Engineering Mechanics

Prof. U. S. Dixit

Department of Mechanical Engineering

Indian Institute of Technology, Guwahati

Introduction to vibration

Module 15 Lecture No. 39
Vibration of rigid bodies Part-2

We will continue our discussion on vibration of rigid bodies. Last lecture, I discussed about bifilar suspension. Now, we discuss a case of trifilar suspension. Sometimes, it is convenient to find the moment of inertia of a disc by observing the frequency of oscillation. When the disc is suspended by three wires symmetrically about the axis of the disc, then that is called trifilar suspension. This arrangement is shown in this figure.

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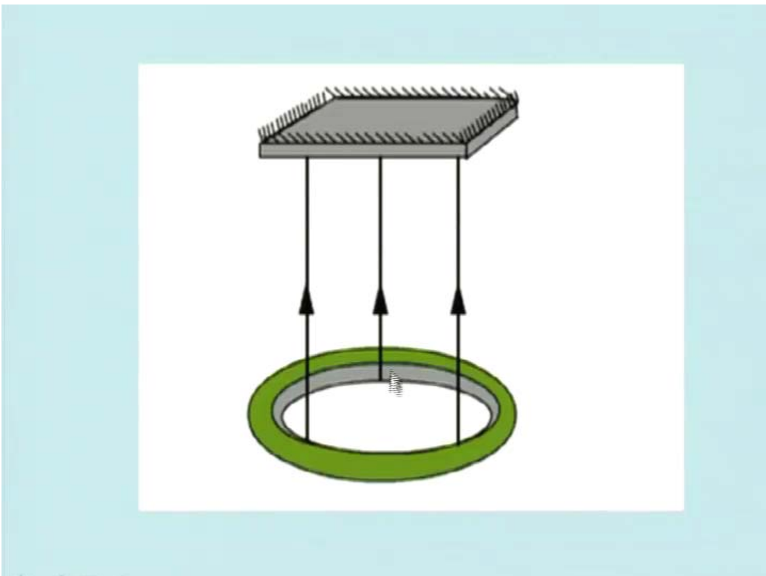

We can find out the moment of inertia of the disc by seeing its oscillations.

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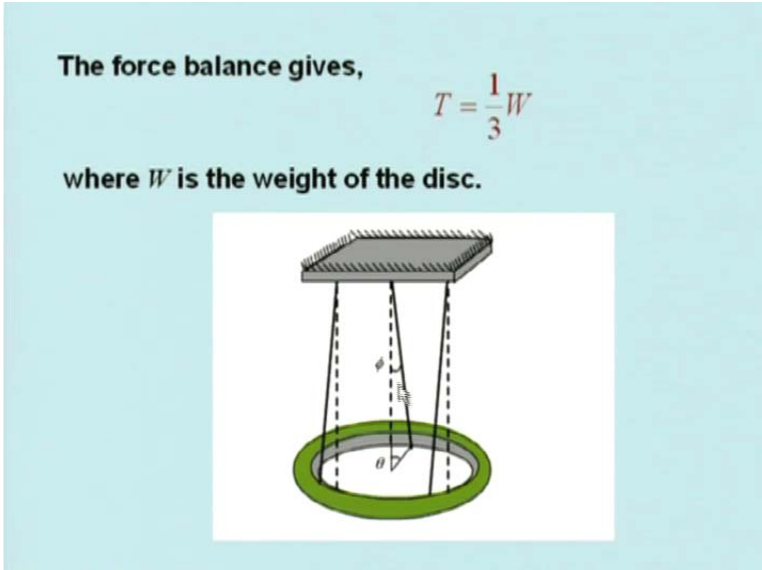

This force balance gives, T is equal to 1 by 3 W. Disc is acted by a gravity force W, and then there is a tension T in the strings which is 1 by 3 W. W is the weight of the disc. We can provide a small twisting in this plane of the disc. That means, disc remains in its own plane, but this radial point, radius moves to another point that is theta. Therefore, these strings get inclined. So, this is phi.

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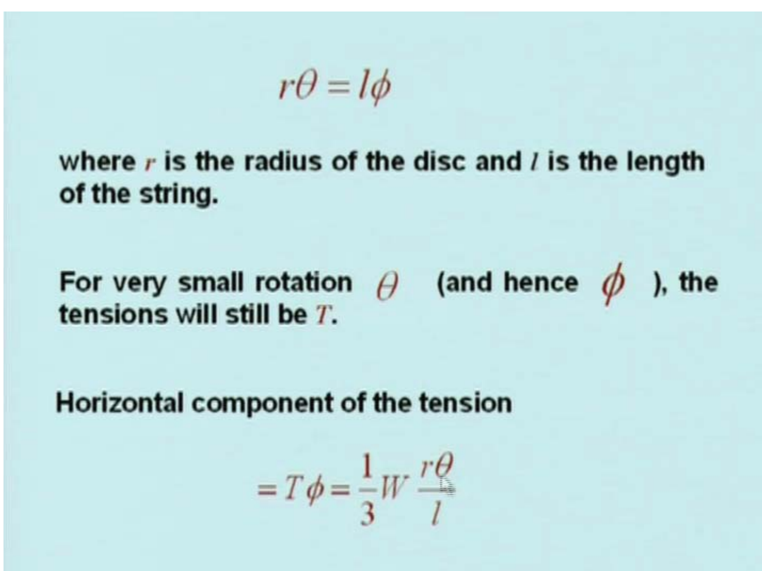

Therefore, r theta is equal to l phi. In this case, r theta will be equal to l. phi l is the length of this thing. So, r theta is equal to l phi, where r is the radius of the disc and l is the length of the string. For very small rotation theta, and hence if theta is small then phi is also small. The tension will still be T. No change in the tension then the horizontal component of the tension is equal to T phi. Actually, it is T sin phi but sin phi becomes equal to phi. So, that is equal to 1 by 3 W r theta by l.

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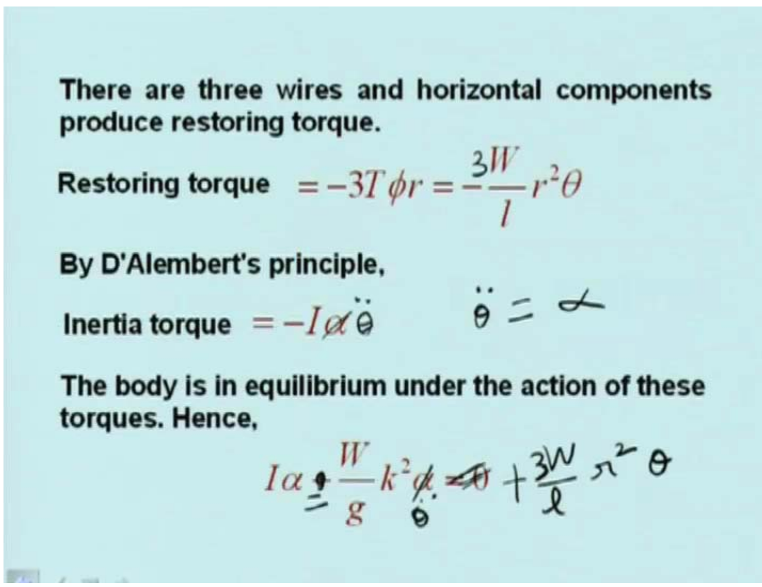

There are three wires and horizontal components produce restoring torque. Therefore, restoring torque is equal to minus 3T phi times r. So, this is 3T phi into r. So, that is minus W by l r square theta. Now, by D’Alembert’s principle, inertia torque is equal to minus I alpha. That must be that and restoring torque together should keep the body in equilibrium. The body is in equilibrium under the action of these torques. That means, I alpha plus W by l r square theta is equal to this one. Here, this is actually, instead of I alpha this is basically I theta. This is the thing that it is I theta. I theta, if I say that disc rotates by theta and this is W by l, this is r square.

This is W by I theta. Inertia torque is I theta double dot theta dot. There, you know that theta double dot is basically same as angular acceleration alpha. Therefore, this is the difference in the notation. At alpha is equal to theta dot and this is basically I alpha can be written as W by g k square alpha, because I is equal to W by g k square and plus you add W by l here. This is 3T here. Also, it is must be 3W by l r square and this is theta, this is theta double dot.

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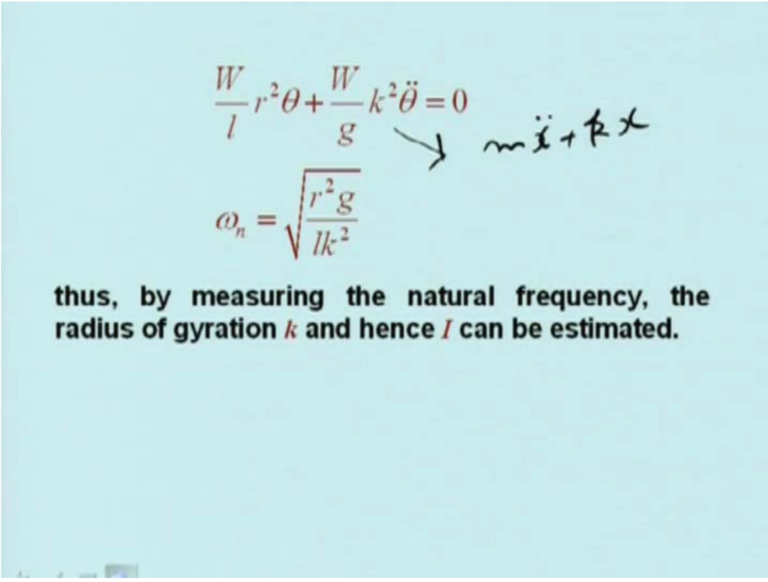

This is basically, you get equation like this: This is W by l r square theta plus W by g k square theta double dot is equal to 0. This is compared with mx double dot plus kx is equal to 0. So, it is basically, the same way that W W gets cancelled; it becomes r square g lk square.

This is r square by l and g by k square. Thus, by measuring the natural frequency, the radius of gyration k and hence, I can be estimated. We can find out, if you measure the natural frequency omega$_{n,}$ you know the gravitational constant, you can find out its r. l is known. So, one can find out this thing.

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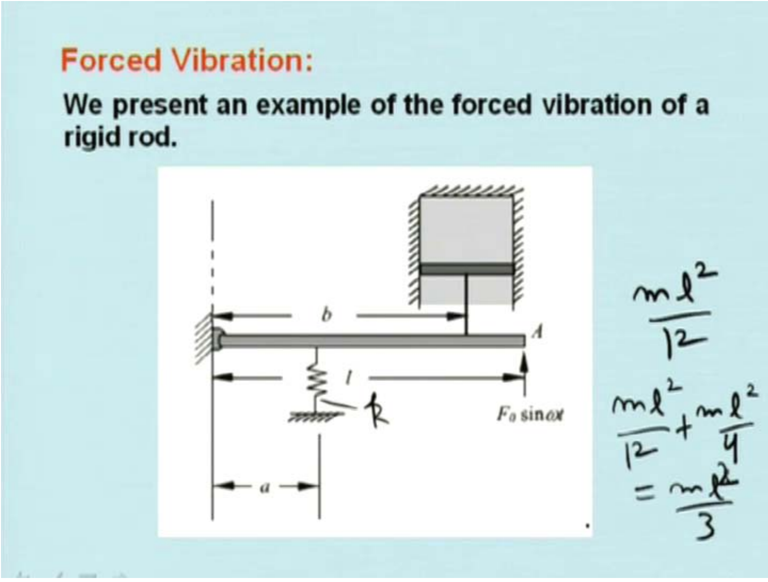

Now, we have talked about the free vibration problems. Let us go to the forced vibration problem. An example of a forced vibration of a rigid rod is like this; this rod is hinged at this point and an excitation force F_0 sin omega t is provided at point A. Dashpot is at a distance of b. This distance is l. This is the spring. Here, the spring stiffness is k. This is a, this is b. Now, this is d. This is a dashpot, linear dashpot.

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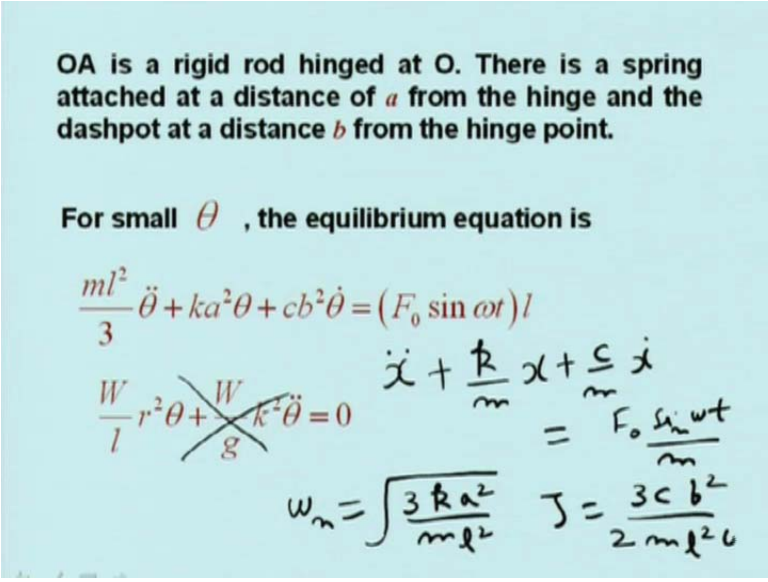

OA is the rigid rod hinged at O. There is a spring attached at a distance of a from the hinge, and the dashpot at a distance b from the hinge point. So, dashpot is that. For a small theta, the equilibrium equation is ml square by 3 theta double dot plus ka square theta. ml square by 3 is the mass moment of inertia of the cylinder rod about point O.

Mass moment of inertia of a uniform rod about the center of mass is ml square by 12 about its one end. It is actually ml square by this thing, because apply the parallel axis theorem, ml square by 12 plus m into l by 2; distance is l by 2 l square by 4. This is 3 by 3l square by therefore, this is ml square by 3. For a small theta, the equilibrium equation is ml square by 3 theta double dot plus ka square theta; that is the restoring force due to spring. When you displace it by theta, this spring gets stressed by a theta. So, k times a theta is the force and it acts at a distance a from the hinge. So, k square. Similarly, the dashpot is moving. So, b theta is the displacement of the dashpot, b theta dot is the velocity and therefore, the force is equal to c times b theta dot but it is acting at a distance of b.

Therefore, it becomes cb square theta dot and this is equal to F_0 sin omega t multiplied by l, l is the distance from here. Therefore, you get W by g by l. This will be ml square by 3 theta dot and this becomes this one. There is no need to write this equation, F_0 sin omega t into l. Actually, compare this equation.

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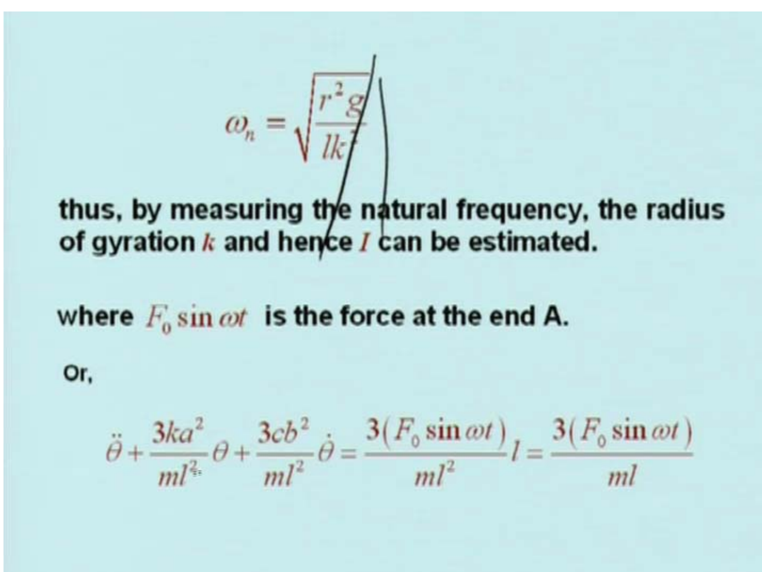

Now, compare this equation with equation x double dot plus k by mx plus c by mx dot is equal to F_0 sin omega t by m, which was spring mass and dashpot system. So, by this you can find out, omega_n is equal to natural and damped frequency, omega_n is equal to under root 3 ka square divided by ml square and damping factor zeta is equal to 3c b square divided by 2 m l square omega n and omega d is equal to omega_n under root 1 minus zeta square. Therefore, F_0 theta dot.

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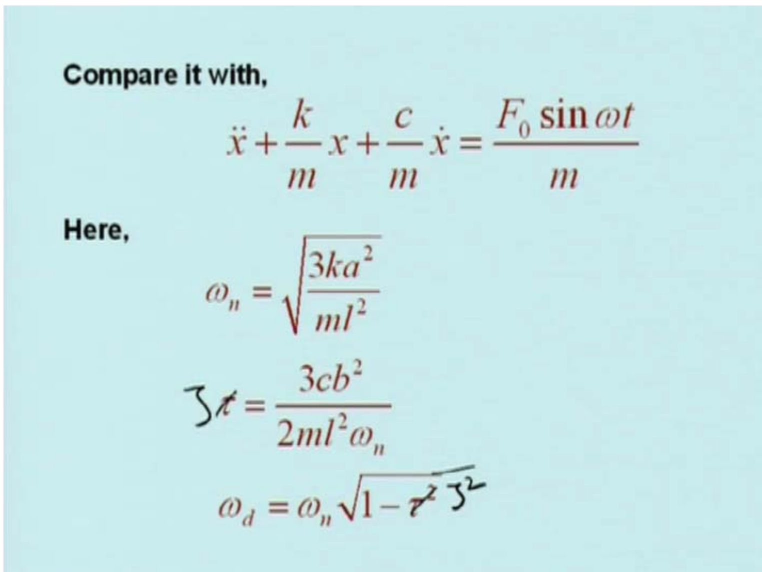

Now, omega_n is equal to square root 3k a square by ml square. This is zeta equal to 3c b square divided by 2m l square omega_n and omega_d is equal to omega_n under root 1 minus, this is tau square.

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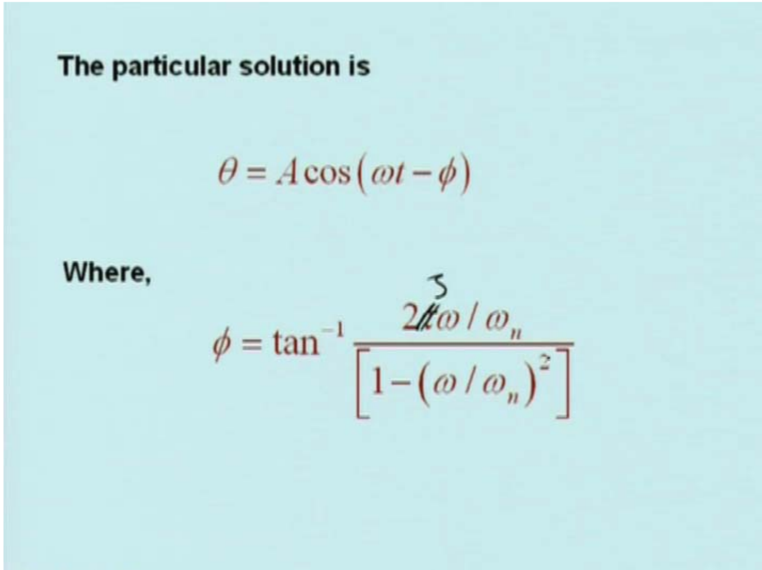

The particular solution of this is, theta is equal to A cos omega t minus phi, where phi is equal to tan inverse 2 zeta omega by omega$_n$ divided by 1 minus omega by omega$_n$ whole square.

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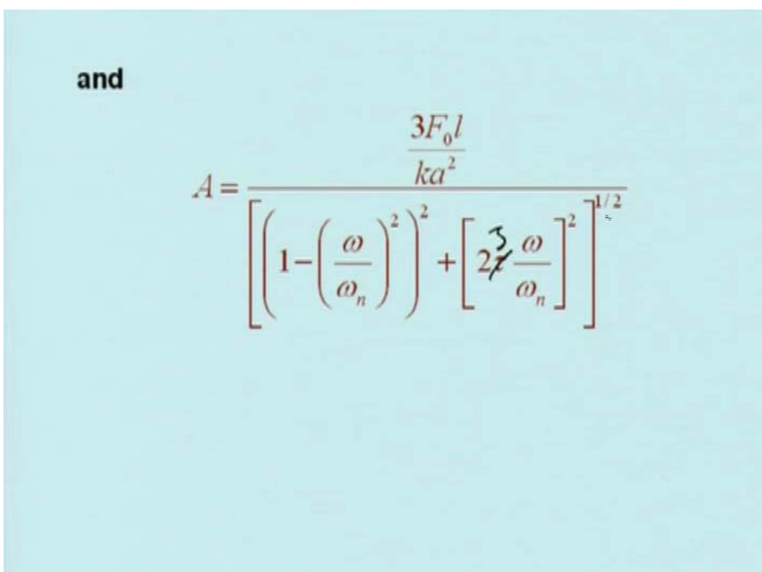

A is equal to $3F_0$l by l k square. This is 1 minus omega by omega$_n$ whole square. This is whole square plus 2 zeta omega by omega$_n$ whole square this is 1 by 2.

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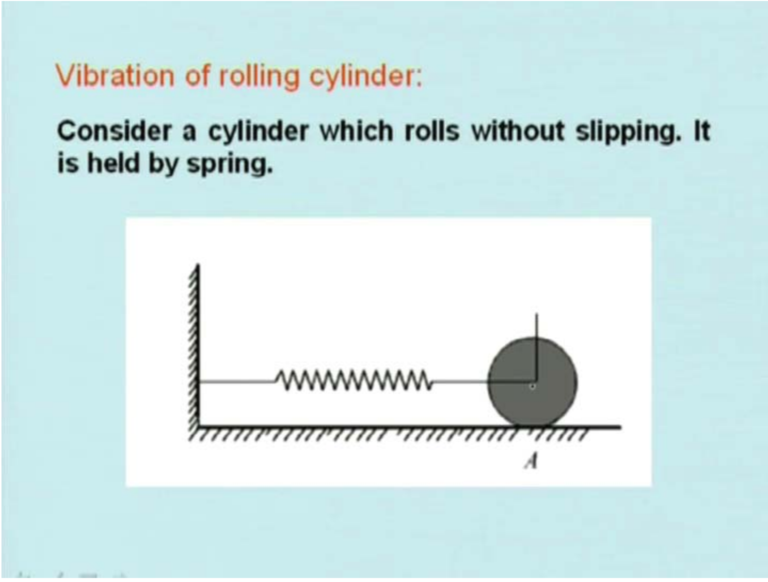

Let us discuss another problem; vibration of a rolling cylinder. A cylinder can, when it rolls on a straight line and if it is having a pure rolling motion then there is only 1 degree of freedom, because its angular displacement and the displacement of the center are related. r into theta is equal to the displacement of the center. However, if it is not a pure rolling motion, if there is some slipping, then you know this type of thing is not correct; that means, there is no relation between the velocity of the center or and the angular velocity.

Suppose, you give the angular displacement of the cylinder, we cannot know how it much has moved, because there is some slippage. Therefore, one degree of freedom is not enough. We have to have 2 degree of freedom.

Therefore, we discuss the case of that motion in which the cylinder rolls without slipping. This cylinder is held by spring. This is the spring and it is attached at the centre and it is displaced slightly.

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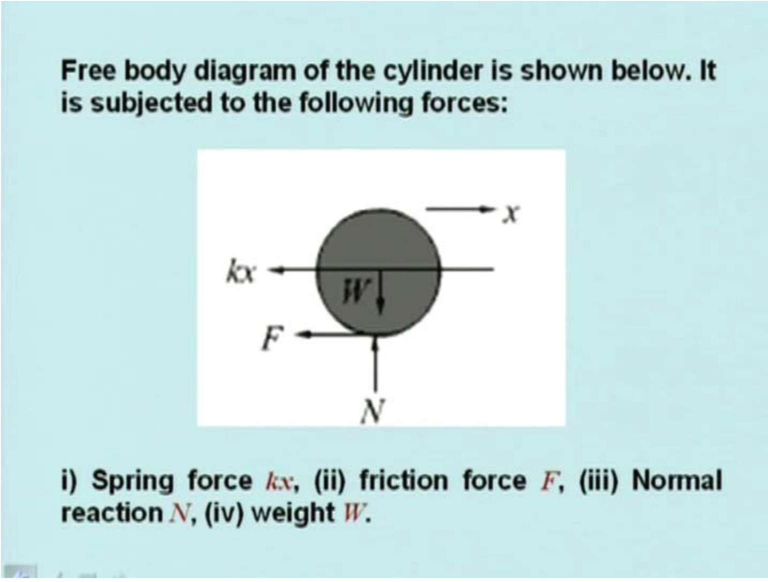

Now, make the free body diagram of the cylinder which is shown here. I have shown the cylinder. Make the free body diagram. What are the forces? Let us discuss the forces coming on the centre.

One is the spring force, restoring force of the spring when the cylinder center has moved by distance x then the spring gets stretched by amount kx. So, kx is the force of the restoring force of the spring. W is the weight of the cylinder acts downward. N is the vertical reaction of the floor and F is the force due to friction. We have got these four forces: spring force kx, friction forces F, normal reaction N and weight W. Now, N if there is no vertical motion then N must be equal to W, but then there is a horizontal direction there is motion.

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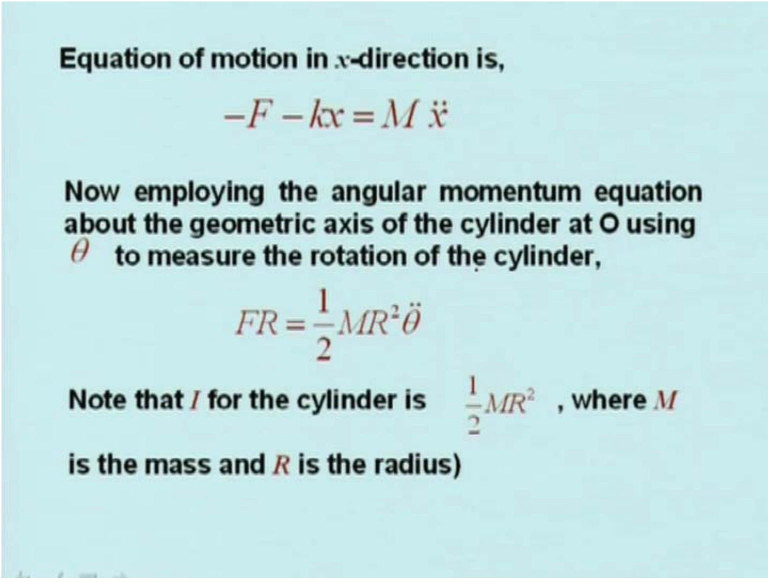

So, equation of motion in x direction is minus F minus kx is equal to Mx double dot, because F is the friction force which we do not know as yet, but we have to write equation minus F minus k. If you employ the angular momentum equation about the geometric axis of the cylinder at O using theta to measure the rotation of the cylinder, using theta about this, therefore, it is F into R is equal to 1 by 2 MR square theta.

If you take about the geometric axis, when you take the moment about this point, kx will not produce any moment because it is passing through O; W will not produce, N will not, only F will produce a moment. That magnitude is given as F into R and that is equal to half MR square theta. Because, if it is a solid cylinder, its moment of inertia is half MR square; otherwise, if it is a hollow cylinder then it can be MR square theta. I for the cylinder is half MR square, where M is the mass and R is the radius.

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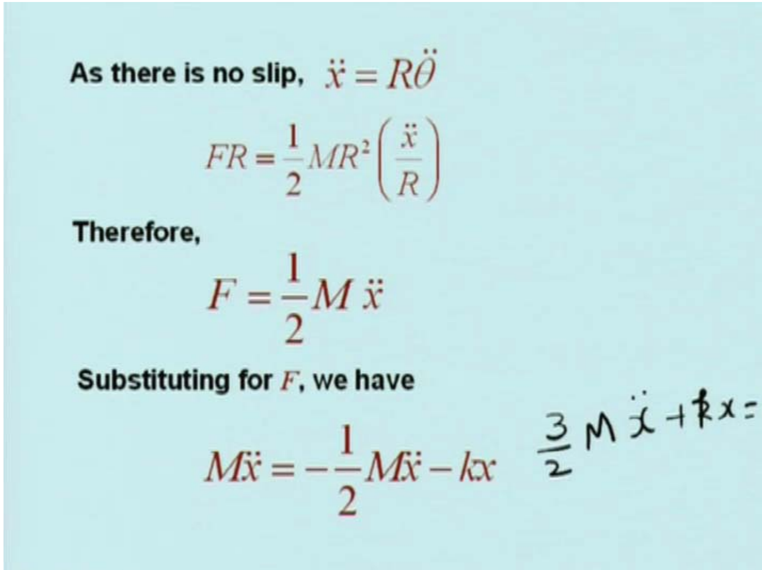

As there is no slip, we can write that equation, x double dot is equal to R theta double dot; that is, the linear acceleration is equal to R times the angular acceleration. Therefore, F into R is equal to half MR square into x double dot by R and therefore, F is equal to half M x double dot. Now, substituting for F, we have M x double dot is equal to minus half M x double dot minus kx.

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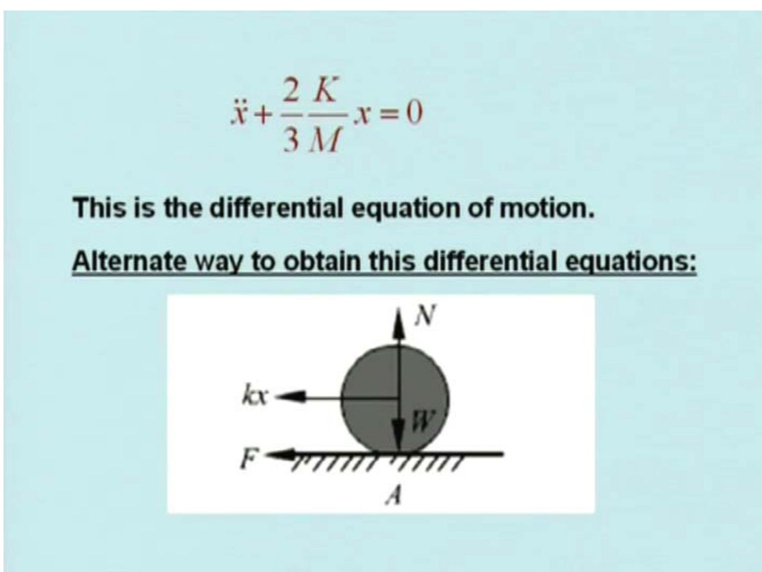

Therefore, x double dot plus 2 by 3 M. x is equal to 0, because here, this will be 3 by 2 Mx double dot plus kx equal to 0. Compare this equation with the spring-mass system, where we had Mx double dot plus kx equal to 0. We are having 3 by 2 Mx double dot plus kx. Therefore, effective mass is actually more. Therefore, x double dot plus 2 by 3 km plus x equal to 0. This is the differential equation of the motion of a cylinder held by the spring.

There is one alternate way to obtain this differential equation.

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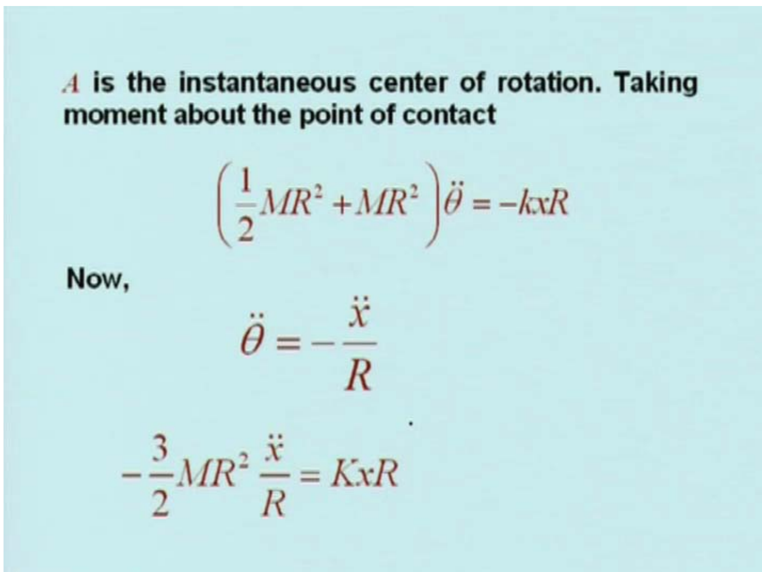

In this, we draw the free body diagram like this, where A is the instantaneous center of rotation. We take the moment about this fixed point of contact. About this point, mass moment of inertia is equal to half MR square plus MR square using parallel axis theorem; because, the mass moment of inertia about the center of the cylinder is MR square. In that, you add half MR square and theta double dot is equal to minus Kx times R. Now, theta double dot is equal to minus x double dot by R, as it is a pure rolling motion. In pure original motion, angular acceleration and linear acceleration of the center are related. This is, minus 3 by 2 MR square x double dot by R is equal to Kx by R.

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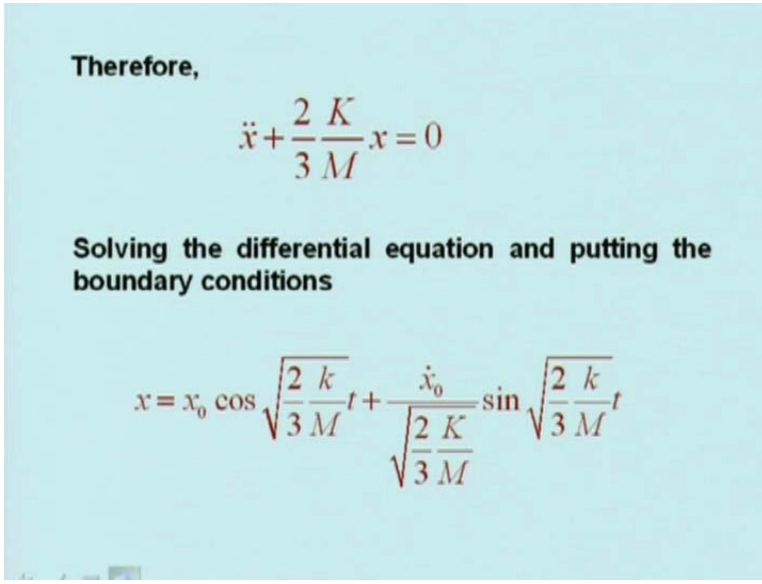

Therefore, x double dot plus 2 by 3 K by M x equal to 0. We get the same type of differential equation. This differential equation can be solved. Solving this differential equation, omega will be naturally under root 2 by 3 K by M x_0 cos under root 2 by 3 by M t plus x double dot x_0 under root 2 by 3 K by M sin under root 2 by 3 k by M t, t is outside the square root sign.

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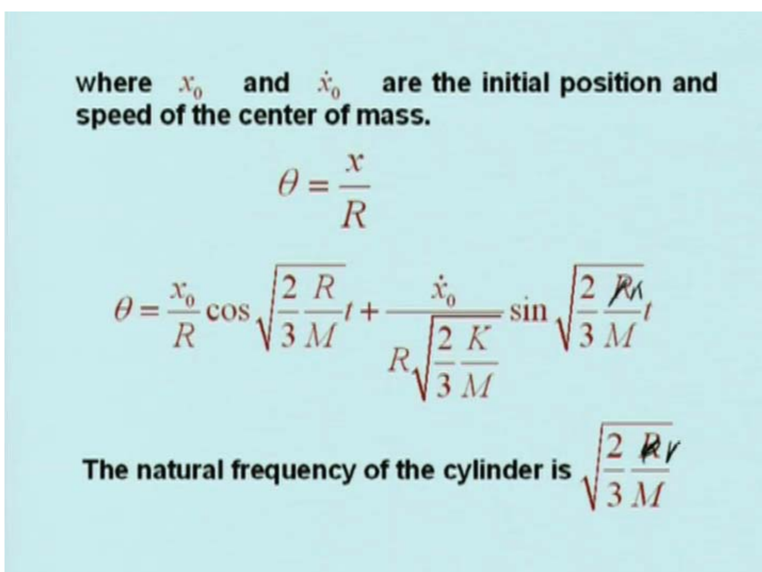

x_0 and x dot $_0$ are the initial position and speed of the center of the mass, theta is equal to x by R. Now, theta is equal to x by R. We put, in terms of that theta is equal to x_0 by R cos under root this one. Similarly, here you will put $x_{0;}$ but x dot $_0$ by R under root 2 by 3 K by M sin under root 2 by 3 R by M t. Natural frequency of the cylinder is 2 by 3, this is K by M. This is the motion of that cylinder. So, we have explained about this.

Now, we have discussed these problems. Let us go to another problem. We can solve the vibration problems by another method, by energy method. That portion also, we will discuss here. We know that in the conservative system, the total energy is constant and the differential equation of motion can be established by the principle of conservation of energy. For the free vibration of an undamped system, the energy is partly kinetic and partly potential.

The kinetic energy is stored in the mass by virtue of its velocity, whereas the potential energy is stored in the form of strain energy in elastic deformation or work done in a force field such as gravity. You can have spring mass system, where the strain energy is stored in the spring and we can have simple pendulum, where the energy is stored in the force field like gravity. Now, total energy is always constant in the free vibration. Therefore, the rate of change is 0; therefore, what happens? We can write this equation.

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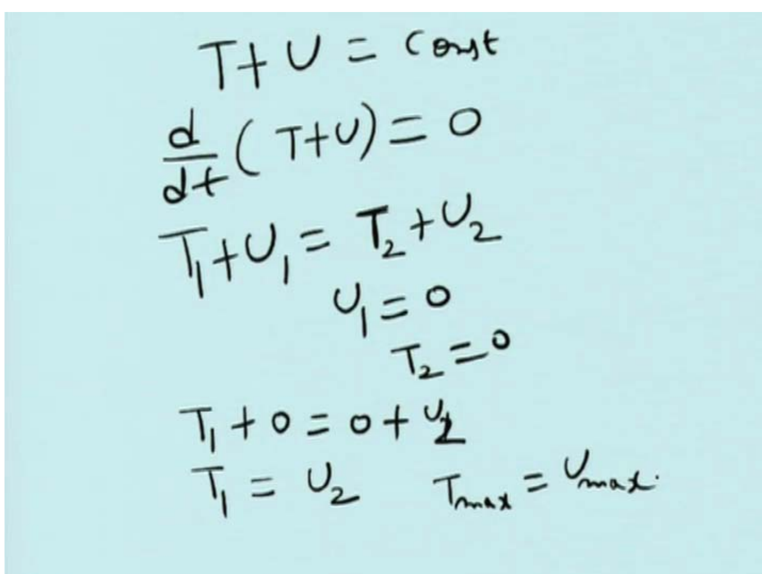

T plus U equal to constant in a free vibration. T is the kinetic energy, U is this one. This means that basically, d by dt T plus U is equal to 0; that means, if you take the derivative with respect to time, you should get 0. Although in the kinetic energy you get the velocity term which may be function of time, and in the potential energy you get displacement which is also function of time. If our interest is only in the natural frequency of the system, it can be determined like this. From the principle of conservation of energy, we can write for two instants T_1 plus U_1 is equal to T_2 plus $U_{2,}$ where one and two's subscripts represent two instances of time.

Let one be the time when the mass is passing through its static equilibrium position. Choose U_1 is equal to 0 as the reference for the potential energy. You know kinetic energy is 0 when the velocity is 0, but for potential energy, there is no such displacement is 0 then potential energy is 0. You have to take a datum. With respect to that datum, you measure the change in potential energy. Therefore, when U_1 is equal to 0, as the reference for the potential energy U_1 is equal to 0 at equilibrium position and T_2 be the time, corresponding to the maximum displacement of the mass. At this position, the velocity of the mass is 0, because extreme position that is the maximum displacement has occurred. It cannot go beyond that. So, velocity must be 0; otherwise, it will go in that direction. So, T_2 is equal to 0; therefore, T_2 is also 0. You have a relation; then, T_1 plus 0 is equal to 0 plus $U_{1,}$ or we have got like this T_1 is equal to $U_{2,}$ this is $U_{2.}$ So, this is T_1 is equal toU_2.

If this system is undergoing harmonic motion then T_1 and U_1 are of maximum value; that means, at the center, T_1 will be maximum when the potential energy is 0. So, obviously that T_1 is maximum and then, here U_2 is of maximum. Therefore T_{max} is equal to U_{max}.

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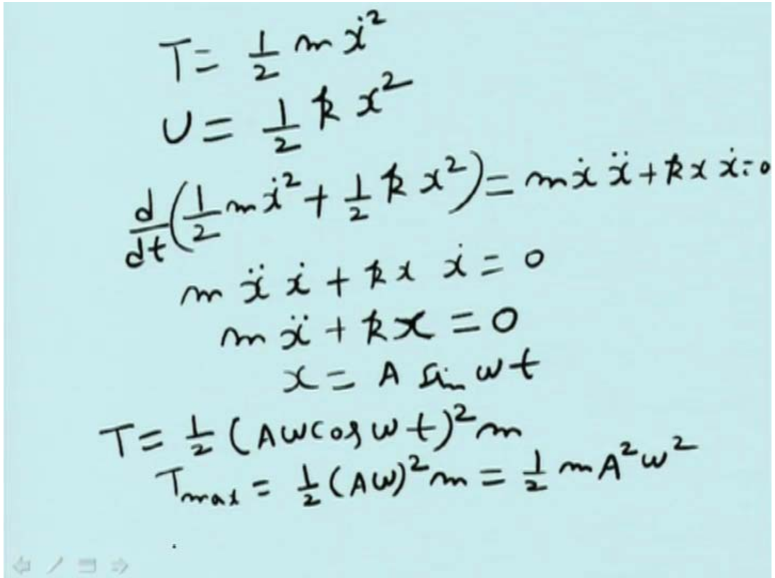

Let us see how you can apply this condition for a spring mass system. T is equal to half m x dot square. At any instant, its kinetic energy is given by half m x dot square and U is equal to half k x square. Now, T plus U is constant; therefore, half m x dot square plus half k x square d by dt is equal to m x dot x double dot. Differentiate it; plus k x x dot equal to 0, or you get the equation m x dot double dot x dot plus k x dot equal to 0. Therefore, we get a differential equation mx dot double dot plus kx equal to 0.

This is because x dot can be eliminated except at the extreme position, where x dot is equal to 0. Therefore, you get the differential equation that is mx double dot plus kx is equal to 0. This differential equation is equal to like this. Now, for finding out the natural frequency, let us assume that x is equal to A sin omega t. If you assume that x is equal to A sin omega t, then we get t is equal to half A omega cos omega t whole square multiplied by m. As the maximum value of cos omega t is 1, therefore, T_{max} is equal to half A omega square into m is equal to half m A square omega square.

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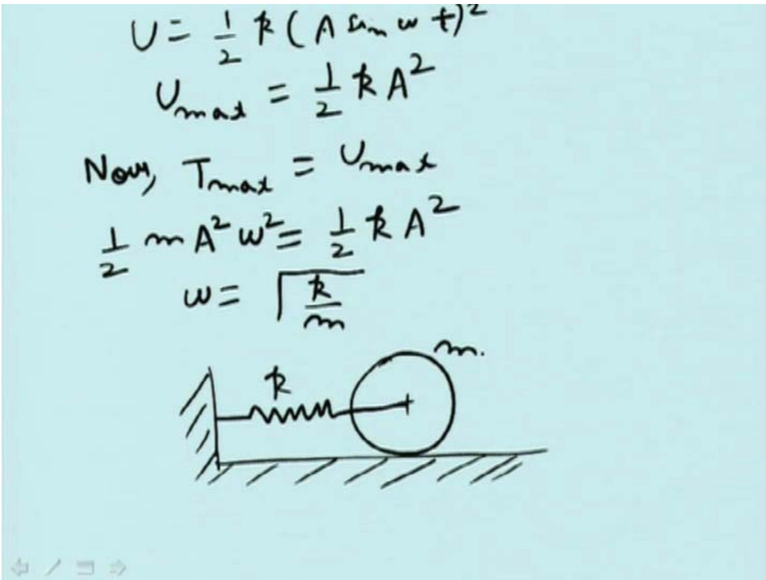

Similarly, the potential energy is given by U is equal to half k A sin omega t whole square. As the maximum value of sin omega t is 1, the maximum potential energy U_{max} will be half k A square; maximum value is half k A square. Now T_{max} is equal to $U_{max.}$ Therefore, half m A square omega square is equal to half k A square.

From this equation, we get omega is equal to under root k by m which is the expression for the natural frequency. Therefore, you get omega is equal to under root k by m. Like that, we can find out. In the last lecture, we have solved one problem, vibration problem. The cylinder rolls without slipping on this thing k and this is the mass m. Now, it is a cylinder which rolls without slipping and is held by spring. The total energy of the system is given by half mV square. So, let me write the expression.

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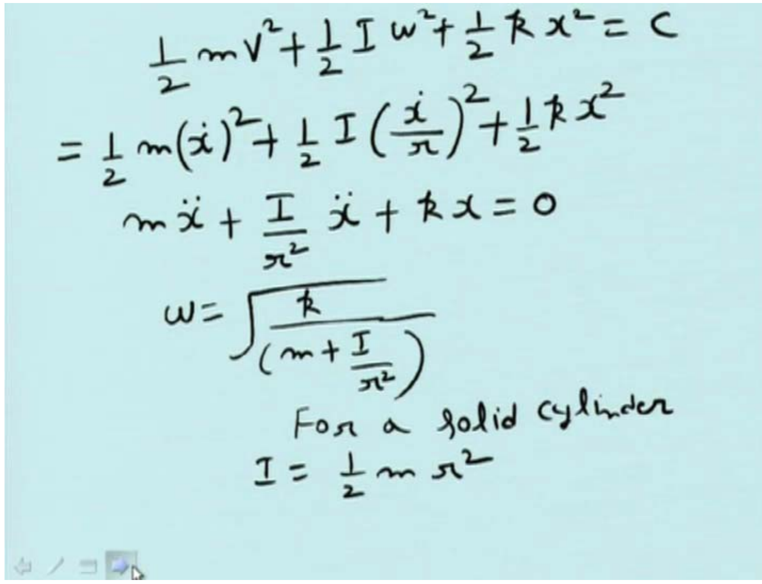

Here, half mV square plus half I omega square plus half kx square is equal to constant. Total energy, you know is constant. In this case, this can be written as half m x dot whole square plus half I x dot by r whole square plus half k x square. If you differentiate this equation for total energy with respect to time, you get mx double dot plus I by r square x double dot plus kx equal to 0. Note that, we could write omega is equal to x dot by r, because of the rolling motion, this is pure rolling motion.

So, omega is equal to under root k divided by m plus I by r square. You get this term. For a solid cylinder, I is equal to half m r square.

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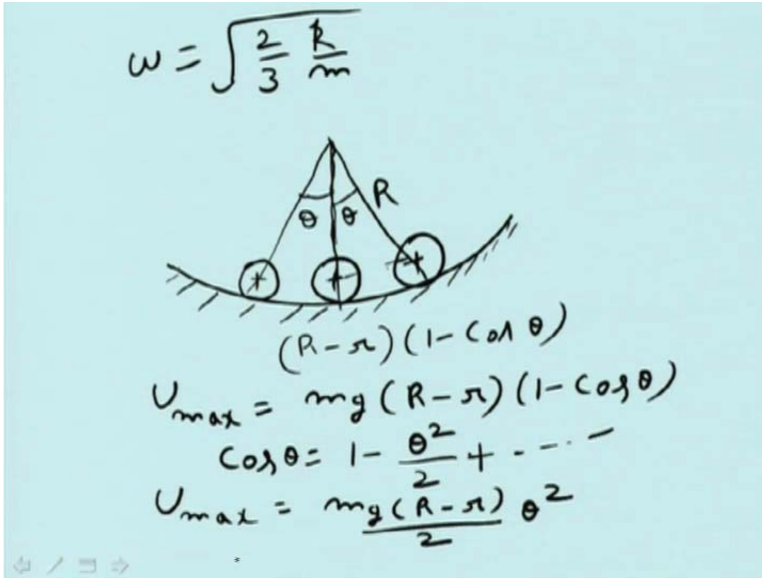

Therefore, omega is equal to under root 2 by 3 k by m; that is, the same expression. Let us take another example, on the use of the energy method. Here, if we have got a concave surface like this and you have kept a ball just like that. Now, displace it slightly. So, it goes like this here, Then its potential energy increases. At this point, it tries to come back but by that time it has attained velocity, it will go on the other side. Neglect the friction. You see that this goes this side then again it tries to come and like that it keeps oscillating.

Here, it attains the maximum velocity. After that, comes back and after that goes in this direction. Again, its velocity becomes zero; then, this is theta and this is R. If this is the maximum angle of the theta and increase in the height of the centre of the cylinder is basically R minus r 1 minus cos theta, then that is the increase in the height of the center of the cylinder. Cylinder is of radius r, this distance is R minus r. Therefore, this is R minus r into 1 minus cos theta.

Assume that the potential energy at the equilibrium position in the middle is 0. So, maximum potential energy, U_{max} is equal to mg R minus r into 1 minus cos theta, theta is the angular amplitude. Here, if we assume that theta is small, then cos theta can be written as 1 minus, expression for cos theta is 1 minus theta square by 2, plus other terms, where theta is meridian.

Therefore, 1 minus cos theta is theta square by 2. Therefore, U_{max} is equal to mg R minus r divided by 2 theta square or small theta.

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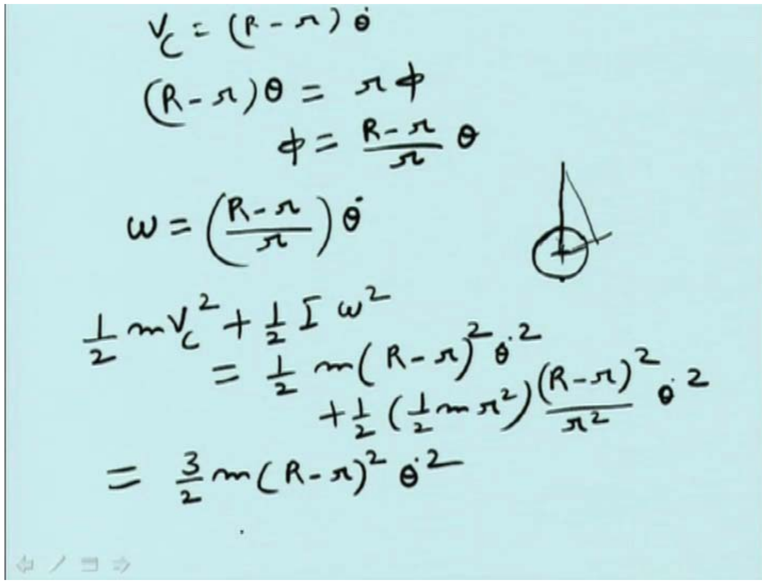

For a small rotation, the velocity of the mass center, V_c is equal to R minus r theta dot, a small rotation. Let the angle of rotation of this cylinder is phi, then R minus r into theta; that is, the moment of the center is equal to r phi, where phi is the angle of rotation of the cylinder about its own center. Therefore, phi is equal to r phi. Phi is equal to R minus r divided by r theta. Hence, the angular velocity of the cylinder is equal to omega R minus r divided by r into theta dot, or else, see it like this that there is a cylinder, this is R and this is moved by some distance, center moves R minus r into theta.

Therefore, about instantaneous center, the distance is r. Therefore, omega is equal to R minus r by r theta dot. The total kinetic energy of the cylinder is equal to half mV_c square plus half I omega square is equal to half m R minus r square theta dot square plus half half m r square R minus r square by r square theta dot square, which is equal to 3 by 2 m R minus r square theta dot square.

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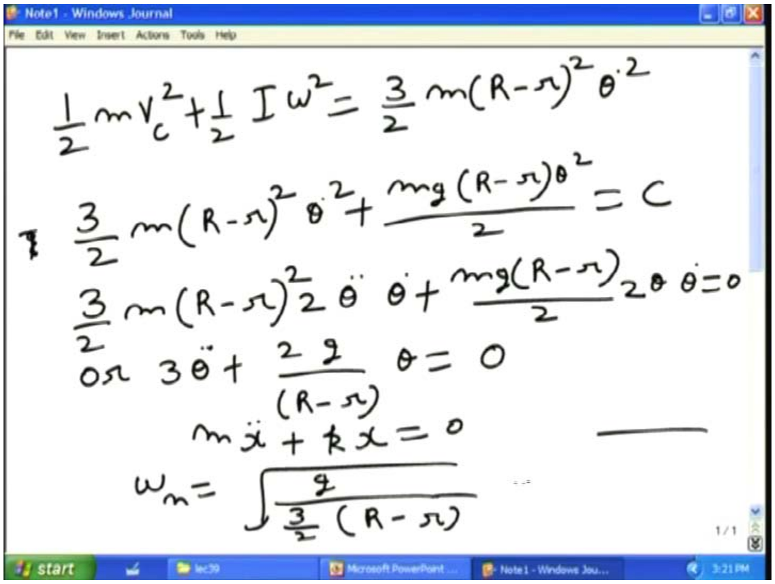

Total kinetic energy of the cylinder is half mV_c square plus half I omega square which we got 3 by 2 m R minus r square theta dot square. Total energy comes out to be 3 by 2 m R minus r square theta dot square plus mg R minus r theta square by 2 is equal to constant. Differentiate this expression with time; this must be 0. That means, 3 by 2 m R minus r square 2 theta double dot theta plus mg R minus r by 2 theta theta dot equal to 0 or theta dot dot plus 2g divided by R minus r into theta equal to 0.

Compare it with m x double dot plus kx equal to 0. We see that ω_n is equal to under root g divided by 3 by 2 R minus r. We get this type of expression. If R is equal to infinite, that means that it is plane surface. Then, there is no vibration and if R is equal to, instead of that if this is just a ball, if a point mass particle is there then it will be g by 3 by 2 R minus r. This is the velocity of that.

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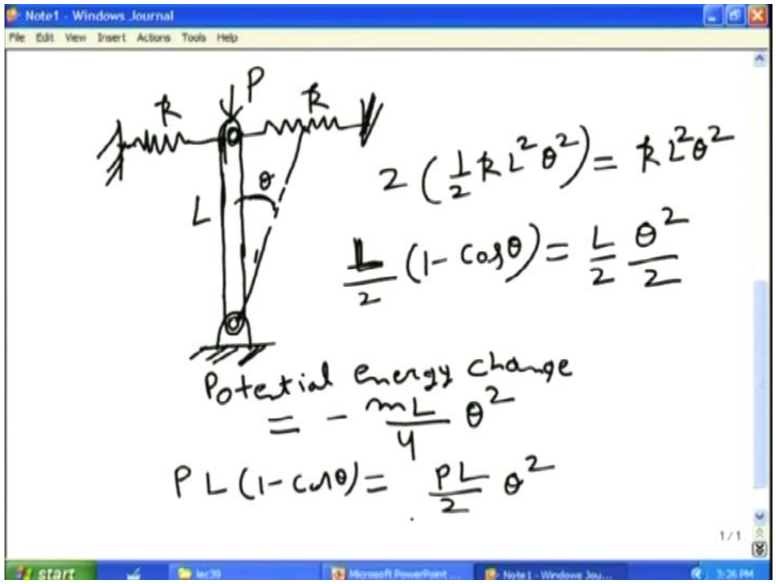

If you take a rod and there is a spring attached here, k and this spring is attached on this side as well this is k, this is P and this is L. Let a rod of mass m and length L is restrained in the vertical position by two identical springs having spring constant k, a vertical load P acts on the top of the rod. This is acting if given a small displacement theta to the rod, one spring compresses by amount L theta.

If you just give a displacement theta, so I am showing very exaggerated, theta, this is L theta and the other spring stretches by the same amount that is this one. So, strain energy in the spring is equal to, 2 half k L square theta square, because this is the thing that is k L square theta square. When the rod moves like this, then the center of the gravity of the mass, moves by L by 2 into 1 minus cos theta. That means, again that is L by 2 into theta square by 2, provided theta is very small.

The potential energy of the rod changes, and potential energy change due to this is given by potential energy change is equal to minus m L by 4 theta square. Now, this is due to this one but the force P which is acting, also gets displaced. That also comes down. So, potential energy change due to rod P is PL, because this is PL, P into L 1 minus cos theta or this is PL by 2 theta square.

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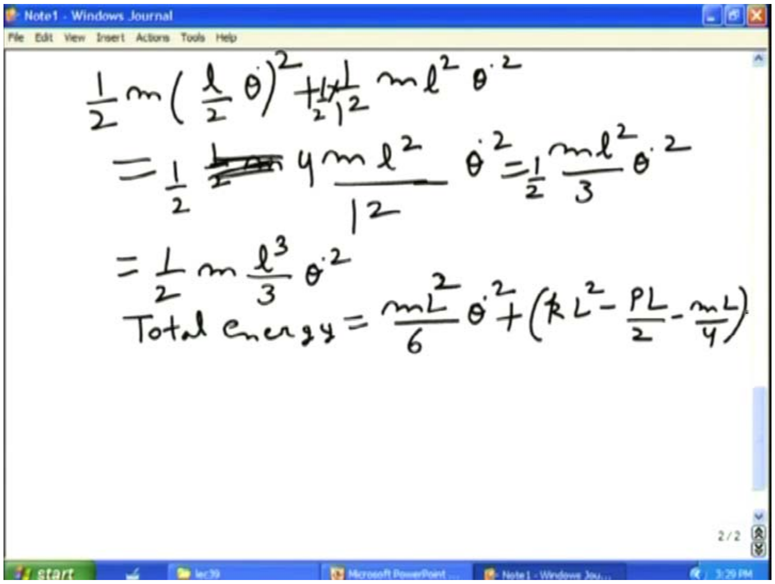

So, kinetic energy of the rod is equal to half m l by 2 theta square plus 1 by 2 ml square theta dot square, or basically it will come l square by 4 and this is 1 by 12. It is I theta square, because this will become equal to half m l square by 12 and this is 4. So, you can take 4 common and this becomes l square ml square by 12. This is 3 plus 1, 4 ml square by 12 theta dot square or ml square by 3 theta dot square into half 4 ml square.

This is equal to half m l cube by 3 theta dot square. Therefore, total energy is equal to m L square by 6 theta dot square plus k L square minus P L by 2 minus m L by 4 theta square. This is total energy, previous 2 by 3.

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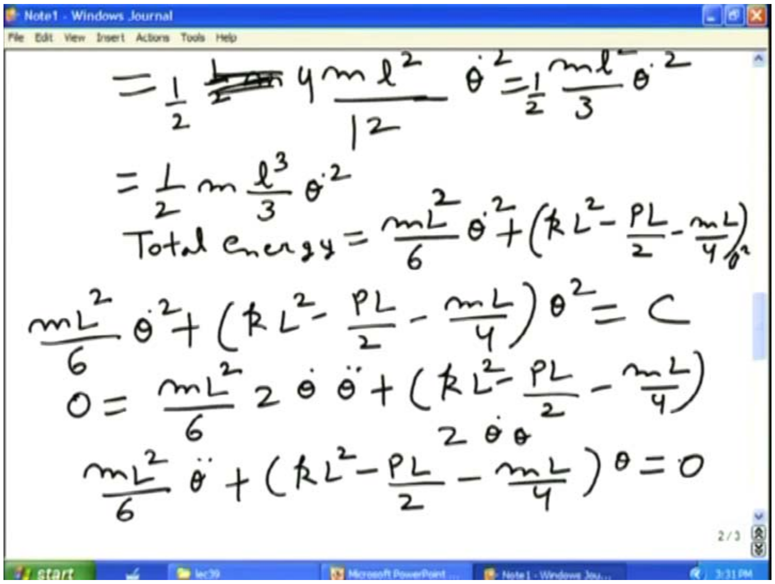

Total energy is equal to, this is the expression theta is this is theta square that means let me write it again. Total energy is m L square by 6 theta dot square plus k L square minus PL by 2 minus mL by 4 theta square. Now, differentiate. This is equal to constant. So, differentiate it with respect to time, this expression can be differentiated with respect to time and then you get, 0 is equal to m L square by 6 into 2 theta dot theta double dot plus k L square minus P L by 2 minus m L by 4 into 2 theta dot theta.

Therefore, theta dot can be cancelled throughout. Therefore, you get m L square by 6 theta double dot plus k L square minus P L by 2 minus m L by 4 theta is equal to 0. Now, observe this expression that you have written m L square by 6 theta double dot plus k L square minus P L by 2 minus m L square.

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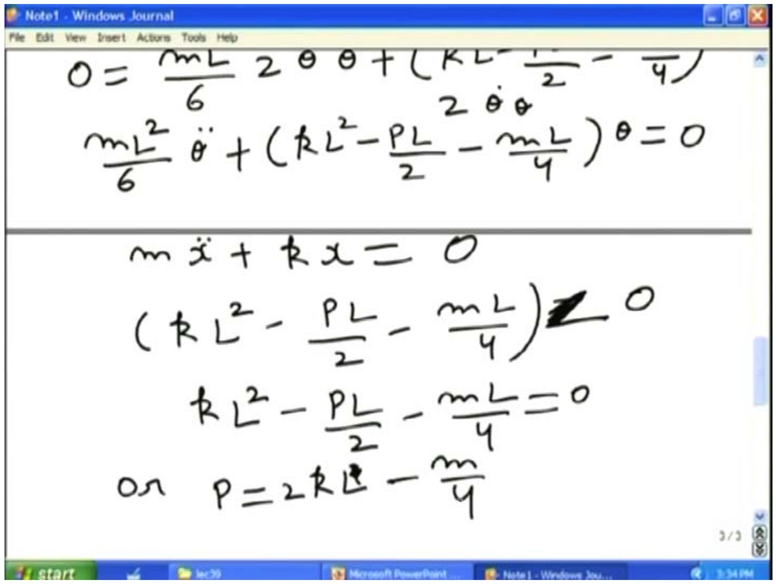

Compare this expression with mx dot plus kx equal to 0. When you compare this expression, you notice that this was positive, then only we got k is positive. Therefore, this expression in the parenthesis must also be positive, then only the system will vibrate. If the expression is not positive then the system will not be vibrating. Therefore, the condition for this is that, this expression must be this one then it will be vibrating, and restoring force will be able to bring to original position.

However, if you go on increasing P then you can say this expression P L by 2 k L square minus because that expression pay attention to this expression k L square minus P L by 2 minus m L by 4. What happens if this expression is less than 0? If we increase P, what will happen? In that case, there will be a value of P, at which this expression will become negative. Beyond that, if you put that also, at that point it will be unstable. The rod will not be able to come back when you displace it. It will go in the other direction exactly when k L square minus P L by 2 minus m L by 4 equal to 0. At that point, you can give any displacement and the rod will remain at that position. It will not try to come back.

So, P can be called critical load, critical load at which this will become unstable. There will be not be any vibrations but this is the thing and if P is more than that, naturally it will have tendency to go to the other direction. Therefore, the critical load is k L square minus P L by 2 minus m L by 4 equal to 0, or P is equal to k L square 2 k L square 2 k L because that is minus m by 4.

At that point, this rod will try to comeback to that original position. That means, if P is less than k, 2 k L minus m by 4 the rod will try to come back to the previous position. This is how we have obtained the expression for that vibration. Here, to summarize what we have done; we have shown that there is another approach to obtain the differential equation of a motion that is not by solving the system forced by this, but by the energy consideration.

In the free vibration case, total energy of the system remains constant. Therefore, by that consideration, we have obtained these expressions and we can find out the angular speed. That is another way of solving the vibration equation. In many cases, this system becomes very simple.

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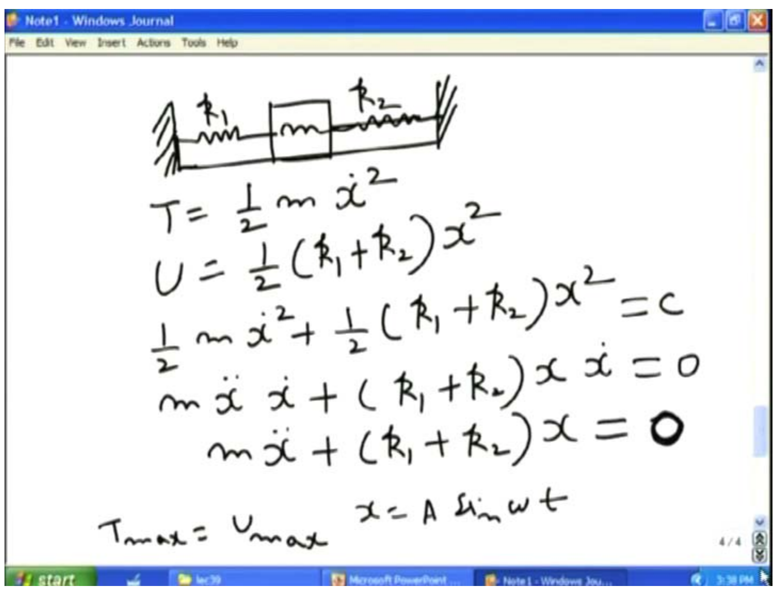

Let us quickly do one more problem and then conclude this thing. Supposing you have the vibration of this rod m and this is kx k_1, this is $k_{2.}$ Then if you put kinetic energy T is equal to half mx dot square then U is equal to half k_1 plus k_2 x square, because half k_1 x square. Energy is stored in both and therefore, half mx dot square plus half k_1 plus k_2 x square is equal to T plus U, that is constant. Differentiate it with respect to time. You get, mx dot x dot plus k_1 plus k_2 x to x dot is equal to 0. You get the differential equation, mx dot plus k_1 plus k_2 into x is equal to 0. In this case, the equation of this one, if you quickly want to find the frequency, we can always put, x is equal to A sin omega t and then we can find out x dot. Then by equating T_{max} is equal to U_{max}, we can find out the expression for angular omega that means circular frequency. Therefore, I stop at this point.