Engineering Mechanics

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Kinematics

Lecture 26
Plane kinematics of rigid bodies

Today, I am going to discuss plane kinematics of rigid bodies. Till now, we have discussed kinematics of particles. A rigid body is composed of a number of particles; in fact, it may consist of infinite number of particles. However, that rigid body has a fixed degree of freedom because all the particles are not free to execute their motion at their will. They are all constraints; that is distance of two particles on the body remains same.

We will discuss about the kinematics of rigid body, but we start with a plane motion in which the rigid body is moving in a plane. Therefore, it has got three degrees of freedom. We can fix up a point on the rigid body say mass center and say what is the x co-ordinate of it? What is the y co-ordinate of it? Then, what is the rotation of the body about this point? You have x, y and theta.

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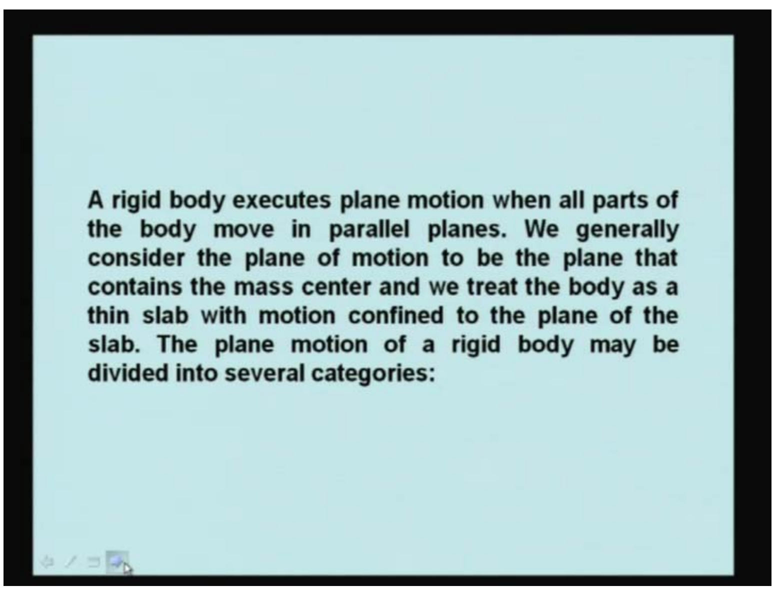

Therefore, we say a rigid body executes plane motion when all parts of the body move in parallel planes. We generally consider the plane of motion to be the plane that contains the mass center and we treat the body as a thin slab with motion confined to the plane of the slab. So, actually, we will take that mass center and pass a plane through that. The motion will be taking place in that plane. The plane motion of a rigid body may be divided into several categories. We discuss some of these.

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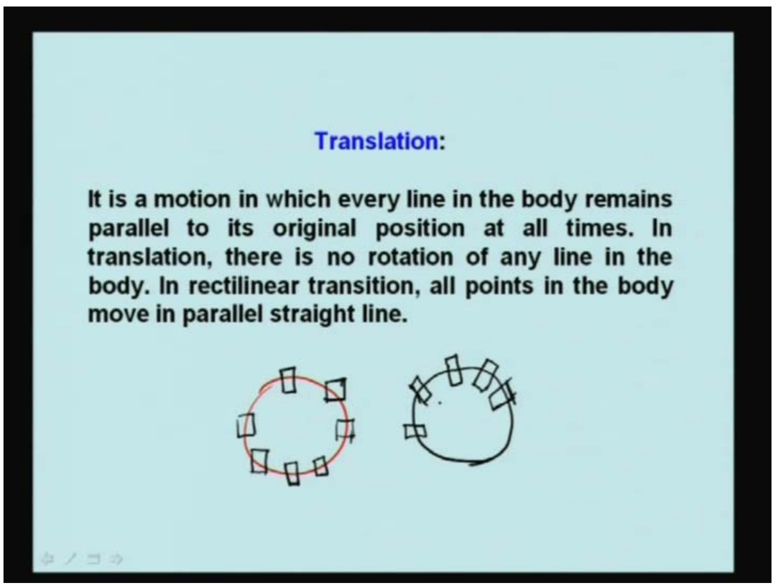

Translation: It is a motion in which every line in the body remains parallel to its original position at all times. There is no rotation of the lines in the body. In translation, we have two types of translation: one is rectilinear translation in which all points in the body move in parallel straight line. Then we have curvilinear translation in which is the points do not move in parallel straight lines, but they move in parallel curves. Now, the body may move in a circular path also.

For example, this body, here this one is moving on a circular path that red circle and this is the thing but body is not rotating. Body is not changing its orientation. Therefore, this is not a rotation, rather it will be called translation. On the same path, if the body moves this thing like this, so that it is radially, that its orientation is changing (Refer Slide Time: 04:59). Then, this motion will not be called translation like this. So, in rectilinear translation all points in the body move in parallel straight line.

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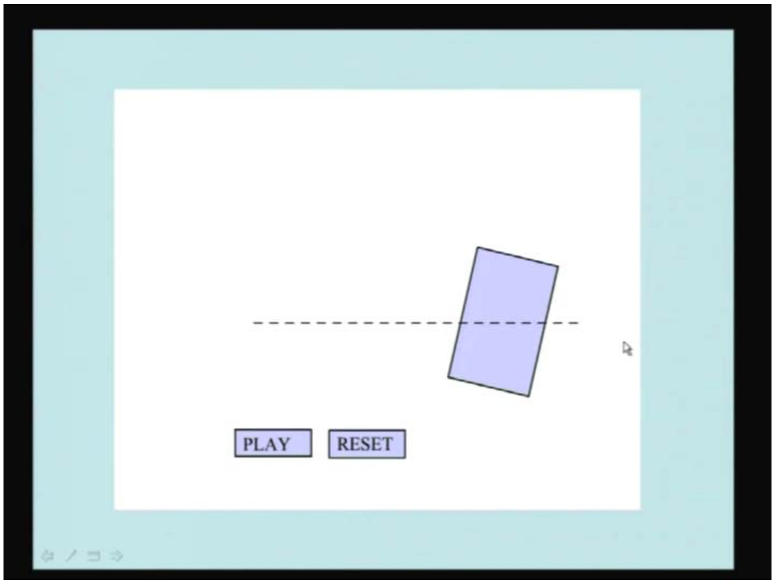

See this animation. Here, the body is moving in parallel straight line. If we know the velocity of any one point on the body, we have known the velocity of whole body; because, all the particles are moving in the same way. So, the velocity of all the particles is same in direction as well as magnitude in this case.

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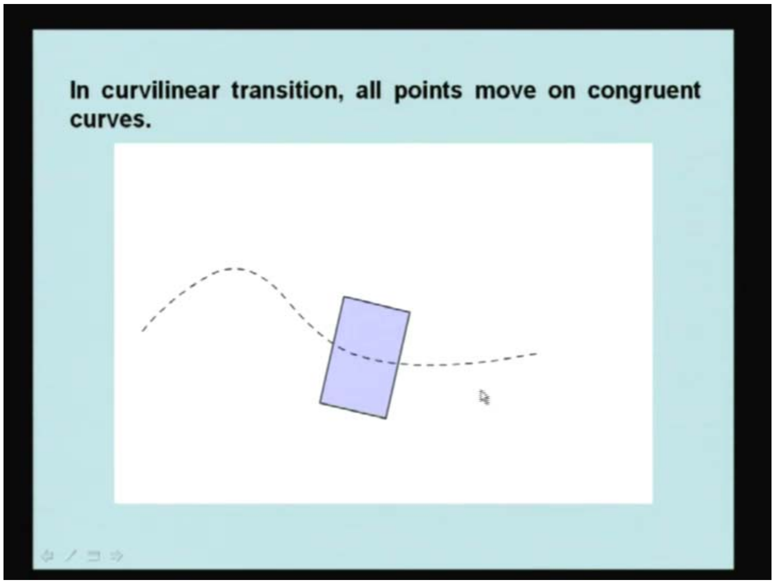

In curvilinear translation, all points move on congruent curves. This is an example of curvilinear translation. However, even in this case, the velocity is same for all particles. So, in this case the velocity may be changing, the velocity here may be small. Then, it is this thing. Body may be accelerating or decelerating that is a different issue but velocity of one particle is same as the velocity of any other particle of the body. So, this is an example of that curvilinear translation.

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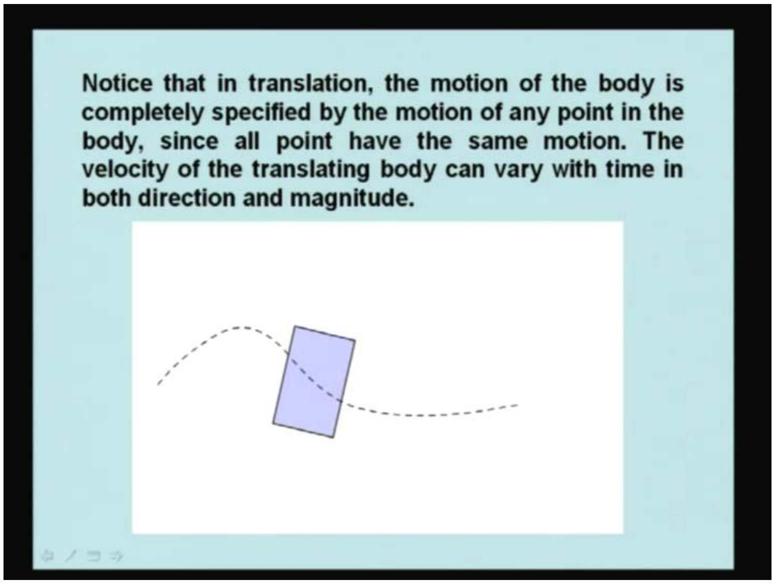

Notice that in translation, the motion of the body is completely specified by the motion of any point in the body, since all points have the same motion. The velocity of the translating body can vary with time in both direction and magnitude. However, the velocity of all those points will be same. Generally, we speak of the mass center. We specify the velocity of the body by the velocity of mass center. In translation, the velocity of mass center will provide the velocity of any point on the body.

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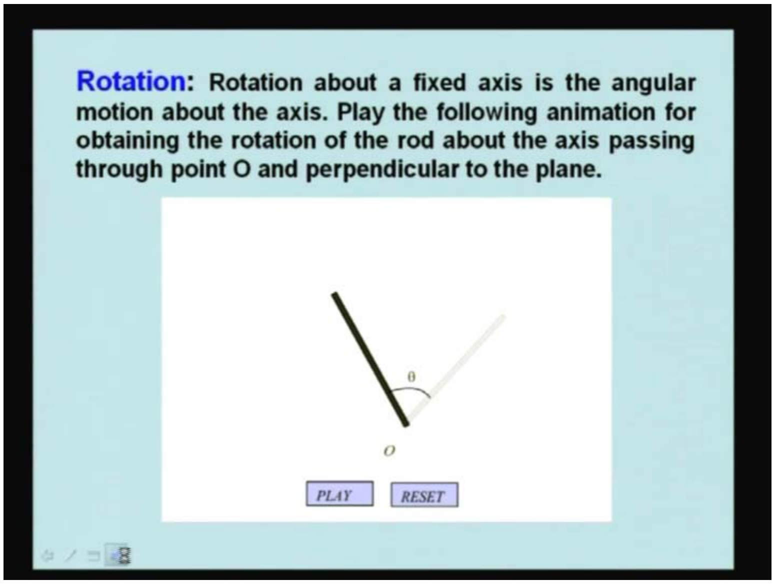

We discuss about rotation; rotation about a fixed axis is the angular motion about the axis.

Consider a point O and draw a perpendicular to your screen; that is the axis of rotation is perpendicular to your screen. See the following animation for obtaining the rotation of the rod about the axis passing through point O and perpendicular to the plane. In this case, this line does not remain parallel, like in translation. Instead, it rotates by an angular theta; this is the initial position. With respect to the initial position, it has rotated by theta.

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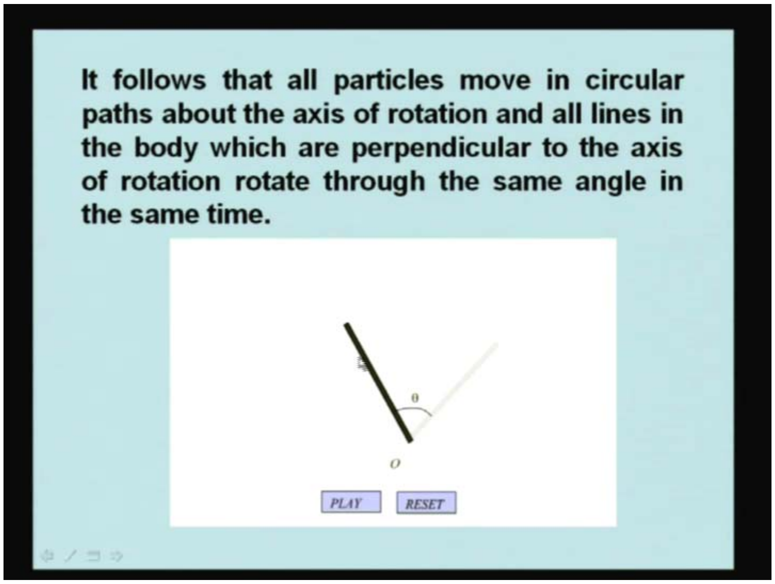

It follows that all particles move in circular paths about the axis of rotations and all lines in the body that are perpendicular to the axis of rotation, rotate through the same angle, in the same time. Here, it is rotating. Now, all the particles are moving in a circular path. Also, all the particles which are in this line have rotated by theta. That means, if we consider this line, this is theta; this is also through the same angle in the same time.

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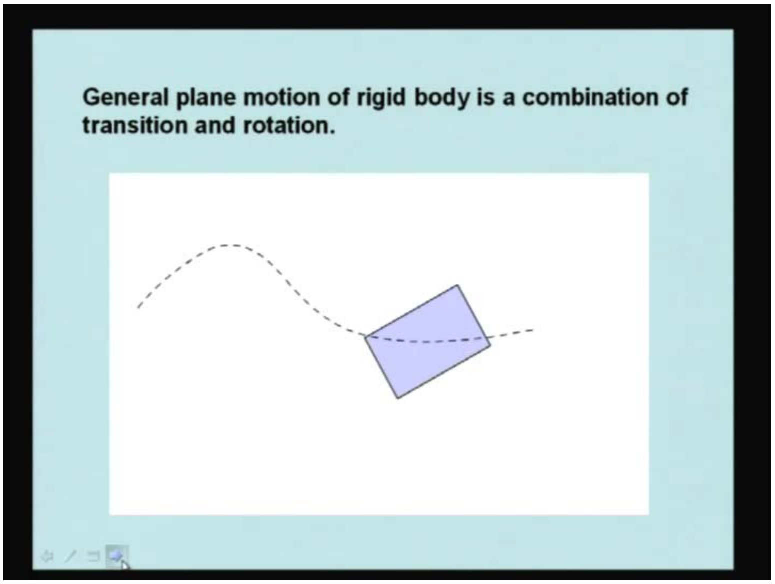

General plane motion of a rigid body is a combination of translation and rotation. A body can have translation as well as rotation. You can reach to some point and translate the body to some point, then you rotate or you first rotate and then translate, you will get the same thing. General plane motion of a rigid body is a combination of translation and rotation.

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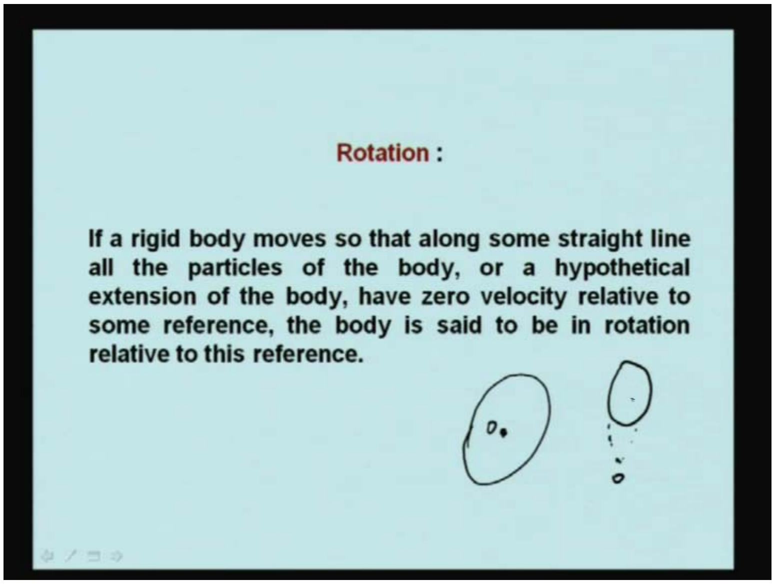

If a rigid body moves so that along some straight line all the particles of the body, or a hypothetical extension of the body has 0 velocity relative to some reference, the body is said to be in rotation relative to this reference. That is, if a rigid body is moving, if it is moving about this point, if it is rotating about this point O and you draw a perpendicular to the plane passing through O, then on that axis all the particles have 0 velocity. Therefore, the body is rotating about this point O.

Consider another case. The body may be rotating about this point O, which is not in the body. Hypothetically, if we extend the body up to that point, it is rotating about this point. Therefore, we can have axis of rotation outside the body also. In that case, we make a hypothetical extension as if there is a body and its particles follow the same kinematics.

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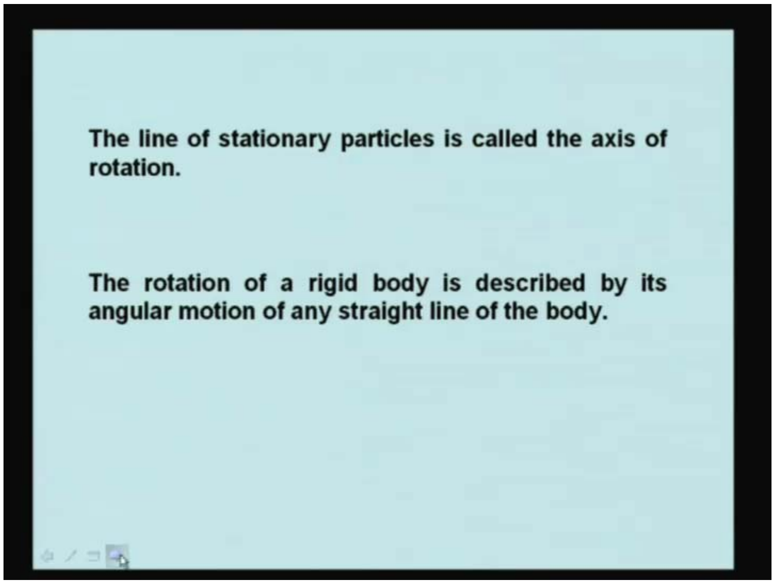

The line of stationary particles is called the axis of rotation. So, when the body rotates there is one axis of rotation. The rotation of a rigid body is described by its angular motion of any straight line of the body.

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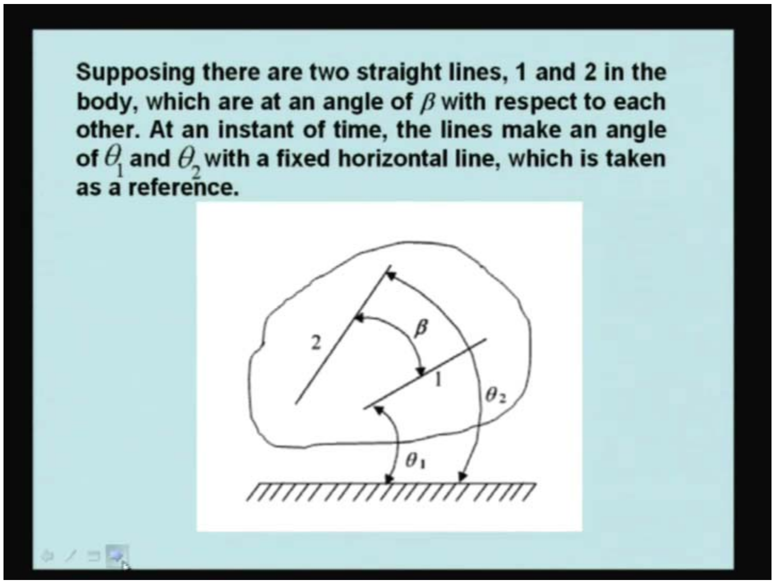

Any straight line, why we say any straight line? Supposing there are two straight lines; 1 and 2 in the body which are at an angle of beta with respect to each other. A body line 1 is there and line

2 is there. Both are making an angle beta. Take a reference horizontal line. At any instant of time, the lines make an angle of theta$_1$ and theta $_2$ with a fixed horizontal line, which is taken as a reference. So, this is theta$_1$ and this is theta$_2$. So, this is what it is flowing.

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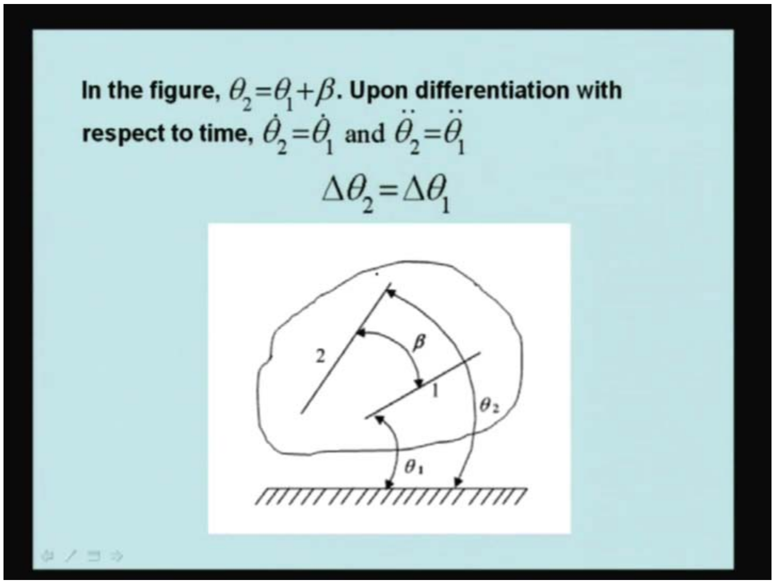

In the figure theta$_2$ is equal to theta$_1$ plus beta because beta is the angle. You can see theta$_2$ is equal to theta$_1$ plus beta. Upon differentiation with respect to time, theta$_2$ dot is equal to theta$_1$ dot because angle beta is a constant. In a rigid body, angle between this line and this line is not going to change. Otherwise, the body is deforming. Therefore, beta has to be constant with time. Therefore, theta$_2$ dot is equal to theta$_1$ dot and theta$_2$ double dot is equal to theta$_1$ double dot. Similarly, delta theta$_2$ that means change in the angle of this line is same as delta theta$_{1.}$

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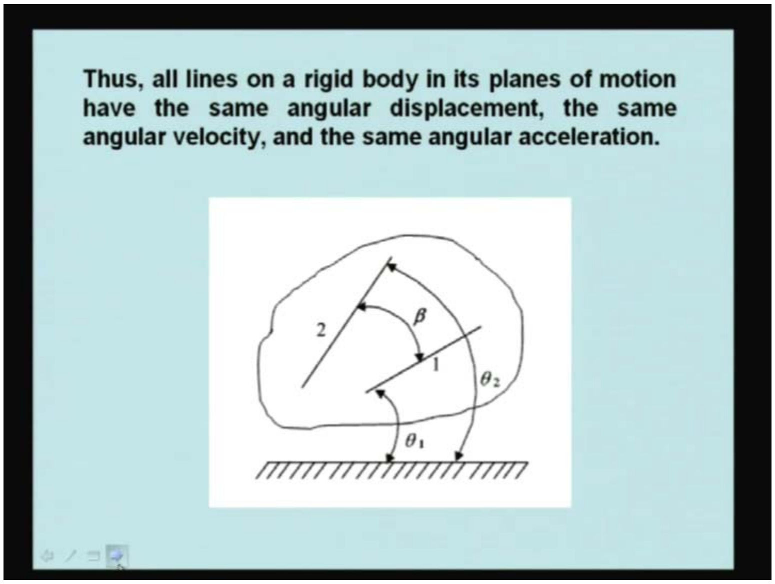

Thus, all lines on a rigid body in its plane of motion have the same angular displacement, the same angular velocity and the same angular acceleration.

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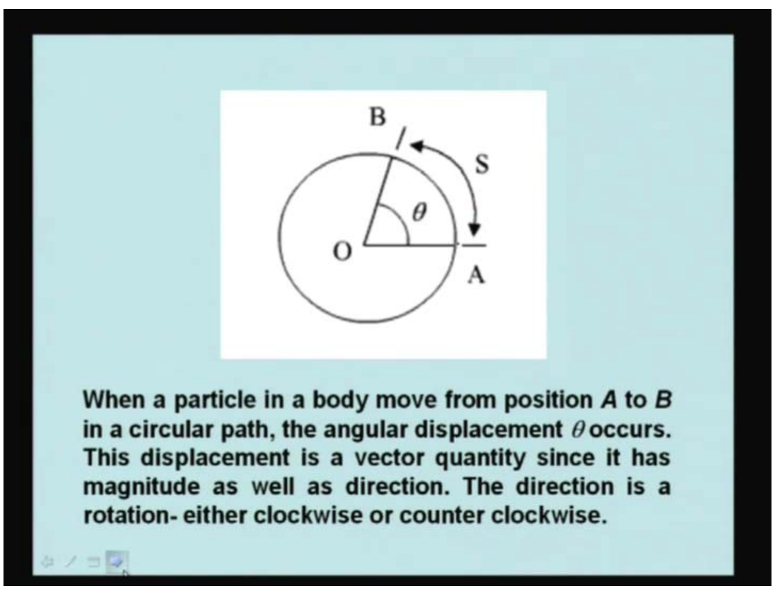

When a particle in a body moves from position A to B in a circular path like this, the particle A has gone from A to B, the angular displacement theta occurs. This displacement is a vector quantity, since it has magnitude as well as direction. The direction is a rotation either clockwise or counter clockwise and this is the distance S travelled by the particle.

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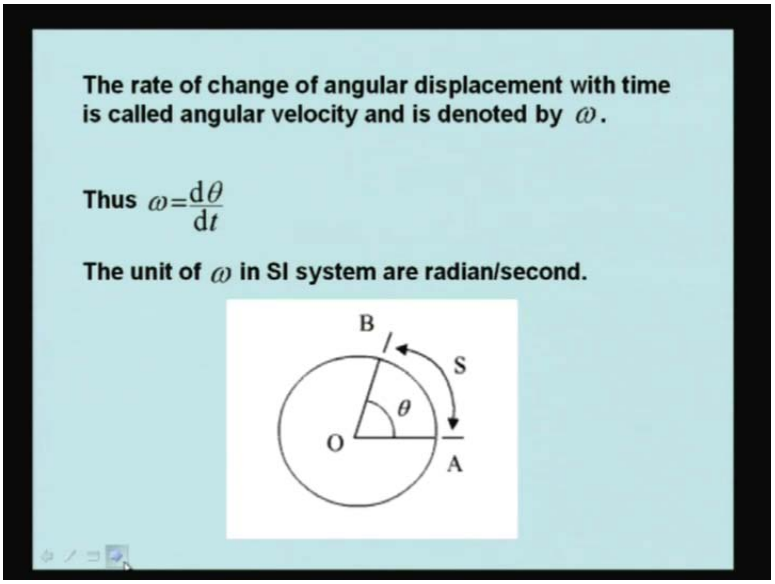

The rate of change of angular displacement with time is called angular velocity and is denoted by omega. In this case, when the body is moving in the plane, that direction will be indicated by normal. By convention if the body is moving anticlockwise, its direction of rotation will be in the direction of k, where k is a unit vector normal to the plane. If the body is moving in the clockwise direction, the direction will be in the direction of minus k, where k is the unit vector which is perpendicular to the plane. The angular velocity will have the same direction. That means, clockwise angular velocity will be minus k direction; anticlockwise velocity will be in the plus k direction.

Since all the bodies moving in a particular plane it will have the velocity either in plus k direction or in minus k direction, angular velocity will be like that. Therefore, we generally orbit k. Instead, we indicate the angular velocity by just either a positive number or a negative number. We can manage it by just algebraic addition and subtraction. The rate of change of angular displacement with time is called angular velocity and it is denoted by omega. Thus, omega is equal to d theta by dt. The unit of omega in SI system is radian per second.

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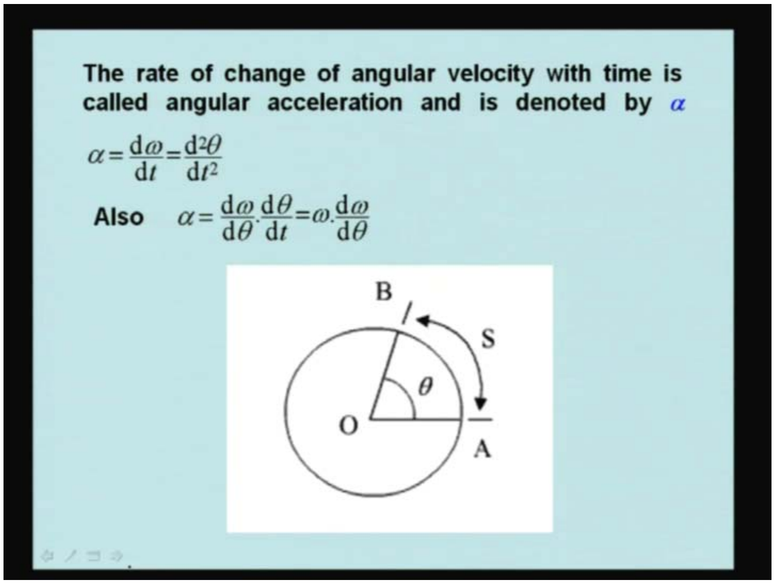

This rate of change of angular velocity with time is called angular acceleration and is denoted by alpha. Alpha is equal to d omega by dt, angular acceleration. Its units are radian per second square or it is same as d square theta by dt square. You have to take double derivative. Also, alpha can be written as d omega by d theta into d theta by dt, that means this becomes d theta by dt is omega. So, it is omega into d omega by d theta. Therefore, if omega is expressed in terms of theta, then angular acceleration can be found by multiplying d omega by d theta and omega.

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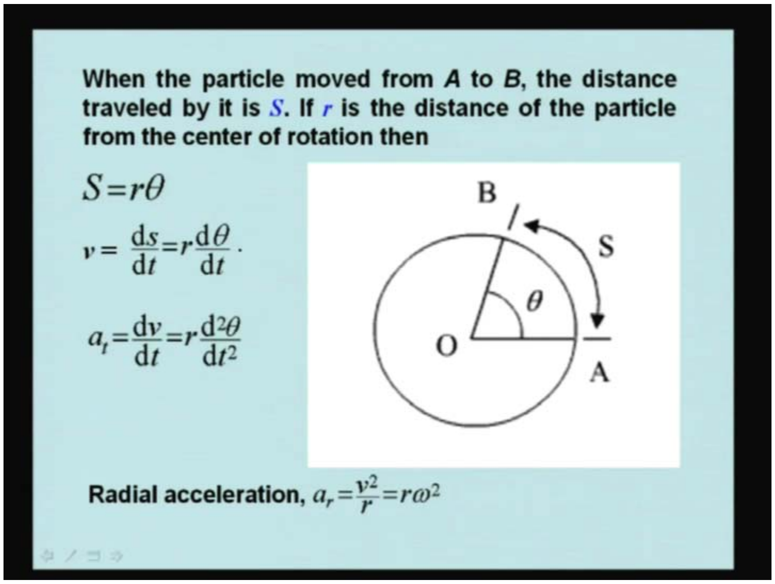

When the particle moved from A to B, the distance travelled by it is S. This is like this. If r is the distance of the particle from the center of rotation, then S is equal to r theta and v is equal to ds by dt; that means this is r d theta by dt. Therefore, the linear velocity v is r times omega. We can find out the tangential acceleration. Tangential acceleration will be dv by dt, which is nothing but rd square theta by dt square, v square theta by dt square is equal to alpha, that is angular acceleration. Therefore, it is equal to r times alpha.

Radial acceleration a_r will be v square by r; putting v is equal r omega, we get r omega square. Therefore, as the distance of a particle increases from center O, its acceleration increases.

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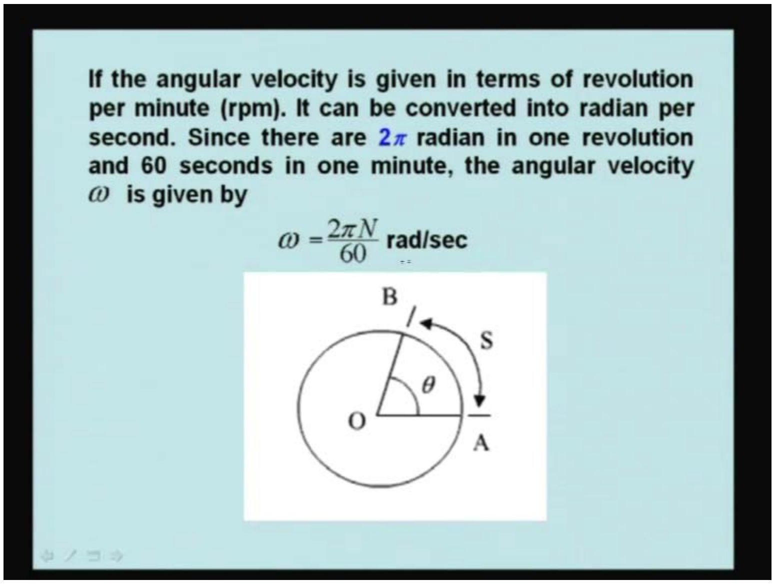

If the angular velocity is given in terms of revolution per minute that is rpm, most of the machinery items or motors will be specifying the revolution per minute; that is rpm; say a motor rotates 1450 rpm like that. It can be converted into radian per second. Since, there are 2 pi radian in 1 revolution and 60 seconds in 1 minute, the angular velocity omega is given by omega is equal to 2 pi N divided by 60 radian per second.

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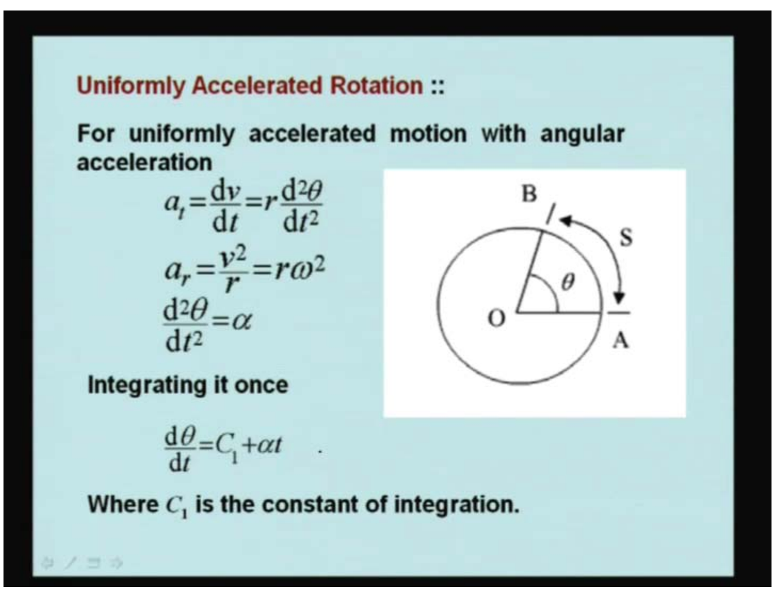

Uniformly accelerated rotation: For uniformly accelerated motion with angular acceleration, a_t is equal to dv by dt which is equal to rd square theta by dt square. a_r is equal to v square by r, r omega square. Let us say, d square theta by dt square is equal to alpha, which is constant. If we integrate it once, you get d theta by dt is equal to C_1 plus alpha t, where C_1 is the constant of integration or omega is equal to C_1 plus alpha t. Constant C_1 can be found from the initial condition.

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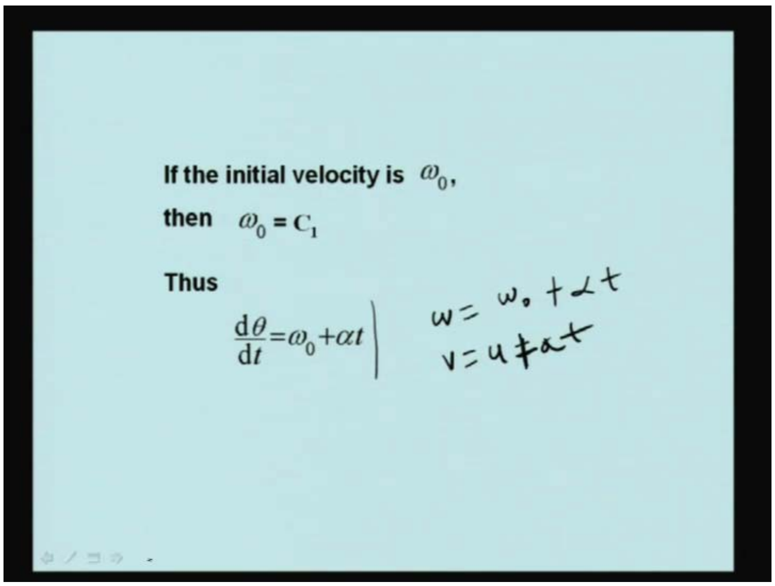

If the angular velocity is $omega_0$ then at time t is equal to 0, omega is equal to $omega_0$; thus, C_1 is equal to $omega_0$ is equal to C_1. Thus, d theta by dt is equal to $omega_0$ plus alpha t, or omega is equal to $omega_0$ plus alpha t. This is the same way as you get equation of motion; v is equal to u plus at.

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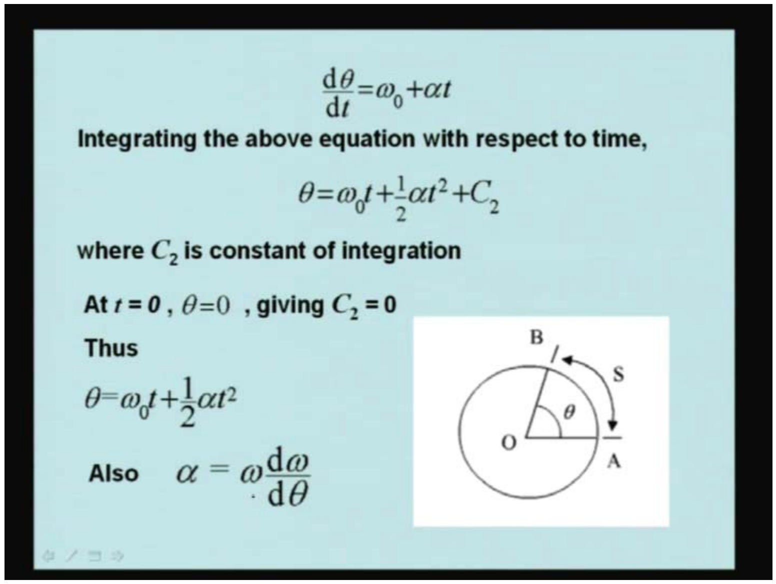

Once we have established that d theta by dt is equal to omega$_0$ plus alpha t, we integrate the equation with respect to time and theta is equal to omega$_0$ t plus half alpha t square plus C_2. This had been obtained by integrating the expression d theta by dt is equal to omega$_0$ plus alpha t, where C_2 is a constant of integration. At time t is equal to 0; theta is equal to 0 or C_2 is equal to 0. Thus, theta is equal to omega$_0$ t plus half alpha t square. Also, alpha is equal to omega d omega by d theta, alpha can be written as omega d omega by d theta.

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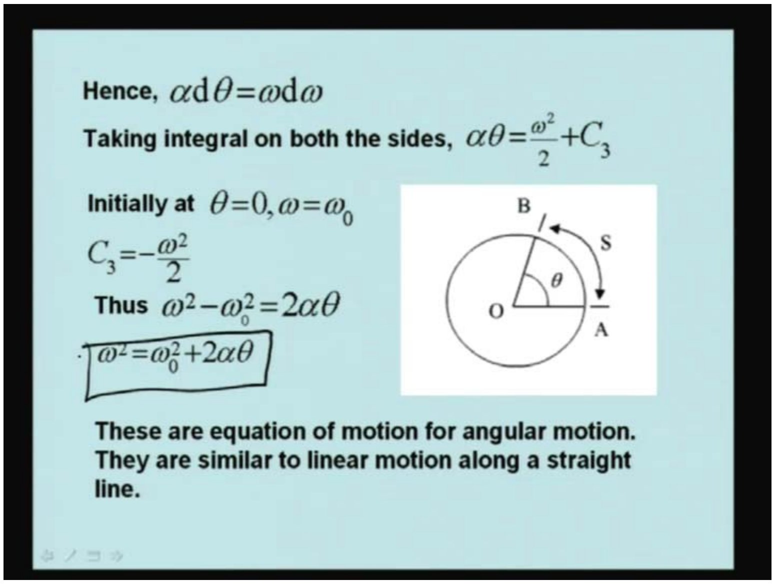

Hence, alpha d theta is equal to omega d omega. Taking integral on both the sides, we get alpha theta is equal to omega square by 2 plus C_3. Initially, at theta is equal to 0, omega is equal to $omega_0$; that means C_3 is equal to minus omega square by 2. Thus, omega square minus $omega_0$ square is equal to 2 alpha theta or omega square is equal to $omega_0$ square plus 2 alpha theta. These are equations of motion for angular motion. They are similar to linear motion along a straight line.

These are the equations of motion: one is omega is equal to $omega_0$ plus alpha t; second is theta is equal to $omega_0$ t plus half alpha t square; the third is omega square is equal to $omega_0$ square plus 2 alpha theta. These are the three equations of motion.

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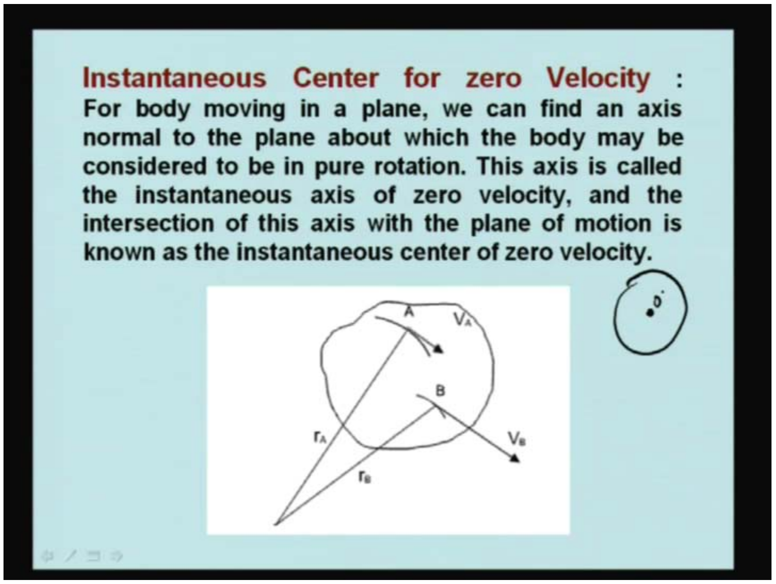

Now, we will discuss another point, the instantaneous center of zero velocity. For a body moving in plane, we can find an axis normal to the plane about which the body may be considered to be in pure rotation. We can always find out a point about which body can be considered to be in pure rotation. This axis is called the instantaneous axis of zero velocity. The intersection of this axis with the plane of motion is known as the instantaneous center of zero velocity.

If a disc is rotating about an axis, then O is its instantaneous center. An axis perpendicular to the plane and passing through is called instantaneous axis of rotation. In this case, instantaneous center of rotation does not change with time. We will see the ways to find out the instantaneous center of rotation. Finding out instantaneous center of rotation is of practical importance. If you know the instantaneous center of the rotation of a body, you can find out the velocity of any particle by multiplying the angular velocity of the body with the distance from the center of rotation. The direction of the velocity will be perpendicular to the line joining the instantaneous center of the body with the particle at that time. Of course, the instantaneous center may keep changing from time to time.

This shows a body in which there is a particle A and particle B. Instantaneous center is O. The particle A is at distance of r_A from the instantaneous center, particle B is at a distance of r_B from the instantaneous center. Therefore, the velocity of A is perpendicular is to the line OA and velocity of the particle B is perpendicular to the line OB. The magnitude of V_A is equal to omega times $r_{A,}$ magnitude of B is omega times $r_{B.}$ Therefore, the magnitude of the velocities is in proportion to the distance from the instantaneous center.

In this case, the figure has not been drawn to the scale $r_{A.}$ This gives us a clue to find out the instantaneous center of rotation; if one body is moving and the particle A is having velocity, draw V_A here. Similarly, particle is having the velocity B; draw and show a velocity V_B. I can draw a perpendicular to $V_{A.}$ Therefore, instantaneous center will lie somewhere on this line. Similarly, I can draw perpendicular to V_B and the instantaneous center will lie somewhere on this perpendicular. Both these lines intersect at A which provides us the instantaneous center of zero velocity.

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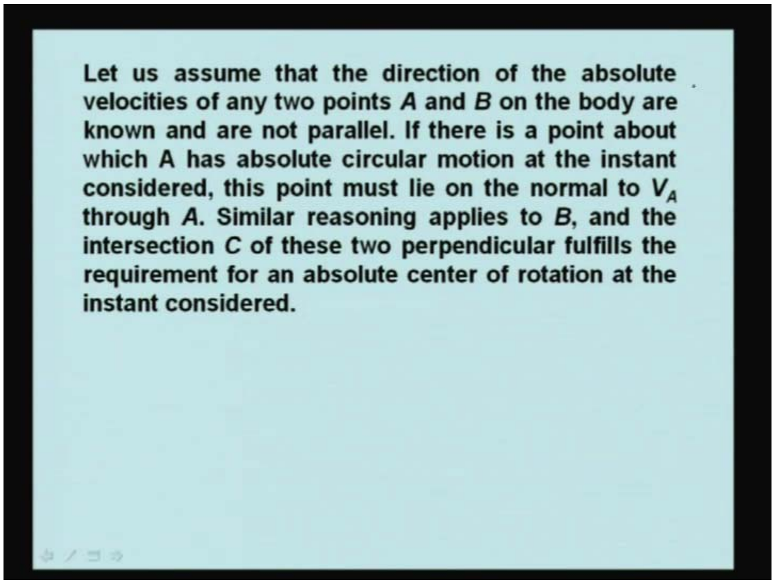

If velocities of any two points A and B on the body are known and are not parallel, then this method is very handy. If there is a point about which A has absolute circular motion at the instant considered, this point must lie on the normal to V_A through A. Similar reasoning applies to B and the intersection C of these two perpendicular fulfills the requirement for an absolute center of rotation at the instant considered.

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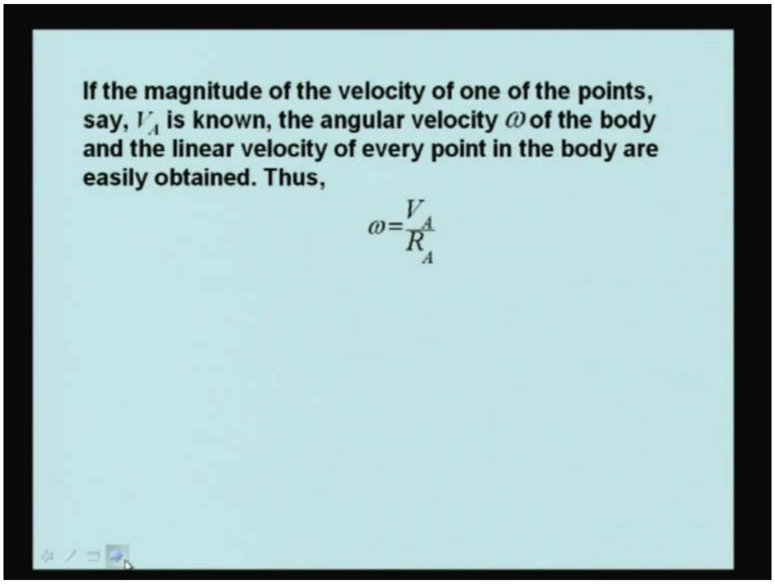

If the magnitude of the velocity of one of the points, say V_A is known, the angular velocity omega of the body and the linear velocity of every point on the body are easily obtained because omega will be equal to V_A by R_A. Once you have found that angular velocity of the point, then after that it is very easy to find out the velocity of any other point.

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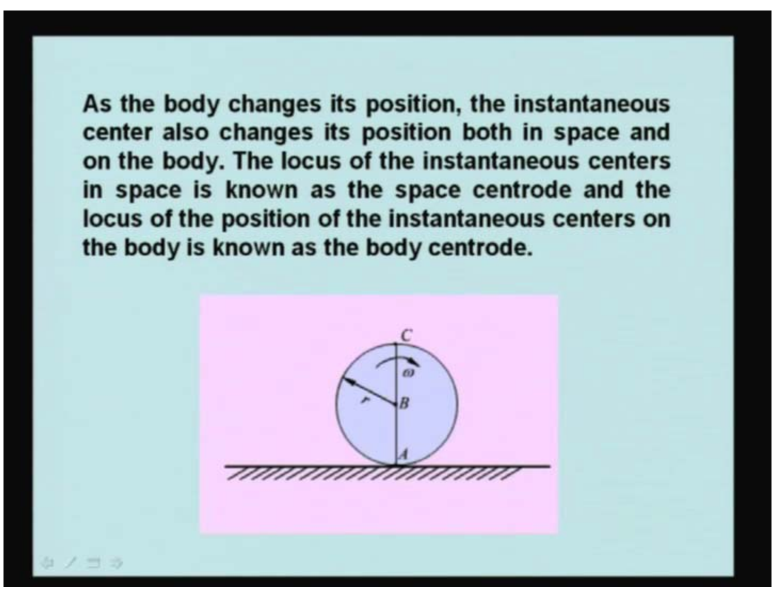

As the body changes its position, the instantaneous center also changes its position both in space and on the body. The locus of the instantaneous centers in space is known as the space centrode. If we draw the locus of the instantaneous centers in space that will be known as space centrode and the locus of the positions of the instantaneous centers on the body is known as the body centrode.

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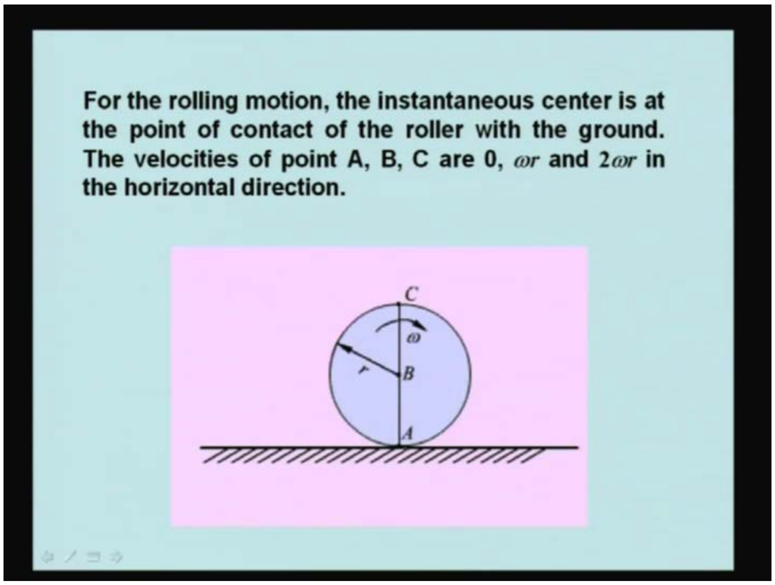

For the rolling motion, suppose the body is undergoing rolling motion then the instantaneous center is at the point of contact of the roller with the ground. It is in contact with the ground. The velocities of points A, B and C are 0, omega r and 2 omega r in the horizontal direction. If the body is not undergoing pure rolling, that means there is a slipping also; in that case, A will not have a 0 velocity and therefore, A will not be called instantaneous center.

In this case, this instantaneous center is changing its position. Its space centrode is always in the straight line because the position of A keeps on changing on the horizontal line. In the pure rolling motion, the body is not having any relative motion with respect to the pure. Therefore, A is instantaneous center. Let us discuss some other cases.

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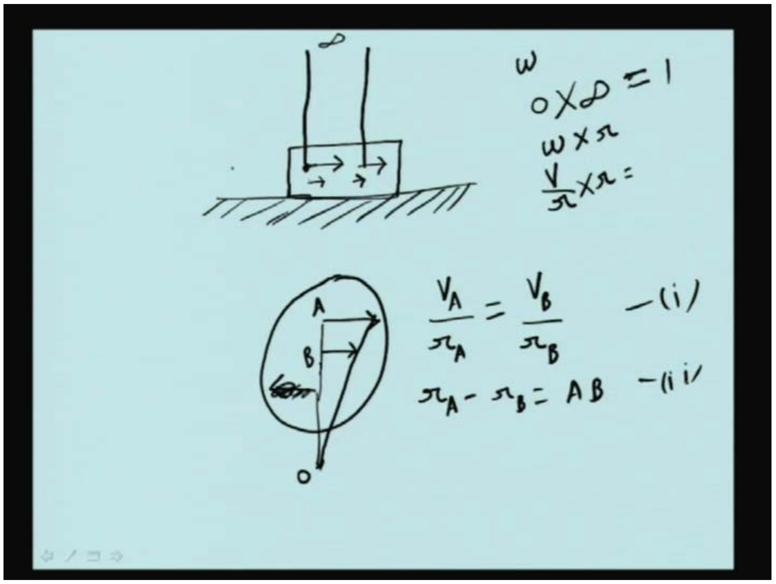

If a body is undergoing translation like this, then all the points on the body move with the same velocity. If we pick up two points on the body and draw lines perpendicular to the velocity of the point and passing through the point, the two lines will be parallel. Here, one perpendicular line, another perpendicular line; the two lines are parallel and they even intersect. However, we can consider that they intersect at infinity. Thus, the instantaneous center of rotation of a translating body is at infinity.

The angular velocity of the body, translating body is obviously 0; because, any line in the body does not rotate. So, the rotation, its angle, there is no change in the angular position. A typical particle in the body has the magnitude of the velocity equal to the product of angular velocity and the distance of the particle from the instantaneous center. Thus, the magnitude of the velocity in the case of a translating body will be 0 times infinity; 0 into infinity omega r, which gives a finite velocity. The product 0 into infinity may be any real number; actually, the velocity is omega into r.

If you say omega is equal to V by r into r, as r tends to infinity this tends to finite velocity V. So, 0 into infinity can be considered as 1 also, it can be considered 2 also. Thus, the translating body may have any velocity with 0 angular velocity and instantaneous center at infinity. In case you choose two points on the body on which the velocities are parallel but different in magnitude, for example this is the case; in this case, suppose this is the instantaneous center then the situation will be like this.

The instantaneous center will show you the line passing through point A and also through B. Let instantaneous center V at a distance r_A from B from A and a distance r_B from A, then $V_{A \text{ by}}$ r_A is equal to $V_{B \text{ by}}$ r_B and also $r_{A \text{ minus}}$ r_B is equal to AB, the distance which is known to you. From these two equations, if you know the ratio V_A by $V_{B,}$ you can determine r_A and r_B.

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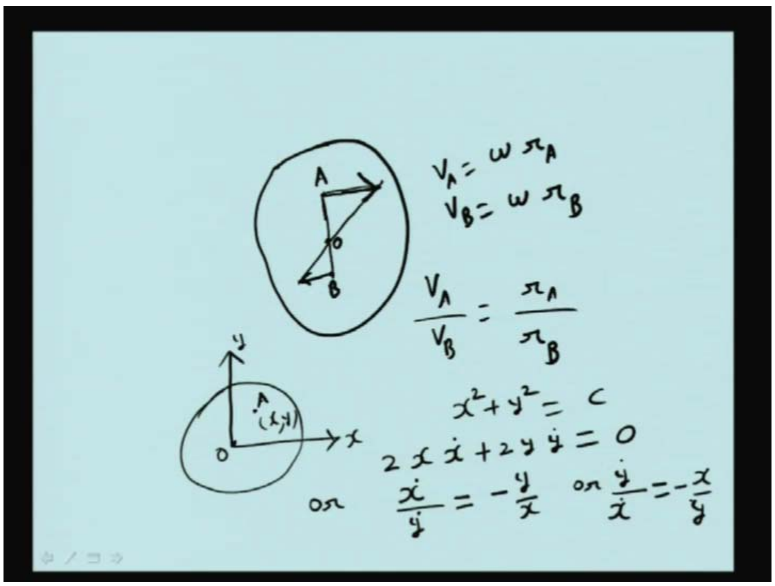

Consider the case, if the velocities of two points are parallel. A is a point, velocity is like this and B is another point in which the velocity maybe like this. They are parallel but opposite in direction. Their magnitudes may be different. In that case, join AB. The instantaneous center will lie somewhere in this line AB. It will divide the line in proportion to the magnitude of the velocities. If you join the tip of the velocity vectors like this, here is the instantaneous center O. Therefore, V_A is equal to omega times $r_{A,}$ V_B is equal to omega times r_B or $V_{A \text{ by}}$ V_B is equal to r_A by r_B where r_A is the distance of point A from O and r_B is the distance of point B from O. So, with this you can find out this thing. Finally, it is to be noted that only rotation causes the relative velocity between the two points in the body.

In translation, the relative velocity between the two points in the body is 0. It can be easily shown that in a rigid body the relative velocity between two particles of the body is in a direction perpendicular to the line joining the two points representing the body. For example, if there is a body and there is a point O on the body. Another point A on the body, whose coordinates are xy when the axis system is passing through O; therefore, in this case the distance OA is equal to x square plus y square. We have x square plus y square is equal to constant. Differentiate this with respect to time, you get two x x dot plus two y y dot is equal to 0. Or x dot is equal to x dot by y dot is equal to minus y by x or y dot by x dot is equal to minus x by y. We have got a relation like this.

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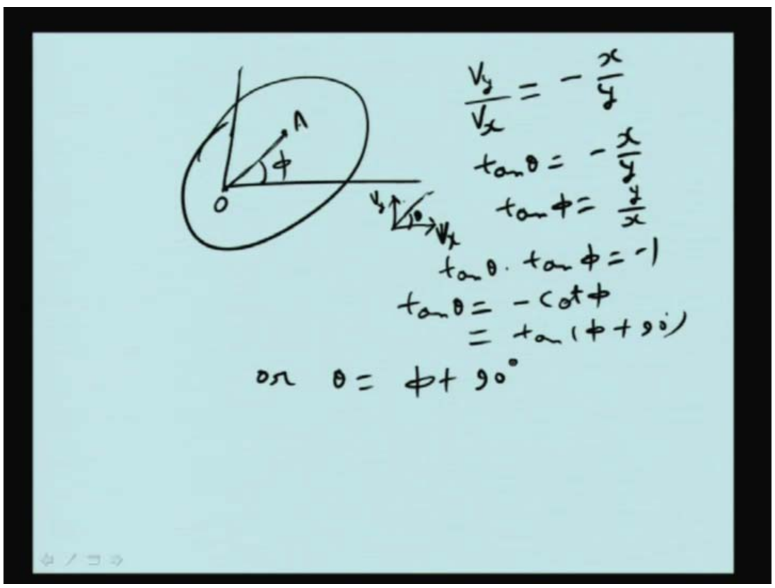

This is a body O; this is a point A. y dot is the V_y component of the velocity of point A. So, V_y and similarly, x dot is V_x. So, V_y by V_x is equal to minus x by y, which is the direction of the velocity. Suppose you have a velocity like this, this is V_y this is V_x. The velocity is making an angle theta from the x-axis, This tan theta is equal to minus x by y, the line joining OA. This is angle is phi. The tan phi is equal to y by x. Therefore, tan theta into tan phi is equal to minus 1. This shows that tan theta is equal to minus cot phi is equal to tan phi plus 90 degree or theta is equal to phi plus 90 degree.

That means, the velocity of the particle A is at 90 degree with the line joining particle A with O. Therefore, the relative velocity between any two particles of the body is in a direction, perpendicular to the line joining the two points representing the particles. We will carry out some simple examples based on this lecture.

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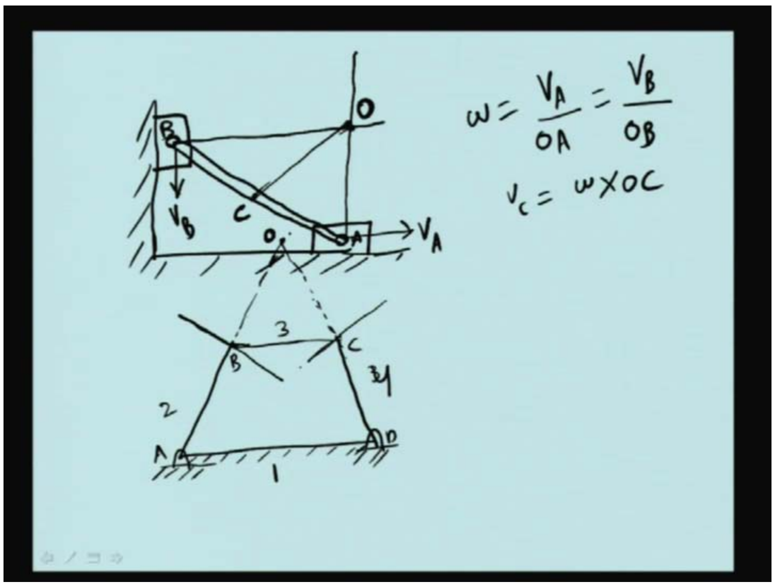

If you are having a mechanism like this, there is a slider here; join this by pin joined with a link. This one, this is slider A and this is B. A, can slide on the horizontal floor and B slides on the other, this perpendicular.

In this case, if you have a rod AB joined by pin, you are required to find out the angular velocity of the rod AB. How do we find out the instantaneous center of the rod? We know that the point A which is on the slider as well as on the rod, is having the velocity only in the horizontal direction. Therefore, this is V_A. Similarly, another point B which is on the rod; but since it is also on the block, it is constrained to move in the vertical direction. This is $V_{B.}$ Draw a perpendicular from $V_{A,}$ draw a perpendicular from $V_{B.}$ These two lines intersect at point O. Therefore, O must be the instantaneous center of rotation of the rod AB. Instantaneous center of rotation has been found.

If you want to find out the angular velocity of the rod AB, it is simple. It is V_A divided by OA. Omega is equal to V_A divided by OA or it is the same way that is V_B divided by OB. If you want to find out the velocity of any point C in between, this is nothing but omega cross OC. Join O and C. Velocity of C magnitude will be omega cross OC and the magnitude of velocity of C is perpendicular to the line OC. This way you could find out the instantaneous center.

Consider a four bar mechanism, this is the for one link which is fixed, another link third link and fourth link, 1, 2, 3, 4 A, B, C, D. What is the instantaneous center of rotation of link 2? It is obviously at A, because, all the time link 2 is rotating about 2. What is the instantaneous center of rotation of link 4? That is D. How do we find out the instantaneous center of rotation of link BC or link 3? B has got a velocity which is perpendicular to AB. So, it may have the velocity in this direction. C has got a velocity which is perpendicular to CD. So, it may have velocity like this. These two points B and C are also falling on link 3. Therefore, in link three we know the two points that is B and C, of which the direction of velocity is known. Draw the perpendicular to the velocities at points B and C on the link C, that is nothing but the extension of link AB like this and extension of link DC. Both extended meet at a point O, which will become the instantaneous center of BC.

Therefore in the four bar mechanism, you extend the link 2 and you extend the link 4 to meet at a point O; that will be called instantaneous center of the rotation of the link 3. This instantaneous center of course keeps on changing with time. Similarly, let us consider this link.

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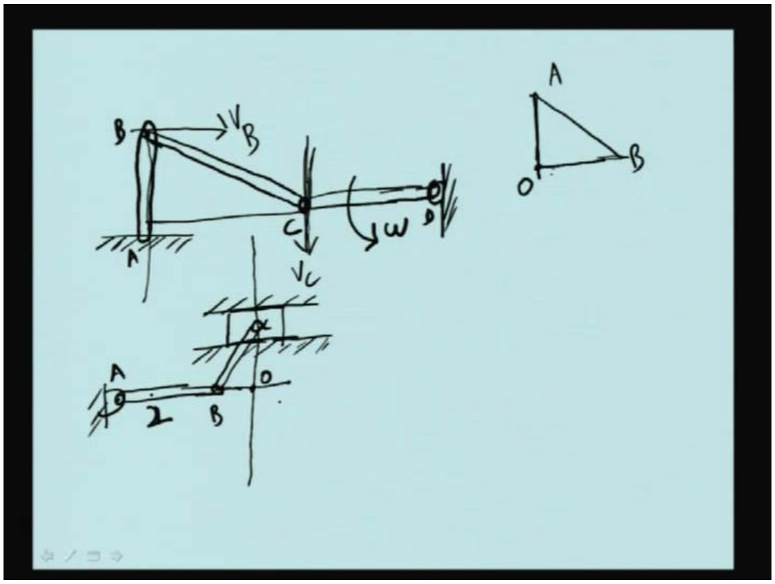

A B, this is second link BC, this is again basically a four bar mechanism shown in a different way; given angular velocity omega to link CD. When you rotate it like this, you have assumed that the rod AB is vertical at the instant depicted. In this case, if you have to find out instantaneous center of BC, how do we go about that? The direction of V_C is in this direction, V_B is in this direction. If you want to draw perpendicular to V_B on the line BC, that is like this, vertical and this is horizontal.

You draw a horizontal line passing through this one; that means, draw a vertical line here, draw a horizontal line here, this is AB. These are meeting at this point. Therefore, this is called the instantaneous center of rotation of this one. What happens, if we make some variant of a four bar mechanism? Second, third link and then you have got a slider. This is moving on a straight line like this. Instantaneous center of rotation of this link 2 is obviously at ABC.

Instantaneous center of rotation of the slider is at infinity. You can draw a line perpendicular to the motion of the slider and say that instantaneous center lies somewhere at infinity at this line. Now, instantaneous center of rotation of BC at the instant, the velocity of AB is in the B point, is in the vertical direction. Therefore, we know that the velocity of point B perpendicular to that can be this BC. Velocity of that point B is along perpendicular; therefore on line BC, I will draw a perpendicular to the velocity which is obviously along AB. Similarly, here I draw perpendicular to the velocity which is along BC. These two meet at the point O. Therefore, in this case O is the instantaneous center of rotation. This is the thing about the instantaneous center of rotation. Today, I have discussed about the instantaneous center of rotation about the plane motion and about the angular velocity. You can do the other numerical problems based on this lecture.

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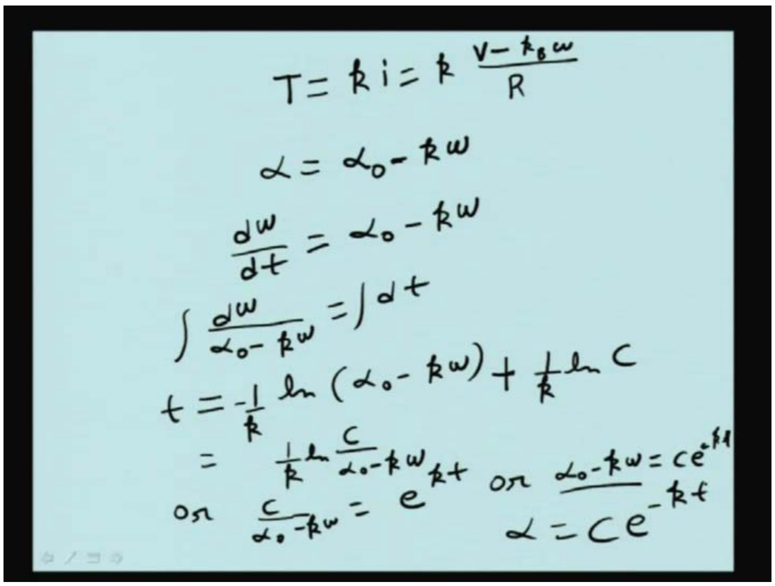

For example, if you know the angular acceleration alpha is equal to V minus omega minus k alpha. Let us say that in a typical motor, when you start a motor you get a torque. This torque is equal to, is proportional to I; that means k times i current. This is k times VB, V applied voltage minus k_V omega divided by R, where this is by that torque divided by moment of inertia gives angular acceleration. So, you get some expression for angular acceleration like this; alpha is equal to minus k omega. If you get that means, its angular acceleration keeps decreasing as the omega increases.

Therefore in this case, if you want to find out whether this particle, this rotating body will approach a steady state motion or not, in this case, it will attain the steady state at time t is equal to infinite. Because, this is d omega by dt is equal to alpha 0 minus k omega. Therefore, you get d omega divided by alpha 0 minus k omega is equal to dt or t is equal to alpha 0 minus k omega minus one by k plus C ln one by k ln C, where C is some constant.

Therefore, this can be written as 1 by k ln C divided by alpha 0 minus k omega or C divided by alpha 0 minus k omega is equal to e to the power kt, or alpha 0 minus k omega is equal to C e minus kt, Which is exponential relation at t is equal to infinite. This portion will become 0 and when alpha 0 will be k omega; that means the final velocity will be reached. At that time, the angular acceleration will be really 0. Angular acceleration alpha at any instant is Ce minus kt. Therefore, in this case, the angular acceleration becomes 0 at infinity. Like that, a number of numerical problems you can solve based on the lecture. In all these things, you may have to use the differential calculus.