Engineering mechanics
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Kinetics-1

Module 11 Lecture - 28
Work and Energy

We have studied dynamics of a particle. For solving the dynamics problem we used Newton's law. Particularly in solving numerical problems, Newton's second law was used. In place of Newton's second law, we can also use D'Alembert's principle. D'Alembert's principle is not much different from Newton's law. It basically considers minus m into a, equal to a force that is called inertia force. Thus, a problem of dynamics can be converted into a problem of a statics, but essentially, both the approaches are making use of the equation F is equal to mass times acceleration, where F is the net resultant force acting on the body and m is the mass, a is the acceleration.

Here, one point has to be noted that the acceleration has to be measured from an inertial frame of reference. An inertial frame of reference will be moving with a constant velocity including zero. It may be moving with a zero velocity with respect to a primary inertial system. A primary inertial frame of reference is that frame of reference which is at rest. Thus, we see the D'Alembert's principle and Newton's second law essentially are the same. Today, we discuss a different approach of solving the dynamics problem that is work and energy.

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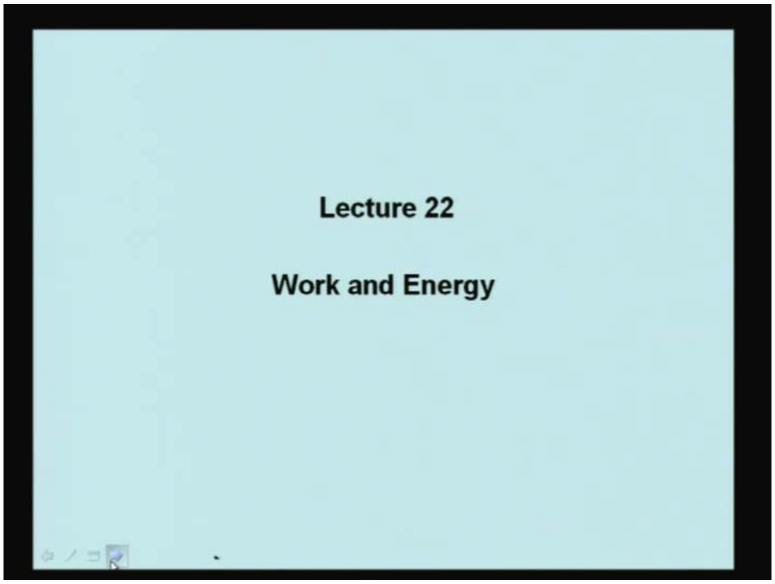

You will see that this formulation is quite different although it can be derived from Newton's second law. In one kind of energy, we will not talk about the same type of things as we talk in Newton's second law. In Newton's second law, one requires acceleration. Here, another approach maybe there. When you require only the measurement of velocities and force and distance, then also the same type of solution can come.

In case, if you have to just find out the velocity after the application of force at some distance then instead of going by Newton's second law, we can directly use work and energy. The Newton's second law provides a differential equation; that is F is equal to m d square r by dt square. Here, you will get a type of integral equation which can be solved in a different manner. Although, Newton's second law and work and energy approach follow different procedures, they are basically derivable from each other. I will explain the basic concepts.

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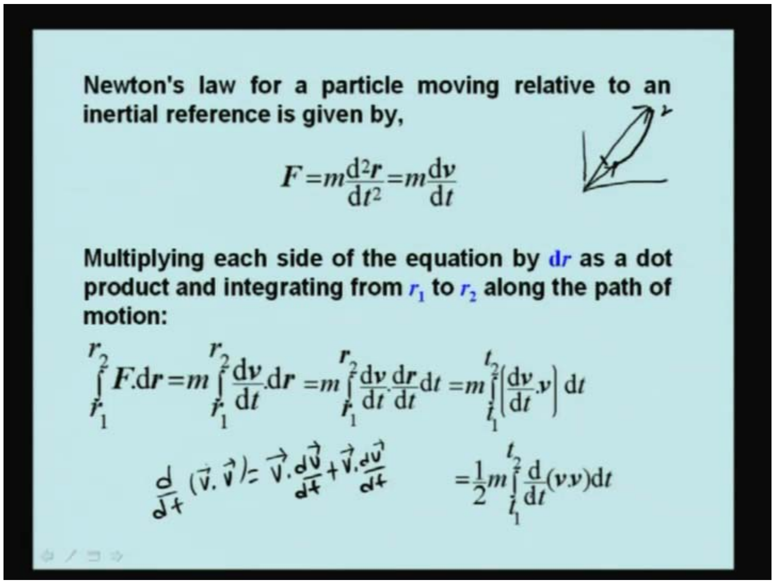

Newton’s law for a particle moving relative to an inertial reference frame is given by F is equal to m d square r by dt square, which is equal to mdv by dt, where r is a displacement vector and v is the velocity vector. Here, I am considering the mass of the particle to be constant. If we multiply each side of the equation by a vector dr as a dot product and integrate from r_1 to r_2 along the path of motion, the particle may be moving from r_1 to r_2.

This is 1 2. Position vector of 1 is $r_{1;}$ position vector of 2 is $r_{2.}$ We can integrate. If dot product of F and dr from r_1 to r_2, which gives us r_1 r_2 F dot dr is equal to m r_1 to r_2 dv by dt dot dr. I have taken m outside the integral sign, considering mass to be constant. Therefore, this has become m r_1 r_2 dv by dt dr by dt into dt is equal to m t_1 to t_2 dv by dt dot v, where dr by dt has been written as v.

This is equal to half m t_1 to t_2 d by dt v dot v dt, where the product r of vector calculus has been used; that is d by dt v dot v is equal to v dot dv by dt plus v dot dv by dt, which gives you half m t_1 to t_2 d by dt v dot v dt.

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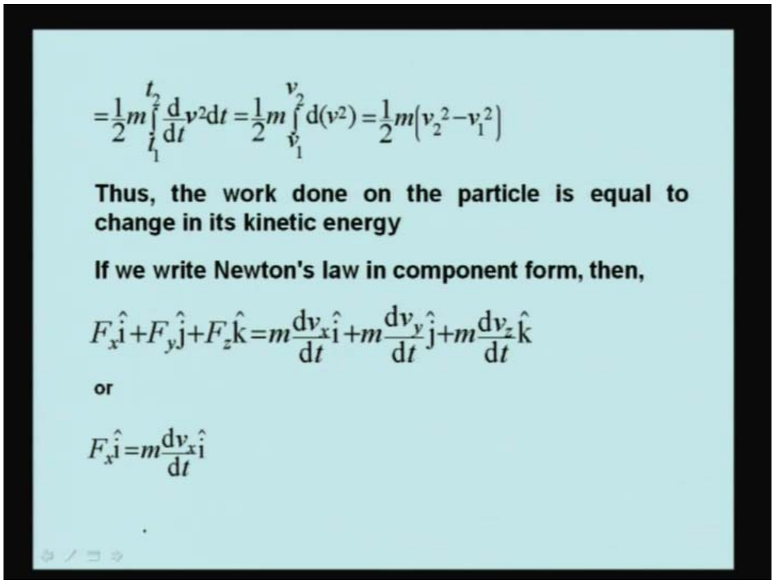

This expression becomes half m t_1 to t_2 d by dt v square dt. See the dot product v dot v is a scalar that is v square which is the magnitude squared of velocity. This becomes half m t_1 t_2 d by dt v square dt is equal to half m v_1 v_2 d v square. That gives, here I have changed the limits of the integration, because the variable dt has vanished from here. This becomes half m v_2 square minus v_1 square. Thus, the work done on the particle is equal to change in its kinetic energy.

Therefore, this essentially has come from Newton’s second law. Therefore, instead of Newton’s second law, one can use this theorem that work done on the particle is equal to change in its kinetic energy. If you know the force acting on the particle and you know the displacement vector, then you can find out the work done. You can equate it to change in the kinetic energy. Thus, you can find out the final velocity, if you know the initial velocity.

We have to see that is it possible for us to find out, if instead of magnitude, can we also find out the components, well just components of the velocity. If we write Newton’s law in component form, then F_x i plus F_y j plus F_z k is equal to m dv_x by dt i plus m dv_y by dt j plus m dv_z by dt k. This Newton’s law has been written in the component form. F_x I is therefore, F_x i is equal to m dv_x by dt i.

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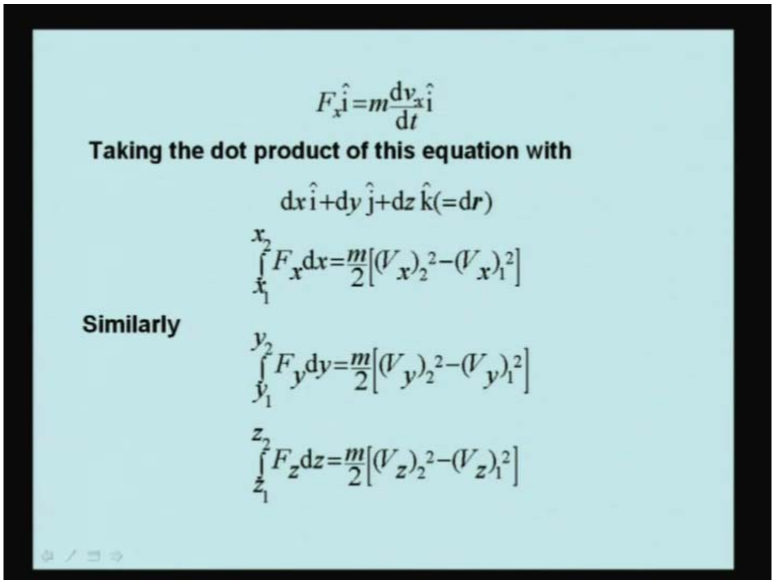

Thus, if we take the dot product of this equation with d_x i plus d_y j plus d_z k is equal to dr, if we take the dot product of equation not the dot product of F with dr, but dot product of F_x i is equal to m dv_x by dt I with this dr, then we will be getting that x_1 x_2, F_x, d_x will be equal to m dv_x by dt into d_x , which, when it will be integrated, it will provide us m by 2 V_{x2} square minus V_{x1} square. Similarly, we can also get y_1 y_2 F_y dy is equal to m by 2 V_{y2} square minus V_{y1} square. Similarly, z_1 z_2, F_z d_z is equal to m by 2 V_{z2} square minus V_{z1} square. Thus, the Newton's law is also valid in the component form.

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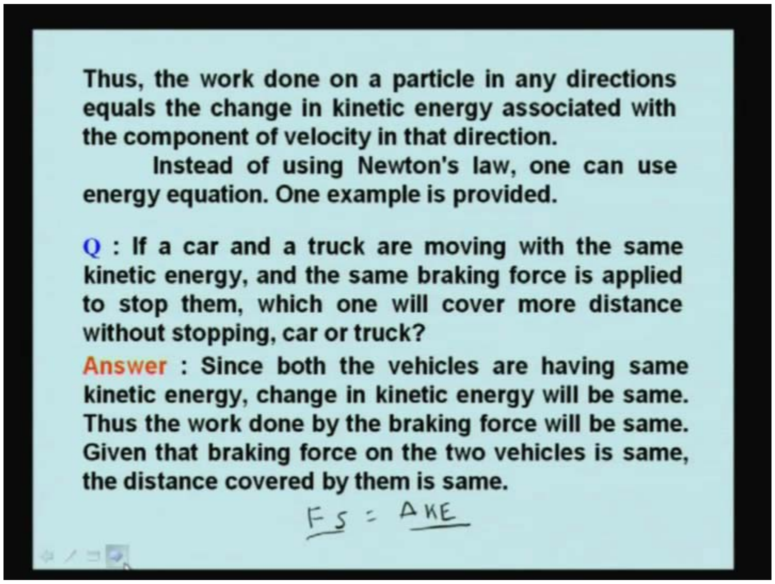

Thus, the work done on a particle in any directions equals the change in kinetic energy associated with the component of velocity in that direction. This is very important because we can find out the velocity components by using the work and kinetic energy equation. One example is provided. If a car and a truck are moving with the same kinetic energy and the same breaking force is applied to stop them, which one will cover more distance without stopping, car or truck?

In this case, if we want to apply Newton's law, we have to do many arithmetic computations. We have to find out the acceleration of the truck and cars. Even after that, we have to apply equation of motion in order to find out the result, but in this case, we can quickly find out the answer by applying the work kinetic energy theorem. Since, both the vehicles are having same kinetic energy, change in kinetic energy will be same. Thus, the work done by the breaking force will be same. Given that breaking force on the two vehicles is same, the distance covered by them is same.

Here, you get a simple relation that F is the breaking force. F into s is equal to change in kinetic energy delta KE. Since change in the kinetic energy is same, breaking force comes out to be same. This one, this shows that how efficiently we apply to work energy theorem in many cases.

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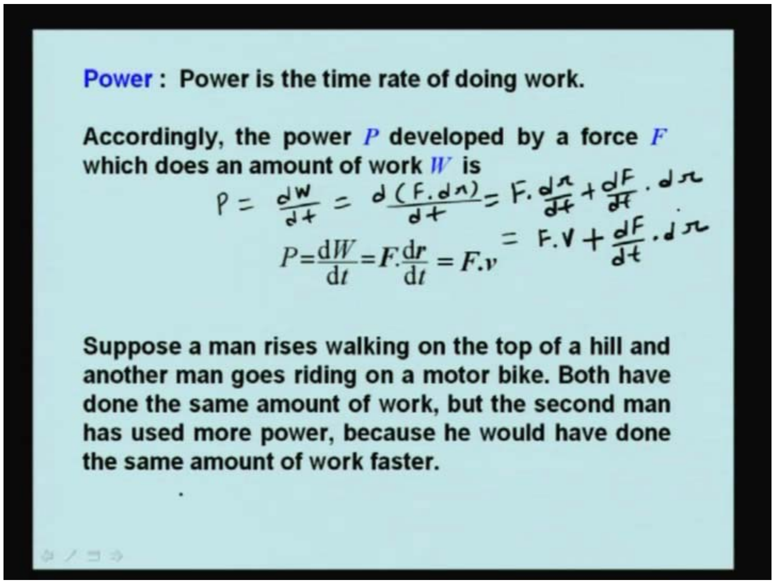

We will discuss something more about these things, but let us go to another definition. We define power is the time rate of doing work. Accordingly, the power P developed by a force F which does an amount of work W, is P is equal to dW by dt; that means, this is equal to F dr by dt that is equal to F dot v. Here, in this case, we have obtained this expression in the following manner; P is equal to dW by dt. dW by dt is dF dot dr by dt, we take F as a constant, which does not depend on time. Take it out and then you get F dot dr by dt.

In case F is dependent on time, then this can be written as F dot dr by dt plus dF by dt dot dr. In this expression, we get additional term. F dot dr by dt surely can be written as F dot v. However, we will get extra term; that is dF by dt dot dr. If F is changing with time, then dF by dt must also be evaluated and multiplied with dr. However, if we consider the dr to be very small, then in that case, this term becomes insignificant.

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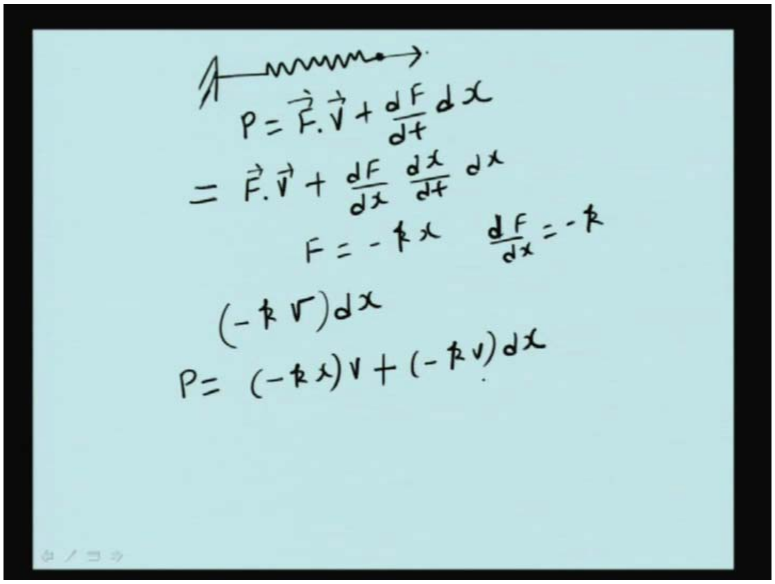

We give one example, where the pole is changing with time or distance. For example, consider a spring. If particle attached with spring is moved, then the spring force is dependent on the extension of the spring. In this case, P will be equal to F times dot V plus dF by dt dot dr, which for one dimensional case, we can write as dx. This can be written as F dot V plus dF by dt, dF by dt can be written as dF by dx; this can be written as dx by dt dx, in which dF, F is equal to minus kx. Therefore, dF by dx is equal to minus k. Therefore, the second term can be written as minus k dx by dt; that is minus kv dx.

Therefore, the expression for power is, in this case minus force into velocity minus kx into V plus minus kV into dx, dx is infinite small. In that case, the power will be minus kx V. However, if force has some singularity, in that case, this term maybe significant. Concept of power is found very useful in certain context. Suppose a man rises walking on the top of a hill and another man goes riding on a motor bike, both have done the same amount of work but the second man used more power because he would have done the same amount of work faster.

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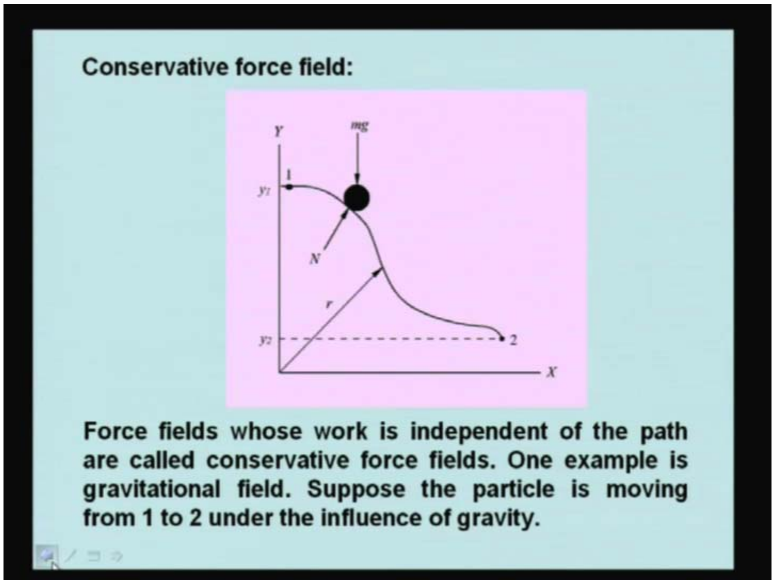

We discuss about the conservative force field. Force fields whose work is independent of the path are called conservative force fields. Force field, we understand the force field, but what is force field? Force can be expressed as a function of x y and z and time also. Therefore, when we express force as a function of x y z, then it becomes a vector field. Force is a vector; this is a vector field. Force field may, in this there can be different type of force fields. Fields, in which work is independent of the path are called conservative force fields. One example is a gravitational field.

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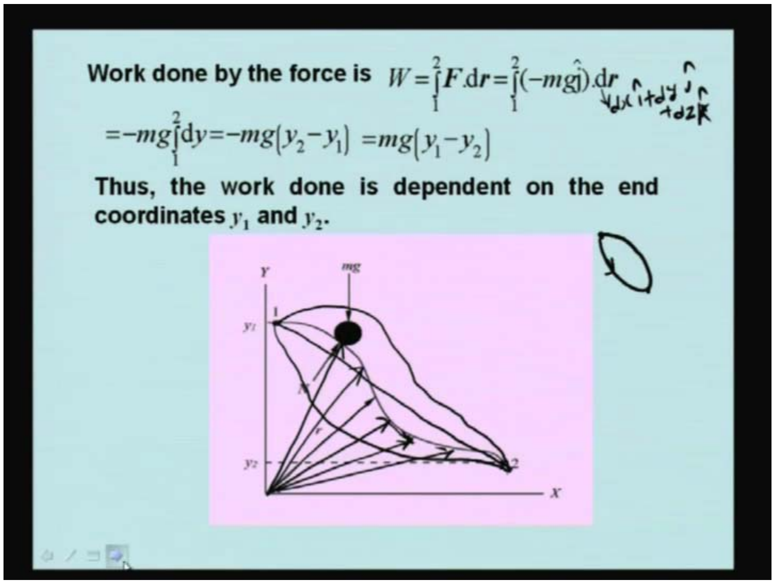

Suppose, the particle is moving from 1 to 2 under the influence of gravity like this, in that case work done by the force is W is equal to 1 to 2 F dot dr, r is the position vector of particle at any instant. r keeps on changing as the particle is moving towards paths, like that. dr is the distance between two position vectors like at this instance, this is dr. Provided the difference is very small it is indicated by dr. Work done by the force W is equal to 1 to 2 F dot dr which is 1 to 2 minus mgj, because the force is always constant and that is directed downwards in the y direction.

Therefore, it is minus mgj dot dr, dr can be written as dx i plus dy j plus dz k. Dot product of j and i is 0, dot product of i and k is 0; therefore, we are left with minus mg dy because j dot is one which gives us minus mg y_2 minus y_1 that is equal to mg y_1 minus y_2. Thus, the work done is dependent on the end coordinates y_1 and y_2 only. It is not dependent on the path of motion. The particle can adapt any path. It could have gone like this to the same point, it could have gone like this or it could have gone in a straight line.

The work done will be same. It is dependent on the end coordinates y_1 and $y_{2.}$ Therefore, gravitational field will be called a conservative force field. Similarly, if the particle goes from 1 to 2, the work done is mg y_1 minus $y_{2.}$ Work done by the gravity on the particle is mg y minus y_2. If the particle would have gone from 2 to 1, the work done by the gravity on the particle will be mg y_2 minus $y_{1;}$ add both the works, you get 0.

In the conservative force field, if a particle undergoes a cyclic motion like this, completes the cycle and comes back to the original position, the network is 0, which is not the case in non conservative force field.

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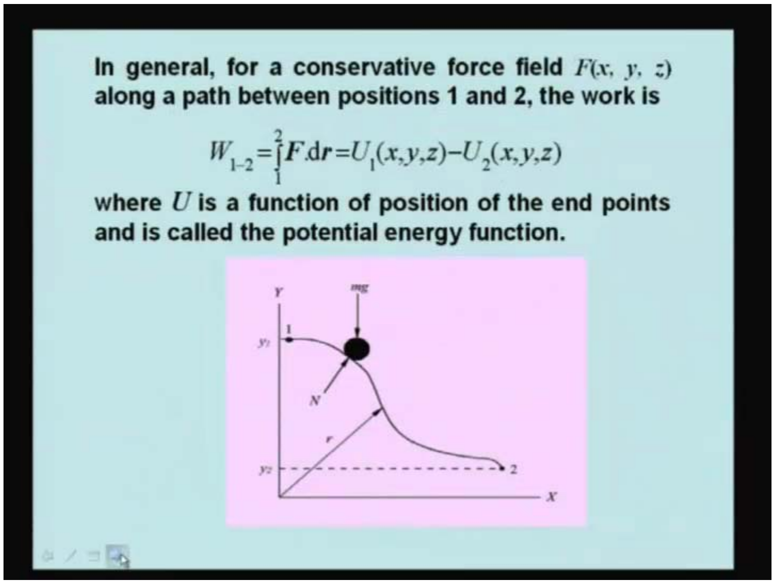

So, in general, for a conservative force field F x y z along a path between positions 1 and 2, the work is $W_{1\text{-}2}$ F dot dr; that is equal to U_1 x y z minus U_2 x y z, where U is a function of position of the end points and is called the potential energy function. In the previous case, you got work done between 1and 2. It is mg y_1 minus $y_{2.}$ So, mg y_1 can be written as the potential energy $U_{1;}$ mg y_2 is the potential energy $U_{2.}$ U_1 and U_2 are functions of x y z. U_1 is equal to y_1 is a function of x y z in which other coefficients are 0. U is a function of position of the end points and is called the potential energy function.

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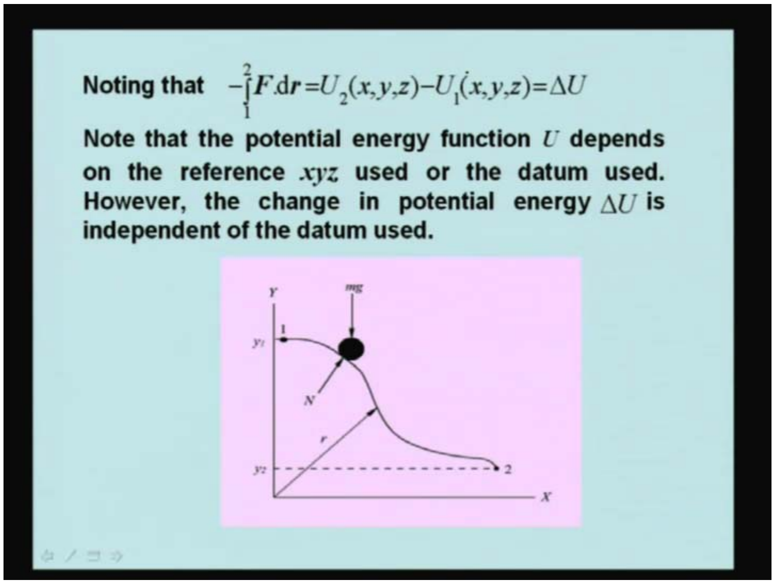

We note that minus 1to 2 F dot dr is equal to U_2, x y z minus U_1 x y z which is delta U. Note that potential energy function depends on the reference xyz used or the datum used. However, we will be concerned only with the change in potential energy delta U, because work done between two points will always be the difference of the two potential energies. Therefore, it is independent of the datum used. Any datum can be used for defining the zero potential energy. In kinetic energy calculation, the zero velocity uses zero kinetic energy, but this is not the case with the potential energy. In potential energy, you can take any datum and you can find out delta U which will be independent of the datum used.

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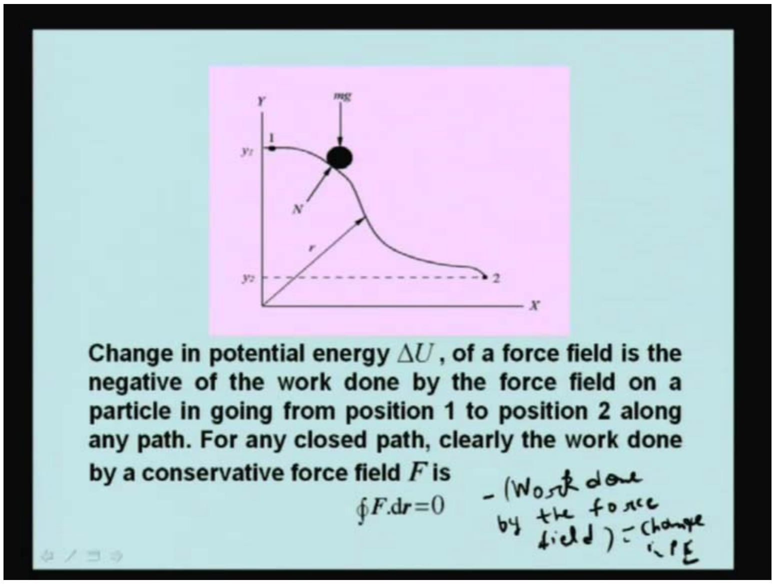

Change in potential energy delta U of a force field is the negative of the work done by the force field or a particle in going from position 1 to position 2 along any path. Therefore, force field, if when the force field does some positive work then the potential energy increases. If the force field does the negative work then the potential energy decreases. Change in potential energy delta U, negative of the work done by the force field is equal to change in potential energy. For any closed path theory, the work done by a conservative force field F is integral F data dr equal to 0.

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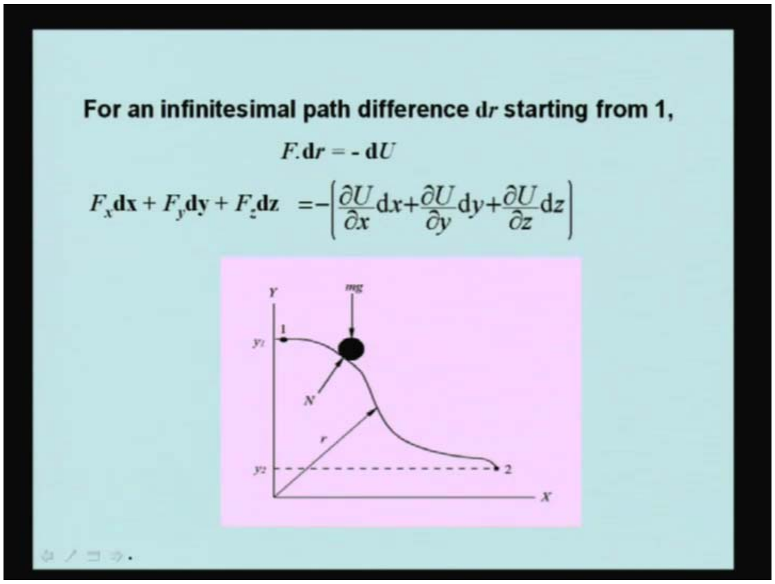

For an infinitesimal path difference dr starting from 1, F dot dr is equal to minus dU; that is F_x dx plus F_y d_y plus F_z d_z is equal to minus delta U by del x d_x plus delta U by del y d_y plus del U by del z $d_{z.}$

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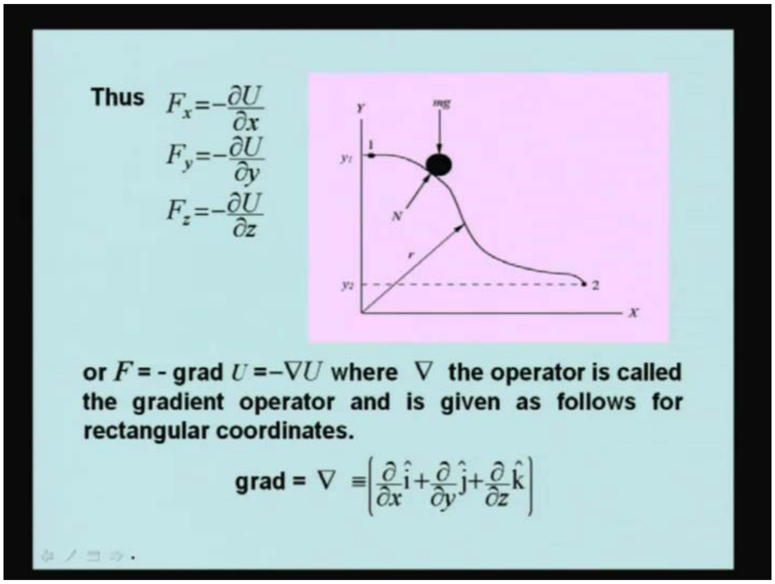

Thus F_x is equal to minus del U by del x, F_y is equal to minus del U by del y, F_z is equal to minus del U by del z, or F is equal to minus gradient of U, where U is a scalar field and this is indicated by minus delta U. Gradient is defined as, this is a gradient operator which is defined by del by del x i plus del y by del y j plus del by del z k. This is in the Cartesian coordinate system. In the same way, it can be defined in the polar coordinate or spherical coordinate system. If you have got a conservative force field, it can be expressed as a gradient of some potential energy function.

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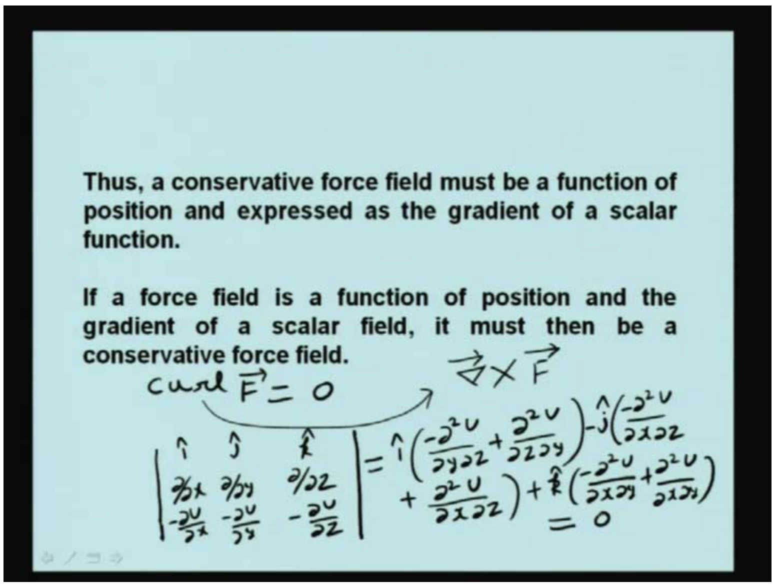

Thus, a conservative force field must be a function of position and expressed as the gradient of a scalar function. If a force field is a function of position and the gradient of a scalar field, it must then be a conservative force field. In some books, it will be shown that for a conservative force field curl F is equal to 0; that is curl of the force is equal to 0 for a conservative force field. Curl F of vector is basically del cross F, where the del operator has been defined here. This can be written as i j k del by del x del by del y del by del z minus del u by del x minus del u by del y minus del u by del z, because a conservative force has got these components.

Therefore, this becomes i minus del square U del y del z plus del square U del z by del y; but for continuous U this term will become 0. Similarly, the other terms will also come out to be 0. minus j minus del square U del x del z plus del square U del x, del z plus k minus del square U del x del y plus del square U del x del y is equal to 0.

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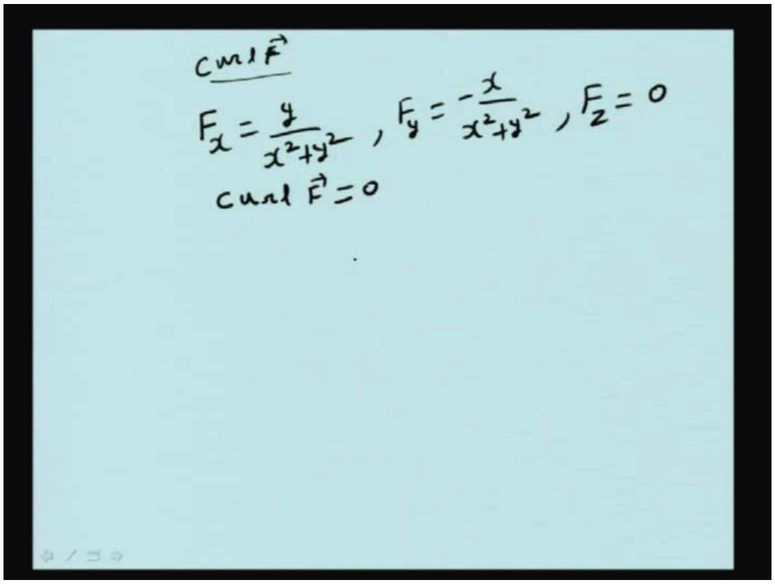

Thus, you can judge whether a force field is conservative or not by finding out the curl of the force. Although if this is too far, if this function which is not having the singularity, in certain cases, like if you have F_x is equal to y divided by x square plus y square F_y is equal to minus x divided by x square plus y square F_z is equal to 0. This force field provides curl F equal to 0. However, this force field is not conservative. Thus, curl F is equal to 0 can be called a necessary condition but not a sufficient condition; sufficient condition is from the basic definition.

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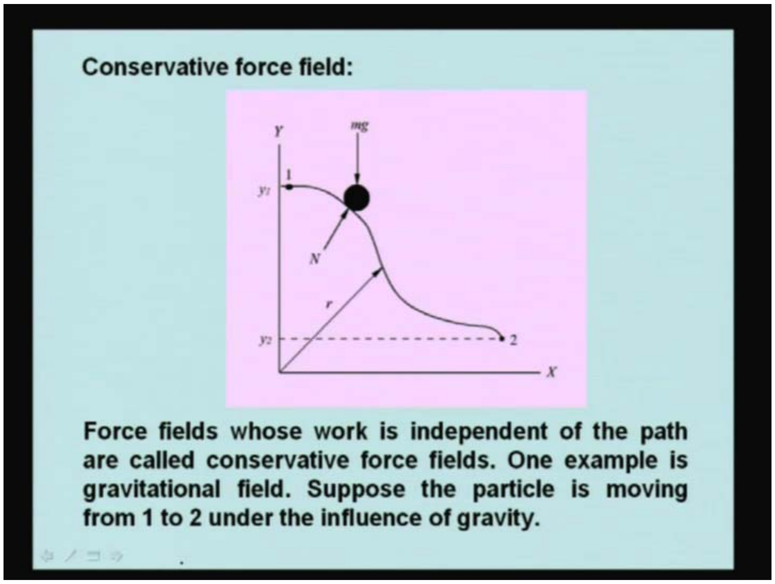

A force field whose work is independent of the path is called conservative force field. What are the examples of the conservative force fields?

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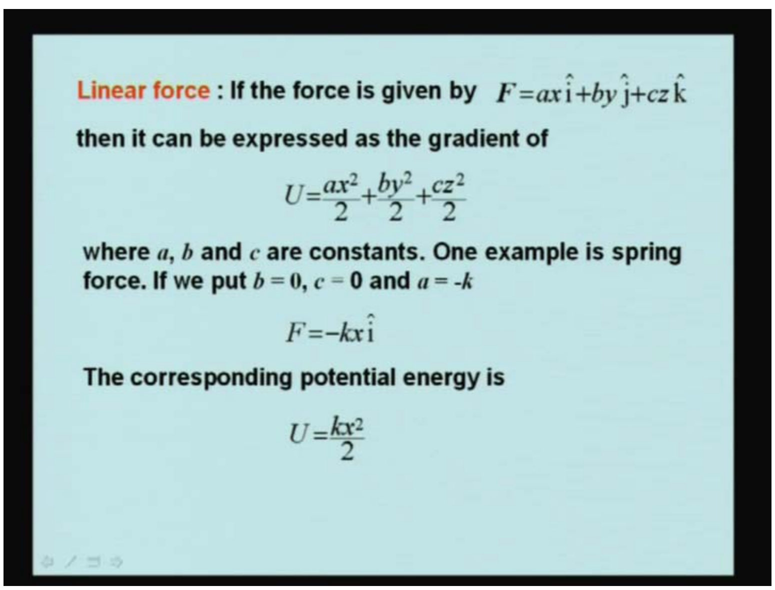

Linear force: if the force is given by F is equal to ax i plus by j plus cz k, then it can be expressed as the gradient of U is equal to ax square by 2 plus by square by 2 plus cz square by 2, where a b and c are constant. If you take the gradient of U, this becomes ax i plus by j plus cz k.

One example is a spring force. If we put b is equal to 0, c is equal to 0 and a is equal to minus k, we get a force field that is F is equal to minus ax i. The corresponding potential energy from this is U is equal to kx square by 2. So, the spring force is a conservative force.

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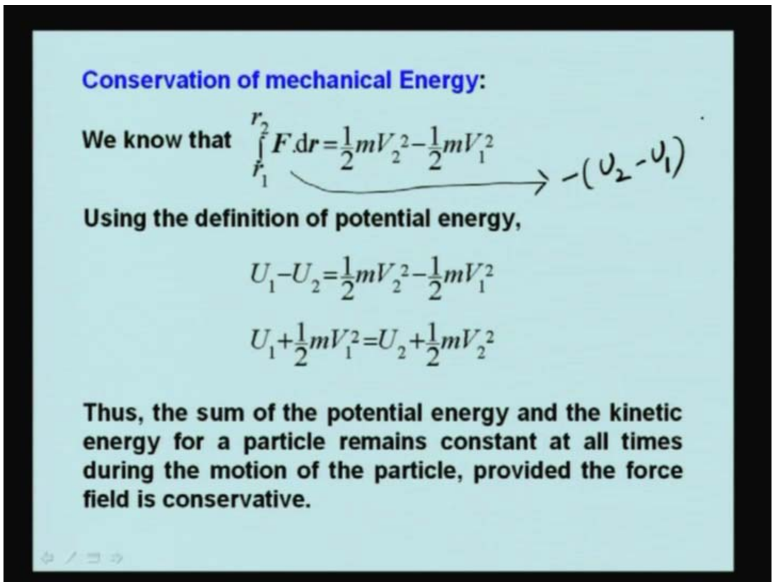

Now, we discuss the conservation of mechanical energy which is valid for a conservative force field. We know, from the basic concepts developed in the beginning that r_1 r_2 F dot dr is equal to half m V_2 square minus half m V_1 square. This has directly come from the Newton's second law. Using the definition of potential energy that is Vr F dot dr is equal to, actually negative of the change in the F is the work done on the particle by the field and it has the tendency to decrease the potential energy. The force field is trying to decrease the potential energy; therefore, r_1 to r_2 F dot dr minus U_2 minus U_1.

If the particle works against the applied force field, if you could raise that particle in the gravitational field, then the potential energy of the particle will increase. But if the particle is allowed to freely fall then the potential energy will decrease. Force field is doing some work on the particle and therefore, the particle's potential energy is decreasing. It will try to make the potential energy minimum. Therefore, if you do not constraint the motion, the particle will keep on moving.

If you drop one object from the sixth floor of a building, it will reach the ground floor. If the ground, was a pit at the ground then the particle would have gone in the pit. If the pit was even deeper, it would how gone even more deep, it is like that. Force field tries to minimize the potential energy. Therefore, this point is important that r_1 r_2 F dot dr is basically U_1 minus $U_{2,}$ not U_2 minus $U_{1.}$ Using the definition of potential energy, U_1 minus U_2 is equal to half m V_2 square minus half m V_1 square.

We can take this to this side and we say, U_1 plus half m V_1 square is equal to U_2 plus half m V_2 square. Thus, the sum of the potential energy and the kinetic energy for a particle remains constant at all times during the motion of the particle provided the force field is conservative. Thus, in a gravitational field that kinetic energy plus potential energy of a particle will remain constant. If you throw a projectile then during the motion of the projectile, this kinetic energy plus potential energy will remain conjunct. It will remain same throughout. Therefore, this conservation principle can be applied at many places.

However, if there are non conservative forces such as friction force, which of course is non-conservative, because it depends on the path of motion. In that case, conservation of energy cannot be implied. However, this basic thing that r_1 r_2 F dot dr is equal to half m V_2 square minus half m V_1 square holds good, because it directly comes from Newton's second law. Condition is that you must identify proper F. If you have, it is not only what you have applied, but also friction forces or the external forces have to be included in this.

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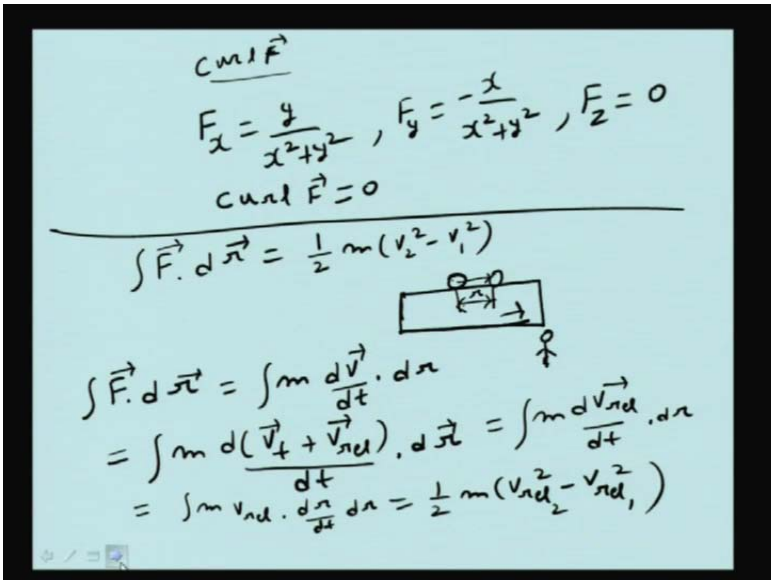

We have learnt that in this case, F dr is equal to this half m V_2 square minus V_1 square is a very basic equation. Let us see whether what velocity should be taken, if I am doing the work on a train. The train is moving with a constant velocity. You have a particle and displaced by some amount dr or some amount r, some work has been done there on this thing. Particle has displaced by an amount of r in a relative frame of reference; that means in the frame of reference of the train. In the absolute frame, for an observer who is standing outside the particle has displaced even a greater distance.

Similarly, the velocities, the question is that in this case, which velocity should be taken? Whether the velocities relative to the frame of reference or the absolute velocity of the particle in calculating the kinetic energy. Let us go to basics for understanding this. F dot dr is equal to m. Here, that force m dV by dt, this V is the absolute velocity and this dV by dt is the absolute acceleration; that means as observed by standing in the inertial frame of reference, which is ground in this case and this is dr.

This can be written as m d $V_{t,}$ that velocity of the train plus V is relative V_{rel} by dt dot dr. Let us consider that we are taking this r is a relative r. So, F dot dr is the work done in the relative frame of reference. Work done will depend on the reference frame chosen. If you have applied a force

F on the particle, the work seen by the outside observer maybe more but in this case, it is F dot dr in the relative frame. So, m dot d m dVt plus V_{rel} by dt.

If V_t is constant, then in that case this equation just simplifies to m d V_{rel} by dt into dr, which can be written as m V_{rel} dr by dt dot dr by dt dr. dr by dt is again $V_{rel,}$ which will give you half m V_{rel2} square minus V_{rel1} square. Therefore, this relation that work kinetic energy theorem can be applied in the inertial frame of reference; that means, which is moving at a constant velocity; provided, you take the work done in the relative reference frame. That means, force multiplied by the displacement in the relative frame and is equal to change in kinetic energy in the relative frame.

In this case, V_2 and V_1 should be the relative velocity in the frame not the absolute velocity. What happens if the frame is non-inertial?

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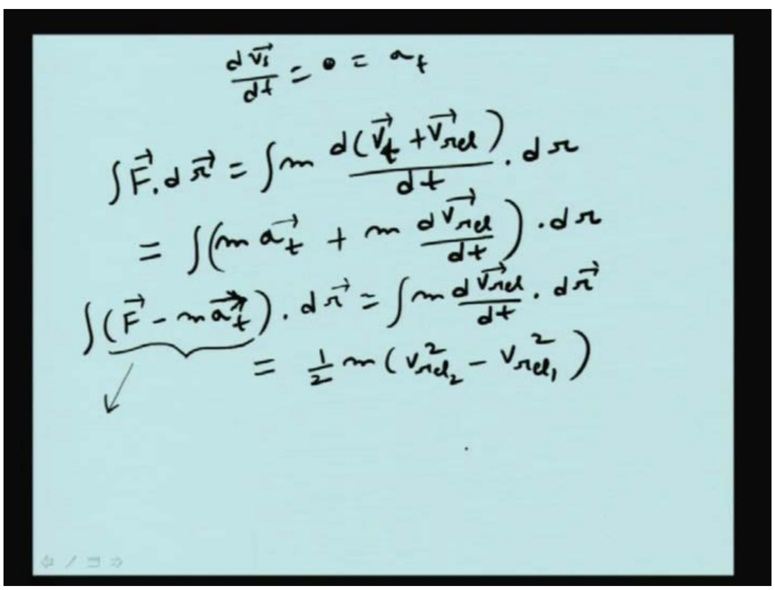

That means dVt by dt is not 0. Train is having some acceleration and that is indicated by a_t. Acceleration of the train, in that case it becomes m F dot dr is equal to m d V_a V_t V_t plus V_{rel} by dt dot dr, which will be equal to m a_t plus m d V_{rel} by dt dot dr. I can take this m a_t to this side and can write that F minus m a_t dot dr is equal to m d V_{rel} by dt dot dr, which gives you this half m V_{rel2} square minus V_{rel1} square. Thus, the work energy theorem in the case of non inertial frame can be implied, provided you replace the force by effective force; that is F minus m a_t, where a_t is the acceleration of the frame. Then, in this case it will be V_{rel2} square minus V_{rel1} square. These are the relative velocities. Let us discuss some interesting things about the energy approach. We have discussed the work energy theorem that work done, if you do a work on this one that is change in the energy.

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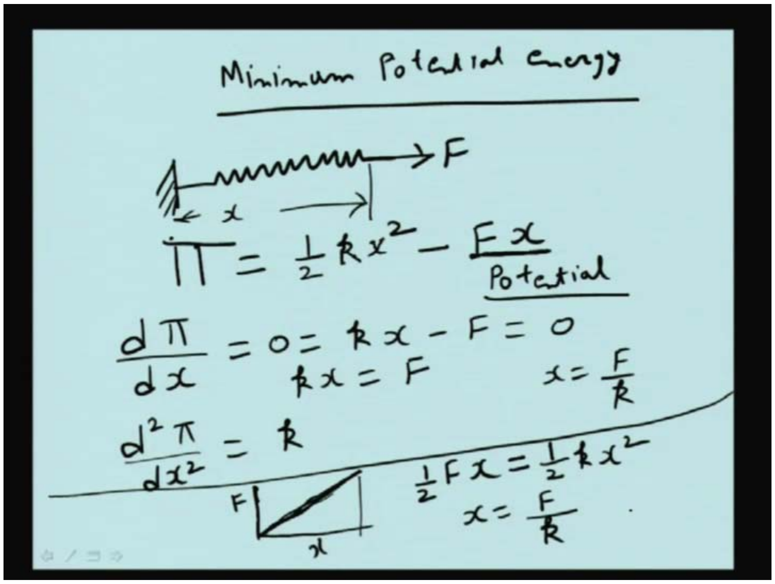

We also have a principle for conservative system that is principle of minimum potential energy in statics. When a conservative force field is applied, the potential energy of the system tries to be minimum. If you take a spring and apply a force F, in this case, if you give a displacement x, potential energy as a function of x can be written as this; U is equal to total potential energy. If I indicate it by big pi, big pi is equal to half k x square strain energy of the spring minus F_x. Remember, the potential associated with the force has been written as minus F_x because if you take the negative gradient of minus F_x you get equal to plus F.

Here, many times, students may get confused. Then, they put half kx square minus plus F_x. Similarly, sometimes the students may put half F_x considering that, the force is gradually applied and therefore, the work done is equal to half F_x. Remember that this is not a work done, rather it can be called a potential; this is potential.

This pi is the value of x which minimizes the potential energy will be the solution of the problem. For that, the necessary condition is that d tau by dx is equal to 0. Thus, kx minus F is equal to 0 and we get kx is equal to F. If the potential energy gets minimized then it is a stable equilibrium situation. In this case, we can see that d square pi by dx square is equal to k. As, k is positive, this is minimization problem. This is a minima and therefore, x is equal to F by k provides the stable equilibrium of the spring force system.

If the same problem is done by the work energy concept then it can be done in this way. If the force is equal to, in this case, force is F; however, this F is applied very gradually. Spring force is linear; therefore, the force is also linear. At each and every stage, applied force and the stretching force is, it is a Cauchy's static situation. No dynamic effects are there and therefore, it is linear. In this case, the work done will be equal to half $F_{x.}$ This work done on the spring is stored as the potential energy of the spring.

The potential energy of the spring increases because external force is doing a work against the conservative force field; that is in the opposite direction, minus $k_{x.}$ This half F_x will be half k_x square and which gives us the solution as x is equal to F by k. We get the same thing by the work energy theorem. Similar to the case of statics, we have the principle in the dynamics too.

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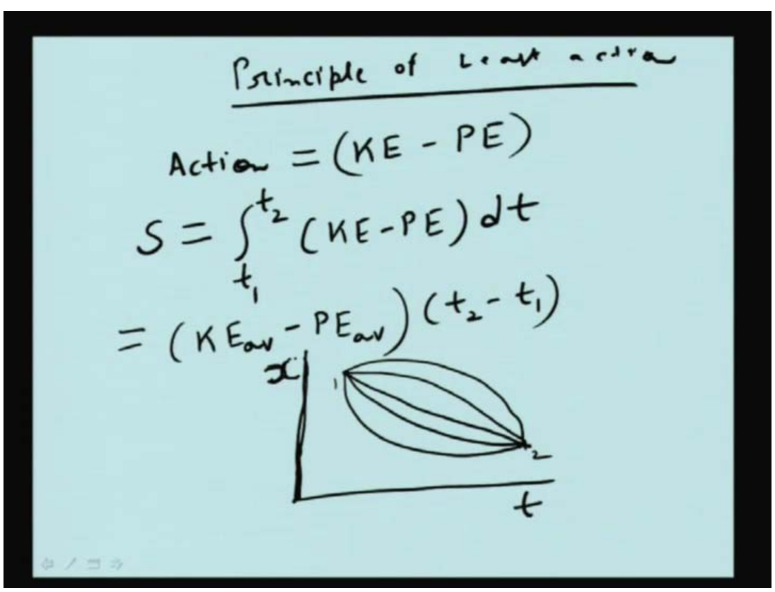

That is called principle of least action. Action along the particle trajectory is defined as kinetic energy minus potential energy; KE minus PE and this is instantaneous action, the total value of action. A class the trajectory from t_1 to t_2 time, t_1 to t_2 is KE minus PE dt. One can write the same as, KE average minus PE average, where the average is appropriately defined. Average is not just the arithmetic average, it is by the integration. If now, the particle follows that path which minimizes its action, this is the thing.

This theorem can be used. Therefore, if a particle is moving from here to here, there can be various paths; but out of all paths, the particle will follow that path which provides the minimum action. We can show from the principle of least action that a free particle that starts from one position at time t_1 and arrives at a different position moves along a straight line here. So, this is the particle's position, this is time and this is particles x. How to find out the minimum path? This requires some advanced study.

It is not possible to find out this by means of, by taking infinite number of paths and evaluating that. Even the fastest computer will not be able to study these types of things. One has to find, study calculus of variation, however that we will not be discussing at this point.