**Engineering Mechanics**

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**Introduction to vibration**

**Lecture No. 40**
**Some Problems of Vibration**

Today, I am going to discuss some typical problems of vibrations. In the last four lectures, we have been discussing about the vibrations of single degree freedom system. We have formulated the differential equation by applying Newton's law or D'Alembert's principle. Also, we have discussed about energy approach of obtaining the equations of vibrations or frequency of vibration. Today, I am going to discuss few problems and some of them of advanced nature and some simple applications of the vibrations. So, we discuss with the case of a simple pendulum again.

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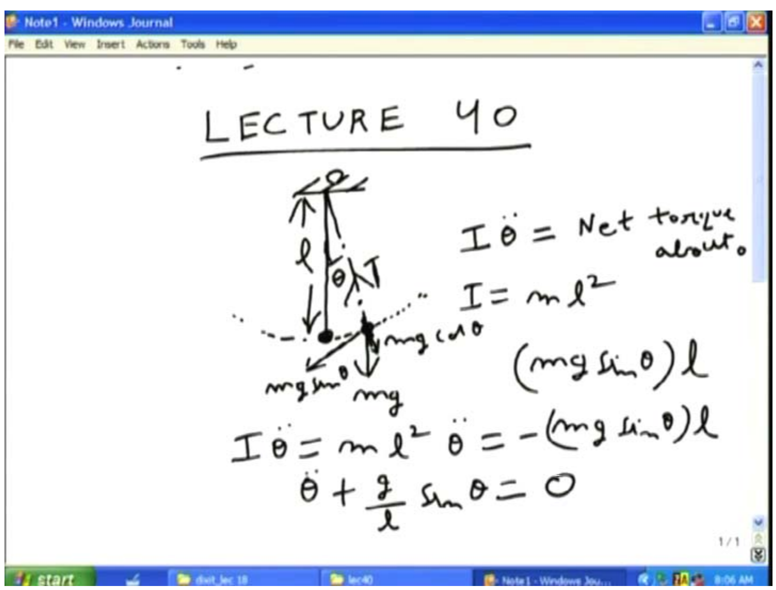


We have already discussed simple pendulum. This is a small bob attached by a string and this is moving in a circular path. This is the next position and that can be like this. This angle is theta.

So, the next position which I show here, like some dotted line, this is the next position; chain dotted type thing.

What is the equation of motion for this? Equation of motion can be obtained by I theta dot equal to net torque about O and I is about this one. Here, I will be equal to, if the length is l, I is equal to ml square.

There is a force acting on the bob and is m times g then there is a tension of the string that is T, mg can be resolved into two components; one is mg sin theta tangential to that circular arc and other perpendicular component that is mg cos theta which is in the direction of the string. If you take the moment about O, then T r mg cos theta will not contribute to moment; only mg sin theta will be contributing to the moment and that moment is mg sin theta into l, where this is the thing so mg. Therefore, it is directed in the direction opposite to theta because it is directed like this. So, it is basically clockwise because if you just place yourself on this line, look towards O, point is towards right hand side.

Therefore, you get the equation I theta dot dot that is equal to ml square theta dot dot is equal to minus mg sin theta multiplied by l. So, m and l can be cancelled from both sides. Then this equation becomes theta dot dot plus g by l sin theta is equal to 0. This is the exact differential equation of the simple pendulum. However, we make assumptions that for small amplitudes, sin theta may be regarded as equal to theta. Therefore, equation becomes theta dot dot plus g by l theta equal to 0. This can be compared with mx dot dot plus kx equal to 0, the spring mass system. So, you get the simple harmonic solution of that, but that is approximate.

Let us see that how the analysis can be done for exact vibrations.

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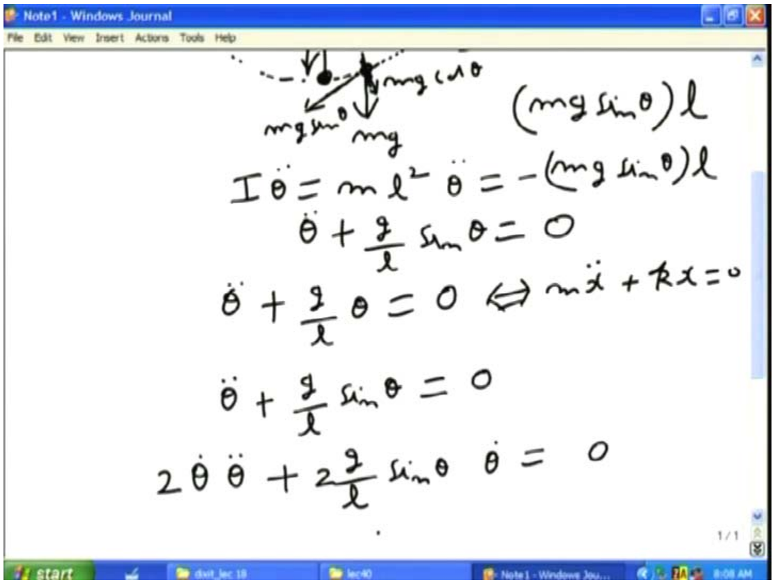


In case I cannot consider theta to be very small, so theta dot dot plus g by l sin theta is equal to 0. How do we solve this differential equation? This differential equation is nonlinear. If there is some solution, suppose x is a solution of this, then that does not mean a constant time x is a solution of this. This is a nonlinear differential equation and for solving it multiply it by theta dot. So, theta dot theta double dot plus g by l sin theta theta dot equal to 0. Let us multiply by 2. So this is 2g sin theta this is the thing.

We want to integrate this equation.

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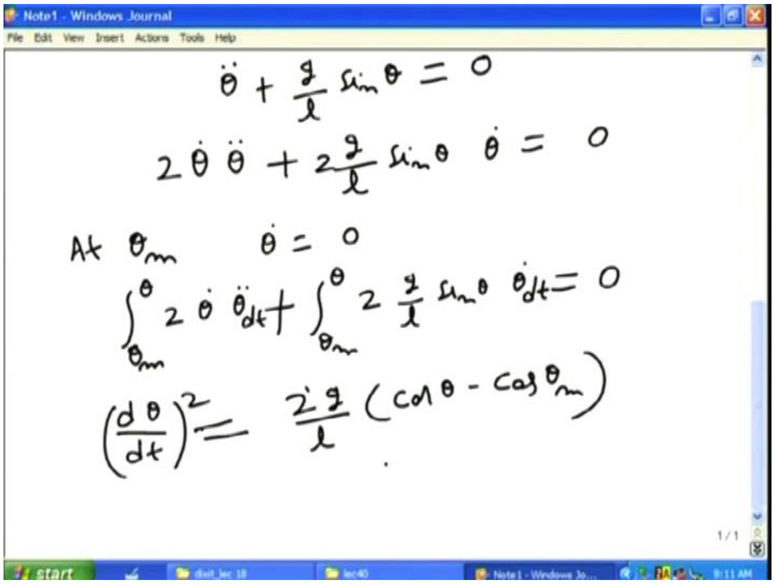


Let us integrate this equation between position of maximum amplitude, maximum displacement that is the last point of the amplitude. Now, theta will be the time. So, at theta m, let us say, theta m is the maximum displacement; so at theta m, theta dot must be 0. Integrating it, theta m theta 2 theta dot theta double dot. So, initial position is theta m and final position is any arbitrary theta plus theta m theta 2g by l sin theta into theta dot is equal to 0.

If you integrate this and put these limits, this will be d theta by dt whole square minus d theta m d theta by dt, d theta at m by dt that means derivative at m, but that is 0. Therefore, I am rubbing here itself this is not required this is d theta by dt square. This one and this thing will become equal to plus. Now, 2sin theta theta dot can be integrated. I must put an integration sign here, dt. This is integration sign dt then it becomes complete. This is equal to 2g by l cos theta minus cos theta m. So, integration of this comes out to be, cos theta minus cos theta m.

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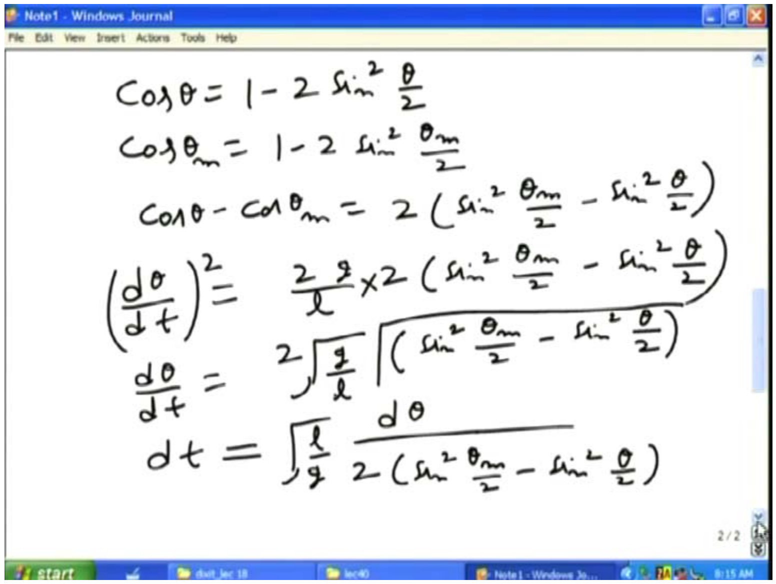


cos theta can be written as 1 minus 2 sin square theta by 2. Similarly, cos theta m is equal to 1 minus 2 sin square theta m by 2. Therefore, cos theta minus cos theta m actually becomes 2 sin square theta m by 2 minus sin square theta by 2. Now, you have got the expression, d theta by dt whole square is equal to 2 g by l cos theta minus cos theta m. In place of that, this can be written as 2 times sin square theta m by 2 minus sin square theta by 2. Therefore, d theta by dt can be written as, 2 under root g by l sin square theta m by 2 minus sin square theta by 2 in bracket that is square. Therefore, this can be written as dt is equal to dt d theta divided by 2sin square theta m by 2 minus sin square theta by 2 then under root l by g.

You can integrate from t is equal to 0 to theta equal to t, where t is equal to t by 4.

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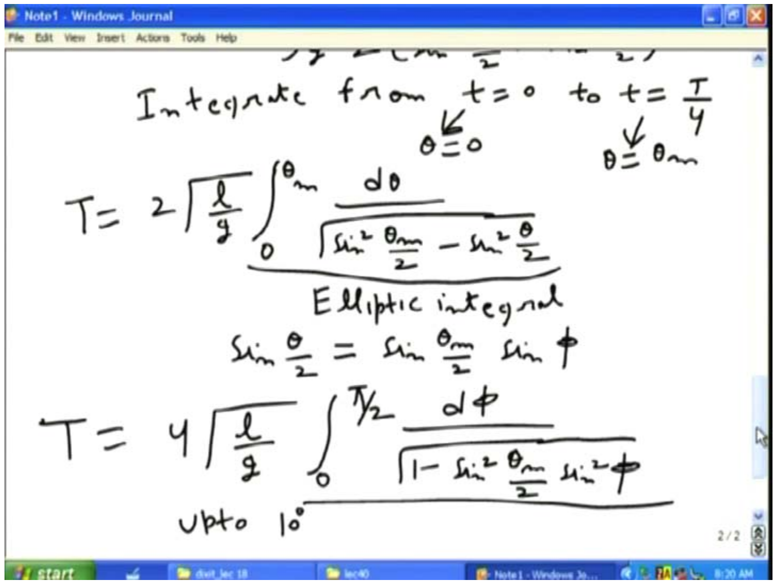


Integrate from t is equal to 0 to t is equal to T by 4. Then, at t is equal to 0; theta is equal to 0 and at t is equal to T by 4, where capital T is the time period, theta is equal to theta m. Doing this, we obtain T is equal to 2 under root l by g 0 theta m d theta by sin square theta m by 2 minus sin square theta by 2. This cannot be further simplified. So, this integration cannot be determined in an exact manner. It can be determined numerically.

This integral is called elliptic integral. One can use the expression sin theta by 2 is equal to sin theta m by 2 sin phi. This type of substitution can be implied so that limits change from 0 to theta m, to 0 to phi by 2, because at theta equal to 0 phi is equal to 0; at theta is equal to theta m, sin phi is equal to 1, that means phi is equal to 0. Therefore, time period T can be written as 4 under root l by g, 0 to phi by 2 d phi under root 1 minus sin square theta m by 2 sin square phi.

This type of integration has come in some standard form, because limits are 0 to phi by 2. Now, the value of the integral depends on the value of theta m. These values have been tabulated which I am not showing here, but they are in available in any book of mathematical tables. Here, these values of the integral have been tabulated. It is observed that up to10 degree, whatever time period is obtained by taking the assumption of sin theta equal to theta is almost same as this one. So, up to 10 degree, simple harmonic motion approximation is reasonably accurate. After that, one has to go for the exact analysis and this is what the example has shown.

I take up the other example.

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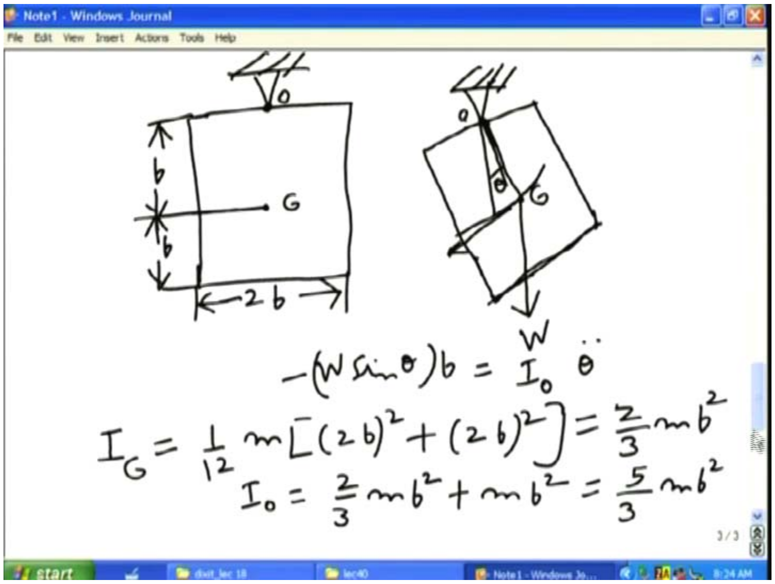


Here, another simple example; let us take a simple square plate, thin square plate which is hanged from this point. This is G, the mass center, then this distance is b, this is 2b. If axis of vibration is passing through this point O and is perpendicular to the plane of the paper what happens? When this is rotated, this will become like this. I have given theta angle and therefore, that time G will be rotated by theta. Therefore, a vertical line is passing here, but this G has gone to this point; therefore this is theta. The weight is W. Therefore, there is a tangential component; that means G moves in a circle. Draw a tangent at this one and find out this component that is W sin theta.

If you take the torque, moment of this about this point will be W sin theta into b. It is clockwise. So, that has been shown. That must be equal to $I_0$ theta double dot, where $I_0$ can be found using the parallel axis theorem. For this plate, moment of inertia about the mass center G is $I_G$ is equal to 1 by 12m, m is the mass. Two b square plus 2b square it is $I_G$ is basically $I_{zz}$, z is perpendicular to the plane of the paper. So, this is $I_x$ plus $I_y$ half m 2b square. So, it becomes 2 by 3 mb square.

Now, parallel axis theorem can be used. Therefore, $I_o$ will be 2 by 3 mb square plus m times b square. Total mass is m; this tends from G to o is mb square, Therefore, this can become equal to 5 by 3 mb square.

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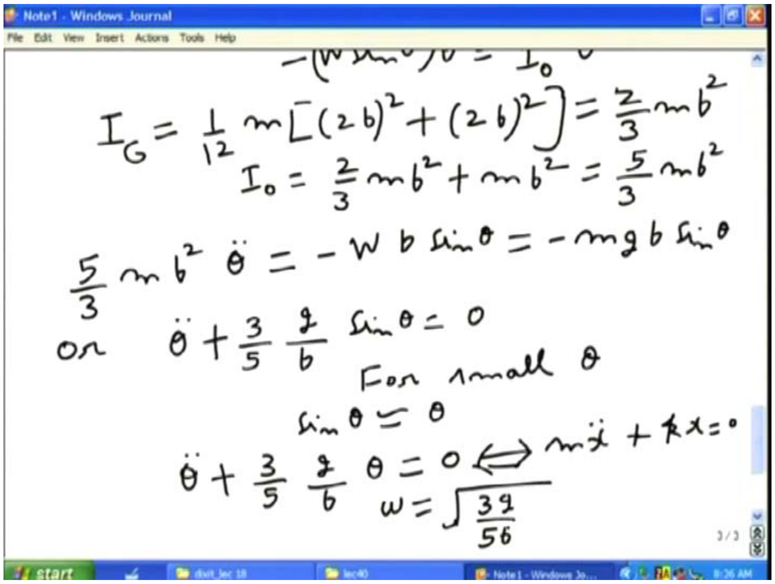


Therefore, equation becomes 5 by 3 mb square theta dot dot is equal to minus restoring thing minus Wb sin theta is equal to minus mg b sin theta. Or theta dot dot plus 3 by 5 g by b sin theta is equal to 0. However, for a small theta, sin theta is approximately theta provided theta is measured in radian and not in degree. So, sin theta. One has to be careful about this, that sin theta is equal to theta provided theta is measured in radian. Then theta dot dot plus 3 by 5g by b theta equal to 0, or this can be compared with mx double dot plus kx equal to 0, for which we know the frequency, under root k by m. Same thing happens here. Therefore, omega will be under root 3g by 5b.

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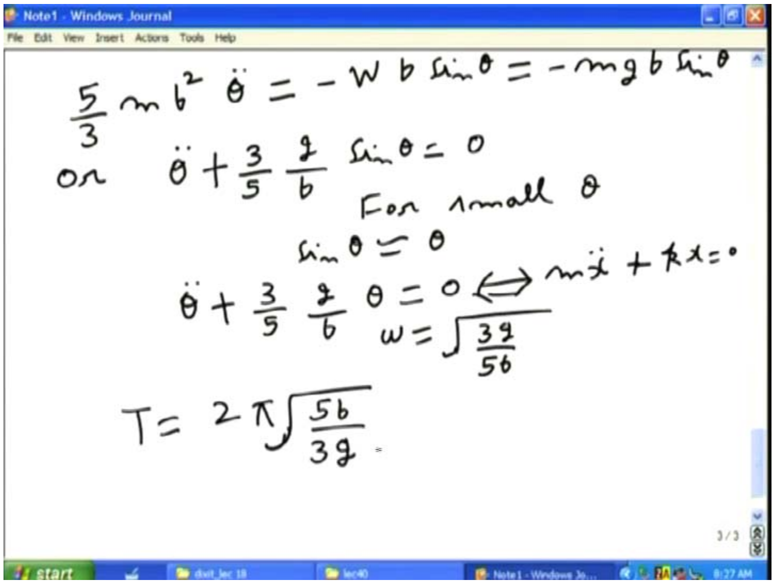


Time period can be written as 2pi by omega; that means, 2pi under root 5b divided by 3g. So, that means more is the size of this plate, more time is required. Similarly, let us discuss some other problems. I will take up, the vibration finds application at many places. Discussing about the vibration of rotating machines, this is one example.

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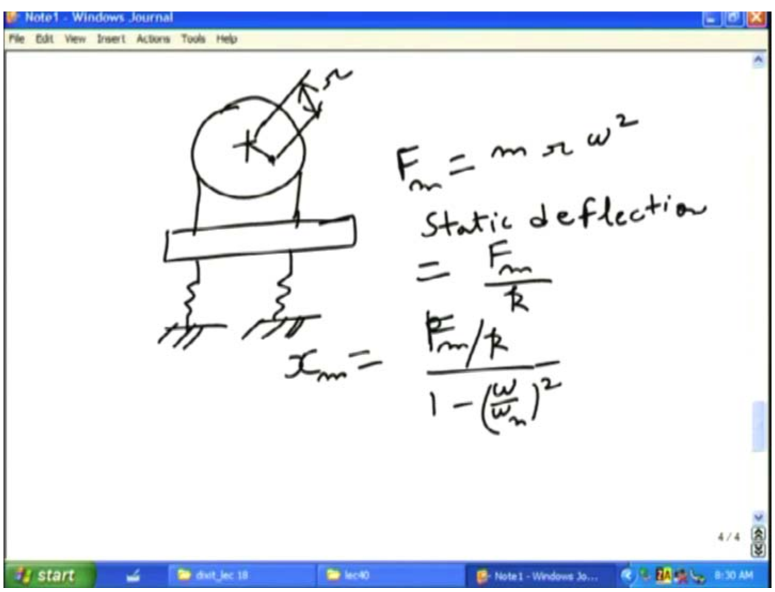

Suppose, you have a motor in which there is a rotor and this motor is mounted on a base plate which is supported by springs. If the rotor is unbalanced like this, that means mass is concentrated not at the center of rotation, but at this point, then there will be a force present here. This is with the amplitude of the force on the springs will be $F_{max}$ is equal to m times r omega square, where r is the eccentricity, r is unbalanced. Static deflection will be equal to $F_m$ divided by k and dynamic deflection will be given by $x_{m.}$ Amplitude of the dynamic deflection, assume that there is no damping. So, this will be $F_m$ divided by $k_1$ minus omega by omega$_n$ Whole Square.

If $r_m$ is very small compared to omega then this term will become very small; that motor vibrations will be less. Also, if omega$_n$ is smaller than omega, vibrations will be out of phase. Out of phase means, when the force becomes very high, at that time the deflection is less. When the force becomes less then deflection is more.

This type of phenomena occurs in dynamics which is not found in the statics. That force is more but the deflection is less. If force is less and deflection is more, then we say that this is the out of phase values. Similar type of that one, that problem can be studied here and that is called whirling of shaft.

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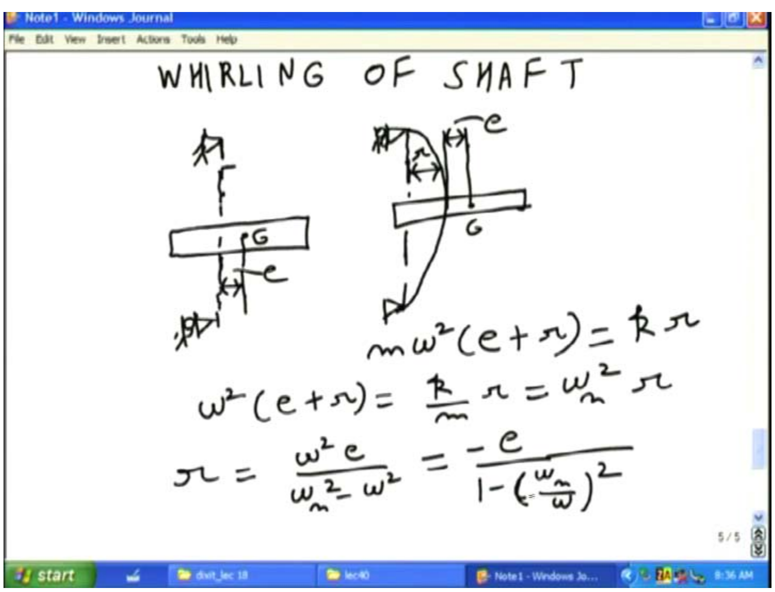

Assume that you have supported a shaft between two simply supported bearings and the shaft is rotating. There is a centrally mounted rotor. However, its CG does not pass through the axis of rotation. In a state, CG is at a distance of e, e is the eccentricity of the rotor. This is in the static situation. The eccentricity is e, the shaft is supported on this thing. Now supposing the shaft starts rotating then because of the eccentricity, the centrifugal forces get a setting they apply a road at the central of the shaft and as a result the shaft gets deflected.

However, when a simply supported beam is subjected to transverse load, it undergoes elastic deflection. As soon as the force is removed, it recovers its original position. So, a restoring moment force is developed because of the elasticity of the shaft. When it is rotating, this is the axis. This is simply supported and bearings may be like this. Then the shaft bends like this. Therefore, the rotor may move here. This is G and this is e. So, this is e and this distance is r.

The m omega square e plus r is the net centrifugal force acting. On this m omega square e plus r and the restoring force is equal to k times r, where k is the equivalent stiffness of the shaft as it has displaced by r, so it is k times r. Therefore, you get the relation omega square e plus r is equal to k by m into r. k by m is equal to $omega_m$ square r, where $omega_m$ is the naturally frequency of the system. Therefore, r is equal to omega square e divided by $omega_n$ square minus omega square, or it can be written as minus e divided by 1 minus $omega_n$ by omega whole square.

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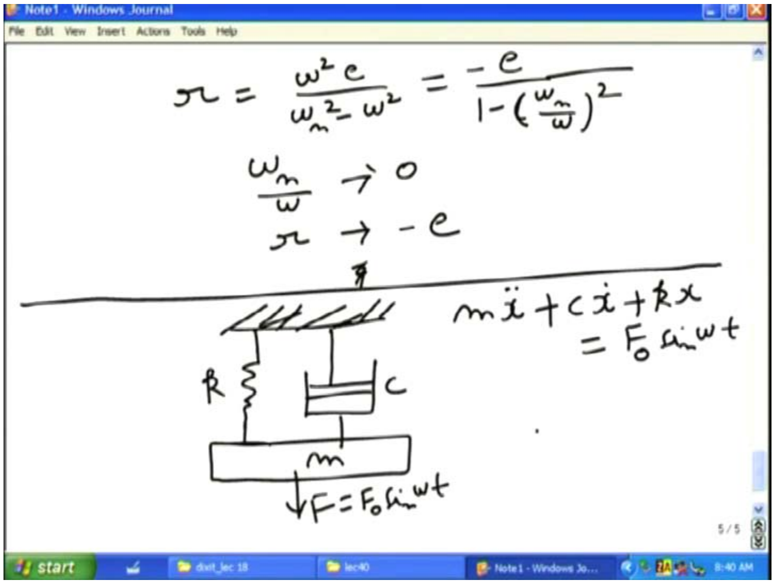


If the operating speed omega coincides with omega$_n$ then r will tend to be infinite. So, the condition of resonance will occur and the shaft may break because of the excessive vibrations. However, if omega is much larger than omega$_n$ then r omega$_n$ by omega tends to 0 and r tends to minus e. In that case, the shaft will bend in the other direction and therefore, this will in fact bend in other direction and this disc will shift. Therefore, the distance of the disc from here is r plus e. In that case also, the distance will be r plus e, but minus e plus e is equal to 0. The disc's mass center will always be at the axis of rotation and a stable motion will be obtained. The operating frequency can be kept higher than the natural frequency. However, at some other frequency, other modes of vibrations may exist; that we are not going to discuss here. This is about the other point.

The other point which I want to discuss that we have discussed about the vibrations of spring mass system. The spring mass and dashpot system is the most commonly implied system for studying the vibrations. Here, this is a dashboard, this is mass m, this is k and this is c. The equation of motion is mx double dot plus cx dot plus kx equal to F. If F is equal to $F_0$ sin omega t, in that case, this will be $F_0$ sin omega t. This is a second order differential equation. So, the systems which are described by these are called second order systems.

First order systems are described by the differential equations of the first order. $0^{th}$ order systems are described by systems of 0 order; that means no derivative is involved. Here, it is a second order system. Studying this equation or the solution of these can help in understanding other things also.

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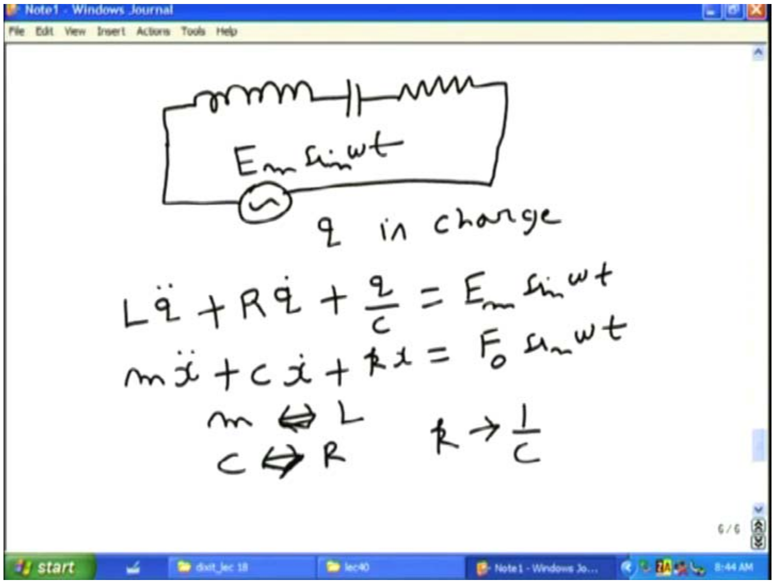


The similar type of thing occurs when spring there, there is a voltage source here, you have inductance capacitance and resistance in the electrical circuit. In this case, if q is the charge then you have the differential equation equal to Lq double dot which is basically L times di by dt, where L is the inductance plus R times q dot, that means basically R times i plus q by c this is equal to $E_m$ sin omega t, where this is the AC voltage that is $E_m$ sin omega t. This is the equation commonly used in electrical and electronics.

Now, compare this equation with the equation of spring mass dashpot system mx double dot plus cx dot plus kx equal to $F_0$ sin omega t. Instead of the force, sinusoidal force we have the voltage. instead of x, we now have variable q, their charge. The m is equivalent to inductance. It behaves in a similar manner, because in differential equation, if the coefficient is same, its behavior is also similar. Because of the inertia, body takes sometime to respond. Here also, the same thing happens. Because of the inductance there is some time to respond.

C is equivalent to R. If C causes energy loss in the system then R also causes the energy loss. It is dissipative; here also, it is dissipative; k is equal to 1 by C. Capacitor is like spring. Spring stores potential energy and at times it releases. Similarly, discharge is stored in the capacitor and then it is released. Therefore, you can have the analogy between these two. Similarly, the other systems and any development made in one field can suitably contribute in the development of the other field.

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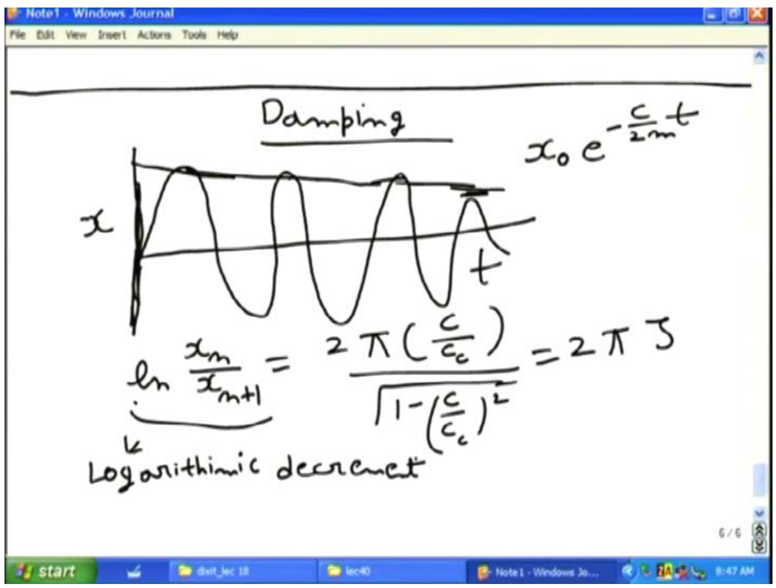


We have been talking about damping. It is difficult to determine damping. Spring stiffness can be easily determined; take a spring, apply some force, see the deflection force divided by deflection and use the stiffness. However, damping is not so easy to determine. One method implied for finding out the damping of a single degree of freedom system is like this. Plot the displacement x versus time, these plots will be like this. We will see that its amplitude keeps on decaying. As we discussed, it is $x_0$ e minus c by 2 m times t. One can take the natural logarithm of the ratio between the amplitudes $x_m$ divided by $x_{m+1}$; that is equal to 2pi. It can be easily shown that it is equal to 2pi c by $c_c$, where $c_c$ is the critical damping divided by 1 minus c by $c_c$ whole square, or if the damping is low, then under root one minus $c_c$ by whole Square can be treated as equal to one. Therefore, this becomes equal to 2pi. This is the damping factor.

Therefore, by measuring the ratios of two consecutive peaks, one can find out the damping. This is called logarithmic decrement. In practice, the ratio between two conservative peaks may be small and may be difficult to measure, means it can create measurement accuracy problems. Therefore, one can take the ratio of a first and k plus $1^{th}$ peak or n and n plus $k^{th}$ peak.

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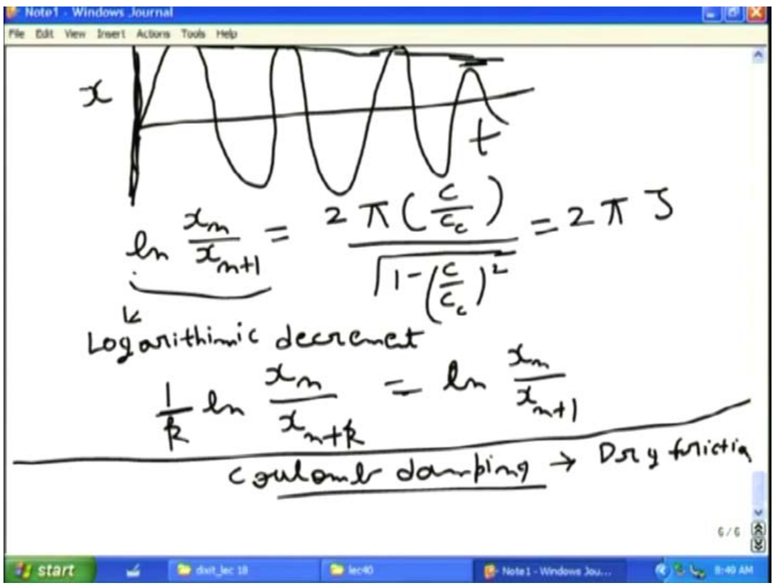


Then the logarithm decrement will be 1 by k $l_n$ $x_n$ divided by $x_n$ plus k. It can be shown that 1 by k $l_n$ $x_n$ divided by $x_n$ plus k is same as ln $x_n$ by $x_n$ plus one, this can be easily shown. It is better to measure the the logarithmic decrement based on the measurement of first, say one peak and after that 4, 5 peaks ahead. So, this is about the logarithmic. Here, we are assuming the viscous damping. One can as well take other type of damping. One is called Coulomb damping; that is damping due to dry friction.

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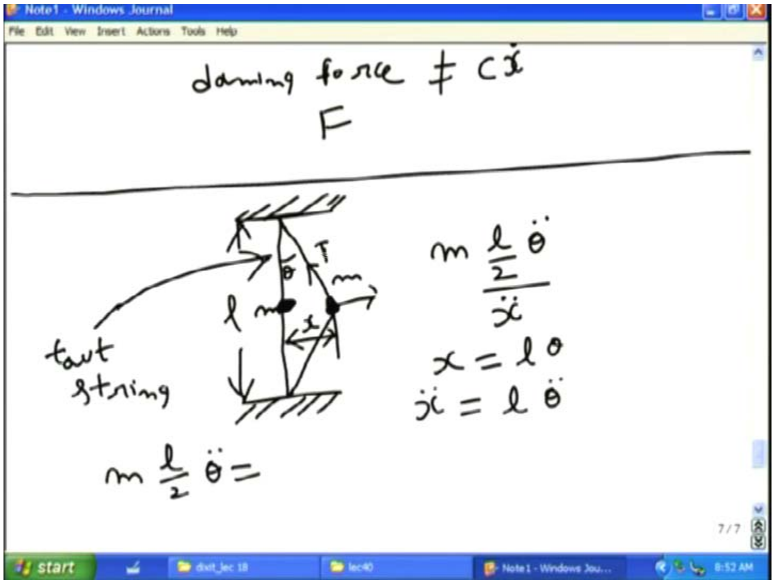


This damping is not dependent on cx. Therefore, damping force is not equal to cx dot. Instead it is constant; that is equal to F. So, that type of damping. Therefore, there will be change in the differential equation and these problems can also be solved. I will give one small example of solving this one. Suppose, let us do one problem. This is a taut string, and you put a mass m at the middle taut string. When you displace this object, by displace the mass, so that this angle is theta and this is mass m. This displacement is x. Then the vibrations start.

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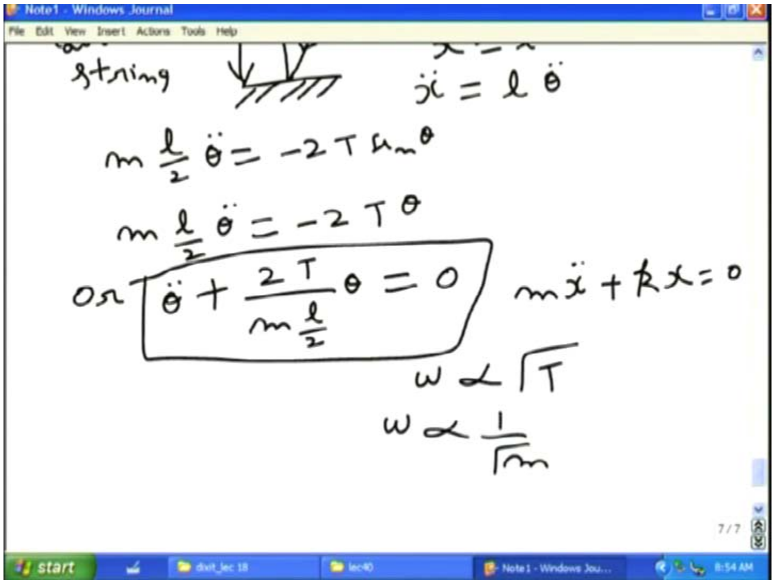


The governing equation for this case is m, this is the mass and this will be equal to l by 2x is equal to l by 2theta double dot. This is basically x dot, because x double dot x is equal to l theta. So, x double dot is equal to l theta double dot. Therefore, equation becomes m l by 2 theta dot dot. Now this is equal to T. This tension of this string is T. Assume that there is no change in the tension. Tension is directed here, T sin theta. So, it will be minus 2T sin theta.

However, if theta is very small, in that case, this can be written as m l by 2 theta dot is equal to minus 2T into theta, or theta double dot plus 2T divided by m l by 2 theta is equal to 0. This is again, the equation of equivalent spring mass system m x dot plus kx equal to 0. In this case, the natural frequency will be proportional to square root of T that means omega is proportional to square root of T. So, more tight the string, more is the frequency. Also, omega will be proportional to 1 by root m. So, we study the spring mass system because other systems also generate the differential equation in that form.

Therefore, this equation becomes very important. Of course here we have considered only the linear system, linear differential equation. Here, k is not changing. It is not a function of x. If k is a function of x then the equation will become nonlinear and you will get nonlinear vibrations.

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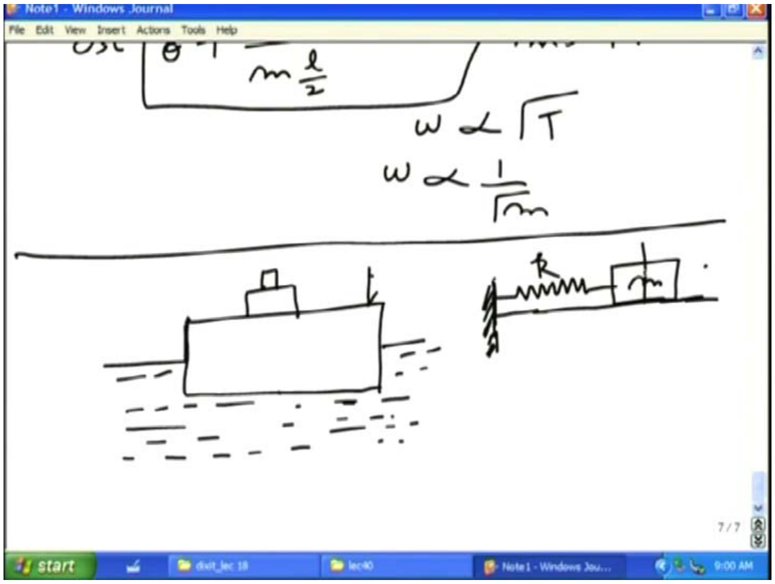


Similarly, other cases are like this; the vibrations of the floating vessels. Assume some container on ship which is like this and it is in water. Up to a certain distance, it is submerged and then this one. If this system is displaced, supposing you apply some force here and so it gets displaced, then it is a center of gravity and the center of buoyancy shift. If the restoring force is more than the disturbing force then it will be able to come back and start vibrating. Otherwise, if the restoring force is in the same direction as the disturbing force then it will become unstable. It will not be able to come back. If the restoring force is less than disturbing force then also it will not be able to come back. Therefore, for stability the restoring force should be equal to more than the disturbing force. Then it will be called stable.

Once, any object, this is so far the stability of ship. However, when it goes to the equilibrium position, by that time it attains some velocity and therefore, it goes on the other side. Again it tries to come back, because of the restoring force and the vibrations start. So, there will be some vibrations. In fact, when the system is stable then the vibrations of the system will be occurring. For example, you have this spring mass system; this is k. In this case, if you disturb it from the position, the restoring force tries to bring it back. Therefore, the system is stable. However there are vibrations present in the system. Suppose, you apply a force which is more than the spring stiffness, then the system will not be able to come to the original position and the spring will break. So this is stability. Now if k is more then it is able to provide more restoring force.

Therefore, it will try to bring it quickly to the stable position and the frequency of vibrations is more in this case. Back to the original position slowly and the frequency of vibration is less.

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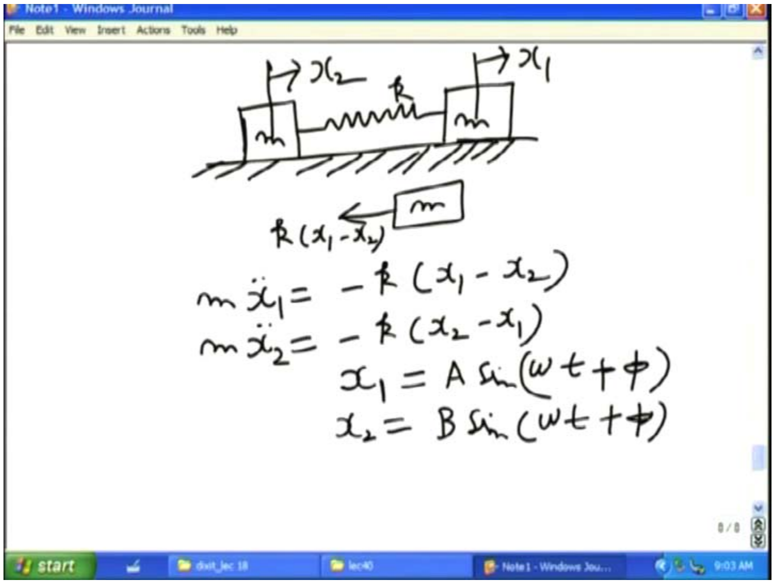


Lastly, we have discussed only single degree of freedom system, but let me just do the simple example of a two degree freedom system so that you can understand that, this can be easily extended to multi degree freedom system. Consider motion of this $x_1$. This is $x_2$. There are two masses m. Both are of equal mass. We are just assuming it. Then this is the spring stiffness m. Let us make the free body diagram of this mass. So, this is mass and then there is a spring force that is $kx_1$ minus $x_2$. So, the equation becomes mx dot dot is equal to k minus k $x_1$ minus $x_2$. For the other mass, the equation will be mx dot dot is equal to minus k $x_2$ minus $x_1$; this is $x_1$, this is $x_2$. These equations can be solved by the two coupled ordinary differential equations. This can be solved by assuming that $x_1$ is equal to A sin omega t plus phi and $x_2$ is equal to B sin omega t plus phi.

When we put these equations, we can obtain two frequencies and this becomes, basically the Eigen value problem which we will not be discussing in this lecture.

Therefore, essentially by solving the differential equations, you can study the vibration. In this problem, there will be two values of omega which will provide the solutions and these will be called two natural frequencies. Therefore, in a multi degree freedom system, you get many degrees of system. If there are n degree freedom systems then you will get n equations. Therefore, the concept starts here. We can easily extend to the multi degree freedom system.