Engineering Mechanics

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Introduction to vibration

Module 15 Lecture 38
Vibration of Rigid Bodies Part-1

Today, I am going to speak on vibration of rigid bodies. In this course, we will be discussing the vibrations of single degree of freedom system, by degree of freedom is meant independent coordinates to describe the system. For example, in the previous lectures, we have discussed the vibrations of a spring mass system with damping or without damping.

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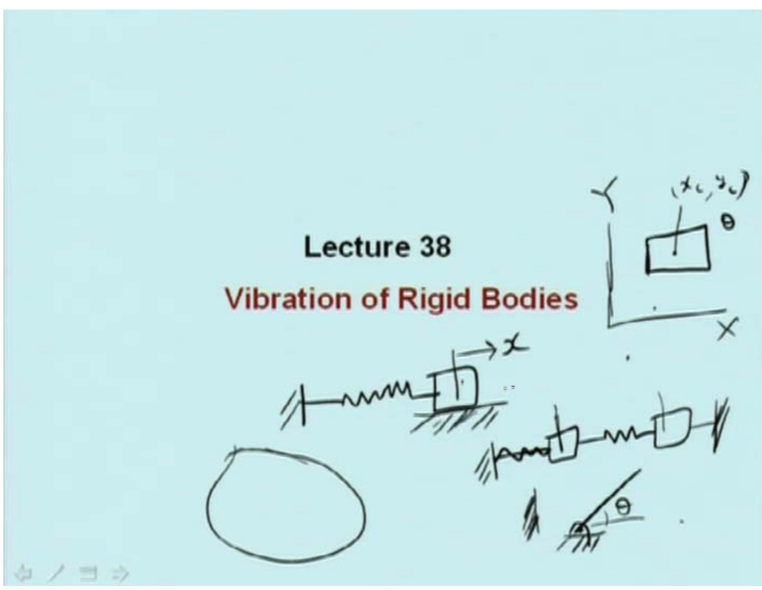

We obtained only one dimensional equation. We had x, position of the mass or that particle is described only by x. So, independent coordinate is x; therefore, this is a single degree of freedom system. If I would have kept like this, suppose there are mass and this was the spring, another spring, another mass; Then this would have become 2 degree of freedom system, because I need to tell the position of this mass, then I need to tell the position of another mass, so, 2 degree of freedom system.

Like that we have 3 degree of system. If there are three masses attached by springs, because I need to locate the position of three masses. We discussed in the first lecture itself the concept of continuum, where we consider that the body is a continuous matter, mass is continuous. Therefore, in this there are infinite number of particles actually. As such, if the body can undergo deformation, then there are infinite degrees of freedom possible.

However, we know that we are studying rigid body, and in rigid body, if a particle is having 3 degrees of freedom and body composed of the particle has basically 6 degree of freedom. In rigid body, the degrees of freedom get reduced and they become only 6. Similarly, the particle we know, although the particle has 3 degree of freedom system, when we discussed the spring mass motion then we assumed that the particle is constrained to move in one direction only. It is not moving in the other two directions. Therefore, it has got 1 degree of freedom. Other 2 degrees are other state.

If the body rigid body is there then that rigid body can undergo 6 degree of freedom and three translations. The center of gravity can move in x y and z direction. Once we fix the center of gravity, then the body can be oriented in three ways by theta phi and another angle alpha. However, if we restrict the body to plane then the body has only 3 degrees of freedom. A rigid body in x-y plane is only having 3 degrees of freedom, because it can be moved along x, it can be moved along y then it can be rotated about the z axis. If we just have its degrees of freedom may be x_c y_c that is coordination and plus its orientation theta.

However, if we somehow constrained the motion of the body and allow it to have only 1 degree of freedom system, then such problems can also be solved, means with whatever we have discussed that means which in the scope of our present course. Now, this rod has been hinged at one point and therefore, it is not able to move in x and y direction. This point is restrained, but it can rotate. So, it is having a single degree of freedom.

Why we cannot study the vibration of these types of things also, it may be rigid bodies they may be undergoing this thing.

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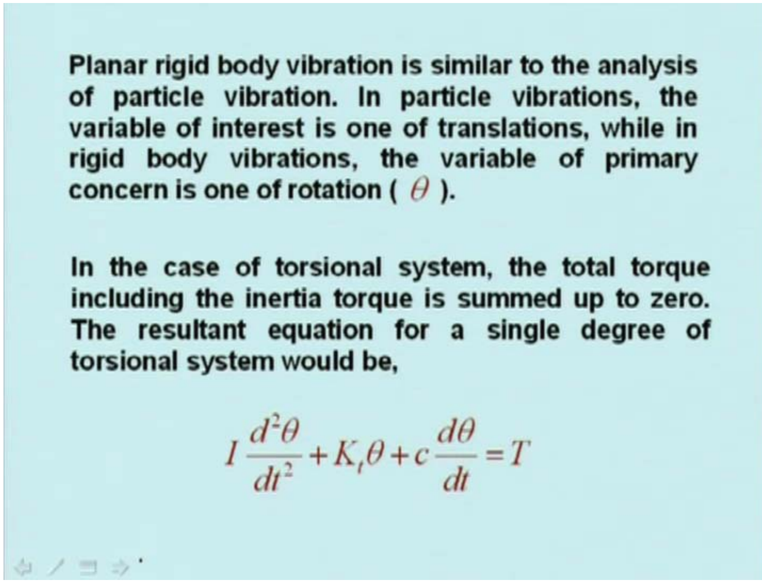

So, planar rigid body vibration is similar to the analysis of particle vibration. In fact, when the particle can be considered as a complete rigid block, also composed of so many particles, because it is under going only translation and all particles are having same velocity and same acceleration, same type of displacement from a particular datum. Therefore, in particle vibrations the variable of interest is one of translation; while in rigid body vibrations, the variable of primary concern is one of rotation theta. You know that, we just go to this one. In the case of when it is theta so, before that let me tell the concept of the torsional system. Let us quickly go to that one.

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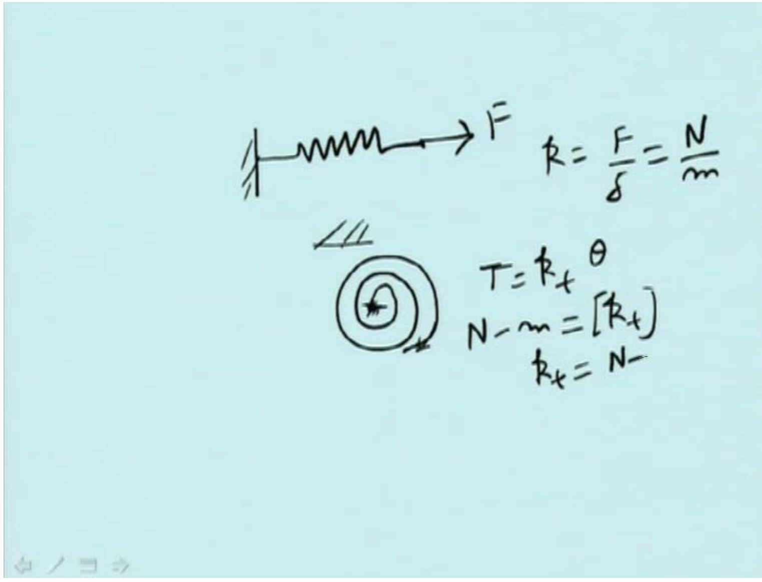

In linear springs, there is a spring and if you apply a force F, the spring gets deflected. Suppose this deflection is delta. Then k is equal to F by delta, where k is called the spring constant and it is having the unit of Newton per meter. In torsional system, this is suppose you have a torsional spring which can be indicated by like this here. This is the spring and if I suppose fix it here and if I apply some torque T here then the spring gets displaced.

Angular displacement is there. We say T is equal to k_t times theta. Now, unit of torque is equal to Newton meter k_t and this is radian per second. This is what happens, the dimension of $k_{t,}$ this is radian only. Therefore, dimension of that thing will be Newton meter. So, k_t will be Newton meter. So, what happens now, going back to the previous slide.

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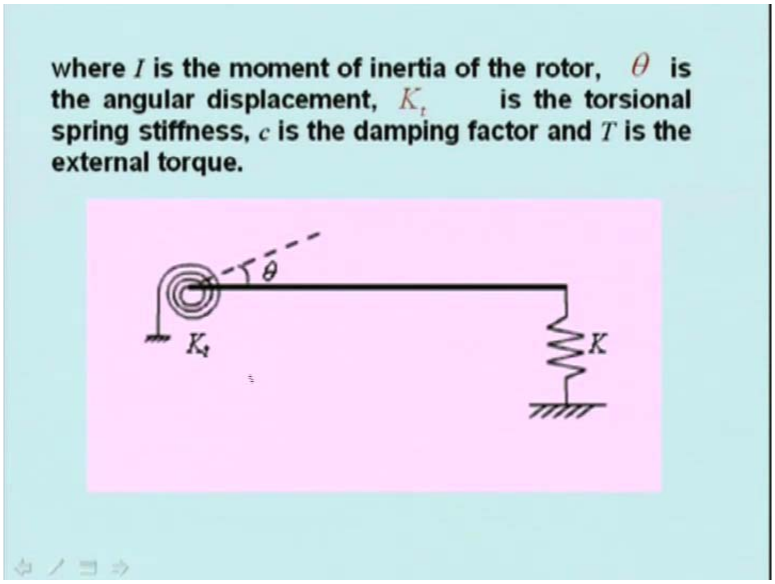

Let us take a rod here. It is fixed here. A torsional spring is with this. When the rod rotates by theta then a restoring moment is developed; that is k_t times theta. That restoring moment will be k_t times theta and this side, we have kept a linear spring, whose spring constant is k.

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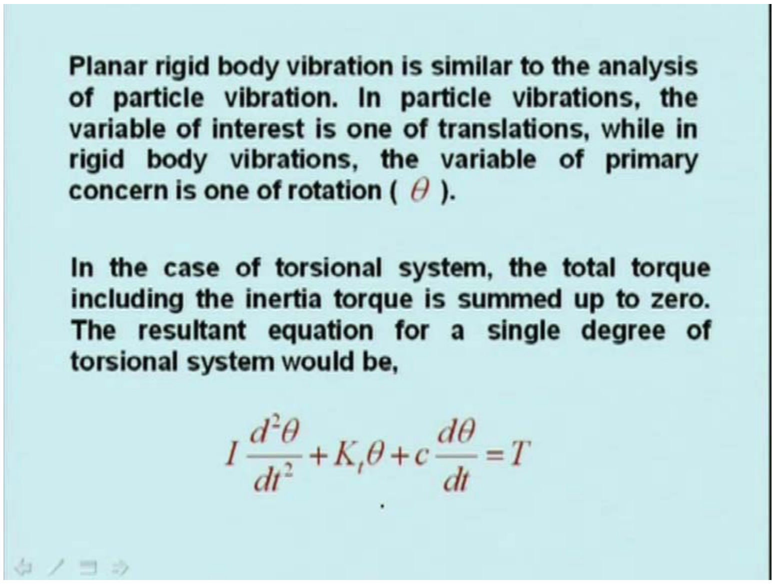

In this case, the total torque including the inertia torque is summed up to 0; that is D D'Alembert's principle, or we can say that we can apply the Newton’s law generalized to that angular coordinates or angular degree coordinates. Then you have I d square theta by dt square plus K_t theta is equal to c d theta by dt is equal to T. I d square theta by dt square is the inertial inertia torque. d square theta by dt square is the angular acceleration with unit of 1 by second d square and I is having the unit of mass moment of inertia. So it is kg meter square. K_t is Newton meter, c is the viscous damping effect.

Just like, there we had c times x dot and here, I am putting c times omega. That means, c d theta by dt must be equal to applied torque. This is the equation; I d square theta by dt square plus K_t theta plus cd theta by dt. If T is equal to 0 then this becomes the case of 3 vibration and you get this thing. The equation is similar to what we have studied in a spring mass system, only theta is replacing x. Therefore, this K_t is called torsional spring stiffness, c is the damping factor and t is the external torque.

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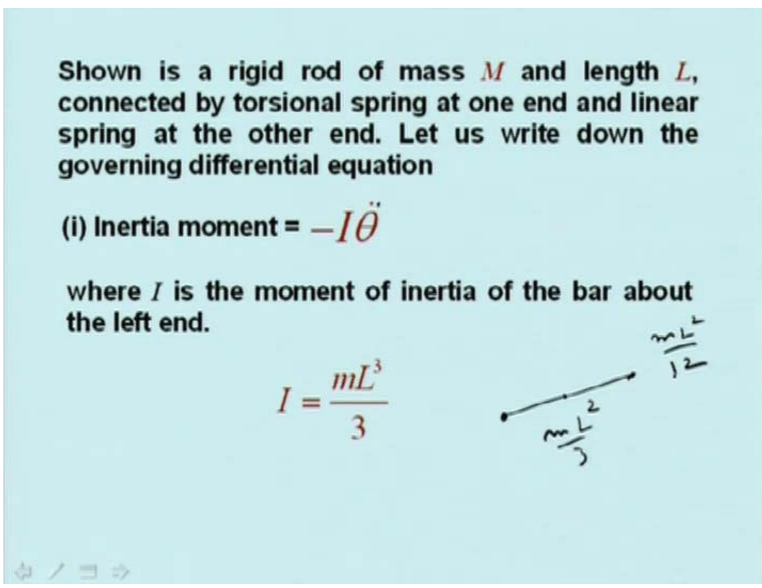

Inertia moment is minus I theta dot dot. I is the moment of inertia of the bar about the left end and this is given by m L square by 3. You have a rod and its moment of inertia, uniform slender rod, its moment of inertia about this point is m L square by 3. About its cg, it is m L square by 12 so I equal to m L square.

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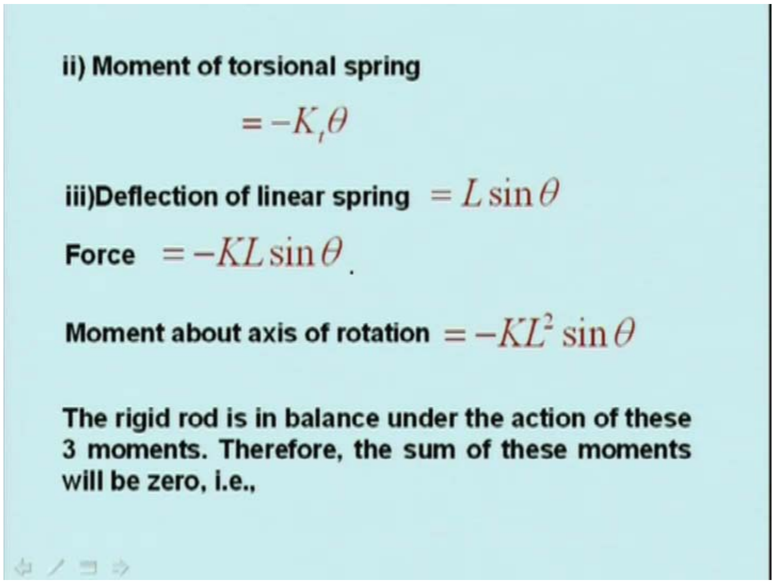

Moment of torsional spring is minus K_t times theta. Deflection of linear spring is actually L sin theta. We have the spring like this and if it gets displaced, this is theta, this is L sin theta. We get L sin theta. Therefore, force is equal to minus KL sin theta linear spring applies a restoring force that that is minus KL sin theta. Now, moment about axis of rotation is minus KL square sin theta. The rigid rod is in balance under the action of these three moments: that is moment of torsional spring, deflection of linear moment due to a linear spring and the inertia moment. Therefore, the sum of these moments will be 0, because there is no externally applied load.

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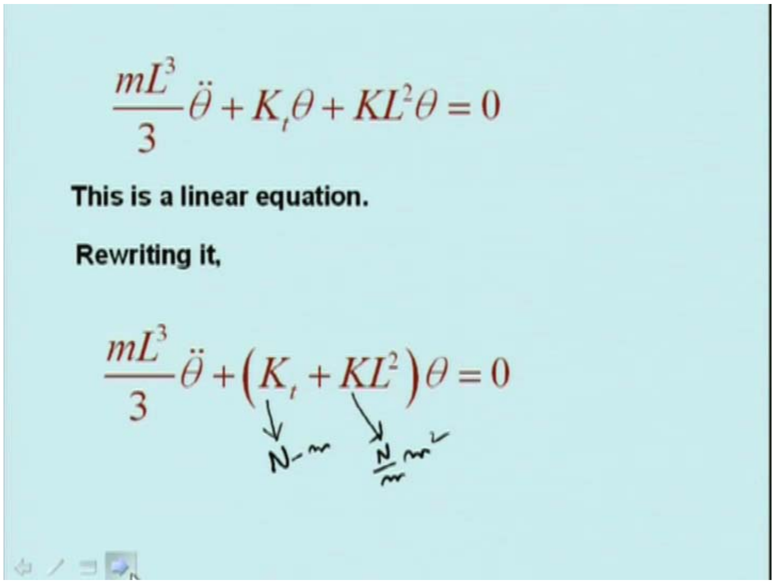

That is mL cube by 3 theta double dot plus K_t times theta plus KL square theta equal to 0. This is a linear equation. That means, if alpha is a solution of this and beta is another solution, then c times alpha plus d times beta is also another solution. Now, rewriting it, we can say mL cube by 3 theta double dot plus K_t plus KL square theta. K_t is the torsional spring constant. Therefore, its unit is Newton meter. This is Newton per meter into L square is Newton square. So, dimensions match Newton meter Newton theta equal to 0.

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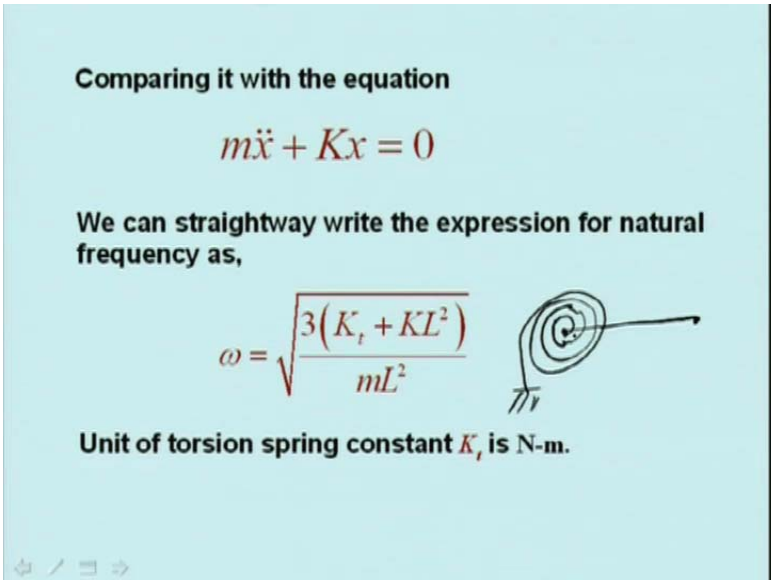

Let us compare it with the equation mx double dot plus Kx equal to 0. We already have solved this equation, why do we solve this again? Only thing is that it is m. Instead of m, I will put mL cube by 3. Instead of K, I will put K_t plus KL square. We can straightway get the expression for natural frequency, omega under root K by m. Sameway, omega under root 3 K_t plus KL square divided by mL square. Unit of torsion spring constant K_t is Newton meter. So, omega is equal to K_t plus KL square divided by mL square.

Of course, K is equal to 0; that means, spring is just supported at one end by torsional spring, but the other end is free. In that case, omega will be under root of 3 K_t divided by mL square. Similarly, if there is no torsional spring that means there this is fixed but then K_t is equal to 0 and it is supported on this spring. If K is there then it will be 3 Q K under root m.

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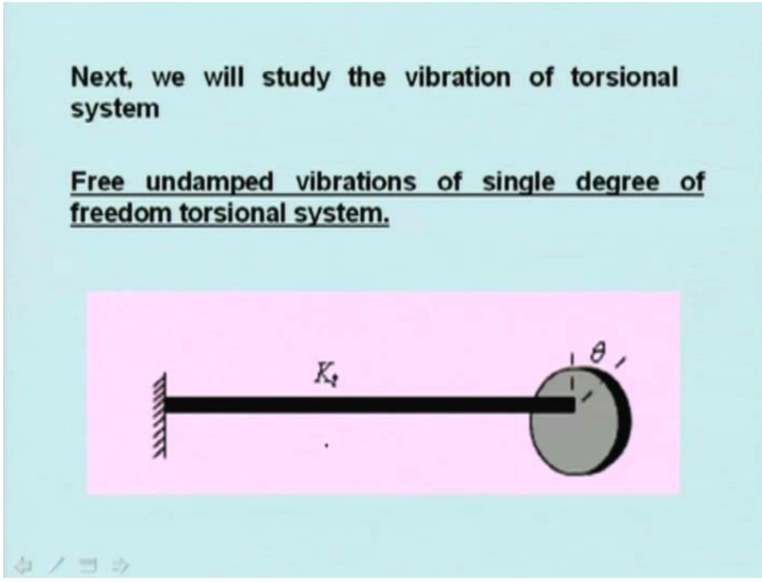

Next, we will study the vibration of torsional system shown in this figure. Here, what happens in this? There is a rod and with this rod, you attach a disc. If you twist the disc, this rod also gets twisted, but this is duly fixed. That means, all cross sections rotate with respect to each other. Therefore, restoring moment is developed and that expression is given, K_t. Now, that can be measured experimentally also. You give a small displacement, angular displacement theta then see how much torque is developed and then K_t will be torque divided by theta.

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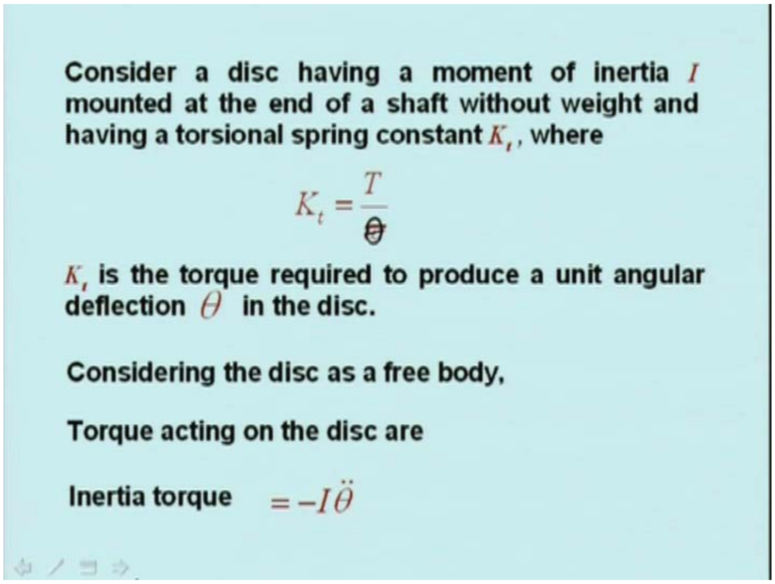

In this system, if the moment of inertia of the disc is I and it is mounted at the end of shaft without weight; that means, weight of the shaft is negligible compared to the weight of the disc and having a torsional spring constant $K_{t,}$ where K_t is equal to T divided by theta. That is K_t is the torque required to produce unit angular deflection theta in the disc. We measure theta in terms of the radian. Consider the disc as a free body. Torque acting on the disc or inertia torque which is equal to minus I theta double dot.

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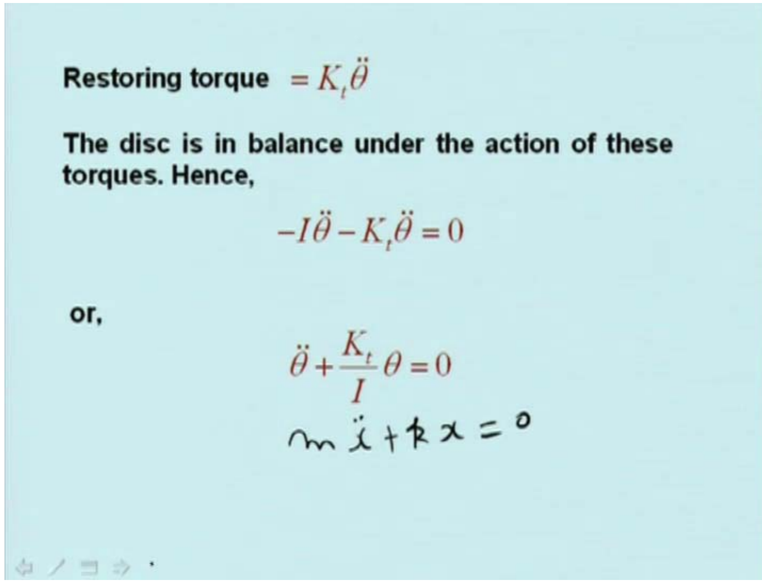

Restoring torque is K_t theta dot. By D'Alembert's principle the rod has to be balanced under the action of these two forces. Therefore, minus I theta double dot minus K_t theta double dot equal to 0, or theta double dot plus K_t by I into theta equal to 0. Therefore, this is natural frequency. You can compare this expression with mx dot plus kx equal to 0.

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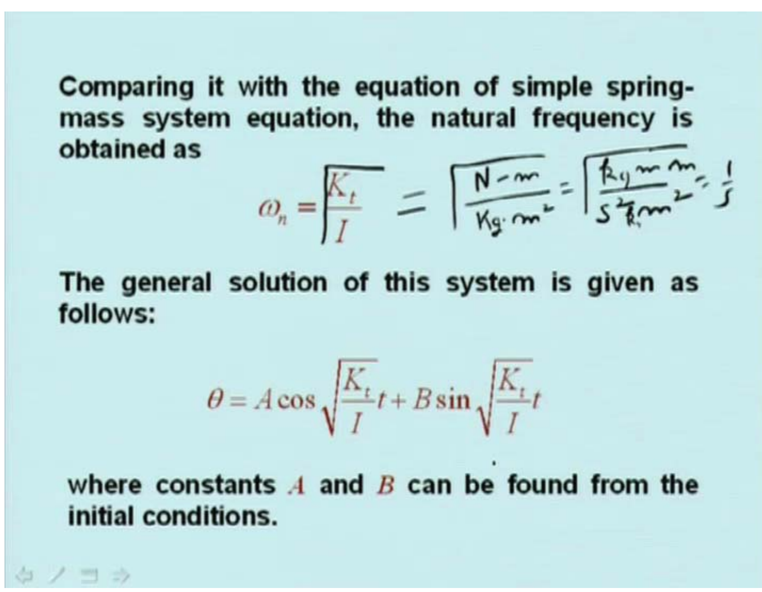

Therefore, omega$_n$ is natural frequency and is obtained. omega$_n$ is equal to under root K_t by I. K_t has the unit of Newton meter. Let us see the units here; Newton meter kg meter square and Newton is kg meter per second square. So, you have second square then one meter was already there, meter meter square. So, meter meter gets cancelled it becomes and this is kg. So, you get 1 by s. So, omega$_n$ means its unit is second inverse.

General solution of this system is given as theta is equal to A cos square root K_t by I into t plus B sin square root K_t by I into t, where constants A and B can be found from the initial conditions. You can have different type of initial conditions. At the starting, the displacement is 0, or you can have the condition at the starting. You give some definite displacement and the velocity is 0, angular velocity 0; this is how it can be done.

Now, let us discuss the case of compound pendulum. That also can be discussed. Before that, a little bit recapitulation of the simple pendulum.

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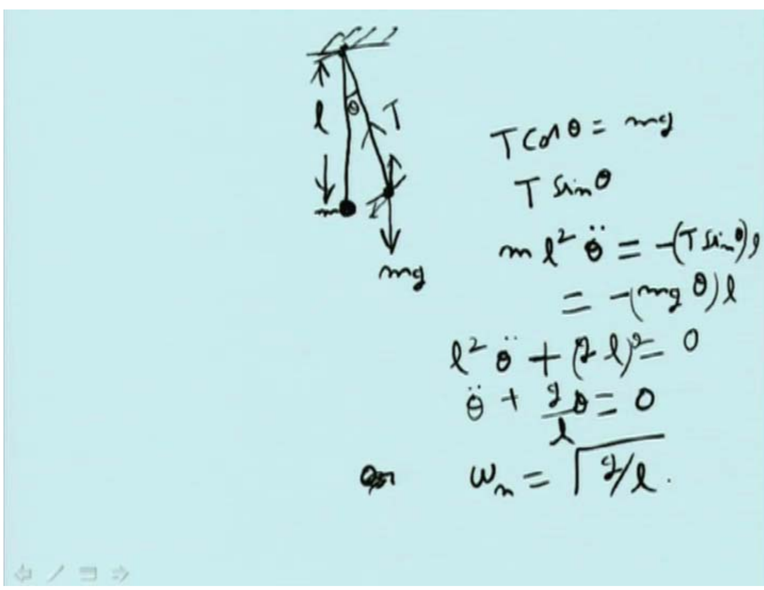

Simple pendulum consists of a string. With that, you attach a small ball. It can be treated as particle. The entire mass is concentrated here and this is l. If you displace it here, there is no spring here, but if you displace it there, its potential energy rises and gravity tries to pull it to another lowest position. So, at this moment, there is a gravitational pull that is mg and then you have got tension. If this angle is theta then T cos theta is equal to 2 mg and T sin theta will be the component in this direction. T sin theta, T vertical component balance and another component is T sin theta which is like this. So, T sin theta.

If we take the moment about this point, then mass m, moment of inertia ml square theta double dot is equal to minus T sin theta. If we assume that theta is very small, in that case, T will be equal to mg and sin theta will be theta. Therefore, this becomes minus mg into theta. Therefore, l square theta double dot plus g by g. This is minus T sin theta. Here, I have to multiply by length. Then, it will become this moment; this is restoring moment l. So, lg theta is equal to 0.

Here, we get theta dot dot plus g by l theta, g by theta equal to 0, or here this system comparing with mx dot double dot plus kx equal to 0, we get $omega_n$ is equal to under root g by l. Therefore, assumption involved here is that theta is very very small. That is why, we could write T cos theta is equal to and we will be able to get the simple harmonic motion.

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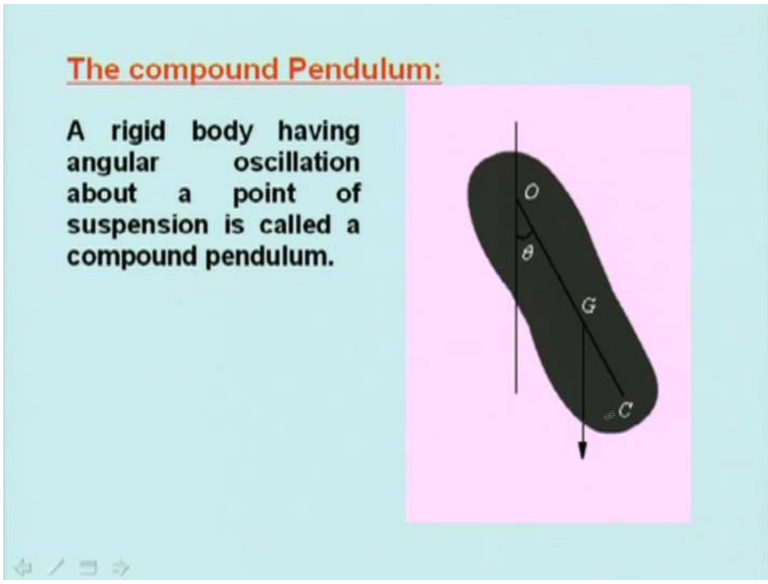

Discussing about compound pendulum, in compound pendulum, mass is not concentrated at one point. In fact, it is distributed. This is the compound pendulum, shown. A rigid body having angular oscillation about a point of suspension is called a compound pendulum. Here, the mass acts through the center of gravity.

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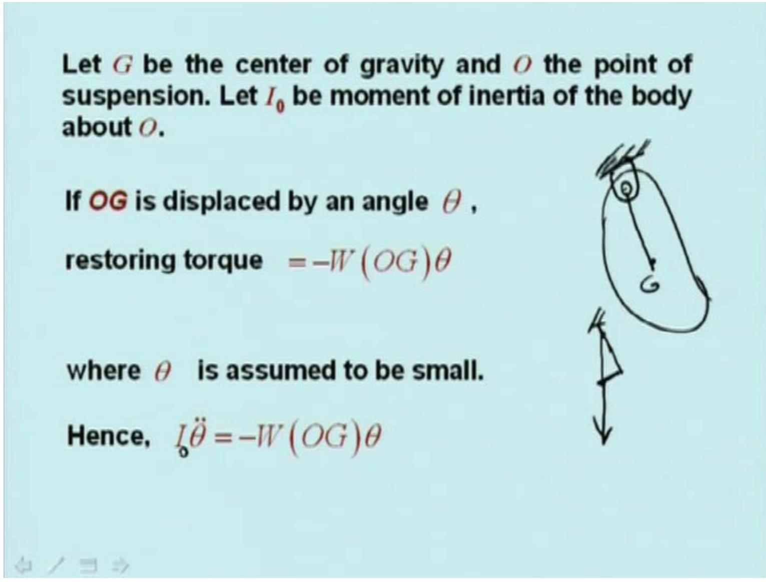

Let G be the center of gravity and O the point of suspension. If I_0 is the moment of inertia of the body about O, you have a pendulum like this; in this case, you have to take moment of inertia about this point. This is O and this point is G, G is the mass. If G gets displaced by an angle theta then you get restoring torque, minus W times OG theta because angle is small. So, here you have W and this point gets displaced. This weight is there and this gets displaced by this one, so it is like this.

This rises here. Therefore, you get a restoring torque as W times OG theta, just like you have got in simple pendulum. Theta is assumed to be small. Hence, I theta double dot is equal to minus W OG theta, where I is basically about $I_{0;}$ that means, it is about a point O which is suspended here.

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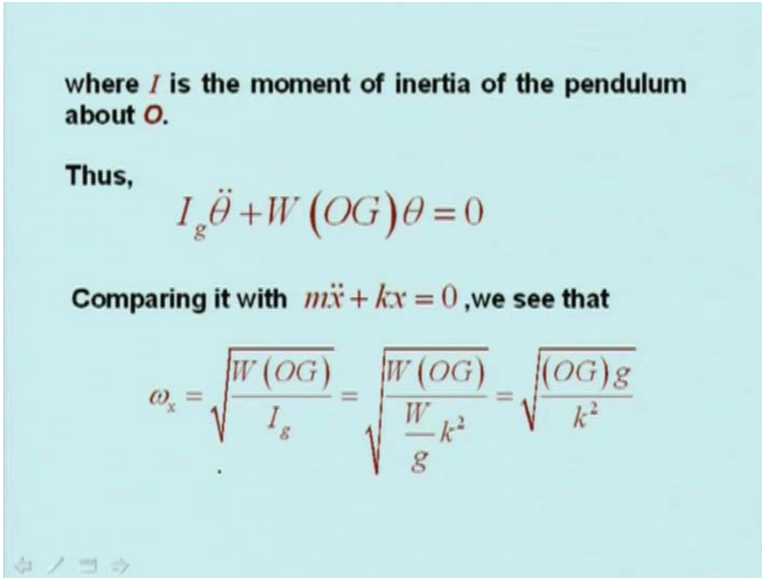

Therefore, I is the moment of inertia of the pendulum about O. Thus, Io theta double dot plus W OG theta equal to 0. Again, compare it with mx double dot plus kx is equal to 0. We see that omega comes out to be omega_n W OG this is I_0 W OG and this can be written as W by g. W by g is basically mass times k square, where k is the radius of gyration. This is radius of gyration about point O. This is equal to OG times g divided by k square.

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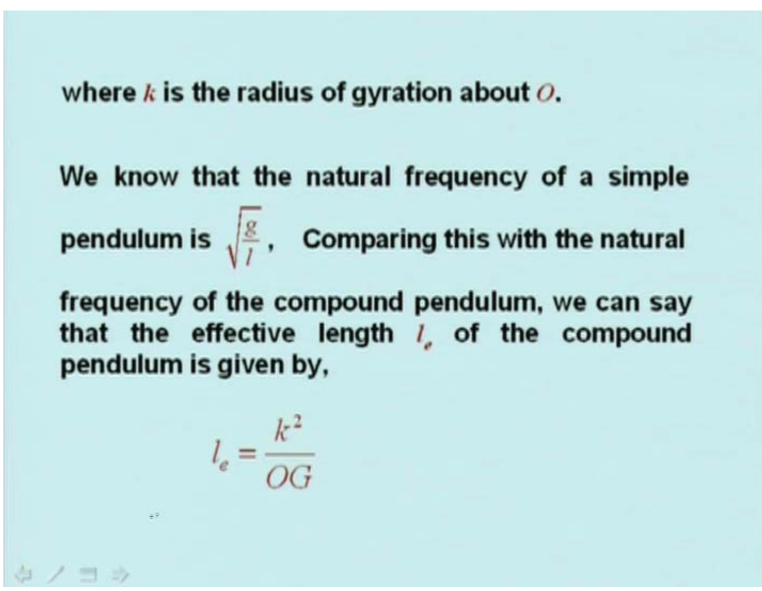

We know that the natural frequency of a simple pendulum is square root g by l. Comparing this with the natural frequency of the compound pendulum, we can say that the effective length l_e of the compound pendulum is given by l_e is equal to k square by OG, because in this expression, this is compared with under root g l_e. Therefore, l_e square l_e is k square by OG.

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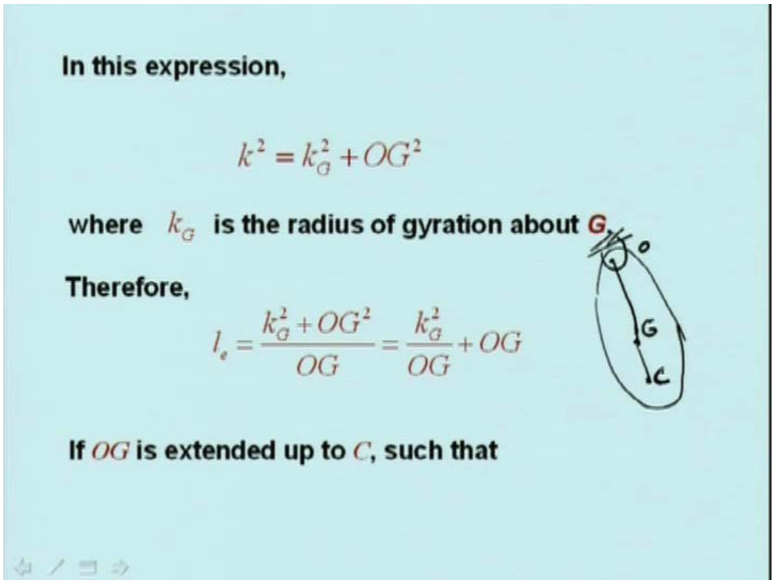

In this expression, k square is the radius of gyration about fixed point O which is equal to k_G square plus OG square. This comes from the parallel axis theorem. k square is equal to k_G square plus OG square, where k_G is the radius of gyration about G. Therefore, l_e is equal to k_G square plus OG square divided by OG. This comes out to be k_G square by OG plus OG. If you have compound pendulum like this and this point is extended up to C, some point C, where OC is equal to OG plus k_G square by OG; that means, GC is equal to L by OG.

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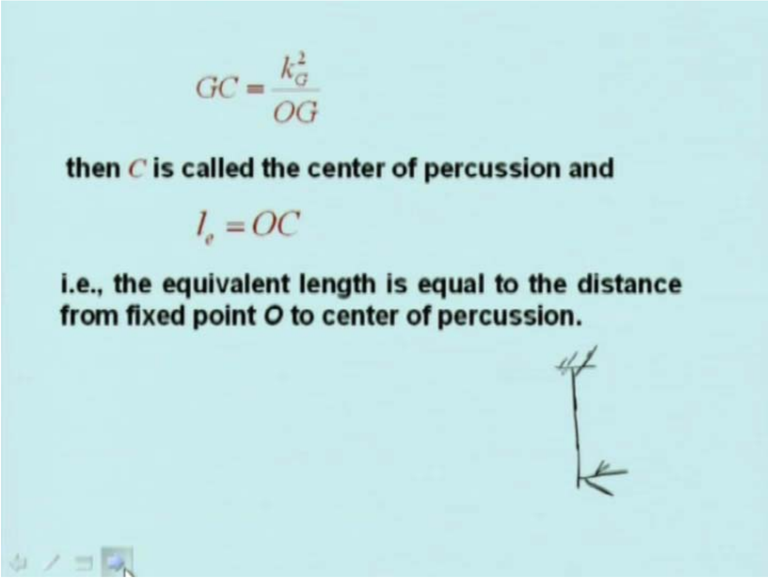

GC is equal to k_G square by OG. Then C is called the center of percussion. We have already discussed about center of percussion in the previous lecture. l_e comes out to be OC; that is, the equivalent length is equal to the distance from fixed point O to center of percussion. Center of percussion is the point, about, if you strike at that point then no reaction is developed in the direction of the striking of that force.

That means, if you know the center of percussion of one body then you can apply a tangential force. Here, it will not produce any reaction in this direction. We have discussed this in one of the previous lectures. Now, we will discuss about the other problem.

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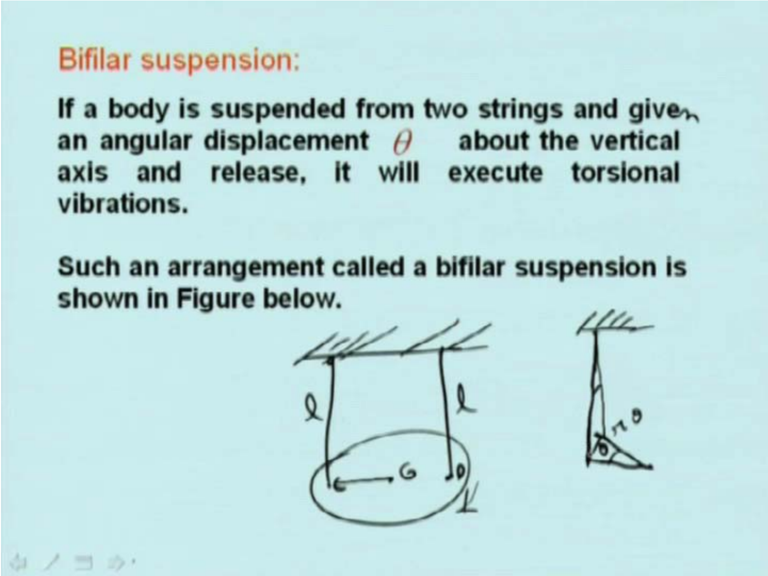

We discuss bifilar suspension. If a body is suspended from two strings and given an angular displacement theta about the vertical axis and released, it will execute torsional vibration. Such an arrangement is called bifilar suspension. I am drawing the figure here. A bifilar suspension will be like this and it is a disc type of thing. It is how, a string is attached here.

This is the G and this is C, this is D then you have l. This arrangement is called bifilar suspension. It is suspending like this. If I twist the disc little bit, if I give the twist here then a restoring moment is developed. Why? Because these strings which are vertical will also be twisted by the same amount and they will become inclined. The inclined strings will provide a vertical as well as horizontal component. Vertical component will support the weight of the disc, whereas the horizontal component will provide the restoring torque.

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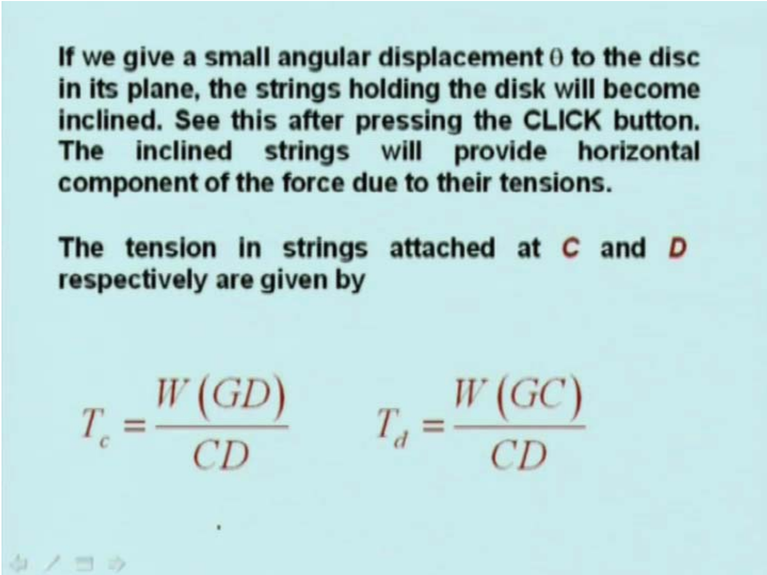

If we give a small angular displacement theta to the disc in its plane, the string holding the disc will become inclined. Now, inclined strings will provide horizontal component of the force due to their tensions. The tension in the strings attached at C and D respectively will be like this: If it is C and it is D, in C it will be W GD divided by CD. Then, this is WGC divided by CD. This is T_d and this is T_c WC. This is because angle theta will become like this. If you say that theta rotates little bit, so that is l. Suppose, you have attached a string like this and this thing here, that G and this becomes theta.

This will be equal to r theta, suppose you give that displacement r theta. This is the height and this is 1. It is basically, r theta and phi is equal to r theta divided by l. You will have r theta divided by l type of thing. So, W GD by T_c is equal to this 1.

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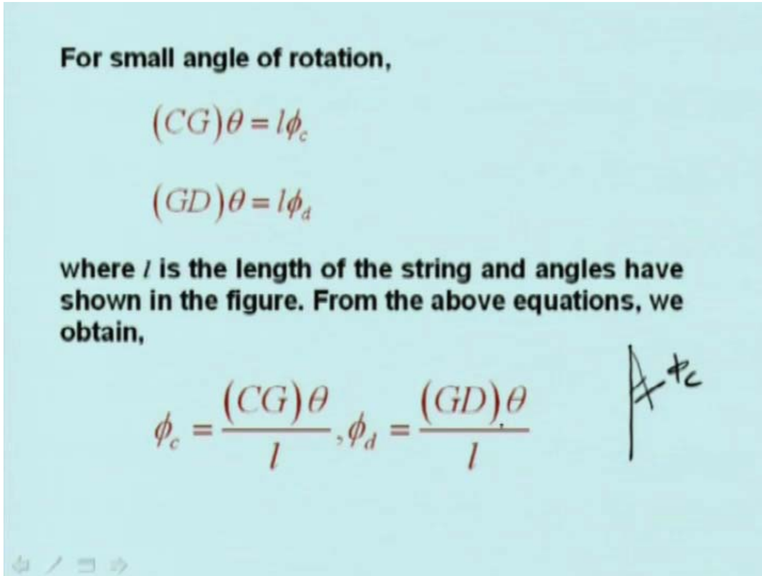

For a small angle of rotation, we can always write, CG times theta is equal to l phi_c. If angular displacement phi_c and GD times theta is equal to l phi_d, where l is the length of the string and the angles are like this, here this is phi_c, like that phi_c. Therefore phi_c is equal to CG times theta divided by l and phi_d is equal to GD times theta divided by l.

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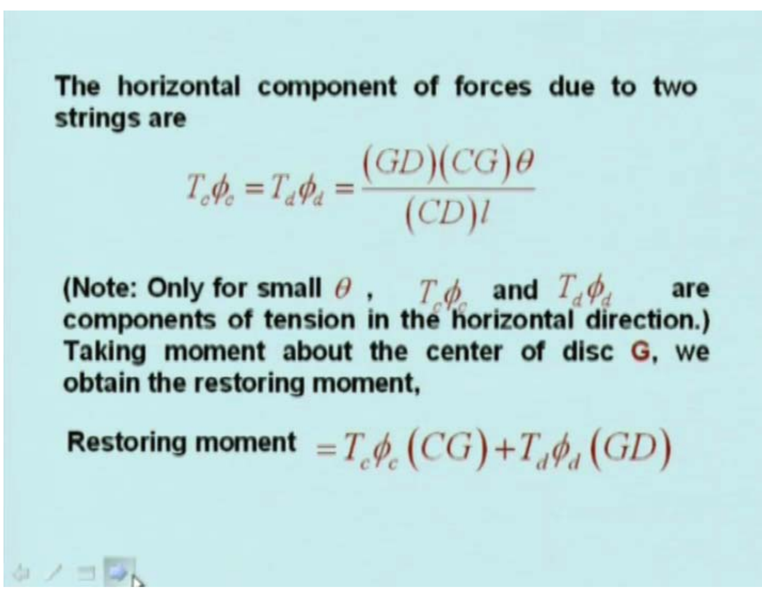

Then, the horizontal component of forces due to two stings will be T_c phi$_c$ and that same thing will be T_d phi$_d$ and these will be GD times CG theta CD by l. Only for the small theta, T_c phi$_c$ and T_d phi$_d$ are components of tension in the horizontal direction. Now, taking moment about the center of disc G, we obtain the restoring moment. So, restoring moment will be T_c into phi$_c$ multiplied by CG plus T_d into phi$_d$ multiplied by GD.

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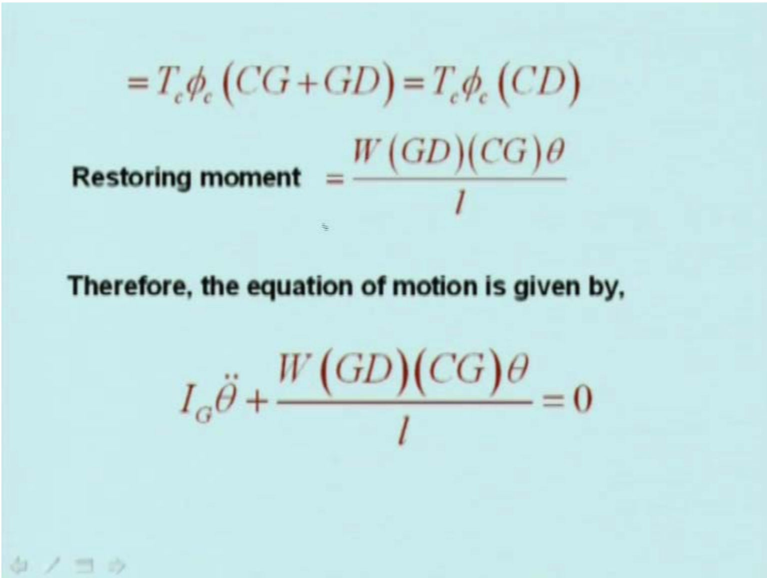

This will be equal to T_c times phi$_c$ CG plus GD, or it will be T_c times phi$_c$ multiplied by CD, where CD is the distance between these two points. Therefore, the equation of motion is given by I_G theta double dot plus W GD into CG theta divided by l into 0. This is I_G theta double dot plus W times GD CG theta l.

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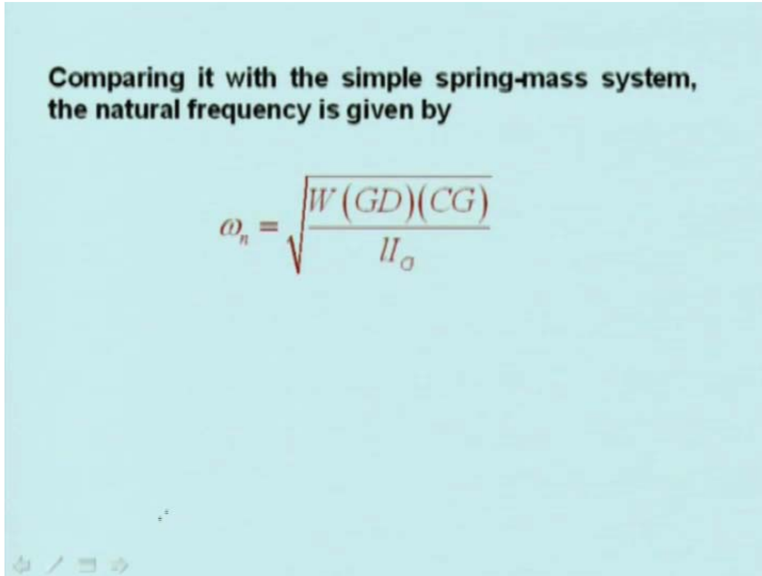

Comparing it with the simple spring-mass system, the natural frequency is given by omega_n is equal to W GD CG and divided by l into I_G. l is the length of the spring. So, this way, we can find out the frequency, natural frequency of this suspension also. Now, I will be doing some simple problems for the system.

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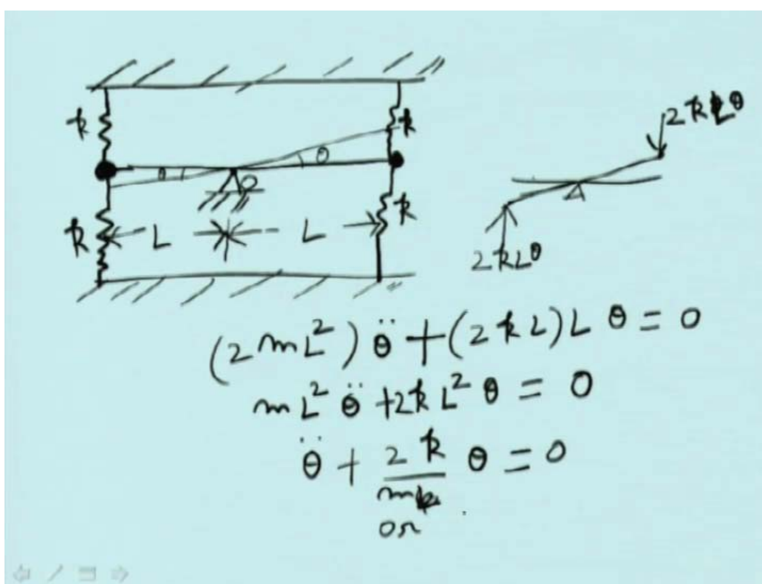

Let us try to solve some other problems. If there is a pivot here and one rod is fixed. At the end of the rod, I took two small balls and their masses maybe m. Then, here you attach, this is the upper that force one right and here another fixed wall and here you attach a spring like this. Assume that, this figure I have drawn is in a horizontal plane. So, gravity effects are not there. It is in a horizontal plane, so, it vibrates.

Now, what will happen? When we rotate it about this hinge point then restoring moment is developed. There is spring k, this spring constant is k. This is also k this is also k. If you give a small angle theta then there is a rotation. This is l and this will be l.

If you give a small rotation theta, this point, this ball gets upward by an amount, l theta. This spring gets compressed by amount k theta; another spring gets stressed by k theta. Therefore, they apply the forces and this will be k theta. So, total force coming on this ball will be basically 2k theta. Why because, this spring gets compressed by about l theta; kl theta. This is kl theta.

Another also gets stressed by kL theta. Both are putting the forces in the same direction. They get added up and this is 2 kL theta. In fact, these two springs are in parallel. Similarly, this side also the spring gets compressed and this gets stretched. Compressed spring puts force like this and stressed spring tries to pull it upward. Therefore, again, this force is 2 kL theta. This is 2 kL theta. Then take the moment about the pivot point, O.

We have to see, what is the total mass moment of inertia about point O. This is mL square plus mL square; that means 2 mL square. You apply that equation 2mL square into theta Inertia torque is equal to plus 2 kL theta. This is actually, a couple 2kL theta 2kL theta acting there and distance between them is 2kL. So, 2kL theta into L that means 2kL. L theta equal to 0 that means mL square theta dot dot plus kL square theta equal to 0, or theta square gets cancelled.

Therefore, theta equal to theta plus 2km 2 kL theta 2kL 2 km theta equal to 0 or omega of this system is given by under root 2k by m. This is how you know that this rigid rod will vibrate. Let us try to do another problem on this system.

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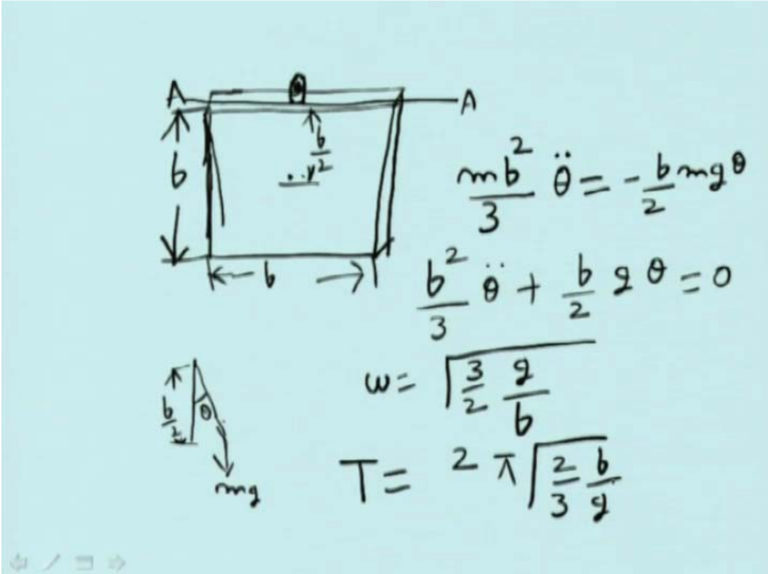

Here, this is rod. If I take a thin laminar and if there is a hinge and you suspend it like this and if it starts vibrating about AA, give a small rotation about AA and this length is b, then we are interested to know what is the frequency of oscillation or what is the time period. When it starts oscillation about axis AA and we have to find out the moment of inertia about axis AA. That is given by mb square by 3 into theta. This is about this one theta dot mb square by 3 is the moment of inertia about AA and this gets displaced. Its cg is at a distance of b by 2 when it gets displaced by an angle theta then you get restoring torque equal to minus b by 2 mg theta.

Therefore, this becomes, b square by 3 theta plus b by 2 g theta minus mg theta into b by 2 is the distance to the torque, because if I make this side view, this gets you theta. So, what happens, this is b by 2; this distance is b by 2 and this mass, mg. So, that one component is basically it is a component along this mg that is tension. So, tension t is equal to mg for a small angle and its component mg sin theta you get is equal to 0 and therefore, omega comes out to be equal to 3 by 2 g by b. So, we get this type of thing; omega is equal to 3 by 2 g by b and time period T comes out to be 2 phi by omega that means 2phi under root 2 by 3 b by g. This is the solution of this problem. This is how it can be done.

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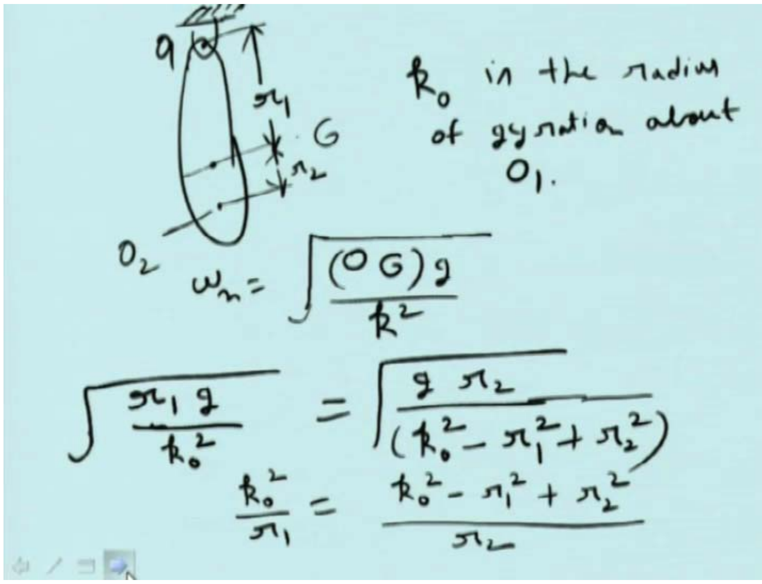

Now, one problem on compound pendulum. A compound pendulum is shown here. This point is O_1, this point is G, this point is r_1 and from here to here, another one point I have marked is O_2 and this distance is r_2. In this problem, it is given that k or k_0 is the radius of gyration about O_1. It is given that if this compound pendulum is hinged at O_1 and then oscillated, you get certain time period. If the compound pendulum is now hinged at O_2 and provided oscillation, then you get time period. Both the time periods are equal.

In that case, what is the relation between the radius of gyration k, k_o, where k_o is the radius of gyration about point O_1 and r_1 r_2. Let us try to solve this problem. Go back to the compound pendulum slide.

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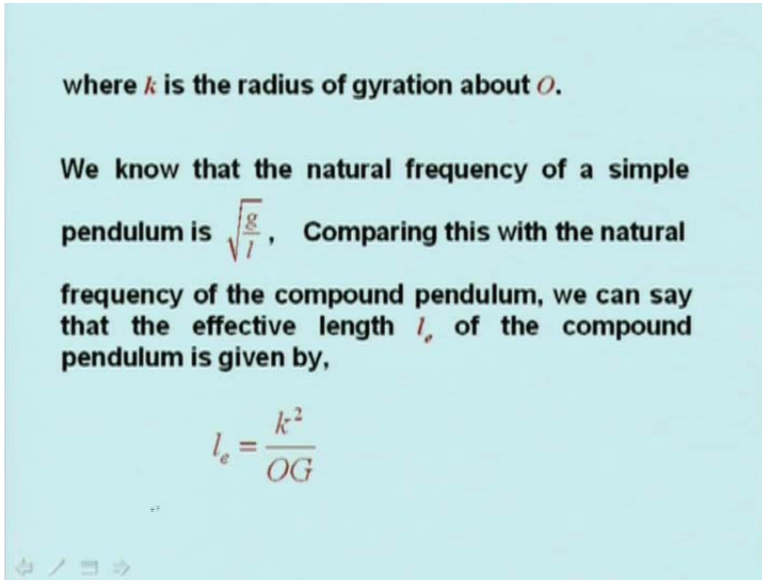

l_e is equal to k square divided by OG. Effective length of the compound pendulum is given by k square by OG.

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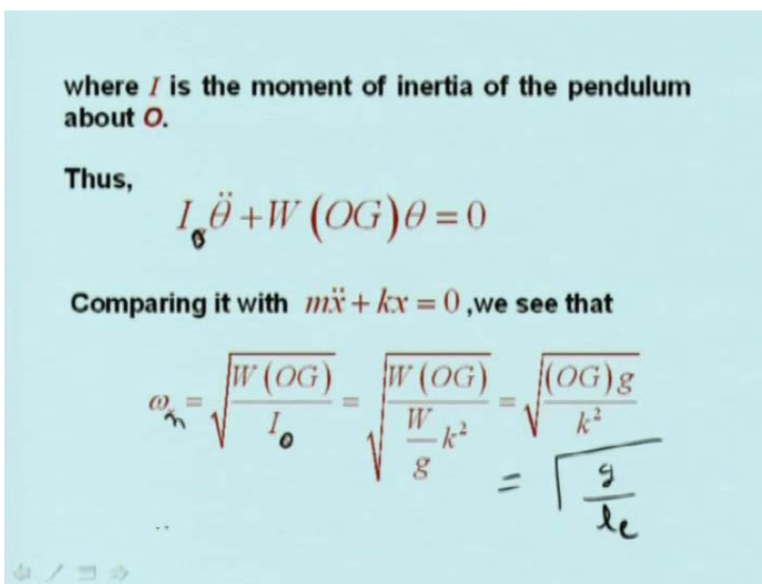

OG into G divided by k square. The same type of thing we have to put here. So, this is 2k. Therefore, time period of the compound pendulum is same. Therefore, angular omega will be same. We will be writing that as, k square. Now, let me use the expression for this thing.

Equivalent length is k square OG omega$_n$ is equal to OG. We had OG multiplied by g. This is k square, that same thing here. For the first case, it is r_1 multiplied by g; r_1 multiplied by g and then divide by k square k_0 square is equal to, I will be writing k_0 is r_2 g g r_2 and this is k_0 square. I am suspending about O_2. Therefore, this will be k_0 square minus r_1 square plus r_2 square, because the radius of gyration about g is now k_0 square minus r_1 square plus I am putting that r_2 square, because I have to find out radius of gyration about this one. Therefore, we get a relation basically k_0 square by $r_{1.}$ Reverse this; so, this must be equal to k_0 square minus r_1 square plus r_2 square divided by divided by r_2.

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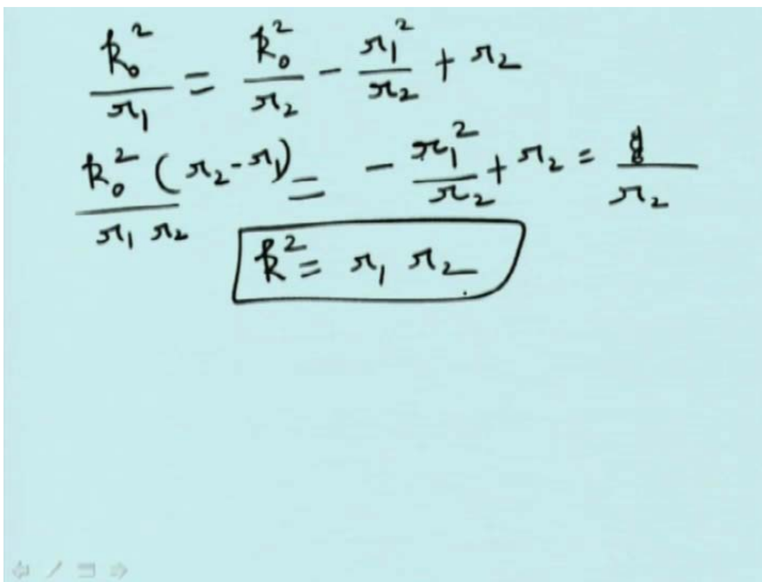

This can be simplified as k_0 square divided by r_1 is equal to k_2 square k_0 square by r_2 minus r_1 by r_2 plus r_2. This can be written as, this is minus r_1 square should be dimensionally balanced to k_0. Therefore, this can be written as k_0 square r_1 r_2 can be taken as common. This is r_2 minus r_1 is equal to minus r_1 by r_2 minus r_1 square by r_2 plus $r_{2,}$ or you take 1 by r_2 common and you take r_2 common here also. This is 1 by r_2 and this will be minus r_1 square by r_2 this is r_2 by r_1. When you simplify this expression, I am now skipping 1 or 2 steps.

Then you should get, k square is equal to r_1 r_2 which is the required relation. So, k square is equal to r_1 r_2. You have been able to solve this problem also. So, this is what happens about this.

In this lecture, let us summarize what we have talked. We have seen that when a rigid body is constrained to move in a particular way so that its position can be described by a single coordinate, it may be the angular position theta, then it becomes a single degree of freedom problem and it can be solved in the same way as the spring-mass system. The differential equation remains basically same. Therefore, mathematics required is basically same. So, we have discussed about this type of motion.

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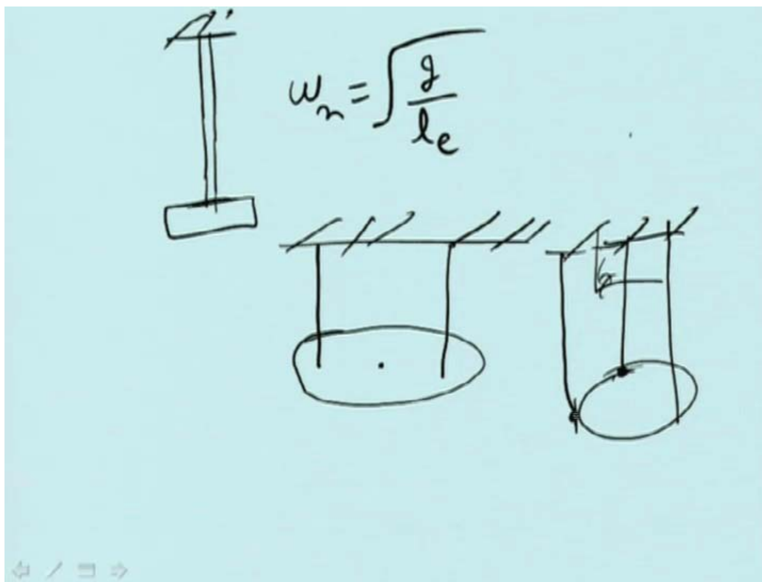

First we took up the case in which a torsional spring was put. After that, we have discussed the case of a disc suspended from a rod or shaft. When we twist it, the shaft gets twisted and due to elasticity, it tries to come back to its position. Therefore, it has some equivalent stiffness. We have discussed that case also.

Then, we have discussed about the compound pendulum problem which can also provide simple harmonic motion. However, ω_n will be basically under root instead of g by l. This will be g by l_e, where l_e is the equivalent length which is the distance from a fixed point up to the center of percussion. This is what we have discussed. Equivalent length is equal to the distance from fixed point O to center of percussion. This we have discussed.

Then, we have done simple problem like one problem we did bifilar suspension. There is a disc which is suspended by the two strings and then it is given some twisting. So, in that case, what happens? That type of motion we have studied.

In the next lecture, we will discuss more about the vibration of rigid bodies. We will first start our discussion with a trifilar suspension where there is a same disc, but it is now fixed from a ceiling by three strings. So, one string may be this, another may be here and the third is here like that. So, this is fixed by this thing and that is called trifilar suspension. We will start our discussion from this in the next lecture.