Engineering Mechanics

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Introduction to vibration

Module 15 Lecture 37
Forced Vibration

(Damped Undamped)

Last lecture, we studied free vibration problems. Specifically, we studied the behavior of two systems.

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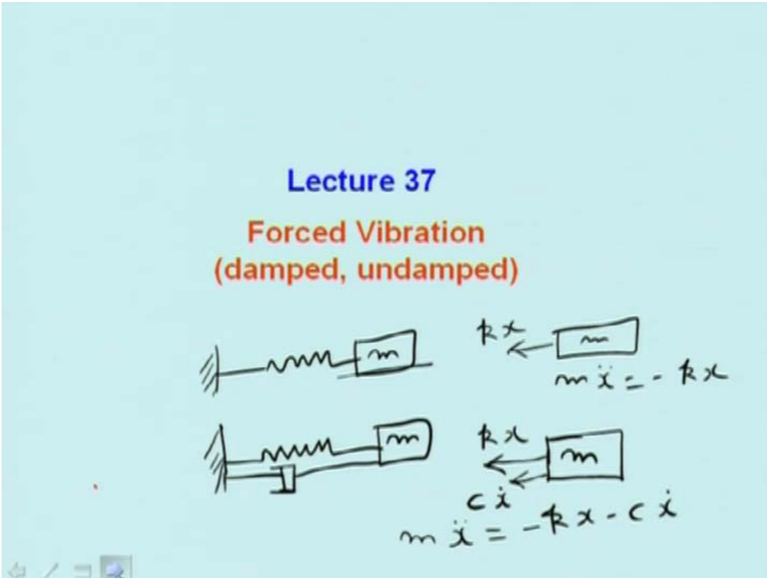

One is a damped natural vibration and another is undamped vibration. In damped one, the damping is present and in undamped, natural vibration damping is not present. Now, we specifically took two systems. This is the spring mass system. We have studied that problem and then another model was that in which we attached a dashpot. We attach a dashpot here and we put like this. Here, if we make the free body diagram of this, this was having kx, the force. Therefore, governing equation was naturally, mx double dot is equal to minus kx. Whereas in this system, this is mass m and this is kx. This will basically be cx dot. Therefore, governing equation was mx dot is equal to minus kx minus cx dot. No external force term was present.

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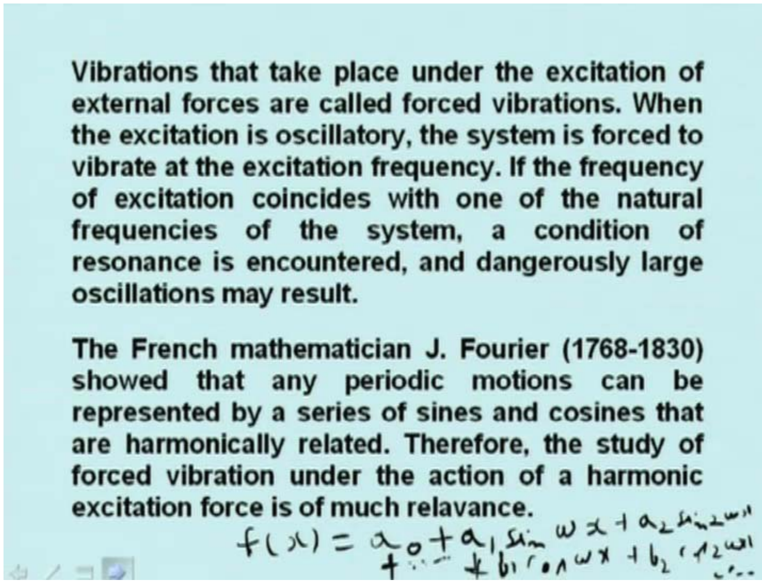

In this lecture, today, we are going to study forced vibrations with damping and without damping. Vibrations that take place under the excitation of external forces are called forced vibrations. When we put that in this one, when the excitation is oscillatory, the system is forced to vibrate at the excitation frequency. What happens, when we apply a force on a spring mass system, suppose we know a mass is there and it is held by a spring then we displace the mass. Naturally, we have created a disturbance from its equilibrium position. So, the natural vibrations may start, but the natural vibrations that is free vibrations, they will decay with time because of the presence of the damping.

If damping is not there, then of course these free vibrations will not decay, but because there is a damping, these decay with time. In any system, damping is always present. There is no system in which there is no damping. All systems, because undamped case is just an idealization, therefore what happens? These vibrations will decay, but because there is a force present that keeps vibrating, we are mostly interested in the steady state response. Then the system will vibrate with the excitation frequency. If the frequency of excitation coincides with one of the natural frequencies of the system then we get a condition of resonance in which dangerously large oscillations may result. Large amount of oscillations take place when there is a condition of the resonance.

Now, we are going to study the mathematical treatment of that forced vibration. French mathematician J Fourier, who was born in 1768 and died in 1830, showed that any periodic motion can be represented by a series of sine's and cosines that are harmonically related. That means you have a periodic function, suppose you have a periodic function fx, it can be represented by a_0 plus a_1 sin omega x plus a_2 sin 2 omega x like that, there may be so many terms upto infinite. Then you have b_1 cos omega x plus b_2 cos 2omega x like that. So, cosine and sin terms are there. In a general case, that Fourier, we will not discuss about the Fourier series method.

However, we know that if we have any periodic motion that can be represented by a series of sine's and cosine's. Therefore, the study of forced vibration under the action of a harmonic excitation force is very important, because suppose if we can find out the expressions for that what happens? When the force f is equal to a_1 sin omega t then we can take another force f is equal to another a_2 sin 2 omega t. Therefore, under the action of combined terms, the response will also be combined, because this system if we take k and m and even damping coefficient c as a constant, then this is a linear system. That means, we can simply find out the displacement by superposition under the two circumstances.

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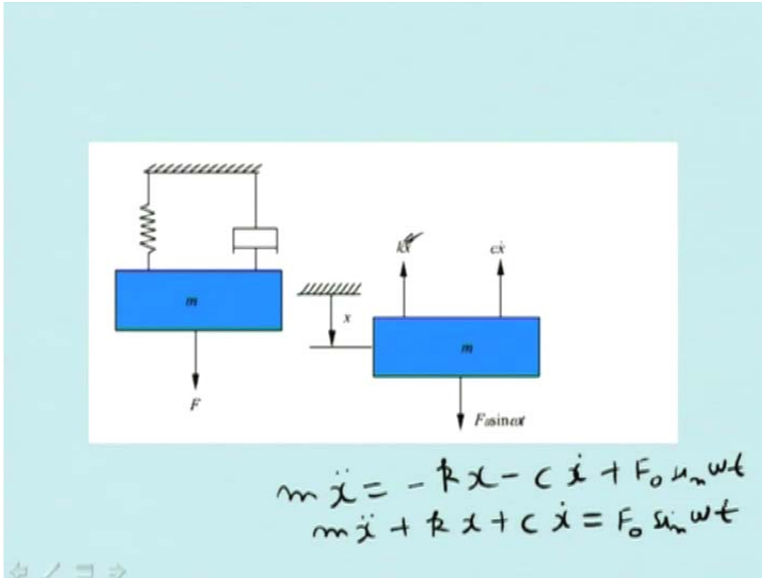

Therefore, let us try to study that. This is mass then this is a spring then this is dashpot and this is x. There is a force F acting here. Now, this will be F_0 sin omega t. Let us assume that F is equal to F_0 sin omega t that means you have got a sinusoidal forcing function. This is kx then this is cx dot; kx is the restoring force and kx is this one. Therefore what happens? Now this is the free body diagram of mass m. There is a force F_0 sin omega t. This is kx and this is cx dot.

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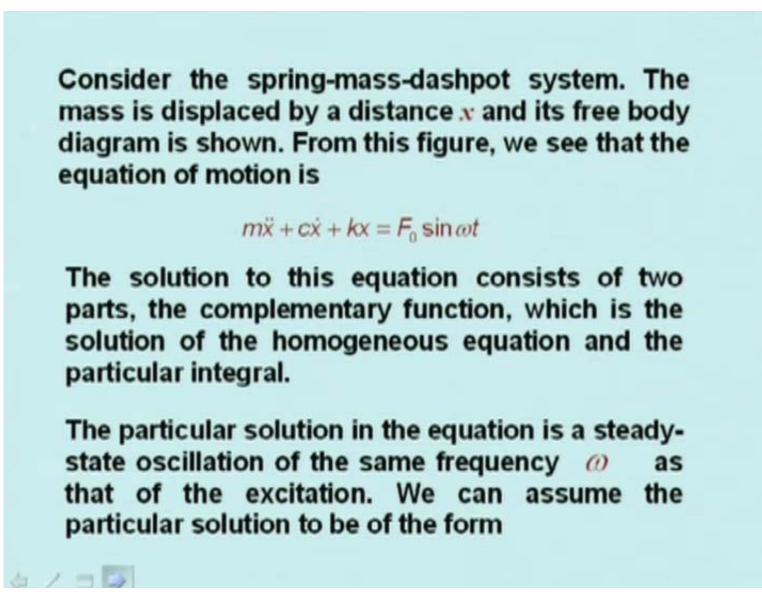

Then consider the spring-mass-dashpot system. The mass is displaced by distance x and its free body diagram is as shown. From this figure, we see that the equation of motion is mx dot dot plus because the equation is basically mx dot is equal to minus kx minus cx dot. Therefore, it can be written as plus F_0 sin omega t. Therefore, this can be written as mx dot dot plus kx plus cx dot is equal to F_0 sin omega t.

Solution to this equation consists of two parts; one is called the complementary function, which is the solution of the homogeneous equation. That means, if we solve mx double dot plus cx dot plus kx equal to 0, we will get one solution that is called solution of the homogeneous equation. This equation mx dot plus cx dot plus kx is equal to 0 is called homogenous, because if x equal to a, is the solution then x is equal to constant times a, is also a solution. That means, if one solution is x then naturally cx is also a solution. So, it is totally homogeneous that it is called complementary function. The other one is that particular integral that depends upon the term on the right hand side. In this case, it is F_0 sin omega t. So, the combined motion is basically just the combination of complementary function and particular integral.

The physical significance of this is that when we displace a mass by some force, then combined effect which we get, is the sum of these two things; that means, natural vibrations, that mass particle starts vibrating in natural mode and then forcing term vibrations are also present. Therefore, the motion is the combination of these two. The particular solution in the equation is a steady-state oscillation of the same frequency omega as that of the excitation.

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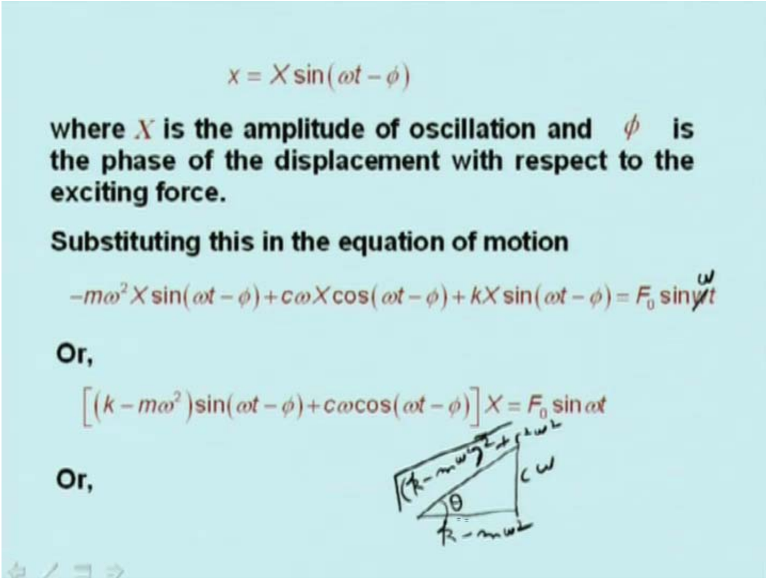

We can assume particular solution to form this one. We just assume, let us see that what happens?

x is equal to let us say X sin omega t minus phi, where X is the amplitude of oscillation and phi is the phase of the displacement with respect to the excitation force. If you substitute this expression in the equation of motion, you get like this; minus m omega square X sin omega t minus phi, because X sin omega t minus phi has been differentiated two times. So, you get minus m omega square X sin omega t minus phi plus c omega X cos omega t minus phi. It has been differentiated one time plus kx sin omega t minus phi is equal to F_0 sin omega t, or k minus m omega square sin omega t minus phi plus c omega cos omega t minus phi into x is equal to F_0 sin omega t, that expression we got.

We have to find out the expression for x. If we carryout trigonometry here, can I simplify this expression, k minus m omega squared sin omega t minus phi. If we just see that if I can represent them in trigonometric form, if this is theta and if this is k minus m omega square and this is c omega and this is under root of k minus m omega square plus omega square plus c square omega square. Finally, we can get sin theta type of term.

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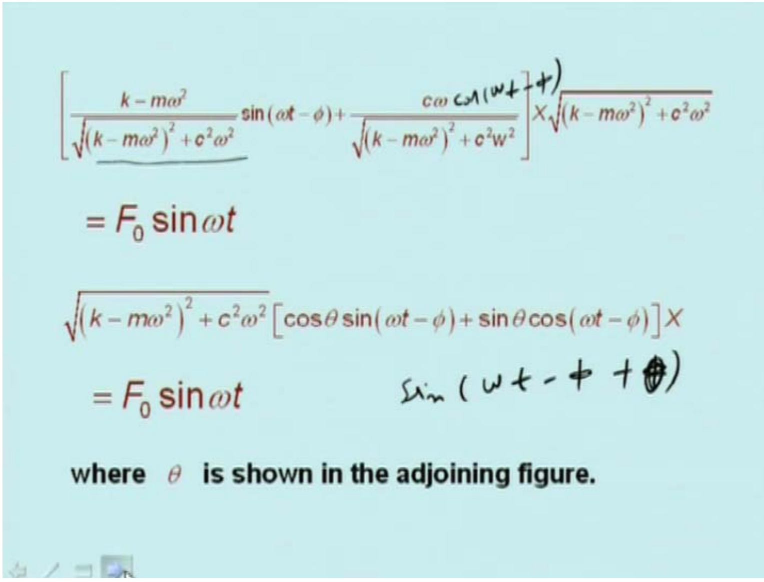

If we divide this whole thing by k minus m omega square divided by that thing this we are dividing this thing by this expression that means square of these two terms this and this then take the under root, we divide by sin omega t. Similarly, here c omega t and here this is cos omega t minus phi, like that multiplied; since we divided by under root k minus m omega square plus c square omega square. Therefore, we have to multiply it also by this thing then this expression remains same. Therefore, the previous expression and this expression is basically same. This is equal to F_0 sin omega t.

We can take this common. Here k minus m omega square whole square plus c square omega square multiplied by cos theta, because this can be written cos theta and this is sin theta cos omega x is equal to F_0 sin omega t where theta is basically like this.

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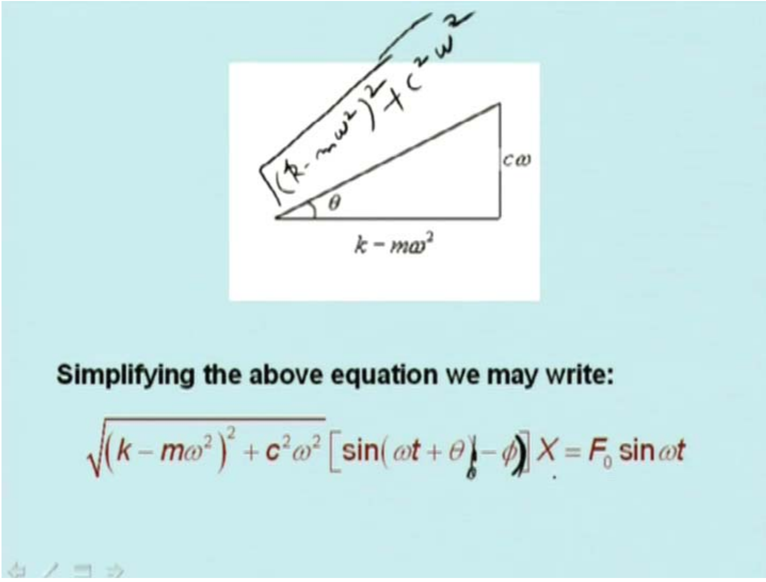

This is k minus m omega square and this is c omega. Therefore, by Pythagoras theorem, this is k minus m omega square plus c square omega square. Simplifying this expression, we may write this as sin omega t plus phi, because this is the formula of sin omega t plus phi. This one cos omega t minus theta this is sin omega t plus theta minus phi basically, because this is a b. So, sin omega t minus phi plus theta. Apply the formula of sin a plus b; so, sin a cos b. So, sin omega t minus phi cos theta sin cos omega t minus phi and this is theta. So, this is sin omega t minus phi cos theta cos omega t minus phi sin theta. So this is like this.

In this expression this is omega t this is equal to like that so sin omega t minus phi plus theta is equal to F_0 sin omega t. We have obtained this expression.

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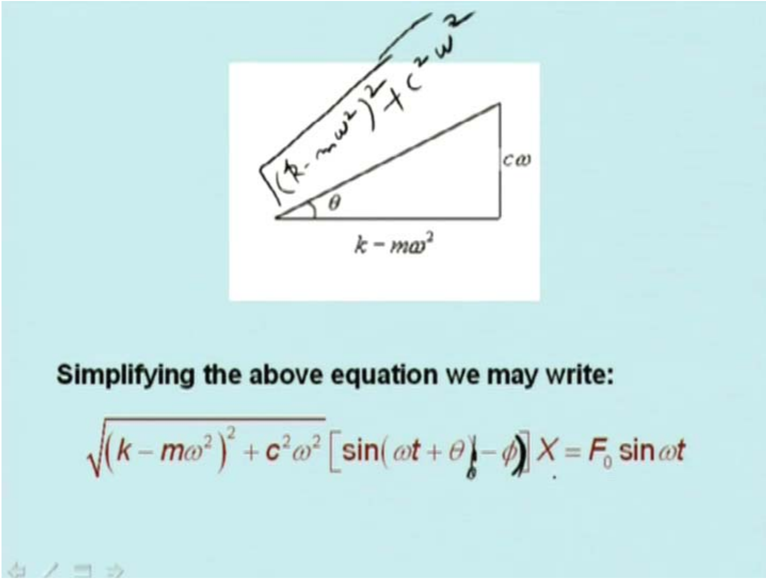

If we compare the right and left hand side of this thing, sin omega t, this has to be equal. So, omega t plus theta minus phi should be equal to 0.

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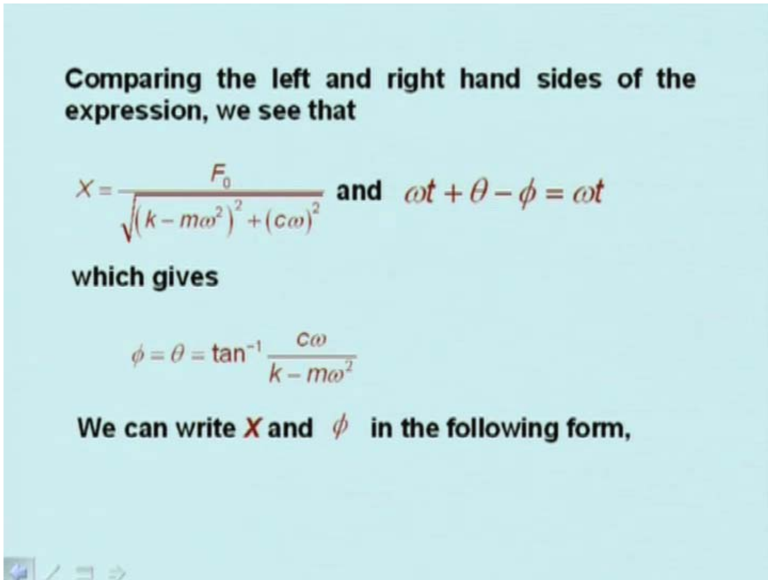

That means we have to say, where X equal to F_0 by this one and omega t plus theta minus phi is equal to omega t. This condition has to be because for all t, if it has to be correct then this relation has to be satisfied which gives phi is equal to theta; that means, which is equal to basically, tan inverse c omega k minus m omega square. We can say tan theta is equal to c omega divided by this thing. So, tan omega theta is equal to tan inverse c omega k minus m omega square. We can write X and phi. So, X is equal to F_0 under root k minus m omega square plus c omega square.

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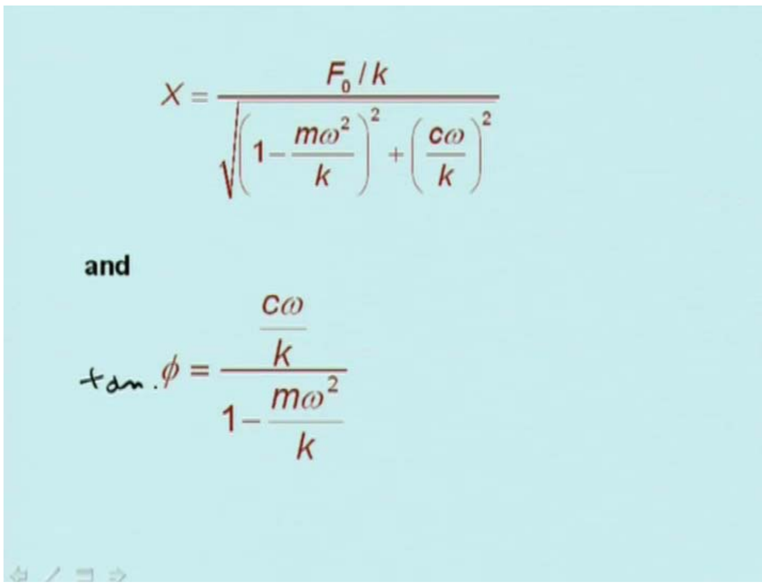

We can write X as F_0 by k divided by k. So, 1 minus m omega square by k whole square plus c omega by k whole square and c omega by k 1 minus m omega square by k. It is to be noted that F_0 by k is the static deflection of the spring under the action of that force F_0 k. Maximum amplitude of the force is $F_{0.}$ So, if you apply maximum force then you will get the static deflection as F_0 by k.

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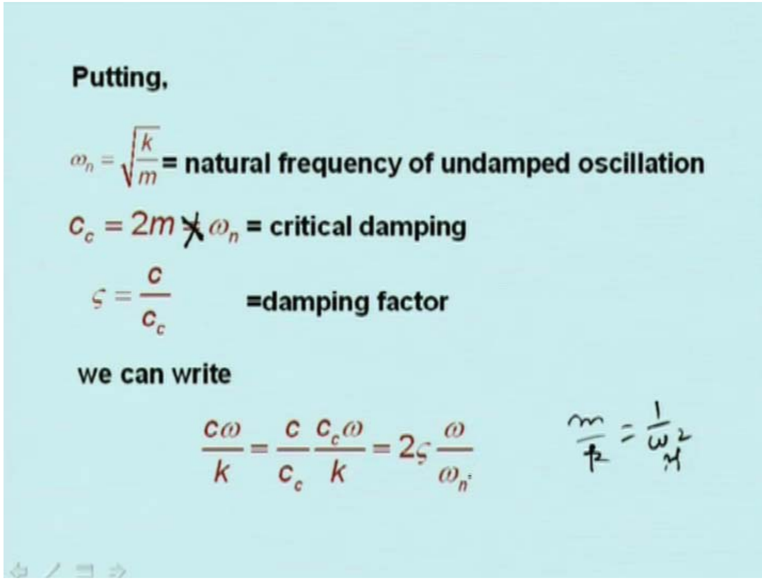

We know that omega$_n$ is equal to under root k by m that is the natural frequency of undamped oscillation. Now, C_c is equal to 2m times omega$_n$ which is called critical damping and zeta is equal to C by C_c, that is called damping factor. We can write c omega by k as c by C_c C_c omega by k, that means 2 zeta omega by omega$_n$, because C_c is equal to 2 C_c by C_c 2g. This C_c is equal to 2 m into omega$_n$ and m by k is basically 1 by omega$_n$ square. We just do that simple. So, we will keep C omega by k is 2 zi where zi is damping factor omega by omega$_n$.

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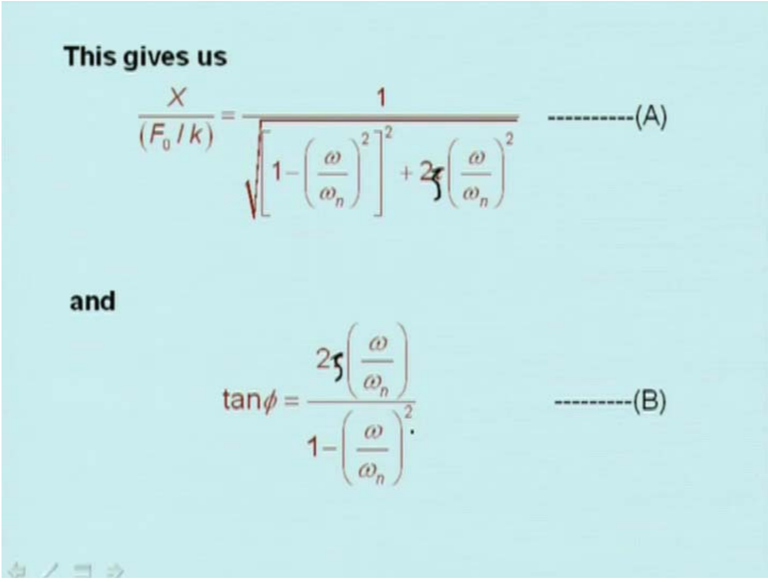

This gives us X divided by F_0 by k is equal to 1 under root divided by under root 1 minus omega by $omega_n$ Whole Square whole square plus 2zeta omega by $omega_n$ whole square. This is also zeta. This tan and tan phi is equal to 2zi omega by $omega_n$ 1 minus omega by $omega_n$ Whole Square.

Why we have written this in this form? As we already mentioned, F_0 by k is the maximum static deflection. X divided by F_0 by k gives the ratio of that dynamic amplitude to static amplitude or maximum static deflection. Similarly, here we get phi, phi is the phase angle, because we started like this.

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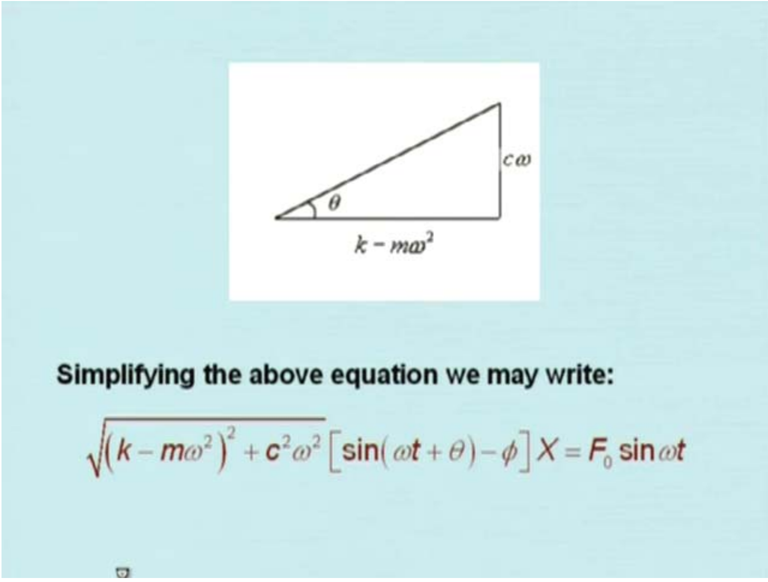

We assumed that the force was F_0 sin omega t but our displacement was sin omega t minus phi; that means, there was a phase difference between two. So, this gives the idea about the phase difference.

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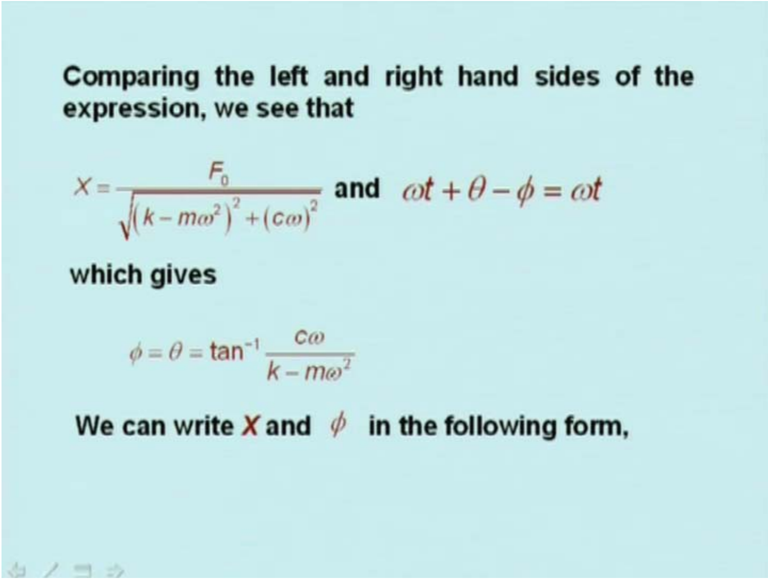

In this case, if c is equal to 0 then there is a no phase difference. There is no damping present, then but if there is a damping present then there is a phase difference between the applied force and the response we get. So, that is because effect of damping when it is like that, so that damping also introduces a force. Therefore, this comes X is equal to F_0 by k under root 1 minus m omega square by k whole square plus c omega by k whole square and phi is equal to C omega by K whole divided by 1 minus m omega square by K. In this, this is tan phi.

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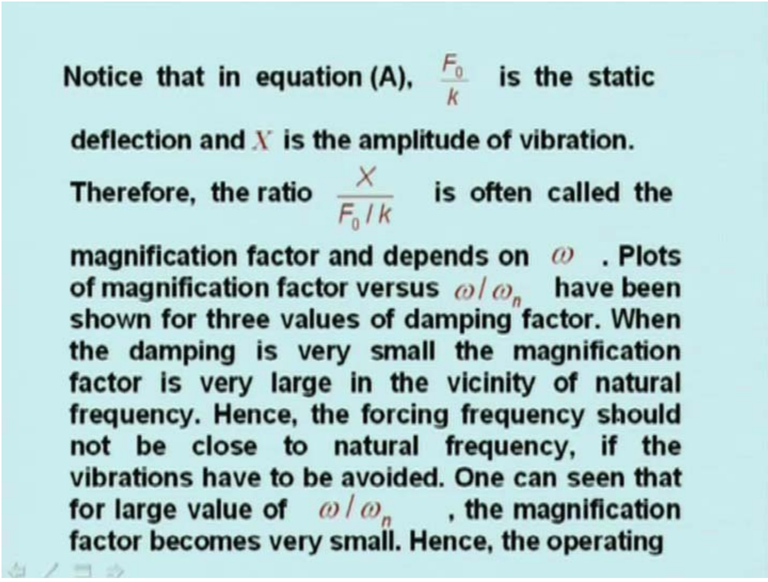

Using this, now notice that in this equation A, F_0 by k is the static deflection and X is the amplitude of vibration. So, amplitude of vibration depends naturally, what is the frequency ratio? That means omega by omega$_{n.}$ What is the natural frequency? Therefore, the ratio X by F_0 by k is often called the magnification factor. How much times the dynamic amplitude is more than the static deflection and depends on omega.

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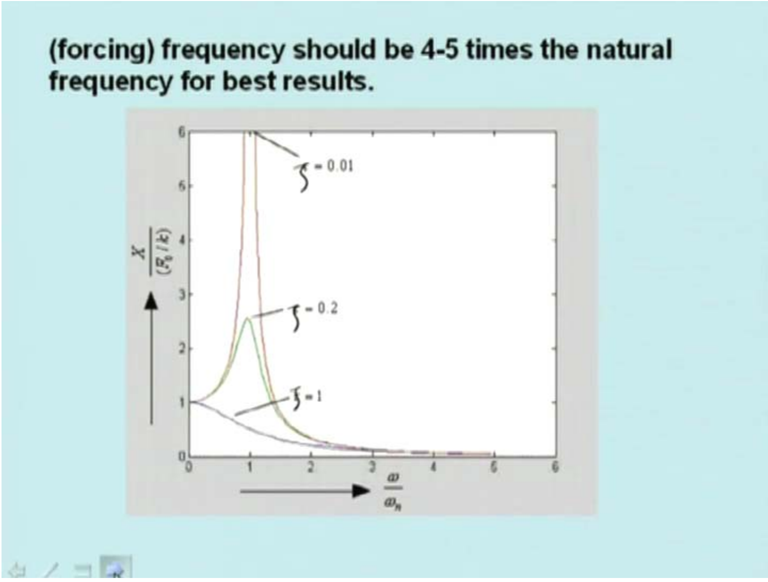

Now, plots of magnification factor versus omega by omega$_n$ have been plotted, we have shown. Actually for three damping values that is; tau is equal to 0.01 that is very low damping, tau is equal to 0.2 and tau is equal to 1. In this case, at omega by omega$_n$ equal to 1. If damping is very low then a resonance type condition occurs. So, we see that the magnification becomes under almost infinity. Otherwise also, nearby this region, near the natural frequencies, the magnification factor becomes very high.

However, if we have critical damping then magnification factor is always less than 1; that means if omega is very high compared to the natural frequency then the magnification factor approaches zero. This is very interesting, that means at omega is equal to 0. That means in static case, if you just apply some force then there will be displacement. That will be basically X by F_0 by k. Therefore, you will get the same type of thing, but if we apply a very high frequency then displacement approaches 0. Why? Because the mass is not able to respond, because of the dashpot it is not able to move at all. Therefore, there approach is 0.

Similarly, but when tau is equal to 0.2 then at resonance it will be very high, but of course, it will not be infinite. If there is a damping present in the system, it will not allow the vibration to become infinite, of course it becomes very high. Similarly, in the case of the low damping, it is like this.

When the damping is very small, the magnification factor is very large in the vicinity of natural frequency. Hence, the forcing function should not be closed to natural frequency, if the vibrations have to be avoided. For very large value of omega by omega$_n$, the magnification factor becomes very small. Hence, the operating or forcing frequency should be four to five times the natural frequency, for best results. That means, if you have the natural frequency of some system is 100 hertz. If you have the forcing function frequency at 400 hertz then you know that it will not. We might have noticed these type of examples, like a fan makes lot of noise when it is run at a low rpm, but when the rpm is increased, then it does not make that much noise because of this phenomena.

Similarly, some cars also, when they run at a low speed then they will show vibrations, but at high speed the vibrations are not seen.

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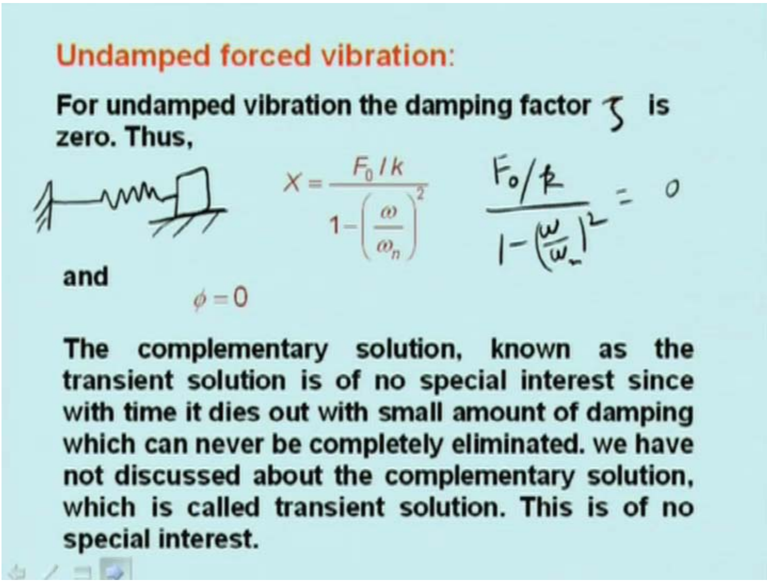

For undamped forced vibration, now undamped, one damping factor zeta is equal to 0. Thus, we get X is equal to F_0 by k 1 minus omega by omega square and we get phi is equal to 0. There is no phase difference between the forcing function and the response. Then we are getting this type of expression. Here also, it is seen that if omega is very small then X is equal to, of course F_0 by k, but if omega is very large then this X will approach 0 because of this term F_0 by k. So, this term will approach 0; so, F_0 by k divided by minus omega by $omega_n$ square. This term is approaching infinity. You get something like, here 0. So, this is the interesting thing.

If you have a spring mass system like this, there is a very flexible spring. If you apply force of a very high frequency, this will not be able to move that means it will not carryout any vibration.

Now, complementary solutions known as the transient solution, is of no special interest since with time it dies out with a small amount of damping which can never be completely eliminated. Therefore, we have not discussed about the complementary solution which are called transient solution. This is of no special interest. However, in undamped case, one can study actually and one can get some insight about the cases in which damping is almost 0.

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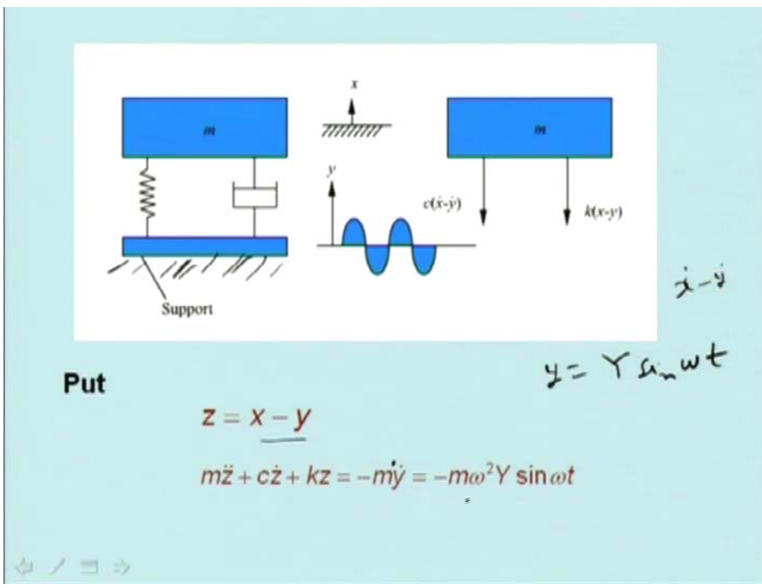

Let us discuss, the other type of problem that is the problem involving the support motion. What is this, that if you have kept this on a support, this is a fixed support like this and this is mass and this is a spring and this is dashpot. Now, support itself is getting excited. Supporting is being excited and support is undergoing the displacement y and mass itself is undergoing the displacement x. This is the equilibrium position of the mass and x is the displacement from this position.

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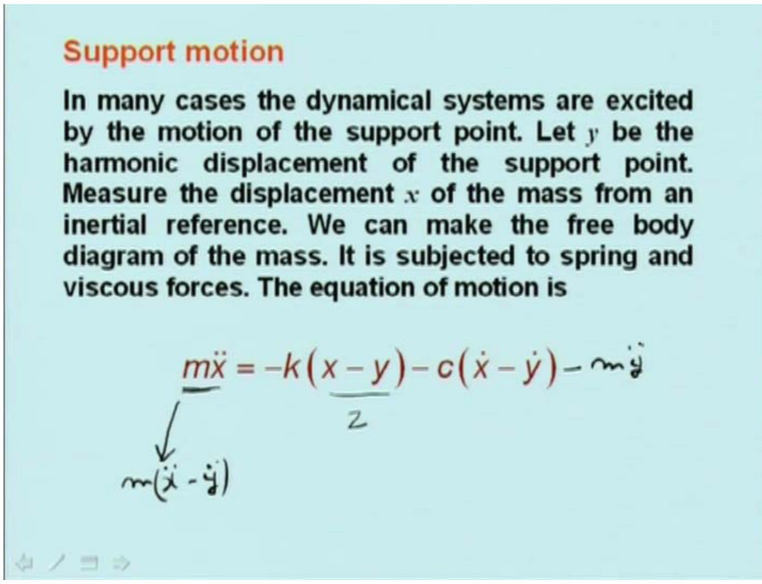

Let y be the harmonic displacement of the support point. If we measure the displacement x of the mass from an inertial reference, we can make from the free body diagram of the mass. It is subjected to spring and viscous force. Equation of motion is like this. This is mass, it is stretched. Suppose, you have pulled the mass by amount x, but the support by that time also has moved by a distance y. So, net stretching of the spring is k(x-y). Therefore, the spring will provide a force k(x-y) in this direction.

Similarly, the net velocity of the mass from inertial reference frame, means from outside observer which is on a fixed frame, which is from that frame he is observing and he says that net velocity of the mass is x dot minus y dot. This is the velocity that is the relative velocity, because otherwise the absolute velocity of the particle is x dot; mass m is x dot. Therefore, support itself is moving with y. Therefore, net velocity is relative velocity between this thing is x dot y dot. That means dashpot is there. So, dashpot's piston is moving with x dot. Cylinder itself is moving with y dot. So, difference is x dot minus y dot.

Therefore, cx dot minus y dot is equal to k(x-y). Now, put z is equal to x minus y. If we put z is equal to x, we get mx dot first, by Newton's law, mx dot is equal to minus k x minus y minus c x dot minus y dot. Now, we are getting the terms x minus y. Therefore, better to write it like z, the y variable gets reduced like that.

You say z is equal to x minus y, we get mz double prime. Here, we do not have any y term. So, if we put that here also, mx dot minus y dot then since we have added here, therefore, from here we can subtract that thing. We can add here also so it becomes minus my dot. Therefore, it becomes mz double dot plus cz dot plus kz is equal to minus m y double dot, and this is y is equal to, if we assume that support is getting excited by y is equal to y sin omega t. Therefore, this is minus m omega square y sin omega t. So, you are getting this thing.

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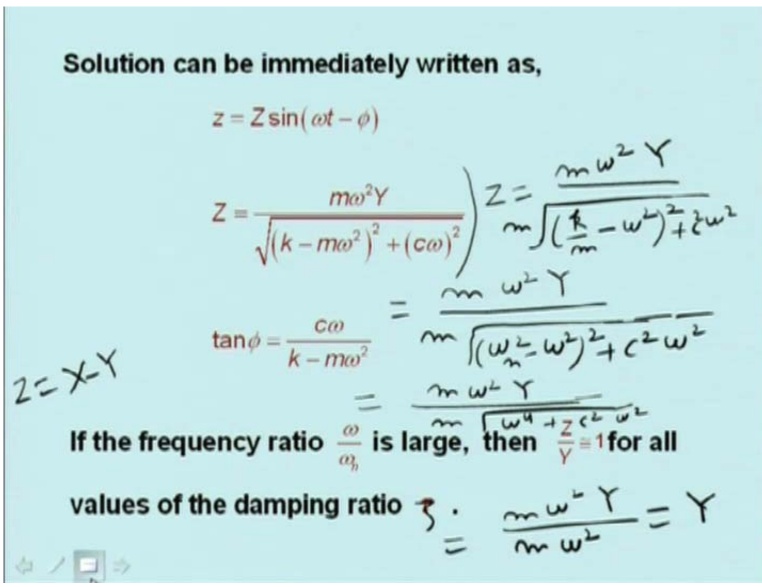

Its solution can be immediately written as z is equal to Z sin omega t minus phi. Because this type of thing we have already done. This is like a force. Instead of the force F_0 sin omega t, we are having m omega square Y. So, m omega square Y is like a force. So, z is equal to Z sin omega and Z is equal to m omega square by Y under root k minus m omega square whole square plus C omega square and tan phi is equal to C omega k minus m omega square.

In this case, if the frequency ratio omega by omega_n is large then Z by Y becomes equal to one for all values of the damping ratio. Let us discuss this point in detail. We have seen m omega square tan phi Z is equal to m omega square Y under root k minus m omega square plus C omega square. Now, tan phi is equal to C omega k minus m omega square.

In this case, Z by Y is equal to one for all values of the damping ratio. How can we prove this one? Let us write Z is equal to omega by omega_n is large; that means, Z is equal to m omega square Y divided by k minus m. So, we take that. Maybe we can take m square common, so, m can be taken common here. This becomes k by m. k by m minus omega square whole square plus C square omega square. This is equal to m omega square Y. This is m and this will become k by m minus omega square omega square. So, this is under root omega_n square minus omega square whole square plus C square omega square.

Now, omega by omega_n is large that means omega_n is smaller, much smaller in fact in comparison to omega square. Therefore, here omega_n square minus omega square can be written as omega square. So, this will become omega to the power 4 and this becomes m omega square Y divided by m under root omega to the power 4 plus c square omega square. Since omega is a very large number, omega whole term is very high in comparison to C square omega square.

First of all, damping is always less than 1 and omega. Therefore, what happens? This I can write as m omega square Y divided by m omega square that is omega square. So, this is equal to Y. Therefore, Z is equal to Y. Irrespective of damping Z is equal to Y if omega by omega_n is large. That means, if the support is being excited by a very high this one then Z will be equal to Y, but then what happens, Z is equal to Y but Z is basically X minus Y. That means, we have said Z is equal to X minus Y and so, that is equal to Y. So, what happens is that Z by Y is almost this one. That means, the relative displacement is almost same as the support displacement.

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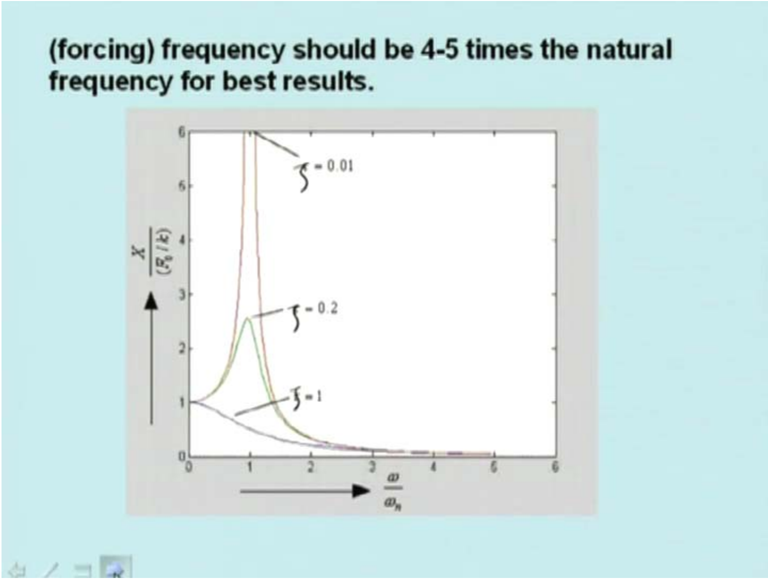

Using the characteristic of this type of response, you know that we have got response and understanding that we can design various types of systems. You know what is our primary interest? Sometimes, you have to design the system for measuring the acceleration. Sometimes, you have to isolate the vibrations. So, these types of things are there.

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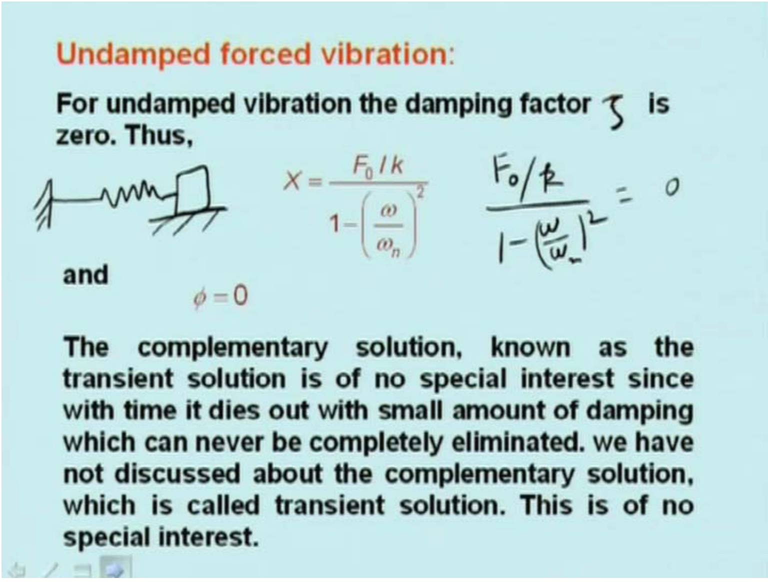

This briefly we have discussed. Let us discuss one or two other interesting things. This factor, which you see here, F_0 by k 1 minus omega by $omega_n$ square whole square, if omega by $omega_n$ becomes equal to 1, this will become 0.

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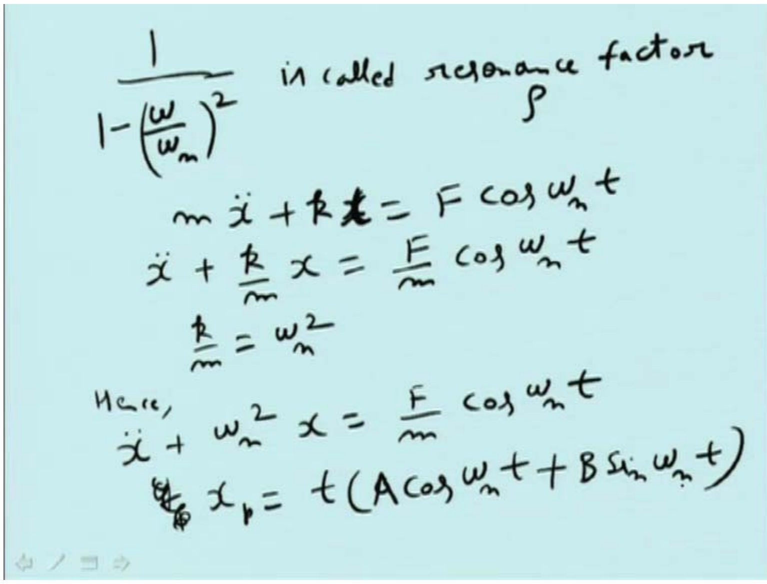

This factor is 1 by 1 minus omega by $omega_n$ square is called resonance factor, rho. We can denote it by rho. Therefore, if omega is equal to 0, the resonance factor is 1. If omega is equal to very large, resonance factor is 0.

If omega is equal to $omega_n$ resonance factor is infinity. We have seen that you perhaps get the idea that at omega is equal to $omega_{n,}$ that amplitude of the vibration becomes very high. Let us solve the differential equation when there is a resonance. We know that the differential equation, suppose mx dot plus kx is equal to kt say x, kx is equal to F cos omega t. Let us say, omega is $omega_{n.}$ So, $omega_n$ t and this can be written as x dot plus k by m x is equal to F by m cos $omega_n$ t. Now, k by m is equal to $omega_n$ square. Hence, x dot plus $omega_n$ square x is equal to F by m cos $omega_n$ t. This is the differential equation which has to be solved.

Here, we can see that, if we just put x is equal to A sin omega t, that will not satisfy this differential equation in this case, because this is the thing therefore, in this particular case, when the you are getting that same $omega_n$ here and $omega_n$ here like that, therefore, you will get a particular solution. In this case, is xp is equal to t times A cos omega$_n$ t plus B sin omega$_n$ t that is it and for this C a. Now, we can have these are the constants. We can have cases.

Let us see, what happens when x is equal to 0 at time t is equal to 0 and x dot is also equal to 0, that means displacement and velocity both component are 0. So, what will happen?

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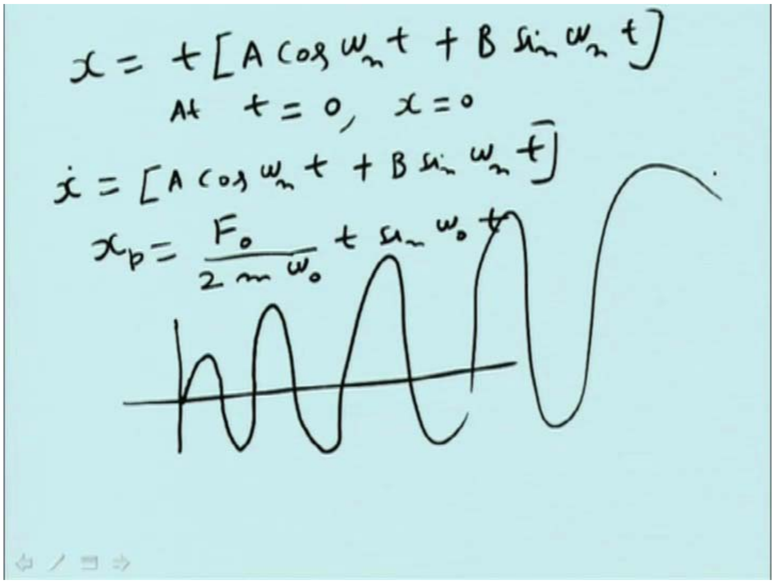

This will become, at t is equal to 0. So, let us write the solution again. x is equal to t and this is A cos omega$_n$ t plus B sin omega$_n$ t. Now at t is equal to 0, x is equal to 0. Let us see x dot is equal to A cos omega$_n$ t plus B sin omega$_n$ t. So, we have to put that condition and after that, we will get some solution which will be equal to x_p is equal to F_0 divided by 2m omega$_0$ t sin omega$_0$ t. That means, at natural frequency the vibrations will always keep increasing. So, this amplitude of vibration will always keep increasing. So, this type of phenomena will come. So, with time the magnitude always keeps on increasing. This is what we observe. This is the type of resonance. So, this condition has to be avoided.

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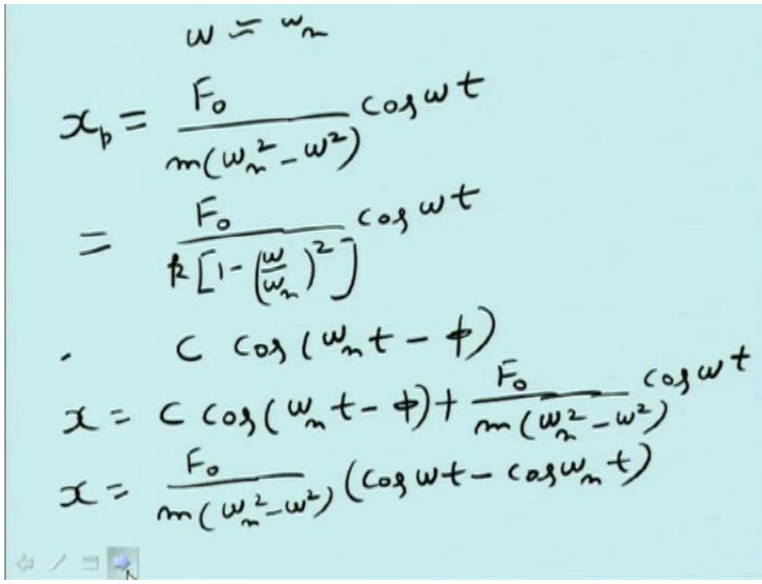

Let us see that if in case damping is very small and your omega is very near to omega 0, that means omega is almost omega_n then x is equal to F_0 divided by m omega 0 square. This is omega omega_n square minus omega square cos omega t. This is the particular integral and this can be written as F_0 divided by A_1 minus omega by omega_n whole square and this is cos omega t.

Therefore we consider that even if there is no damping, let us take the case of no damping, then there will be a homogeneous solution also. That will be the complementary part. Complementary function is given by C is a constant cos times omega_n t minus phi, where phi is some angle. Therefore, x is equal to C cos omega_n t minus phi plus F_0 divided by m omega_n square minus omega square cos omega t. This term has come like this. In this, we put the condition that at time t is equal to 0, x is equal to 0 and at time t dot t is equal to 0, x dot is also 0.

That means, we are applying the forcing function at the time when the particle is at rest and from that equilibrium position. Then we will not show that required algebra. Then C and phi can be eliminated and you get x is equal to F_0 divided by m. This will be equal to omega_n square minus omega square. This is cos omega t minus cos omega_n t. So, this can be written as cos omega t minus cos omega_n t. We have the formula for cos C minus cos t.

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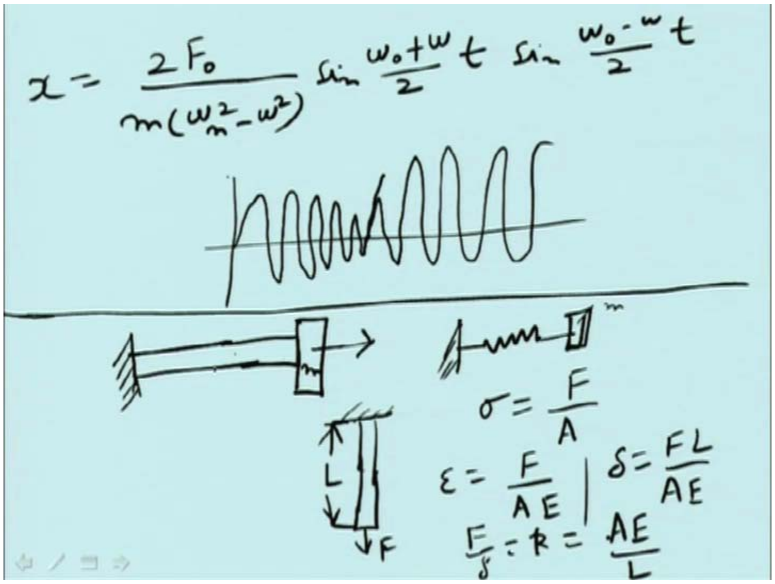

This can be written as, x is equal to $2F_0$ divided by m omega_n square minus omega square sin omega 0 plus omega divided by 2 into t sin omega 0 minus omega divided by 2 into t. So, if we plot this thing here, plot may look something like this and then again it goes like this. What happens, because omega 0 minus omega is very small, therefore, this time period will be very large and we will get the phenomena of beats type of thing. So, here we will be getting the beats type of thing. Like that, we can actually study different type of problems.

We will conclude here. So, today I have discussed about forced vibration problem in which we considered damping and this one. Let us consider the cases, you will always not getting a spring mass system. Sometimes you have got the axial rod, if you have got axial rod and you are applying a force. Here, there may be a mass and you are applying a force here. Even this type of problem can be solved by a spring mass system. So, you model it like a spring and this is the mass. Some portion of the rod mass can also come here and it can be damped at that point and then it becomes like that one.

Supposing you are applying the displacement; this is the force, this is a rod of length L and this is the force. If you apply a force F, cross sectional area is A. Therefore, this will give stress is equal to F by A and strain is equal to F by A divided by A times E. E is the Young's modulus of elasticity. Therefore, displacement delta of the end point is equal to FL divided by AE. F by delta is called stiffness k that is equal to AE by L. In this case, the spring constant can be written as AE by L and you can actually solve this problem. So, axial vibrations of this system will be more, having more frequency if A is more.

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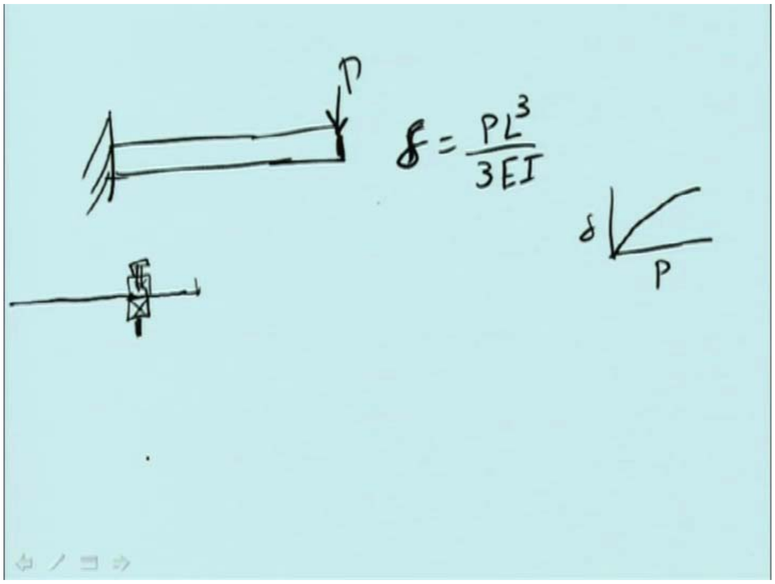

Similarly, if you have some cantilever beam type of thing here and if you apply some load delta here, you have F is equal to PL cube by 3 EI, delta is equal to PL cube by 3 EI. This is known from the strength of material. You will come to know about that thing experimentally. Otherwise, what happens? Take a cantilever beam. Apply some force and measure its deflection. Then you know that one can plot P versus delta P is the force and slope of this gives you stiffness dp by d delta. As the length increases, its stiffness decreases.

Therefore, you can do that one simple experiment. You can make that you take a beam type of thing and here between two adjustable clamps you put it and clamp it properly from some screw. You can always see that when the length was large and if you displace it slightly, you can observe its natural vibrations. It will vibrate with a longer time period that means less frequency. If the same thing you displace and you make like this thing has been displaced this side and now you have like this here. If you vibrate then the frequency of vibration will be very high and this is how it will be doing. Therefore, these types of simple experiments one can do and one can observe. In the next lecture, we will discuss the vibration of rigid bodies.