Engineering Mechanics

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Module - 12

Kinetics -2

Lecture - 33

Method of momentum and analysis of robot manipulator

In this lecture, we will discuss methods of momentum for bodies. We will also discuss one problem of robotic manipulator, where to develop the equation's dynamics finds an important role. We have already discussed the concept of momentum for a particle.

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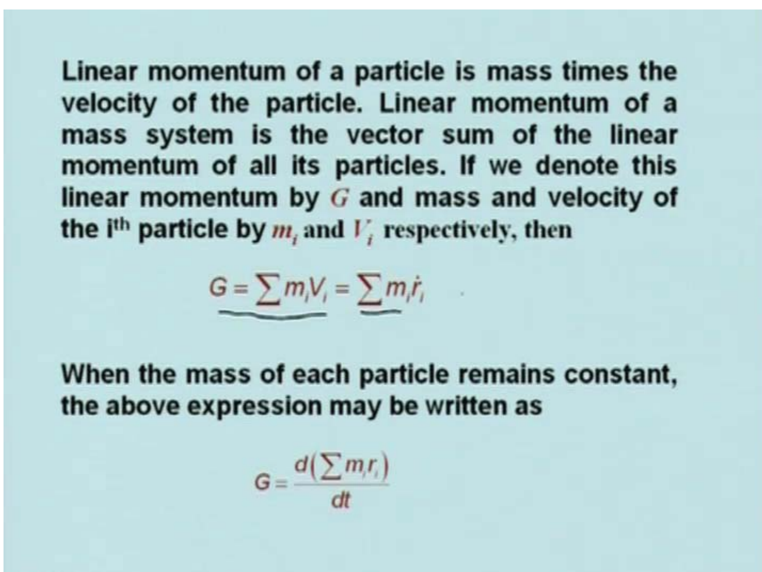

Linear momentum of a particle is mass times the velocity of the particle. It is the product of the mass and velocity. It is a function of mass and velocity; just a simple product. Linear momentum of a mass system is the vector sum of the linear momentum of all its particle. Because momentum is a vector and kinetic energy is scalar, we find out the kinetic energy of this thing by summing kinetic energies of all the particles. Here also, we will do simple summation only. However that radial momentum is vector; therefore, it has to be summed in that way. If we denote the linear momentum by G and mass and velocity of the i th particle by m_i and V_i irrespectively, then G is equal to sigma m_i $V_{i.}$ We get this for a system of particle and this is equal to sigma m_i r_i dot, where r_i is the position vector of the particle.

When the mass of each particle remains constant, this expression may be written like this. G is equal to d sigma m_i r_i divided by dt; d by dt $m_{i.}$ So, m_i can come outside and it becomes the same previous expression.

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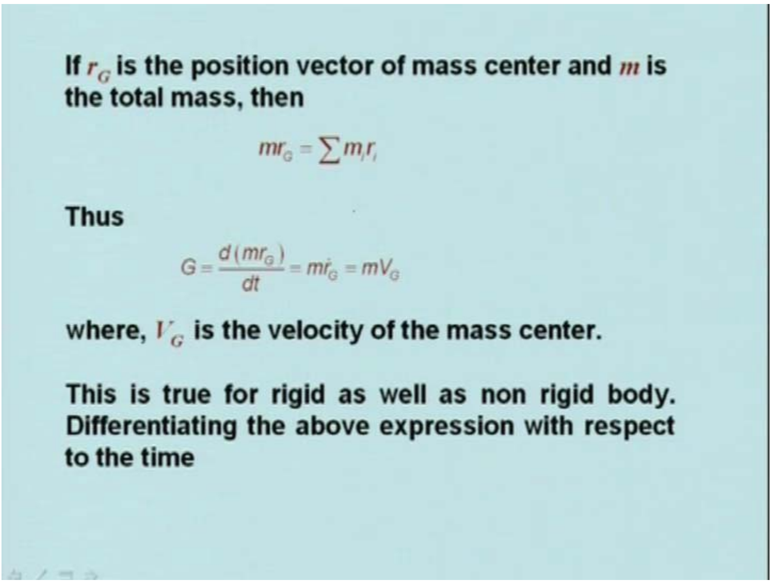

If r_G is the position vector of mass center and m is the total mass, then m times r_G is equal to sigma m_i times r_i. Thus, we have d (mr_G) by dt is equal to mr dot G, which is equal to $mV_{G,}$ where V_G is the velocity of the mass center of the particles. There may be finite number of particles or infinite number of particles making the body or it can be system of mass. So, we get the equation that momentum of a system of particles is basically nothing but the mass times the velocity of the mass center. It is true for rigid as well as non rigid body. Here, we are not making any assumption that the particles cannot come towards each other. Therefore, this equation is valid for rigid as well as non rigid body.

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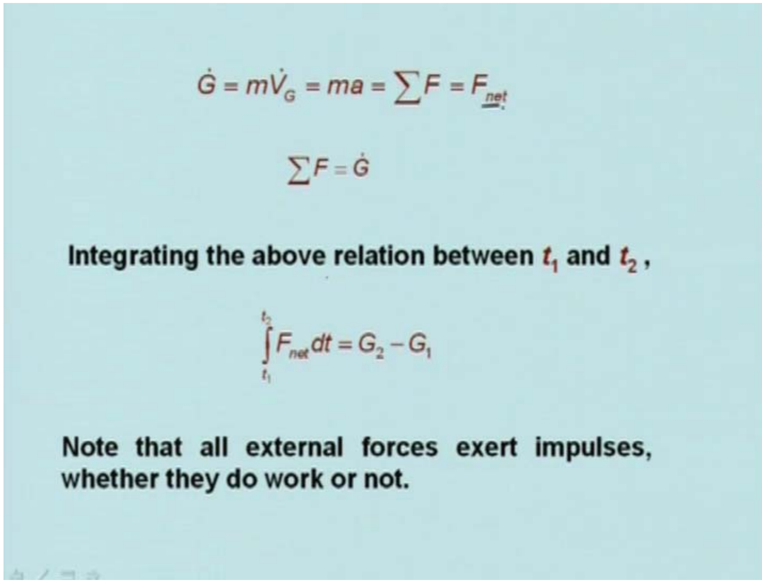

If we differentiate this expression with respect to time, then we get G dot is equal to mV dot G is equal to ma, a is the acceleration of the mass center. This will become sigma F and that will become $F_{net.}$ So, sigma F is equal to G dot. We get one relation that sigma F is equal to G dot. If we integrate the above relation between $t_1$and $t_{2,}$ then t_1 to t_2 F_{net} dt is equal to G_2 minus $G_{1.}$ This is the impulse of the total. Note that, all external forces exert impulses whether they do work or not. So all the forces have to be considered in this case. Then you can find out G_2 minus $G_{1.}$ So, this equation can also be used. This is about linear momentum.

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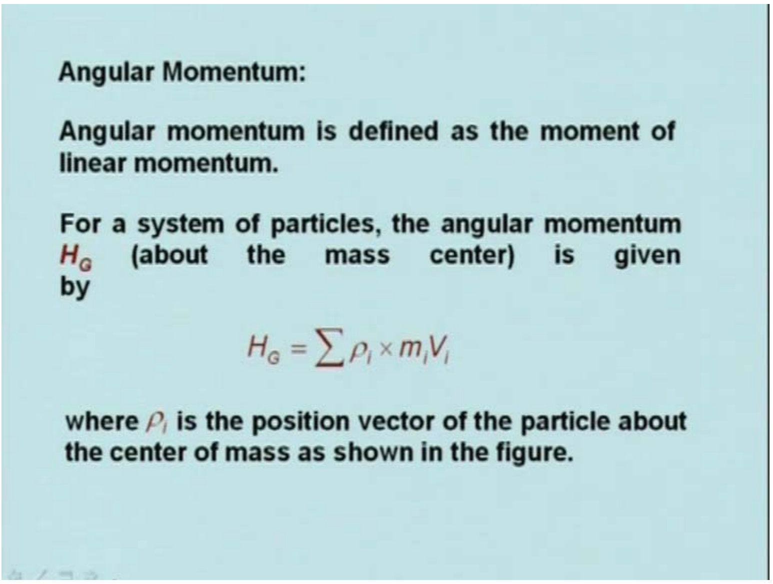

Now, we discuss about the angular momentum. Angular momentum is defined as the moment of linear momentum. For a system of particles the angular momentum H_G about the mass center is given by H_G is equal to sigma rho_i cross m_i V_i, where rho_i is the position vector of the particle about the center of mass and V_i may be the absolute velocity of this.

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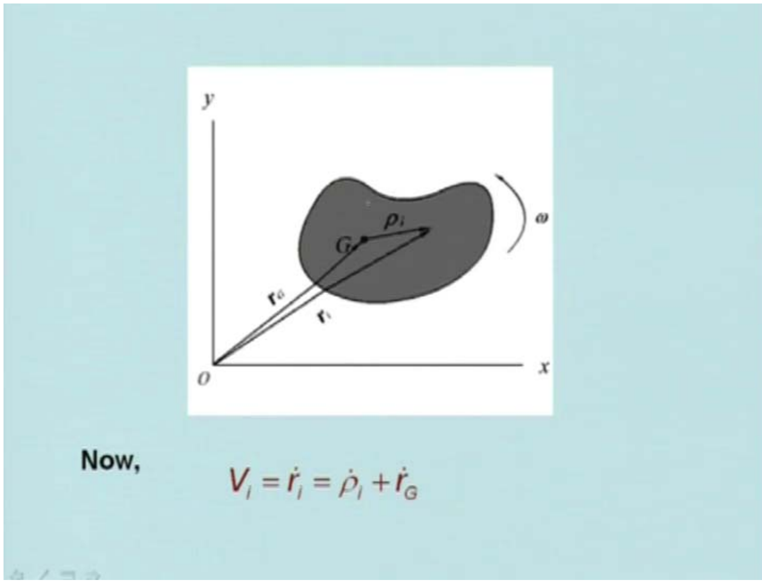

For example, this is the situation. You have body. This is the axis system; G is the mass center and it is rotating with angular velocity omega, and rho_i is the distance of the mass from this and this is r_i and this is r_G. We have the absolute velocity term here m_i $V_{i,}$ but rho_i is the position vector of the particle about the center of the mass. Then you have $V_{i;}$ that means, velocity of the particle is r dot i is equal to rho dot i plus r dot G.

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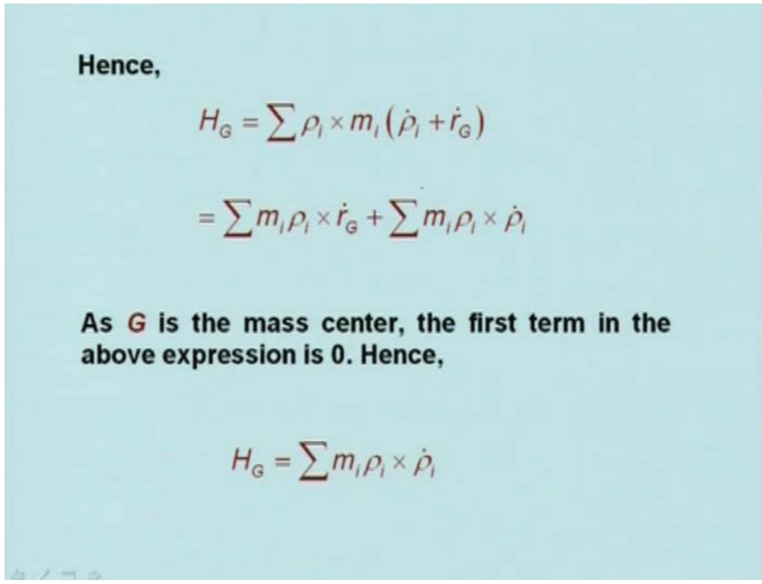

Hence, H_G is equal to sigma rho_i cross m_i rho dot i plus r dor G and this will become sigma m_i rho_i cross r dot G plus sigma m_i rho_i cross rho dot i. As in this case, G is the mass center. So, the first term in this expression is 0 because m_i rho_i, if you will do the summation that will be total mass times the distance of center of mass from that point. So, it is 0 and this portion goes to 0. Therefore, H_G is equal to sigma m_i rho_i cross rho dot i.

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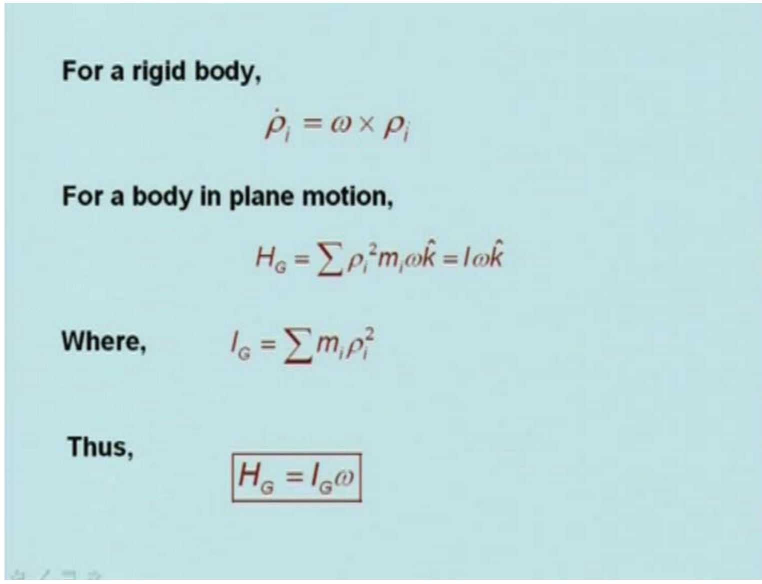

Now, for a rigid body rho dot i is equal to omega cross rho_i; rho_i is the distance of the particle from the mass center. If we differentiate it, magnitude remains same, only rotation is there. So, therefore we get omega plus rho_i. For a body in plane motion, this simply becomes H_G is equal to rho_i square m_i omega k that means this is basically I omega into k. I is the mass moment of inertia of the whole thing because H_G is equal to sigma mi $\text{rho}_{i.}$ In plane motion, it is having only angular velocity omega k. We will get, I_G omega in that k direction and this is what has come.

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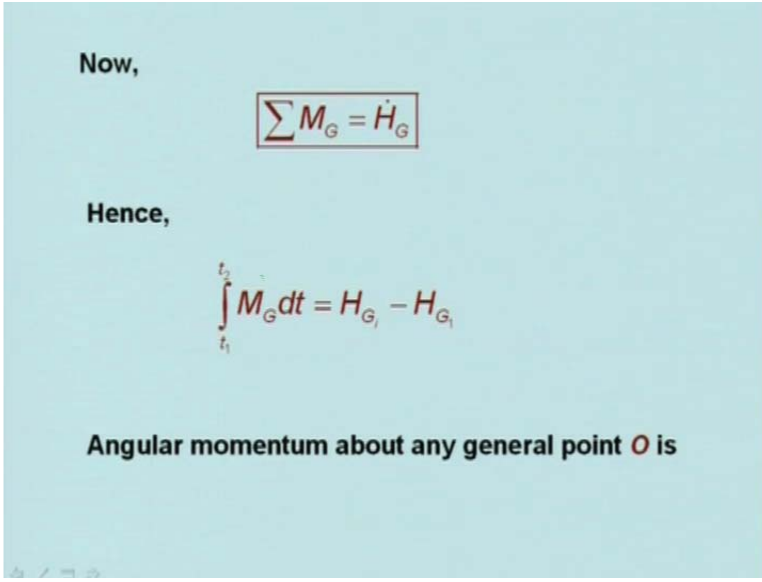

I_G is equal to basically sigma mi rho_i square and that I can write I_G. This is $I_{G.}$ Therefore, H_G is equal to I_G omega is the expression we get in this one. Sigma M_G is equal to H dot G; that we already have. Hence, if we take t_1 to t_2 M_G dt is equal to H_{Gi} minus H_{Gi}. If we can find out impulse of the moments then that is change into the angular momentum.

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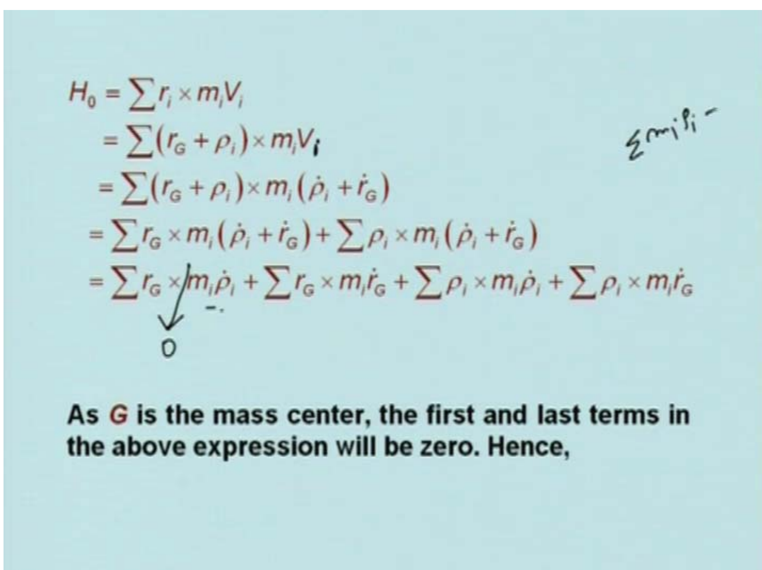

We have discussed angular momentum about the center of mass. Let us discuss angular momentum about any general point O that is H_o is equal to sigma r_i cross m_i V_i that is sigma r_G plus rho $_i$ cross m_i $V_{i.}$ This can be written as sigma r_G plus rho$_i$ cross m_i rho dot i plus r dot G and that becomes sigma r_G cross m_i rho dot i plus r dot G plus sigma rho$_i$ cross m_i rho dot i plus r dot G and this is equal to sigma r_G cross m_i rho dot i plus sigma r_G cross m_i r dot G plus sigma rho$_i$ cross m_i rho dot i plus sigma rho$_i$ cross m_i r dot G. As G is the mass center, the first and last term in the above expression will be 0. r_G is a constant; r_G cross m_i rho dot i is 0, because it is about the mass center we are taking the rate of moment of mass. You take the moment of mass and then differentiate it, so this term becomes 0 and last term also is this one; sigma rho$_i$ cross m_i and mi rho$_i$ sigma and this term will become 0. In this case, m_i rho$_i$ is equal to 0. So, therefore this term also vanishes.

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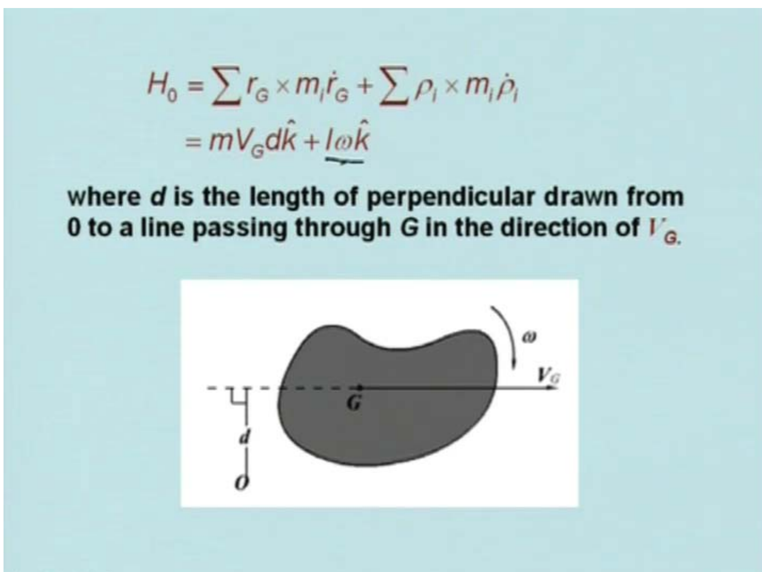

Hence, you have got H_0 is equal to sigma r_G cross m_i r dot G plus sigma rho$_i$ cross m_i rho dot i and this will be equal to mV_G dk, we can write. How does this come? r dot $_G$ is V_G and this is the cross product of r_G into $V_{G.}$ Therefore, if the perpendicular distance is d, so mV_G into dk and plus I omega into rho$_i$ cross m_i is I omega k and d is the length of the perpendicular drawn from zero to a line passing through G in the direction of V_G. This is what we get. H_0 is equal to mV_G in a plane that is this one and i omega into k, for a plane motion.

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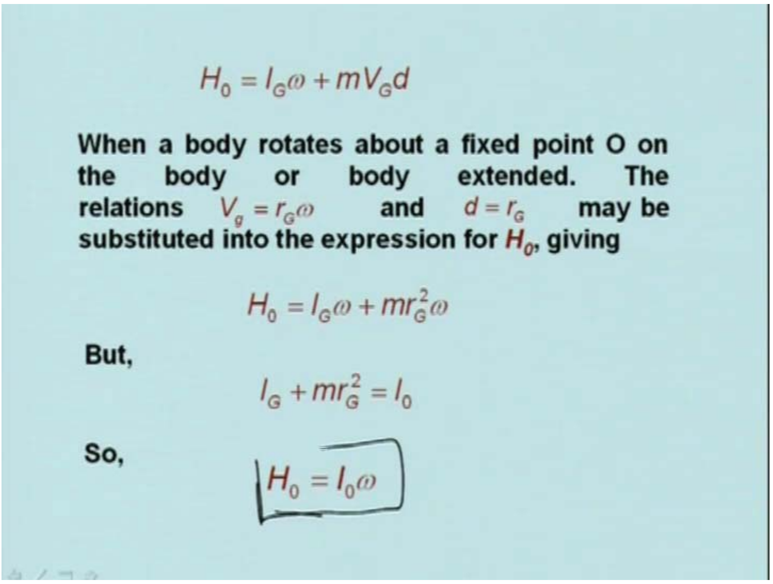

Therefore, we can write H_0 is equal to I_G omega plus mV_G into d. When a body rotates about a fixed point O on the body or body extended, the relations V_G is equal to r_G omega and d is equal to r_G may be substituted into the expression for H_0, giving H_0 is equal to I_G omega plus mr_G square into omega. But I_G plus mr_G square is equal to I_0, therefore H_0 is equal to I_0 omega. So, we get H_0 is equal to I_0 omega.

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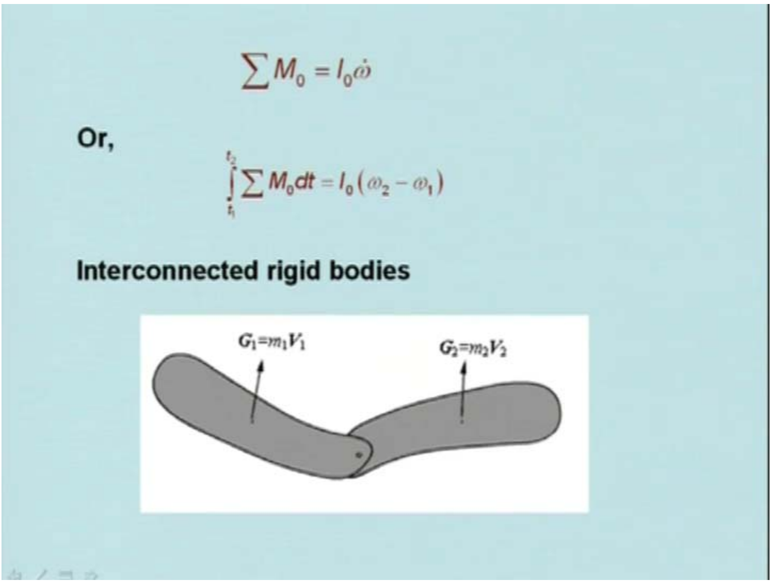

We also have sigma M_0 is I_0 omega dot. So, therefore t_1 to t_2 sigma M_0 dt is equal to I_0 $omega_2$ minus $omega_1$.

Now, we discuss about interconnected rigid bodies. Suppose two bodies are interconnected, then the linear momentum of one body can be written as G_1 is equal to m_1 V_1 and another body G_2 is equal to m_2 V_2. These may be the two links.

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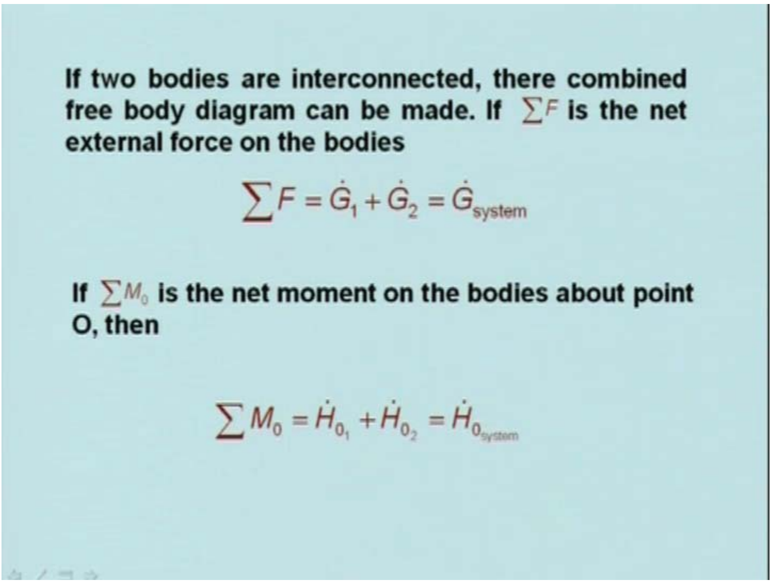

If two bodies are interconnected their combined free body diagram can be made. If sigma F is the net external force on the bodies then sigma F is equal to G_1 dot plus G_2 dot is equal to G dot system. If sigma M_0 is the net moment on the bodies about point O, then sigma M_0 is equal to H dot o_1 plus H dot o_2 is equal to H dot o system.

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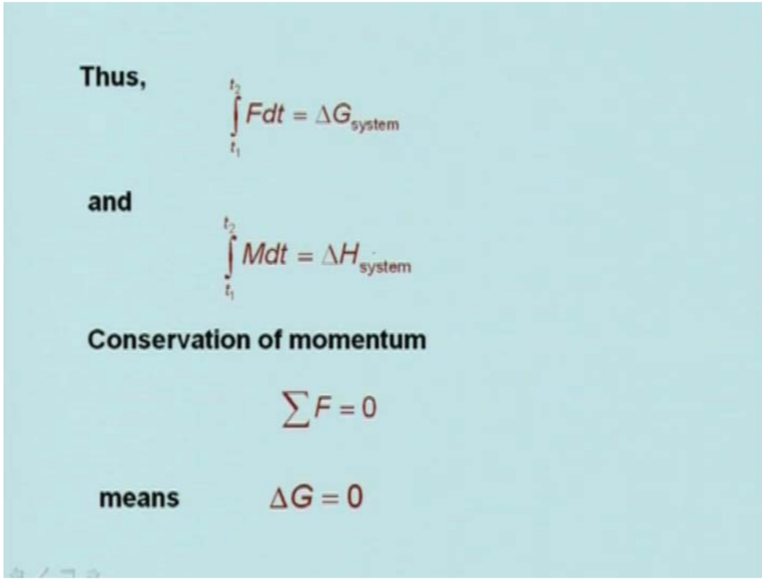

Similarly, t_1 to t_2 Fdt is equal to delta G_{system} and similarly, we can have t_1 to t_2 Mdt is equal to delta H_{system}; combined for system, we can write. Conservation of momentum applies here also, if the net external force sigma F is zero. This means that delta G is equal to 0, combined linear momentum ….. of the bodies or links connected together remains conserved although their individual momentums may change.

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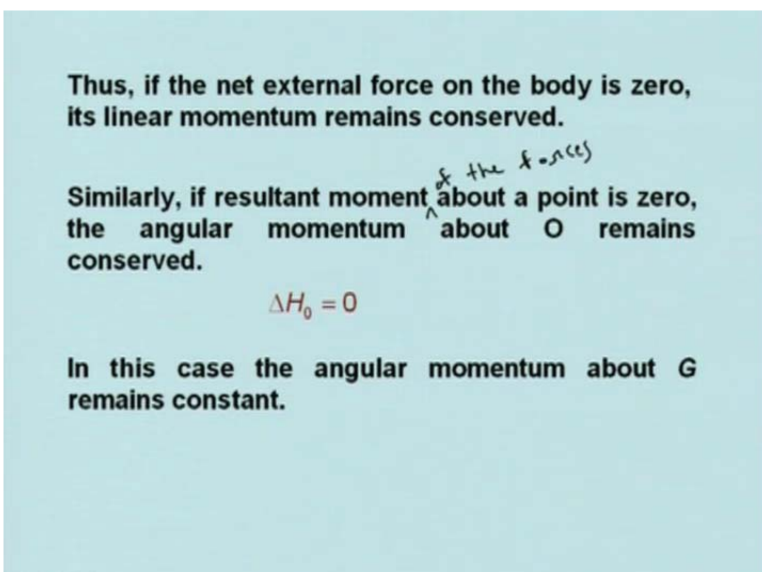

So, combined linear momentum is this one. Thus, if the net external force on the body is zero, its linear momentum remains conserved. Similarly, if resultant moment about a point is zero, the angular momentum about O remains conserved. That is, if the resultant moment of the forces about a point is 0, then the angular momentum about O remains conserved. In this case, the angular momentum about G remains constant. So, that is what you get.

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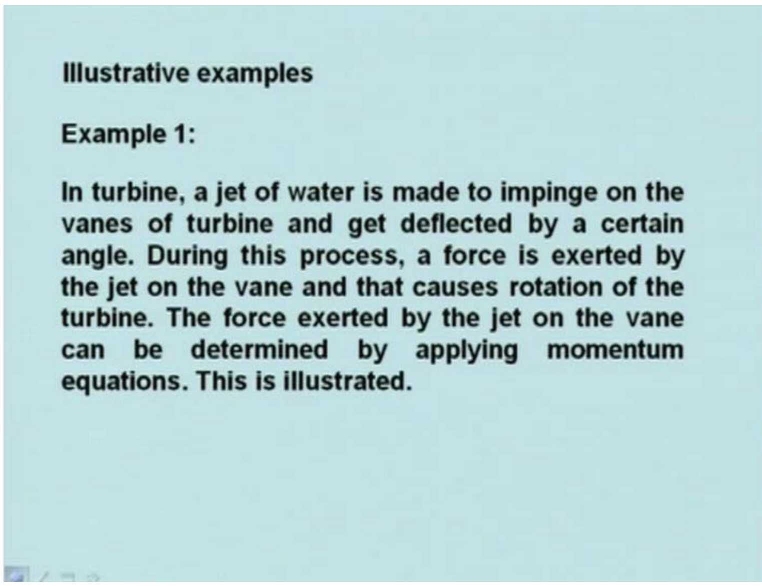

We give some examples; let us take one example of these principles. Now, let us see, one example from turbine.

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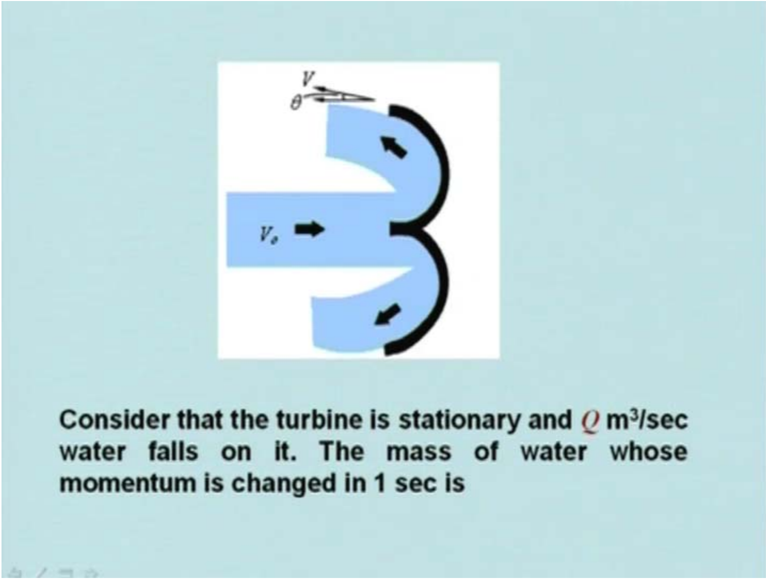

This is a turbine bucket. Suppose, in the beginning it is stationary. You have jet of water impinging with velocity V_0 and it gets deflected. The velocity remains unchanged but it gets deflected at theta, and here also, it gets deflected to theta. Then, in that case, this will be like this. In a turbine, jet of water is made to impinge on the vanes of turbine and get deflected by a certain angle. During this process, a force is exerted by the jet on the vane and that causes a rotation of the turbine. The force exerted by the jet on the vane can be determined by applying momentum equations. Now this can be done like this.

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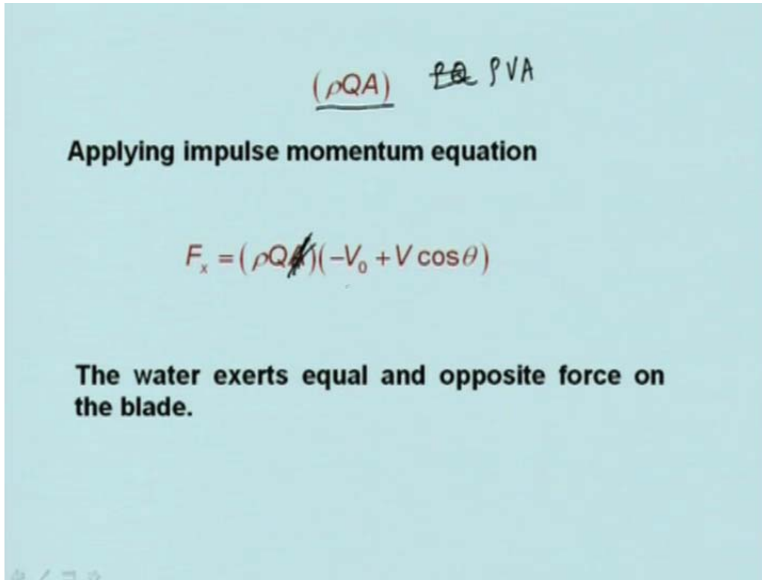

Rho QA is the rate of mass coming on that; Q is the discharge, A is the area and rho is this one. We can say Q or it should be rho times V_A. Then applying momentum equation in this equation, we have rho Q and this is minus V_0 and plus V cos theta, that is the change in the momentum. So, Vcos theta minus V_0, that is F_x. So, water exerts equal and opposite force on the blade. There is no net vertical force in this problem because of symmetry.

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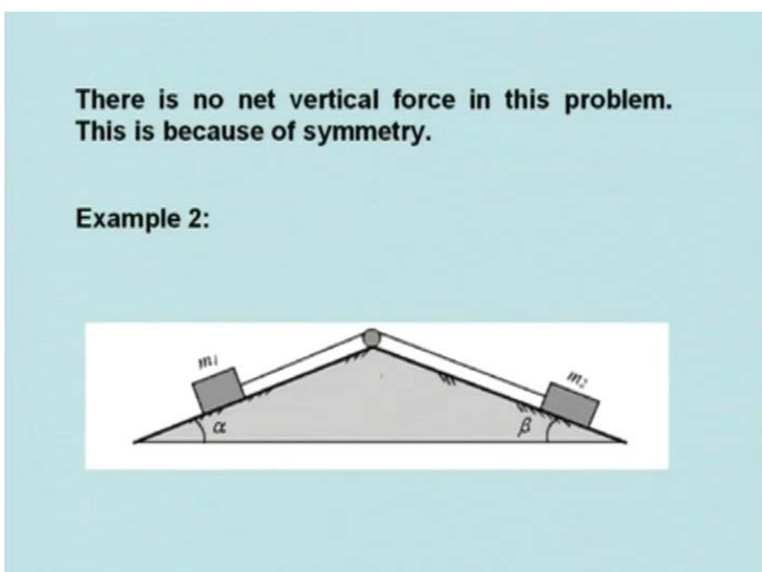

Now example 2 of interconnected bodies, suppose two bodies are acting here. This is M_1 and this is M_2 and they are put on the inclined plane. This is alpha and this is beta, angle inclined with this.

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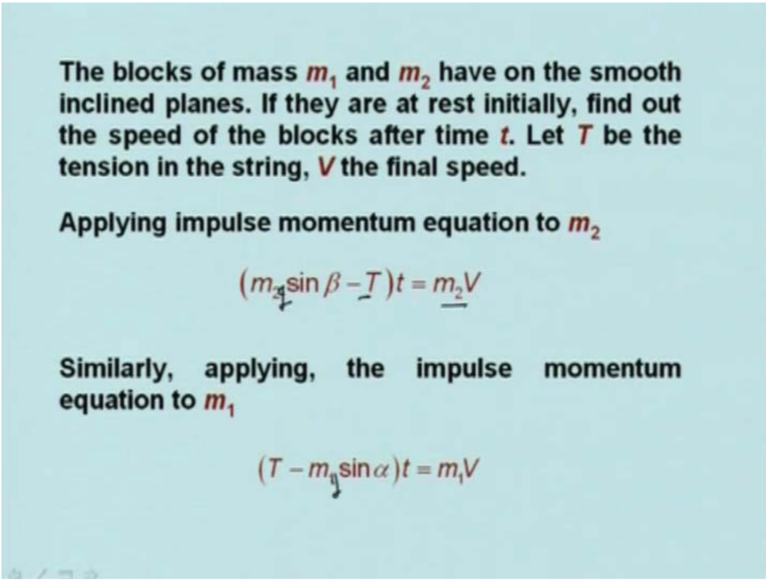

The blocks of mass m_1 and m_2 have been kept on the smooth inclined plane. There are no forces acting, external forces. If they are at rest initially, find out the speed of the blocks after time t. Let T be the tension in the string and V the final speed. So, applying impulse momentum equation to m_2, we have m_{2g} sin beta minus T into t is equal to m_2V. We have to consider this tension T also. Similarly, apply the impulse momentum equation to m_1, T minus m_1g sin alpha t is equal to m_1V.

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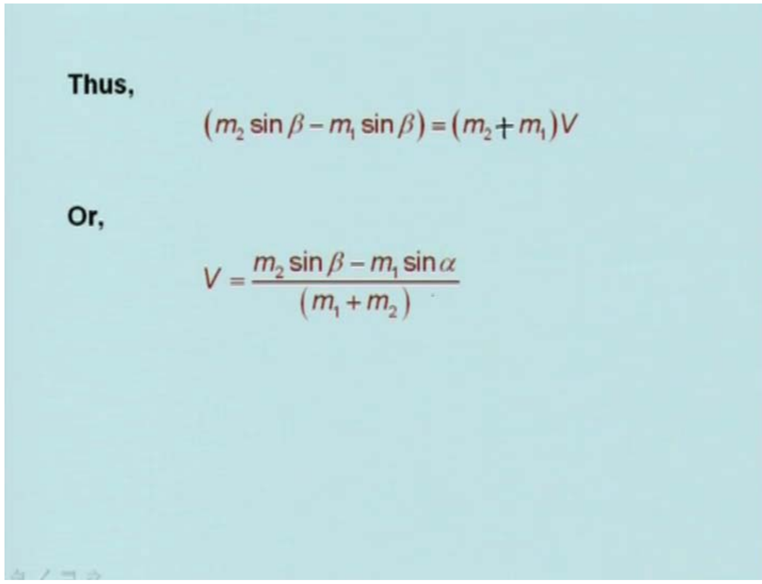

Then what we can do, we can add these two equations. So, we get m_2 sin beta minus m_1 sin beta is equal to m_2 plus m_1 into V or we get V is equal to m_2 sin beta minus $m_1$sin alpha divided by m_1 plus $m_{2.}$

This is how we have found out that the velocity of particle of masses m_1 and m_2, by applying the impulse momentum. Impulse momentum equation has provided us this. We have discussed about the kinetics of bodies and mostly, we have given the examples from the rigid motion of the bodies.

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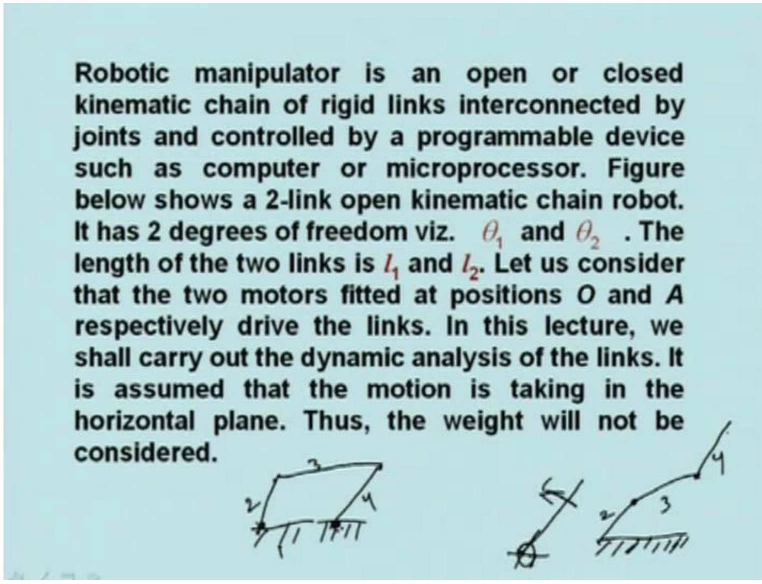

Now, most of the things we have taken are the simple examples so that you can easily solve them with hand calculations and may be with the help of calculator. But actually, the application of dynamics is very broad and you may have to do number of problems. In that, even hand calculations will not be sufficient. One may have to use computer for solving the equations and there are these things. By now, you must know how to develop the equations, for you know dynamic this one.

I have taken one example the dynamic analysis of a two link rigid robotic manipulator. There are two links which are moving on a plane. So, it contains number of very big equation. I am not going to discuss about the solution, but we will form the equations. What is a robotic manipulator? Robot consists of mostly the links. Suppose this simple link may be there.

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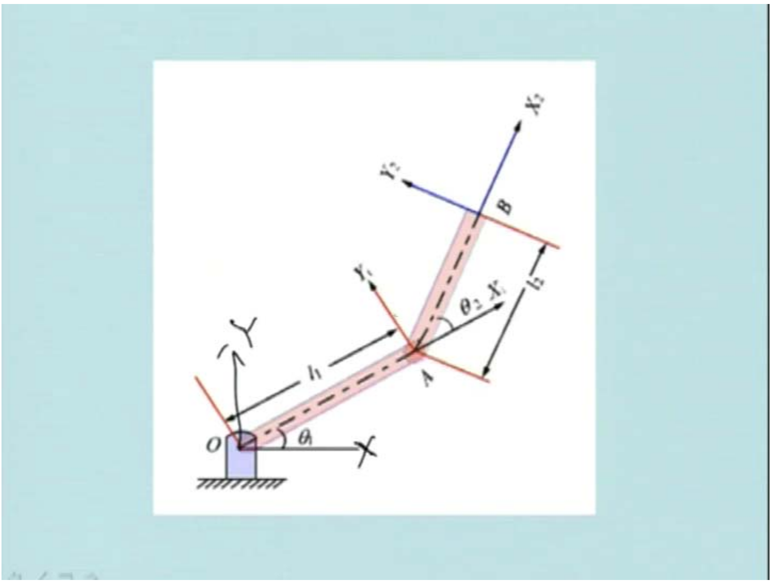

This may be some link. It rotates; when you give command it rotates. This is one link of a robot. In general, a robotic manipulator can consist of number of links. So, it may likely have kinematic chain. That is called four bar mechanism. Here, this link is fixed, another link connected by pin joint, another like this and then this is like this. This is 1,2,3,4. So, this is a four bar mechanism and you see that this chain is closed. This is a chain of links, link 1,2,3,4. Link 1 starts at one point and link 4 closes at that point. This is called closed kinematic chain and its degree of freedom is 1; that means, if we give some rotation to 2, then we can know that how much that 4 will move.

We can have open kinematic chain also. Suppose one link was fixed, but I could have connected to 2 and this becomes 3 and this becomes 4. So this is 2,3,4. In that case, it is an open kinematic, but in robots, these types of things are quite often used. This is link number 2, this is 3, this is 4 and they will have independent motion. I can put a motor, here I can put a motor here; I can put a motor at link number 3. So, this is what can be done. This has got three degrees of freedom, because three independent motions are possible. This is open kinematic chain. So, robotic manipulator is an open or closed kinematic chain of rigid links. Generally, we consider the links to be rigid. Otherwise, flexibility can also be considered; interconnected by joints and controlled by a programmable device, such as computer or microprocessor. This is just linkage; but in robotic manipulator this will be generally controlled by programmable device such as any computer or microprocessor.

We take example of just two link robotic manipulator. Only two links are there. This is open kinematic chain and it has got two degrees of freedom. You see that even this will give us complicated equations; one link is simple case. So, here, that link 1 is of length l_1 and this angle is $theta_1$ and it is pivoted about point O. It can rotate. One motor is put and link 2 is about this one. This is theta $_2$ and this is about this one X_1 and this is Y_2 and this is $l_{2.}$ So, this becomes like this one X_1 Y_1 $theta_2$ l_1 l_2. Then the length of the two links is l_1 and $l_{2.}$ Let us consider the two motors fitted at position O and A respectively, driving the links. O and A are driving the links.

In this lecture, we shall carry out the dynamic analysis of the links. It is assumed that the motion is taking in the horizontal plane. Then we will not have to consider weight. So, motion is just taking place in the horizontal plane; that means there is a motor fitted at O and there is a motor on A also, which is driving this thing. So, that way it can be done.
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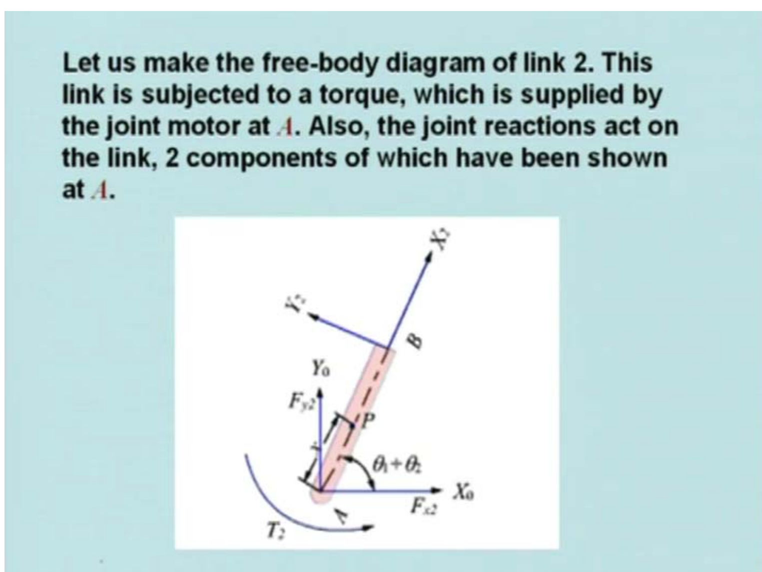

We can make the free body diagram of link 2. Now, this is the link 2. Here, I have shown the axis system also. One fixed axis is like this, X and Y here. Then you have X_1 Y_1 that is X_1 is along the link 1 and Y_1 is perpendicular to that one. So it is a moveable coordinate system that is attached at point 1 and similarly X_2 Y_2 is like this; that along this and this is this thing.

Now, make the free body diagram of link 2. This link is subjected to torque which is supplied by the joint motor at A. The joint reactions act on the link. We have shown that some effects on F_y component, but you know that effects 2 and Y_2, but we are not going to determine this. It is not necessary to determine the joint reactions, but I have shown here. Also the joint reactions act on the link. Therefore, joint reactions are also there and they have been shown just like this.

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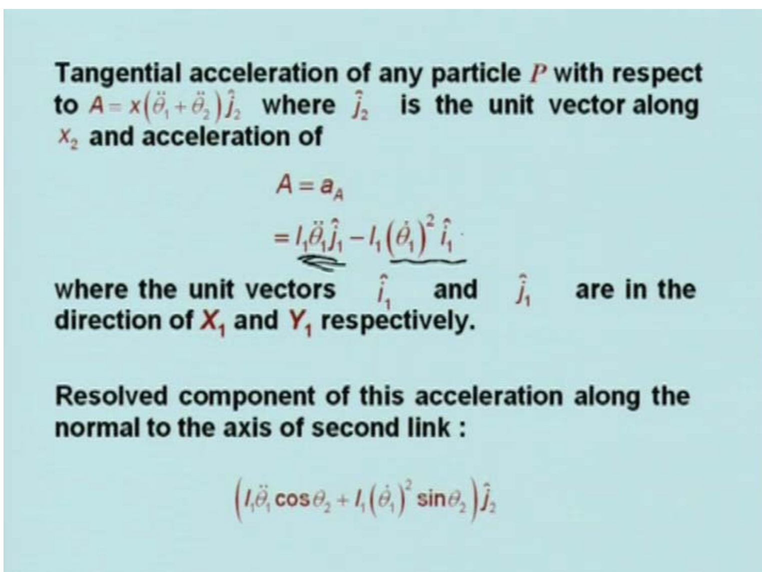

Here, tangential acceleration of any particle P with respect to A is x $theta_1$ dot dot plus $theta_2$ dot dot j_2, where j_2 is the unit vector along x_2. Acceleration of A is basically given that is a_A that is l_1 $theta_1$ dot dot j_1 that is tangential acceleration minus l_1 theta dot l whole square i_1 that is centripetal acceleration, where the unit vectors i_1 and j_1 are in the direction of X_1 and Y_1 respectively. Resolved component of this acceleration along the normal to the axis of the second link are like this; you resolve these components l_1 theta$_1$ dot dot cos theta$_2$ plus l_1 theta$_1$ theta$_2$ into j_2. You get like this. So, you have obtained the resolved component along normal to the axis because that will be.

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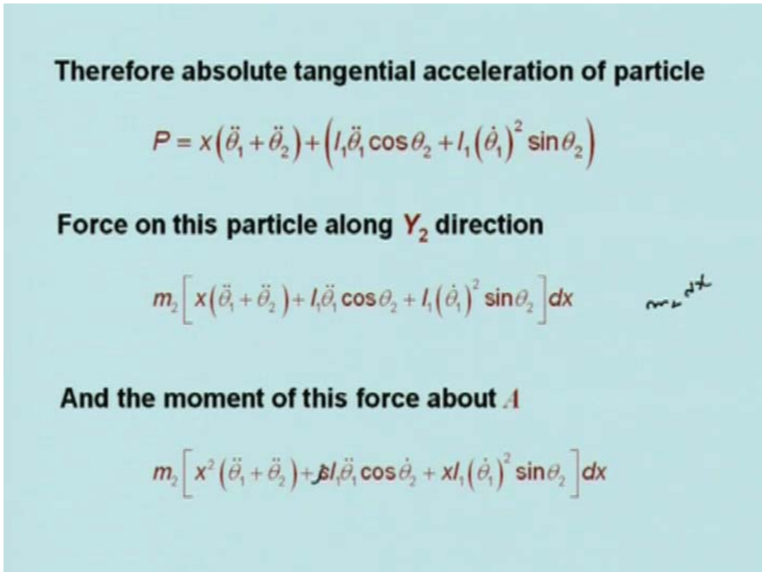

Therefore, absolute tangential acceleration of particle becomes x theta$_1$dot dot plus theta$_2$ dot plus l_1 theta 1_1 dot dot cos theta$_2$ plus l_1 theta 1 dot square sin theta$_2$. We have to do this because, Newton's law can be applied in inertial frame of reference and therefore, we have to talk about the absolute acceleration. Now, force on this particle along Y_2 direction is this and one single particle m_2 x theta$_1$ dot dot plus theta$_2$ dot dot plus l_1 theta$_1$ dot dot cos theta$_2$ plus l_1 theta$_1$ dot square sin theta$_2$ into dx, because the mass of a small element at distance x is m_2 into dx. So, m_2 dx is the small mass, m_2 is the mass per unit length or it can be a function of per unit length, but it can still be a function of x. The moment of this force about A is basically you multiplied by x perpendicular distance. It becomes m_2 x square theta$_1$dot dot plus theta$_2$ dot dot plus this will be l_1 x times this is x times l_1 theta$_1$dot dot cos theta$_2$ dot plus x times l_1 theta$_1$ dot square sin theta$_2$ into dx.

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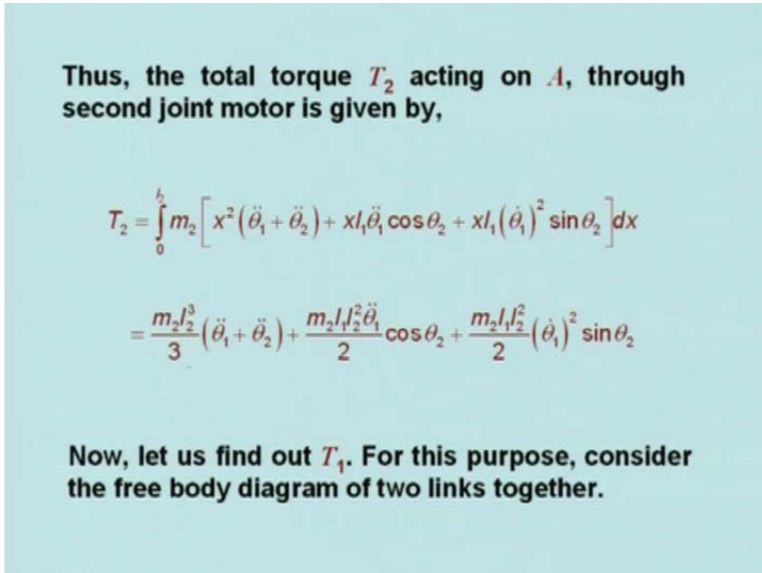

Thus, the total torque T_2 acting on A through second joint motor is given by T_2 is equal to m_2 x square $theta_1$ dot dot plus $theta_2$ square plus xl_1 $theta_1$ dot dot cos $theta_2$ plus xl_l $theta_1$ dot square sin $theta_2$ dx. This we have to integrate from 0 to 1_2. If we integrate this from 0 to l_2, we get by integrating m_2 l_2 square l_2 cube by 3 plus $theta_1$ dot dot plus $theta_2$ dot dot plus m_2 1_l l_2 square $theta_1$ double dot by 2 cos $theta_2$. The third term is m_2 1_l l_2 square by 2 $theta_1$ dot whole square sin $theta_2$. This is the expression for the torque coming on link 2. You see that this contains not only this term m_2 l_2 square, but also it will depend on the other terms like acceleration of this and all these type of things. So we are getting these things. Here you are getting only the mass of link 2; you are not getting the mass of link 1.

Now, let us try to find out T_1, that means torque for link 1. For this purpose, consider the free body diagram of two links together.

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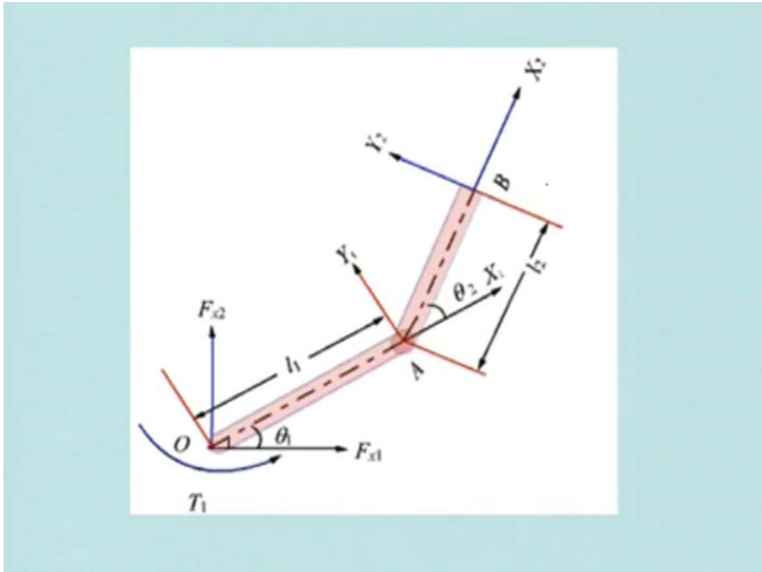

We are considering the free body diagram of the links together. We could have drawn the free body diagram of only the link 1. In that case, the joint reactions will be coming at join A. Those are unknown to us, so we do not want to find out. That is why we are doing like this and in this case, you are showing that joint reactions F_{x1} F_{x2} here and then torque T_1 acting.

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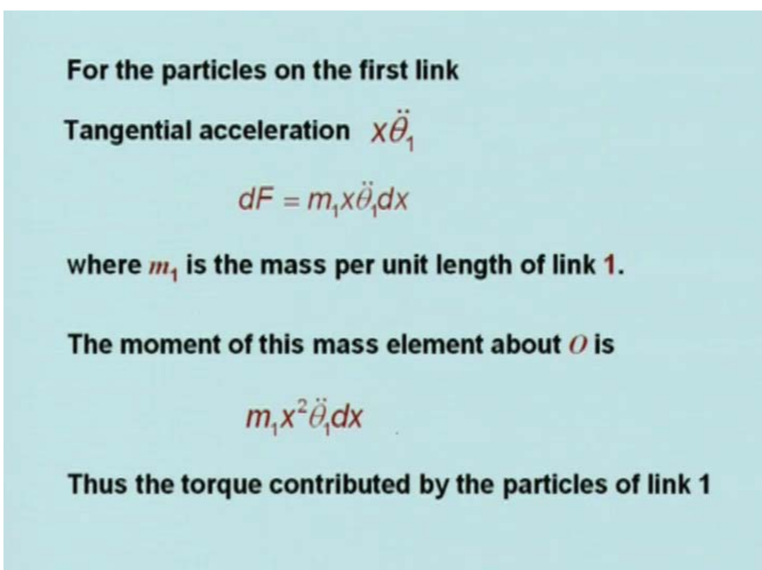

What happens, when we consider it together, we have to consider not only the accelerations of link, 1 but you have to consider the accelerations of link 2 also. For the particles on the first link, tangential acceleration is X times theta$_1$ dot dot and that is dF is equal to m_1 X theta$_1$ dot dot dx where m is the mass per unit length of link 1. The moment of this mass element about O is m_1 X square theta$_1$ dot dot dx. Thus, the torque contributed by the particles of link 1 is basically $T_1$1 particles; 0 to $l_{1,}$ m_1 x square theta$_1$ dot dot dx.

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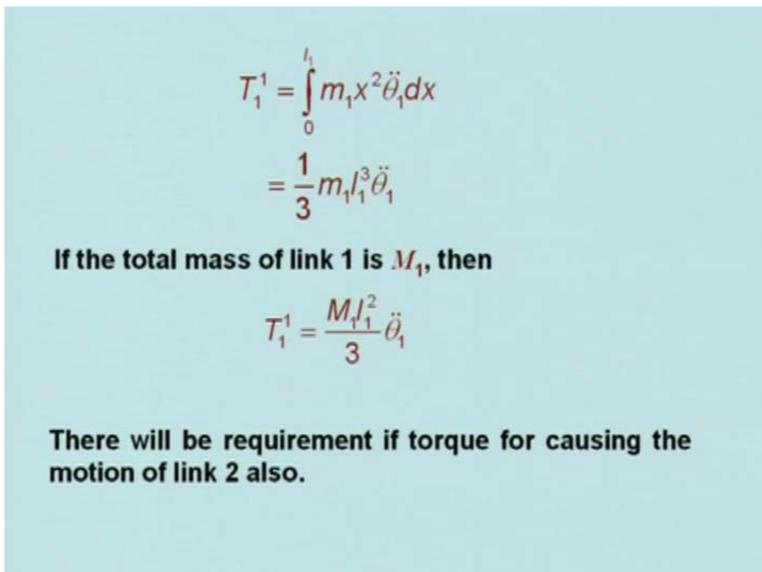

If you integrate between 0 to $l_{1,}$ you will get integration. If you put the limit, 1 by 3 m_1 l_1 cube theta$_1$ dot dot. If the total mass of link 1 is $M_{1,}$ then $T_1$1 is equal to $M_1 l_1$ square by 3 and this becomes theta$_1$ dot dot. So, this is a very nice expression that means, basically theta$_1$ dot dot is equal to alpha. That means T is equal to y alpha, had only the first link been present; other link was not there then matter was finished. Because this is the torque contributed by the particles of first link, motor will supply that torque. However there will be requirement of torque for causing the motion of link 2 also. That also has to be considered and that portion is little complicated to find out, but we will just discuss about that also.

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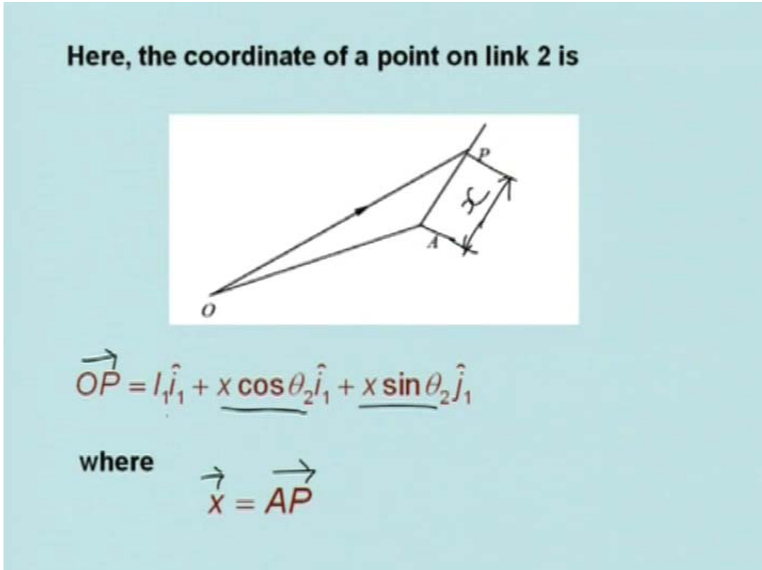

Now, about the point O, the coordinate of point on link 2 is suppose P, so position vector OP is l_1 i_1 plus x cos theta$_2$ i_1 plus x sin theta$_2$ j_1, where basically x is equal to AP. This is x. In this figure, this is x. We want to find out the position vector of point P that means position of point P with respect to O. So if we differentiate this position vector twice, we will get the absolute acceleration of point P. That is what we need.

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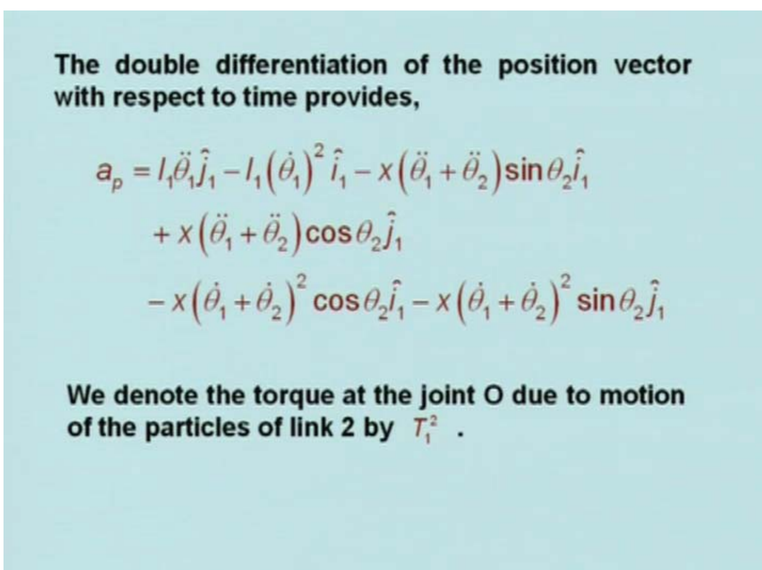

We differentiate it two times and in differentiation, we have to understand that if we differentiate even this one, i_1 with respect to time, because it is rotating it has to be written that $theta_1$ dot j like that. We have to follow the rules of differentiation. We know i dot was that. If you differentiate unit vector i dot like this, then that is omega into j and j dot is equal to minus omega i. These types of things we have to remember and then this differentiation will give this expression; AP is equal to l_1 $theta_1$ dot dot j_1 minus l_1 $theta_1$ dot square i_1 minus x $theta_1$ dot dot plus $theta_2$ dot dot sin $theta_2$ i_1 plus x $theta_1$dot dot plus $theta_2$ dot dot cos $theta_2$ j_1 minus x $theta_1$ dot plus $theta_2$ dot whole square cos $theta_2$ i_1 minus x $theta_1$ dot plus $theta_2$ dot whole square then sin $theta_2$ j_1.

Now, this is the expression we have got. We denote the torque at the joint O due to the motion of the particles of link 2 by T_1 2.

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$$T_1^2 = r \times F$$

$$= \int_0^{l_2} m_2 dx \left[l_1^2 \ddot{\theta}_1 + l_1 x \left(\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos\theta_2 - l_1 x \left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin\theta_2 + l_1 x \cos\theta_2 \ddot{\theta}_1 \right]$$

$$+ x^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos^2\theta_2 - x^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \sin\theta_2 \cos\theta_2 + x l_1 \left(\dot{\theta}_1\right)\sin\theta_2$$

$$+ x^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right)\sin^2\theta_2 + x^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \sin\theta_2 \cos\theta_2.$$

$$= \int_0^{l_2} m_2 \Big[l_1^2 \ddot{\theta}_1 + l_1 x \left(\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos\theta_2 - l_1 x \dot{\theta}_2^2 \sin\theta_2 - 2 l_1 x \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$$

$$+ l_1 x \cos\theta_2 \ddot{\theta}_1 + x^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \Big] dx$$

Therefore, T_1 2 is equal to r cross F, cross product and F will be basically m_2 dx and this thing and we have to multiply it by cross product F. You will get these expressions; m_2 dx l_1 square $theta_1$ dot dot plus l_1 into x $theta_1$ dot dot $theta_2$ dot dot cos $theta_2$ minus l_1 x $theta_1$dot plus $theta_2$ sin theta l_1 x cos $theta_2$ $theta_1$dot dot plus x square $theta_1$ dot dot plus theta$_2$ dot dot cos square theta$_2$ minus x square theta$_1$ dot plus theta$_2$ dot square. Then, sin theta$_2$ cos theta$_2$ plus x l_1 theta$_1$ dot sin theta$_2$ plus x square theta$_1$ dot dot theta$_2$ square sin square theta$_2$ plus x square theta$_1$ dot plus theta$_2$dot whole square sin theta$_2$ cos theta$_2$ which can be written as 0 to l_2 this is m_2 $l_1$square theta$_1$ dot dot plus l_1 cos theta$_1$ dot dot plus theta$_2$ dot dot cos theta$_2$ minus l_1 x theta $_2$ dot dot theta $_2$ dot square sin theta$_2$ minus 2 l_1 theta$_1$ dot theta$_2$ dot sin theta$_2$. This is l_1 x cos theta$_2$ theta$_1$ dot plus x square theta$_1$ dot dot plus theta$_2$ dot dot dx.

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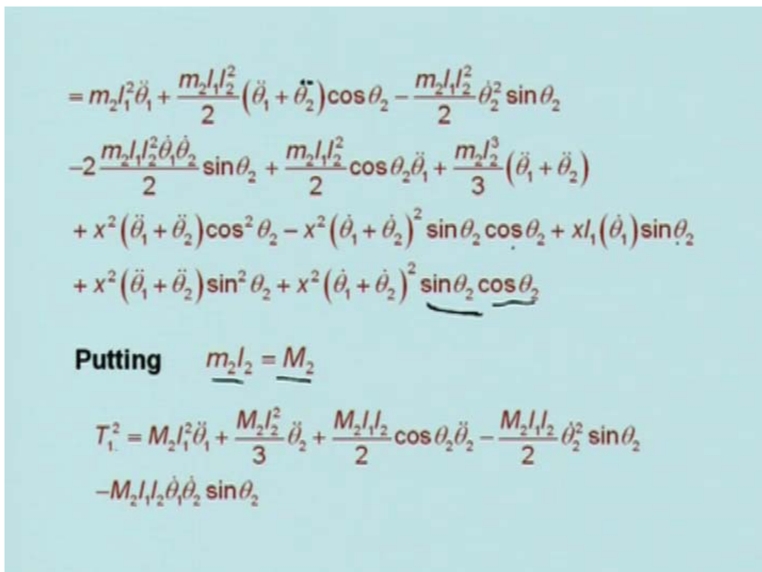

Now, if we do the integration, 0 to l_2, particles are 0 to l_2, we get this expression M_2 $l_1$square theta$_1$ dot plus M_2 l_1 l_2 square by 2 theta$_1$ dot dot plus theta$_2$ dot dot cos theta$_2$ minus M_2 l_1 l_2 square by 2 theta$_2$ dot square sin theta$_2$ minus 2 M_2 $l_1 l_2$ square theta$_1$ dot theta$_2$ by 2 sin theta$_2$ plus M_2 l_1 l_2 square 2 by cos theta$_2$ theta$_1$ dot dot plus M_2 l_2 cube by 3 theta$_1$dot dot plus theta$_2$ dot dot plus x square theta$_1$ dot dot plus theta$_2$ dot dot cos square theta$_2$ minus x square theta$_1$dot plus theta$_2$ dot square sin theta$_2$ cos theta$_2$ plus x l_1 theta$_1$ dot sin theta$_2$ plus x square theta$_1$ dot dot plus theta$_2$ dot dot sin square theta$_2$ plus x square theta$_1$dot plus theta$_2$ dot whole square sin theta$_2$ cos theta$_2$. We are getting very big expressions, lengthy expressions; you must check it by calculations. When you do such big expressions, there are chances of mistakes. That is why, these days most of these integrations are carried out in computer. One can do the numerical integration.

Now-a-days softwares are there which can obtain the close form integrations also, like Matlab etc; in that you can do. Otherwise numerical integration is preferred, but I thought of showing at least one example of developing the dynamics equation. So, I have taken this. Now, you must verify these things. Put m_2 l_2 is equal to $M_{2.}$ m_2l_2 is the total mass M_2, then you have T_1 2 that means torque required at joint O, because of the motion of link 2 that is M_2 l_1 square theta$_1$dot plus M_2l_2 square by 3 theta$_2$ dot dot plus M_2 l_1 l_2 by 2 cos theta$_2$ theta$_2$ dot dot minus M_2 l_1 l_2 by 2 theta$_2$ dot square sin theta$_2$ minus M_2 l_1 l_2 theta$_1$ dot theta$_2$ dot sin theta$_{2.}$

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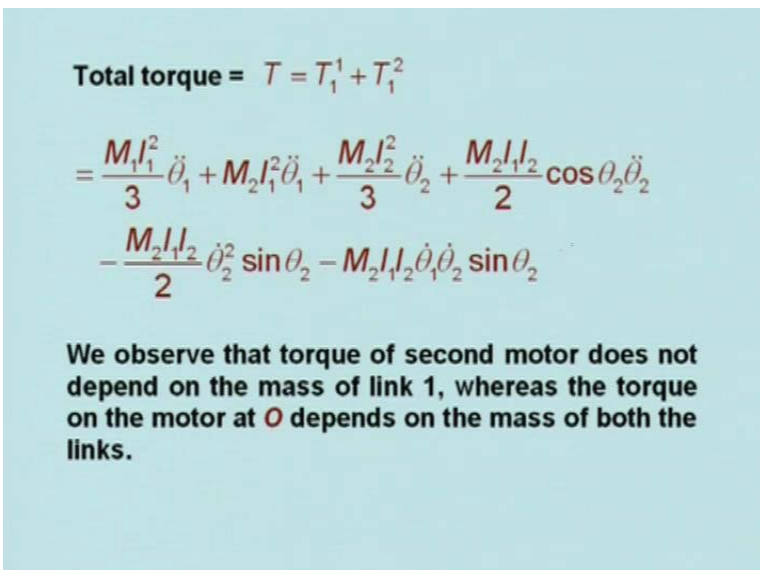

The total torque of this one required at point O, joint O of this one comes out to be T is equal to T_1 1 plus T_1 2 that is $M_1$by 3 l_1 square theta$_1$dot dot plus M_2 l_1 square theta$_1$dot plus M_2 l_2 square by 3 theta$_2$ dot dot plus M_2 l_1 l_2 by 2 cos theta$_2$ theta$_2$ dot dot minus M_2 l_1l_2 by 2 theta$_2$ dot sin theta$_2$ minus M_2 l_1 l_2 theta$_1$dot theta$_2$ dot sin theta$_2$. We see that the torque required is …… by both the motors. We get the equations which contain the square terms also. Nonlinear terms are there. We have theta$_2$ dot square and all these things. So, we get the expression for total torque. We observe that torque of second motor does not depend on the mass of link 1, whereas the torque on the motor at O, depends on the masses of both the links. Therefore, these are the expressions for the torques and these equations can be solved.

For any given configuration solving this type of equation, you may have to use numerical method. One can be direct problem in which somebody may give you $theta_1$ as a function of time, $theta_2$ as a function of time then it is easy. You can just put that, differentiate and get the value of that. In some cases, you may have been prescribed the torque and you have to find out the angular displacements. In that case, the inverse procedure has to be done. So, you have to do some sort of integration because this then becomes a differential equation which has to be solved. Numerical methods can also be used. This is the thing that you have to do.

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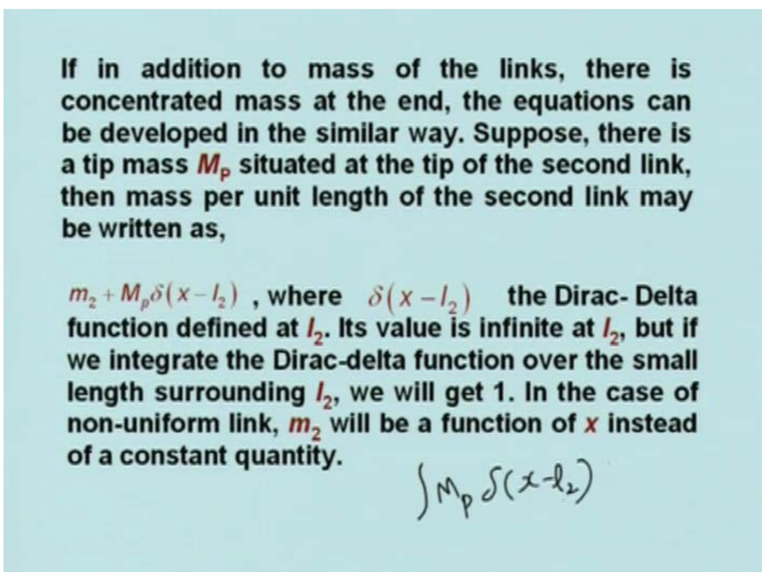

Here we have done simplification in integrations. We have been taking m_1 etc., outside. If it was not so, if m_1 was a function of x, that means i can take out rod and this thing. then, we have to put inside that and some cases we can integrate given m as function of x. In some cases we have to use the numerical methods only. If in addition to the mass of the links, there is a concentrated mass at the end then what should be done? Then equations can be developed in the similar way. If you have got concentrated mass then suppose there is tip mass for example $M_{p,}$ which is situated at the tip of the second link, then same type of expressions can be developed, but the mass per unit length of the second link may be written as m_2 is equal to m_2 plus M_p delta x minus $l_{2,}$ where M_p is the peak mass and m_2 is the mass per unit length. m_2 has got the unit of kg per meter, whereas M_p has got the unit of kg but it has been multiplied by delta x minus l_2, where delta x minus l_2 is the Dirac-delta function which is defined at $l_{2.}$ Dirac-Delta function is at $l_{2,}$ whose value approaches infinity and V, where velocity is 0. If you integrate Dirac-delta function between limit 0 to $l_{2,}$ then you will be getting the value as 1 only. So, Dirac-delta function has got the unit of 1 by meter.

So, its values are finite at l_2 but if we integrate the Dirac-Delta function over the small length surrounding $l_{2,}$ we will get 1. In the case of non-uniform link, m_2 will be a function of x instead of a constant quantity. Therefore, in the integral sign, we can put and that causes no problem. Suppose you have the Dirac-Delta function, we will have the property. Suppose you get some term like M_p, then delta x minus l_2 multiplied by something like x dot dx. So, it will become M_p the value of that function at l_2. That means, if you have a function F(x) which is multiplied by delta x minus $l_{2,}$ this is nothing but the value of the function of at $l_{2.}$ Therefore, that is $H_{G.}$ In the case of non-uniform link, m_2 will be a function of x instead of a constant quantity.

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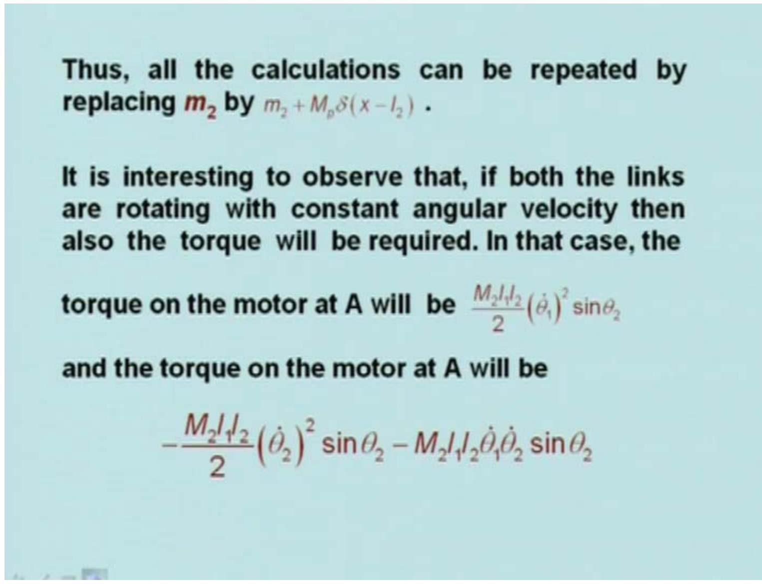

Thus, all the calculations can be repeated by replacing m_2 by m_2 plus M_p delta x minus l_2. It is interesting to observe that, if both the links are rotating with constant angular velocity then also the torque will be required. Suppose both are moving in a single link, it is not required; but in this case it is required. In that case, the torque on the motor at A will be M_2 l_1 l_2 by 2 theta_1dot square sin theta_2 because of the angular velocity of the second link. If the second link is stationary then, that torque will not be same. First link is stationary then the torque will not be required. If the first link is not stationary then this will be required and sin theta_2 and the torque on the motor at A will be minus M_2 l_1 $l_2$by 2 theta_2 dot square sin theta_2 minus M_2 l_1, l_2 theta_1dot theta_2 dot sin theta_2.

We see that although links are moving with constant velocity, still that torque is required because of the second link. We take the torque in inertial system. Now, if m_2 is 0 then there will not be any requirement of the torque.

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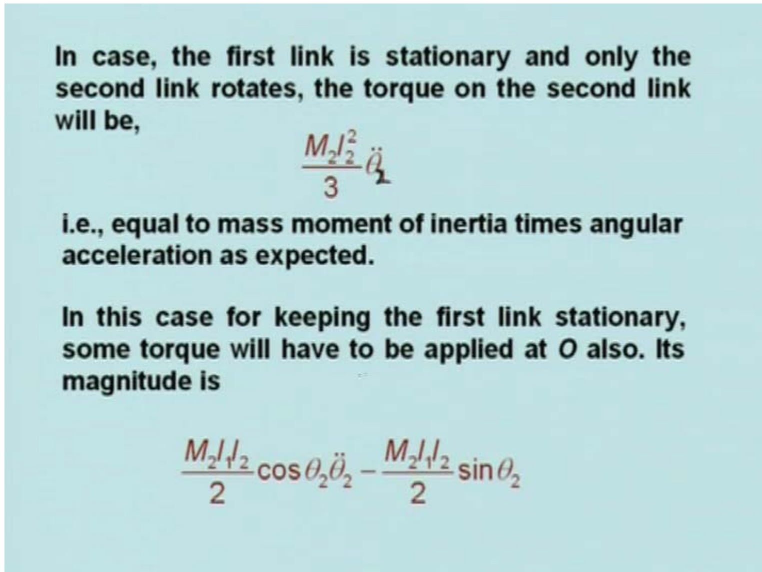

In case the first link is stationary and only the second link rotates, the torque on the second link will be M_2 l_2 square by 3 theta_1 dot dot; that is theta_2 dot dot, basically. That is equal to the mass moment of inertia times the angular acceleration as expected. In this case for keeping the first link stationary, some torque will have to be applied at O also and its magnitude is M_2 l_1 l_2 by 2 cos theta_2 dot dot minus M_1 l_1 l_2 by 2 sin theta_2. This way, by applying only the basic principles of the dynamics, we can solve these typesof problems. Lot of expression have been developed, but if we apply only the Newton’s law to particles and remember the fact that these laws are valid in inertial reference system only, then we can develop the equations for any type of problem.

We have been doing in the previous lectures, basically this. We have been developing other expressions like for rigid body, but basically we apply the equations to particle then we can always get the expressions. It may be simple rod rotating or it may be that robotic manipulator. In that case also, these type of things can be applied. So, that was my purpose of showing this example.