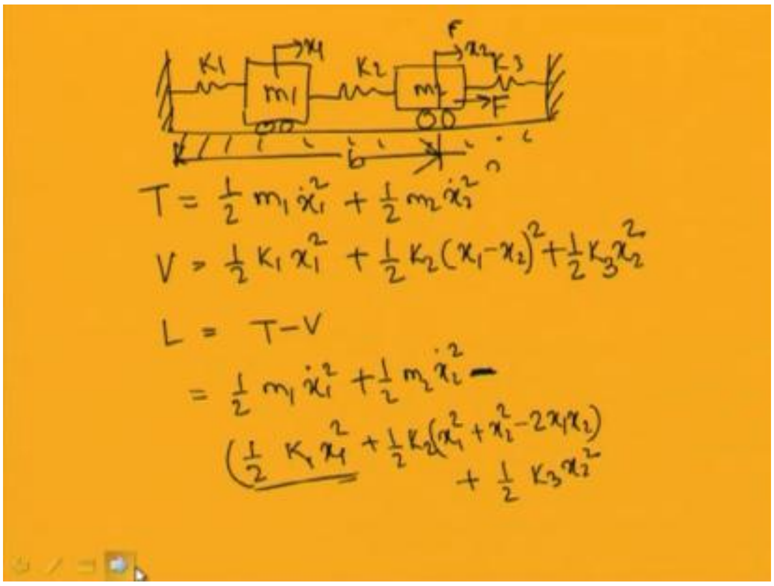


So, let a force F is acting on the second mass. So, you can write that thing. So, the physical coordinates you can take a physical coordinates, already I told you that physical coordinate will be b plus. So, b is the distance from this. So, you can write this is b. So, the displacement will be plus x 2.

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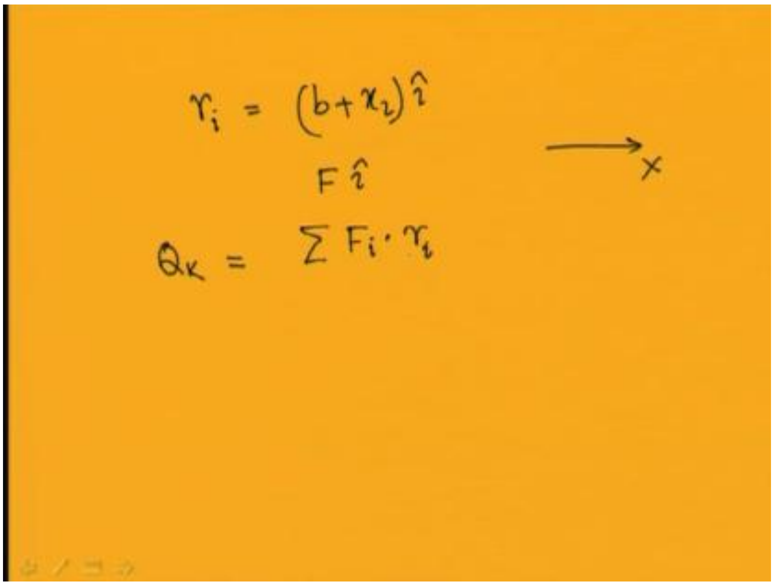


So, in this case the force will have a position vector. So, you can write that is r i will be equal to b plus x 2 i. So, if I am taking that direction as x direction and i is the unit vector associated with that. So, r i I can write equal to b plus x 2 i and this force I can write equal to F I; F i is acting on the second mass. So, by using that thing, I can write Q k. Q k is the first generalized force acting on the system. So, it is equal to summation F i dot r i. So, you just see that expression.

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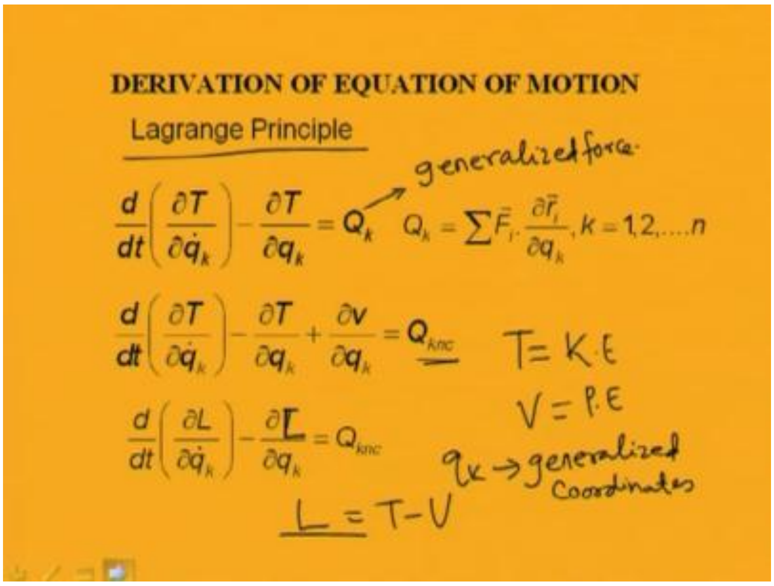


So, from that expression you can find it. So, this is equal to F i or F i into del r i by del q k F i dot del r i by del q k.

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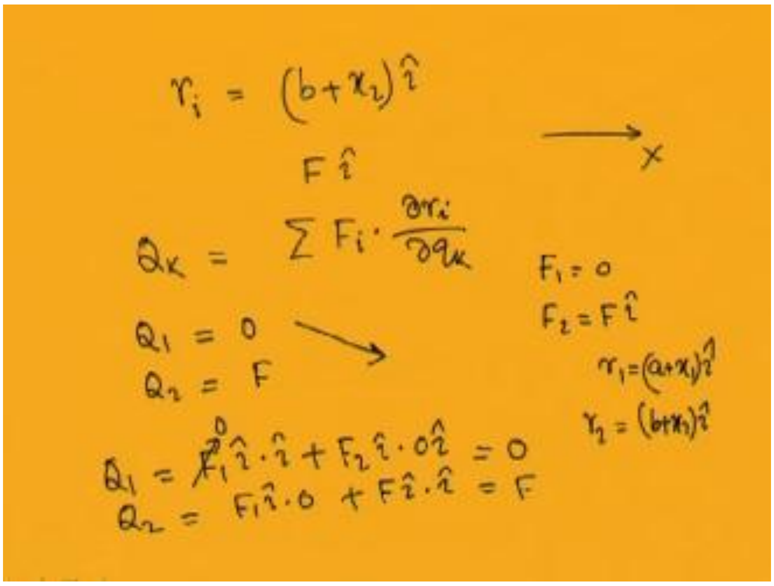

So, from that expression you can write this is equal to F i dot del r i by del q k. So, F 1 equal to 0. So, i equal to 1 to 2. So, you have two cases. So, i equal to 1 and 2. So, in the first case F 1 equal to 0; there is no force acting at station 1 or mass 1 and your F 2 equal to F force acting on this. So, you will have this is equal to. So, if you want to write this Q

1. So, Q 1 will be equal to 0. So, 0 and Q 2 you can find this is equal to equal to F. So, Q 2 will be equal to F 1 equal to 0 into this.

So, in this case you can find in this way. So, you can find this. So, this is equal to F 1 dot del r 1. So, r 1 position vector is. So, your r 1 equal to a plus x 1 i and your r 2 equal to b plus x 2 i. So, here a and b are the position from this reference frame to the equilibrium position of mass m 1, and this is our mass m 2. So, your F 1 dot del r 1. So, del r 1 by del x 1. So, as F 1 equal to 0. So, this becomes F 1 i. So, this is zero. So, this part is 0 as F 1 equal to 0. So, the first part will become 0, and second part I can write it in detail. So, this becomes.

So, while deriving Q 1, it will be equal to F 1 i dot del r 1 by del x 1. So, this becomes i plus F 2 F 2 in i and dot del r 2 by del x 2. So, del x 2 becomes del r 2 by del q k; q k equal to for k equal to 1. So, this becomes x 1. So, if you differentiate this r 2 with respect to x 1. So, this becomes 0. So, already this F 1 equal to 0. So, this is 0. So, this becomes 0. Similarly, when you are finding Q 2 this will be equal to F 1 i dot. So, differentiate this with respect to Q 1, Q 2 is your x 2.

So, if you differentiate del r 1 by del x 2 that is equal to 0 plus F 2 that is equal to F i dot. So, if you differentiate this with respect to x 2. So, this becomes i only. So, this becomes F i dot i. So, this becomes F. So, in this way you can find the generalized force acting on the system.

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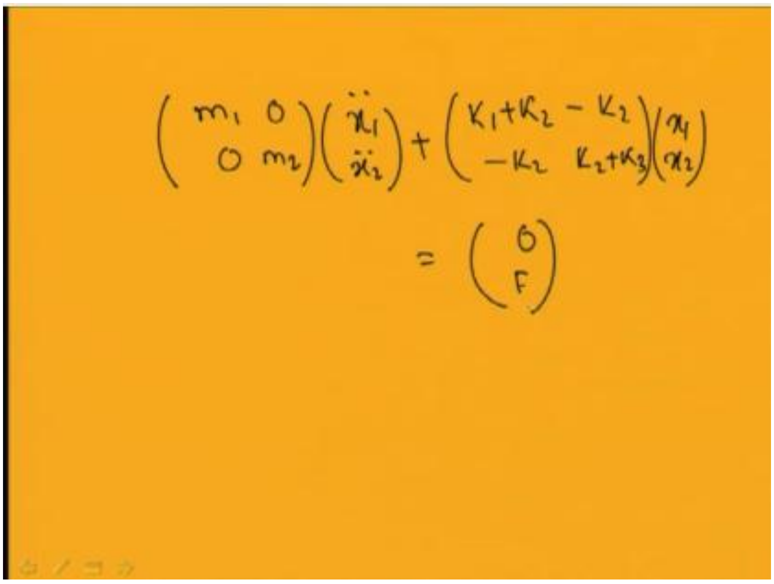

So, by finding this generalized force acting on the system, you can write the equation of motion in this form m 1 0 0 m 2 x 1 double dot x 2 double dot plus k 1 plus k 2 minus k 2 minus k 2. This is k 2 plus k 3 x 1 x 2 equal to 0 F. Similarly, if the force is acting on the first mass, you can write the equation of motion. And in that case you can find the generalized force and put it at the proper place. So, now you know by using the Lagrange principle or d'Alembert principle, you can derive the equation of motion of a system.

Also you may use the extended Hamilton principle to derive the equation of motion of a system. So, in case of extended Hamilton principle, you can also use this kinetic energy and potential energy of the system. In terms of kinetic energy and work done on the system, you can write the extended Hamilton principle in this form.

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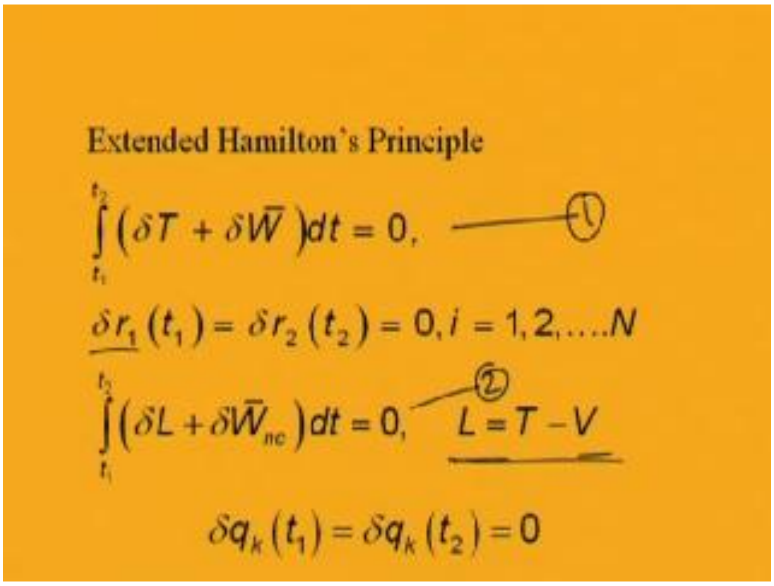


So, where del this is the del operator. So, you can write this del T plus del W bar d t equal to 0, where del r i at t 1 equal to del r 2. So, if we are taking two degree of freedom system. So, you have two coordinates r 1 and 2. So, del r 1 t 1 equal to del r 2 t 2 equal to 0. So, this del r 1 is the virtual displacement by or virtual displacement of mass m 1 and del r 2 is the virtual displacement of mass m 2. So, already you know from the virtual work principle that virtual displacement will be equal to 0.

So, according to extended Hamilton principle, when this virtual displacements equal to 0 at time t 1 and t 2. So, the system will obey this principle. So, that is equal to del T plus del W bar d t equal to 0. So, this W bar as the work is a path function that is why we have written this W bar. So, in terms of Lagrangian you can write the same expression like this. So, already you know your Lagrangian equal to T minus V. So, you can write the same expression in this form. So, integral t 1 to t 2 del L plus del W bar n c d t equal to 0, and using the generalized coordinate you can write this is equal to this del q k at t 1 equal to del q k at t 2 equal to 0.

So, using either this first expression or the second expression, you can derive the equation of motion of a system. So, already you know this del W contains two terms that is the conservative work done and non-conservative work done. So, when you are taking conservative work done. So, you know that conservative work done equal to negative of the change in potential energy. So, by using that principle you can write. So, you can write this del W equal to del W n c plus del W due to non-conservative force, and del W due to conservative force equal to negative of the del potential energy. So, del T minus del V will give you del L, and this will give you del W n c. So, using either this equation or this equation you can derive the equation of motion of a system.

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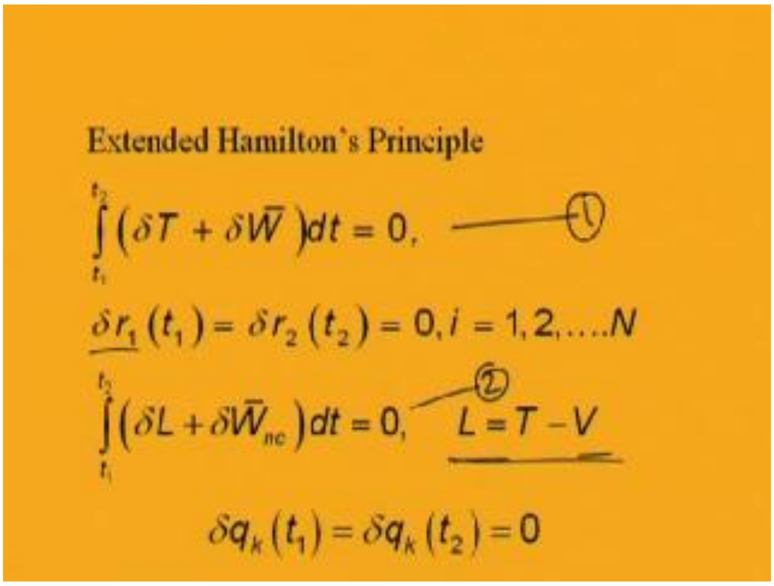


So, for a conservative system you can see that work done by the non-conservative force equal to zero. So, in that case it will reduce to the simple form.

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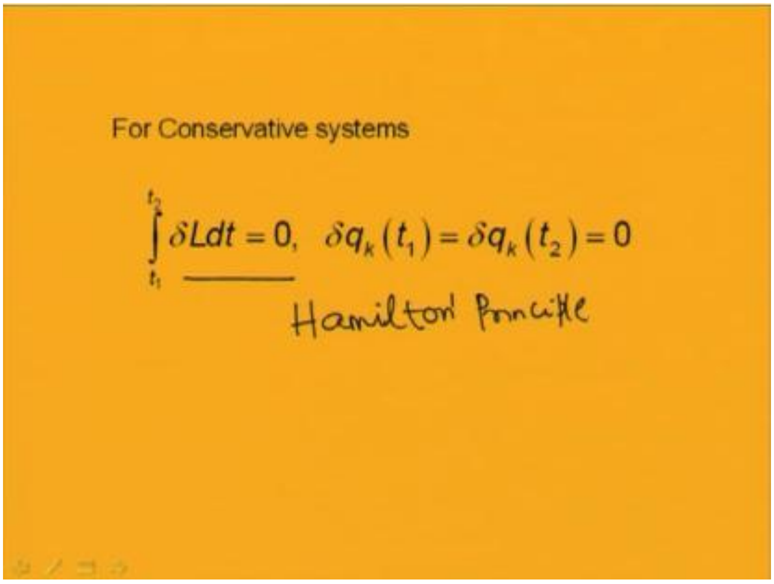


Del L d t equal to 0, where del q k t 1 equal to del q k t 2 equal to 0. So, this is known as the Hamilton principle of this. So, this is known as Hamilton's principle.

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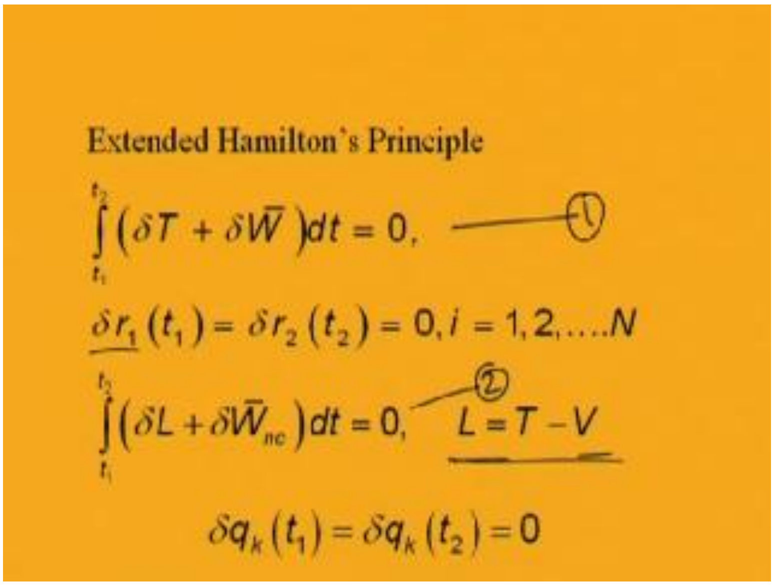


The previous one where you are using the non-conservative force is known as extended Hamilton principle. So, the detail derivation of this extended Hamilton principle or Lagrange principle, you can find from the text book by Myrophis principle and techniques of vibration or elements of vibration. Also you can find the same thing from the other books by Thomson and Balkrishnan also.

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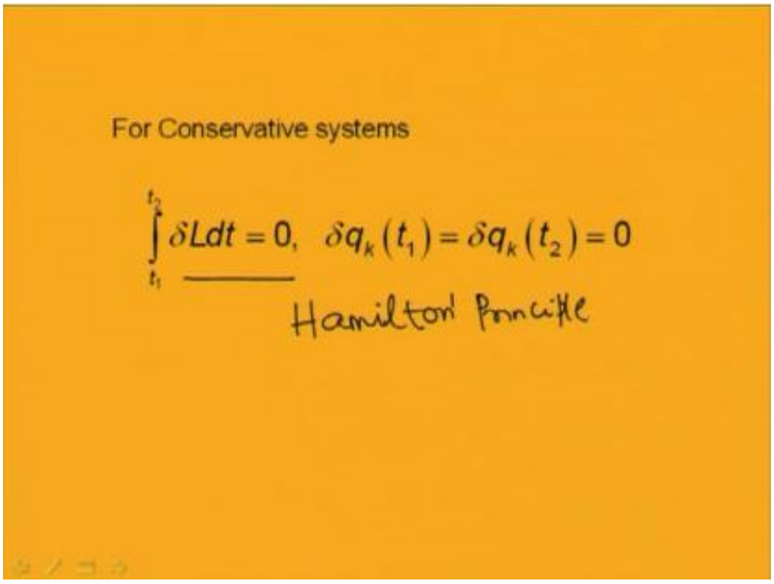


So, let us take one example. So, in this case I have taken a simple system. So, in the system I have just taken a spring and a mass, and I have attached another mass or I have attached a rod which is hinged at this point, and also I have applied a force F in the horizontal direction here. So, in this case you can see that this will reduce to a two degree of freedom system, because this mass will have a motion x 1, or this mass will have a motion x. And in addition to that, this rod will also oscillate during this motion. So, you can express the motion of this total system by writing the displacement x of this mass m and displacement of this rod.

So, this displacement of this rod either you can express in terms of the motion about this point that is using this angle theta or you can express this motion also in terms of its center of gravity c g. So, the c g of this rod I can locate it here. So, either you can express this motion of this rod in terms of this hinged point O or you may express in terms of this center of gravity of this mass. So, let me take a coordinate system; I can take a physical coordinate system.

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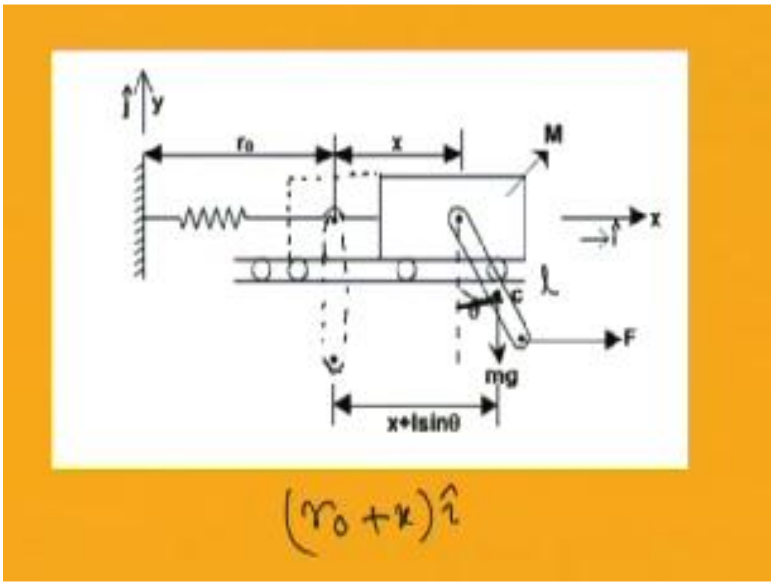


So, this figure you may see. So, in this case this is the original position and this one is the displaced position. So, in this original position let r 0 is the distance from this equilibrium position from this mass, and now it has a displacement x. So, this is the mass M. So, I can take a coordinate system Y with unit vector j in this direction and I can take a unit vector i here; this is j, and this is the displacement. So, first I should write the position vector of all these points. So, you know the position vector of this point equal to r 0 plus. So, position vector of this point will be equal to r 0 plus x i.

Similarly, the position vector of the center of the mass I can write in terms of. So, this will be r zero plus x plus this distance. So, this distance you can find. So, if this length is l. So, this is at a distance l by 2. So, it will be l by 2. So, this angle will be l by 2 sin theta. So, the i component will be equal to r 0 plus x plus l by 2 sin theta I, and I can take a vertical component positive in this direction. So, it will be negative in this direction. So, it will be minus l by 2 cos theta j.

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Position vector of point C=

$$\left(r_0 + x + \frac{l}{2}\sin\theta\right)\hat{i} - \frac{l}{2}\cos\theta\,\hat{j}$$

$$\vec{V}_c = \left(\dot{x} + \frac{l}{2}\cos\theta\,\dot{\theta}\right)\hat{i} + \frac{l}{2}\dot{\theta}\sin\theta\,\hat{j}$$

Kinetic Energy $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\vec{V}_c \cdot \vec{V}_c + \frac{1}{2}I_c\dot{\theta}^2$

So, the position vector of point C.

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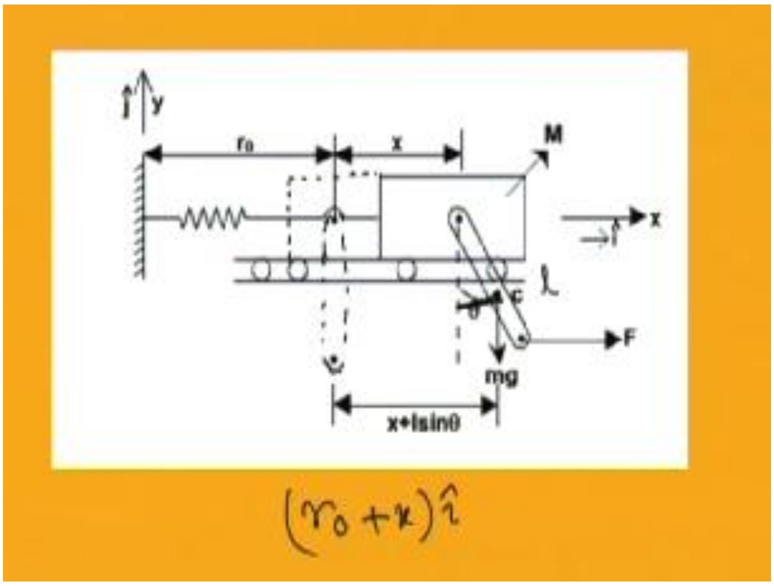


So, this is point C the center of gravity of this rod.

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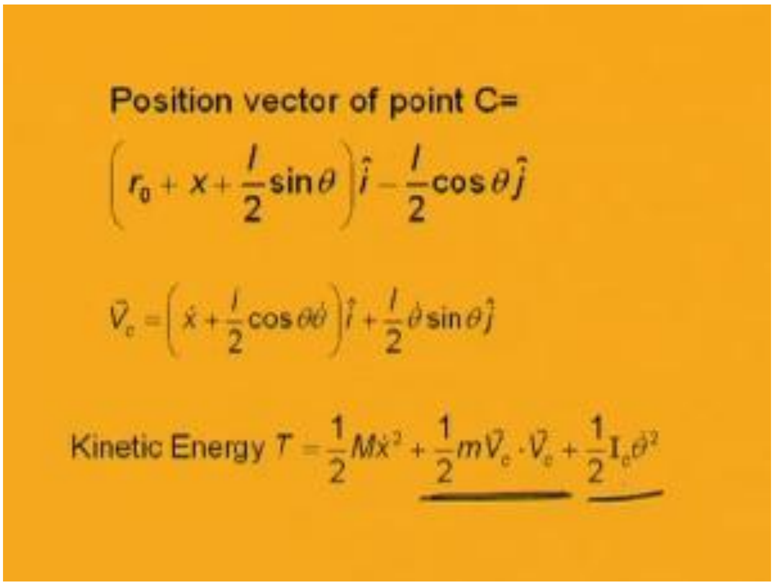


So, i can write that equal to r 0 plus x l by 2 sin theta i minus l by 2 cos theta j. So, the velocity of this, so this is the position vector. So, you can find the velocity of this. So, velocity will be equal to x dot. So, you just differentiate this expression and you can find the velocity as x dot plus l by 2 cos theta. So, sin theta differentiation will be cos theta into theta dot. So, this is I th component and j th component will be cos differentiation will be minus. So, this is equal to plus minus minus plus. So, l by 2 theta dot sin theta j.

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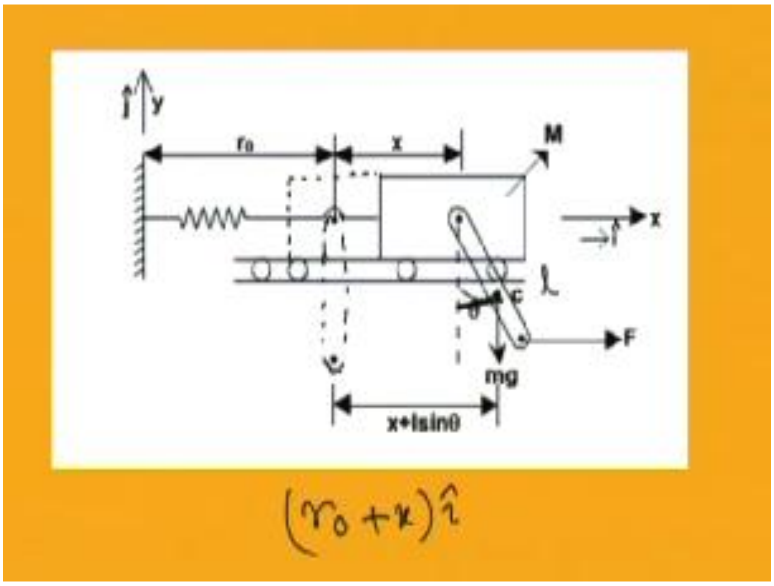

The kinetic energy of the system you can write. So, the kinetic energy of the system will be kinetic energy of this mass and kinetic energy of this rod.

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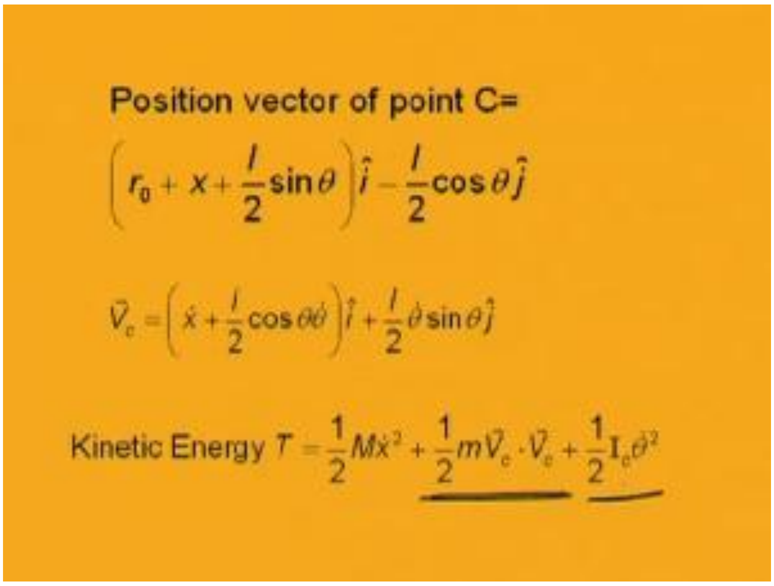


So, the total kinetic energy will be half M x dot square.

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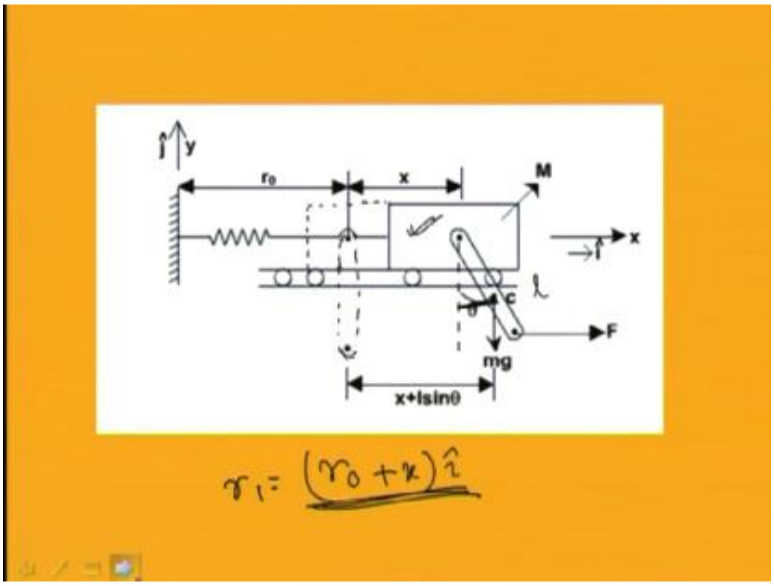


Half M into differentiation of this. So, this is your r 1 physical coordinate r 1. So, differentiation of this will be equal to x dot i.

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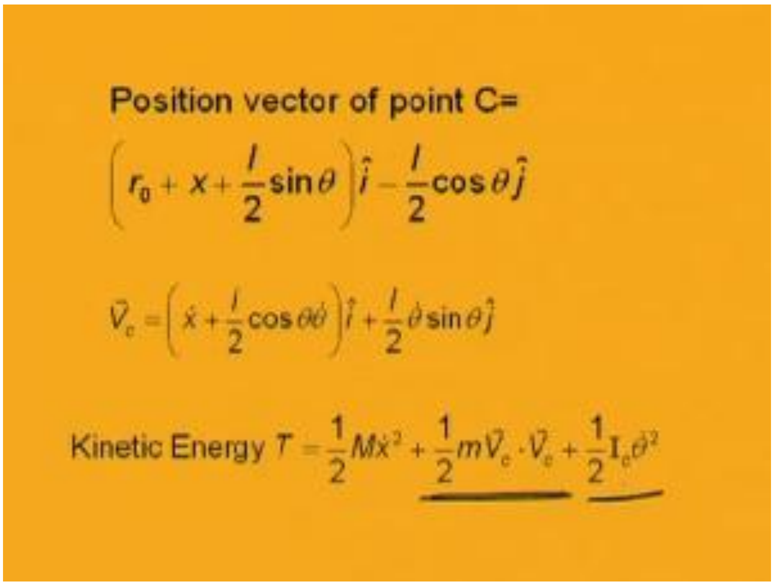


So, kinetic energy will be equal to half M x dot square. Similarly, the kinetic energy of this rod will be half.

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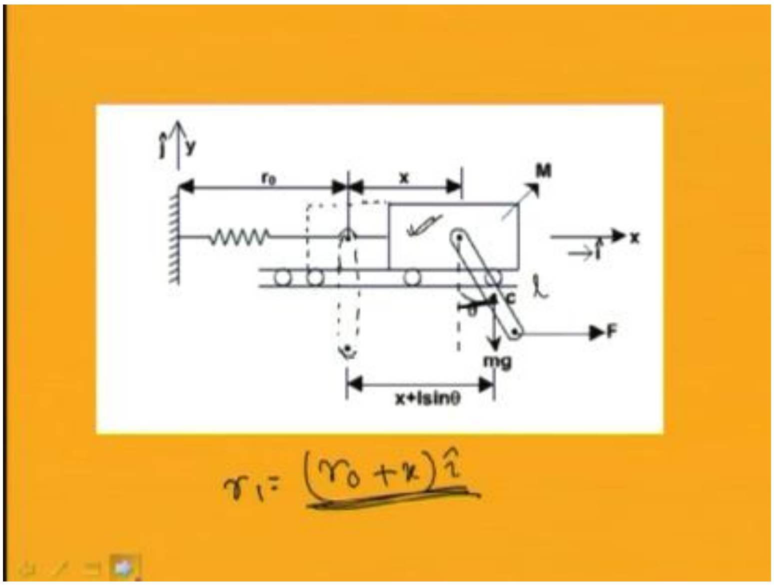


So, if you are taking at this point, directly you can find the kinetic energy of the rod about this point. So, in that case it will be the inertia about this point into theta dot square or you can write the kinetic energy about these points. So, kinetic energy about this point when you are writing this point has both translation and rotation. So, you have to write both translation and kinetic energy and rotational kinetic energy.

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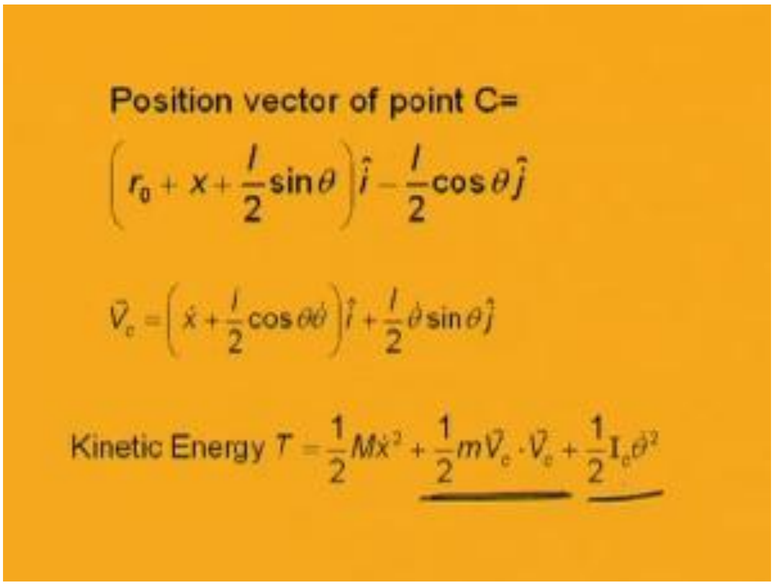


So, this is the transitional kinetic energy that is equal to half m V c dot V c plus rotational kinetic energy will be equal to half I c theta dot square. Already you know that this moment of inertia I c equal to if m is the mass per length of that rod. So, it will be equal to m l square by 12.

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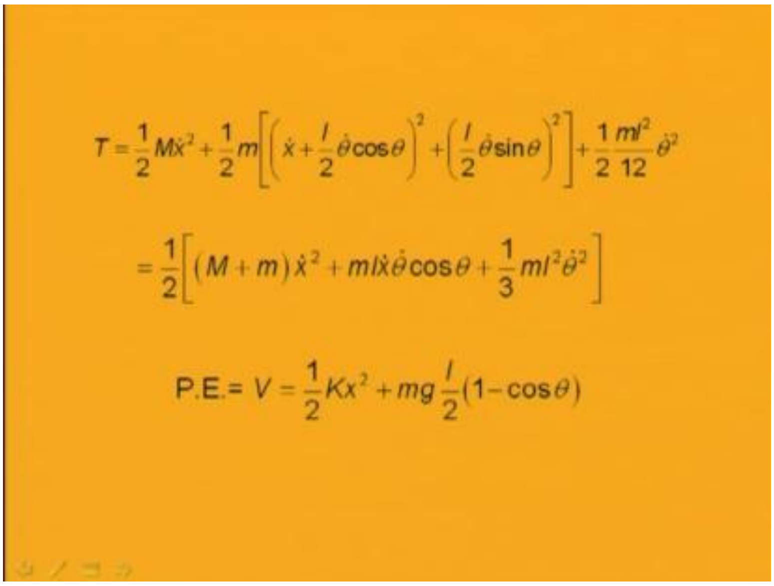


So, this expression will reduce to this. So, it will be equal to half M x dot square plus half m x dot plus half theta dot cos theta square plus half theta dot sin theta square plus half m l square by 2 l theta dot square. So, you can add these terms by using this cos square theta plus sin square theta equal to 1. So, it will reduce to half M plus m x dot square plus m l x dot theta dot cos theta plus one-third m l square theta dot square and potential energy of the system.

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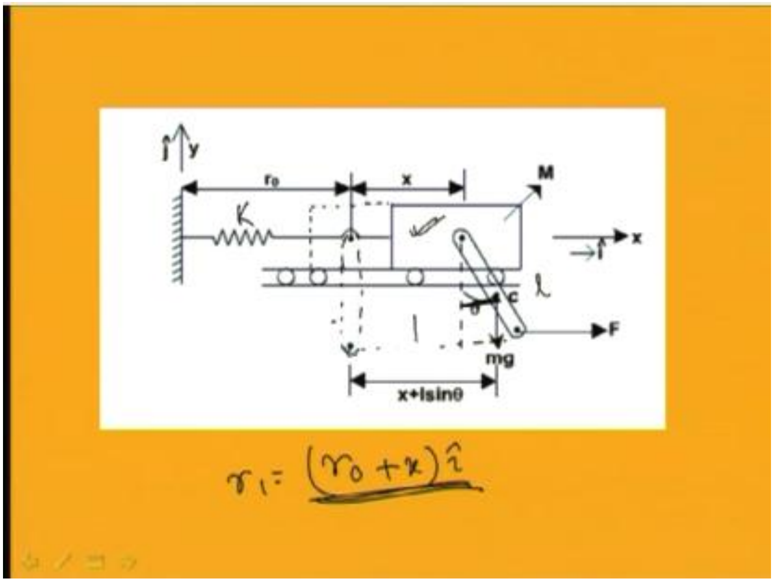


So, potential energy of the system is due to the spring and due to the change in position of this rod. So, due to this spring it will become half k as the change in distance of the spring equal to x or displacement of the spring is x. So, it will become half K x square and change in position of the system if you see. So, it will become. So, this is the original position. Now you draw this and now it is the change in position. So, the change in position becomes l by.

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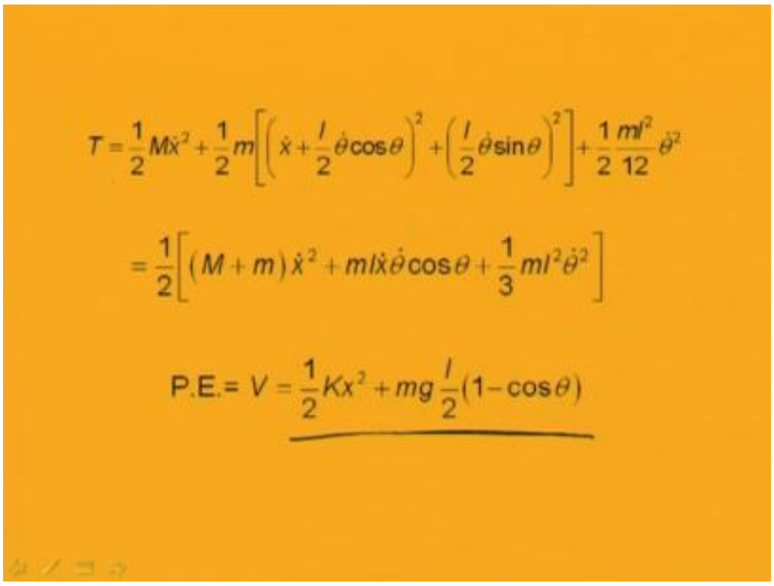


So, the potential energy you can write equal to. So, this becomes l by 2 into 1 minus cos theta. So, total potential energy of the system becomes half K x square plus mg l by 2 1 minus cos theta.

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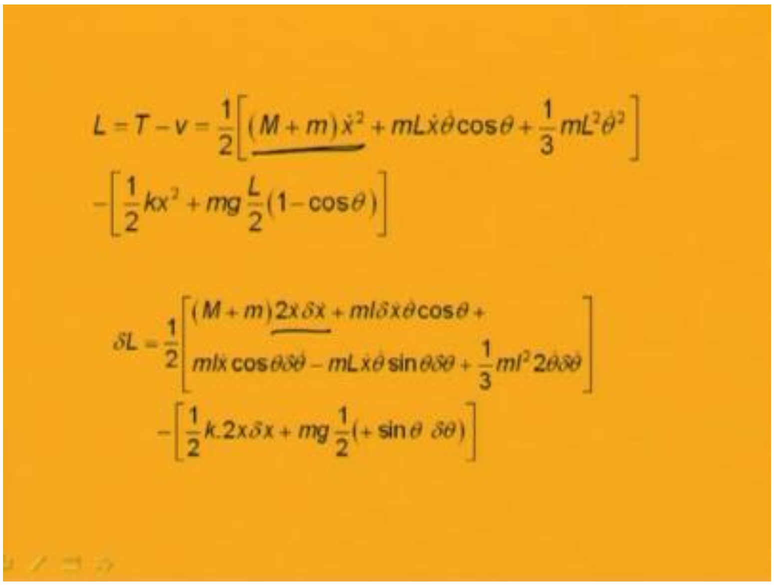


So, by using this kinetic energy and potential energy you can write l equal to T minus V and now you just write this del L. So, you just use this del operator. So, for a single term I will show you how you can do. So, if you use the del operator it becomes M plus m 2 into x dot into del x dot. So, 2 x dot into del x dot. So, in this way you can operate this del operator, and you can write this expression like this. So, you can find this del L.

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$$\delta L = \left[ (M+m)2\dot{x} + ml\dot{\theta}\cos\theta \right]\delta\dot{x} + \frac{1}{6}ml\left(3\dot{x}\cos\theta + 2l\dot{\theta}\right)\delta\dot{\theta} - kx\delta x - \frac{1}{2}ml\left(\dot{x}\dot{\theta} + g\right)\sin\theta\delta\theta$$

So, after finding del L. So, this is the expression for del L.

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$$\overline{\delta W}_{nc} = F\hat{i}\cdot\delta\left\{(r_0 + x + l\sin\theta)\hat{i} - l\cos\theta\hat{j}\right\}$$
$$= F\delta x + Fl\cos\theta\delta\theta$$

From extended Hamilton's Principle

$$\int_{t_1}^{t_2}\left(\delta L + \overline{\delta W}_{nc}\right)dt = 0,$$
$$\delta x = 0, \delta\theta = 0, \quad \text{at} \quad t = t_1, t_2$$

Now you just use the extended Hamilton principle. So, before that the work done by the this non-conservative force will be equal to F i dot this.

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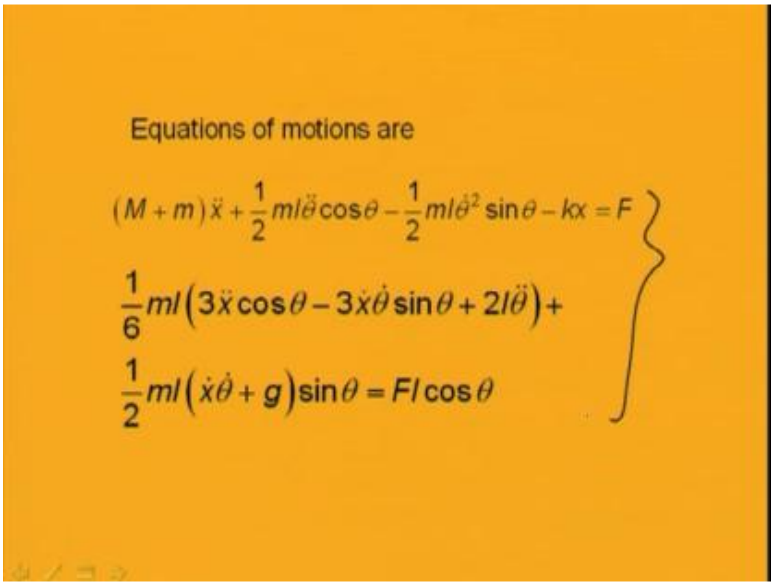


So, using extended Hamilton principle you can find the expression for this equation of motion. So, this becomes the equation of motion of the system. So, today class we have studied about how to determine the equation of motion by using Newton’s principle or d’Alembert principle and also by using this Lagrange principle and extended Hamilton principle. Also we have studied about the coordinate coupling of the system. We have taken three different types of coordinates; one is the physical coordinates of the system, second one is the generalized coordinates, and third one is the principle coordinates of the system. So, next class we will study how to determine the free vibration of a two degree of freedom systems.