Engineering Mechanics

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Module – 9
Virtual method and energy method -2
Lecture - 22
Stability of Equilibrium

In today's lecture we will see a method of analysis for the stability of equilibrium. The potential energy method that we discussed in the last lecture is useful in analyzing the stability of equilibrium of a rigid body or a system of rigid bodies. For your reference, this is the lecture number 22 of the engineering mechanics course.

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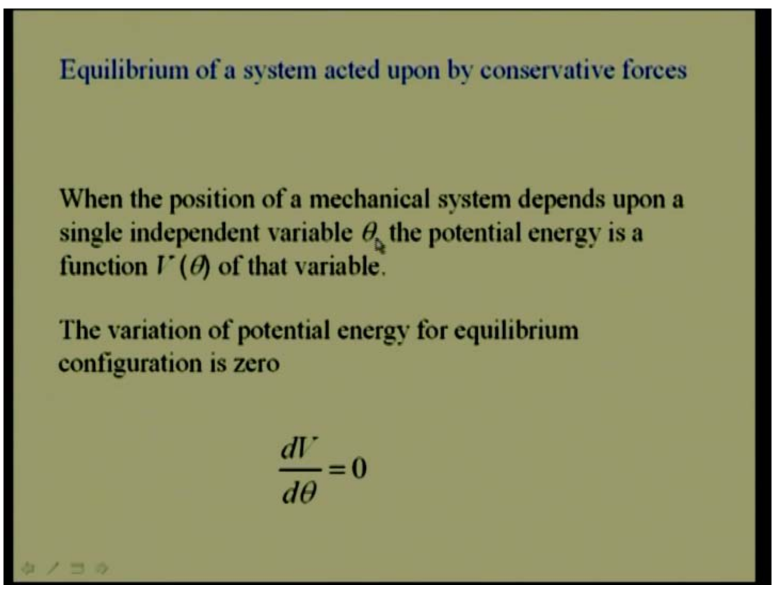

Let us first again go through what we have just learnt in the last lecture that when we have system in equilibrium which is acted upon by only conservative forces like we have seen the gravitational force, the elastic force which are conservative forces. We can associate a potential energy associated with these conservative forces and if the mechanical system depends upon an independent variable theta then the potential energy associated with conservative forces for a particular configuration defined by this theta is a function which is V of theta of that variable. The equilibrium equation can be replaced by this equation that change in the potential energy is 0 that is dV by d theta for the equilibrium configuration is 0 - this is what we saw in the last lecture.

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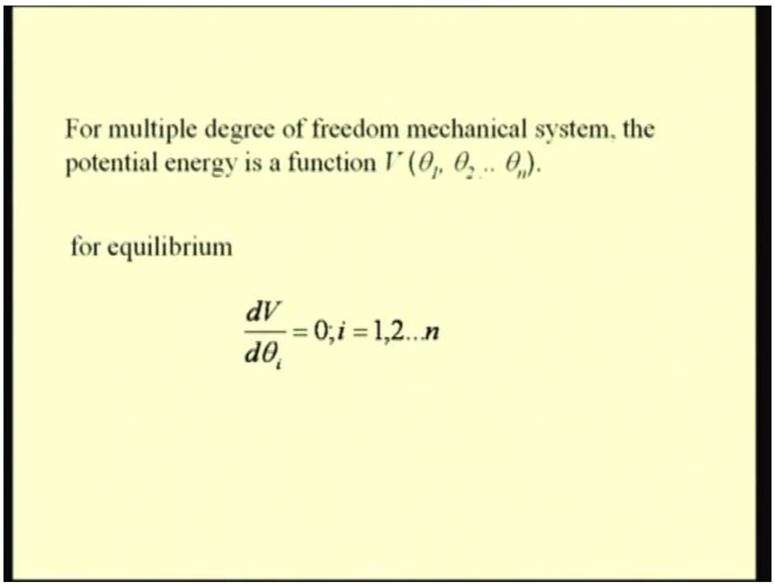

If we have multiple degree of freedom mechanical system then the potential energy is a function of each of those individual independent coordinates which define the configuration of the system. If theta_1 theta_2 up to theta_n are the various independent coordinates defining the configuration of the mechanical system then the potential energy is a function of all these independent variables. The equilibrium equation in that case becomes dV by d theta_i equal to 0 where i is from 1, 2 etc., up to n. We have n equations in case of an n degree of freedom system and these n equations can be used to determine the various forces. This was the discussion that we had in the last lecture. We also saw some problems applying this method of potential energy to solve equilibrium problems. In this lecture, we will see how we use this and extend this to analyze the stability of equilibrium.

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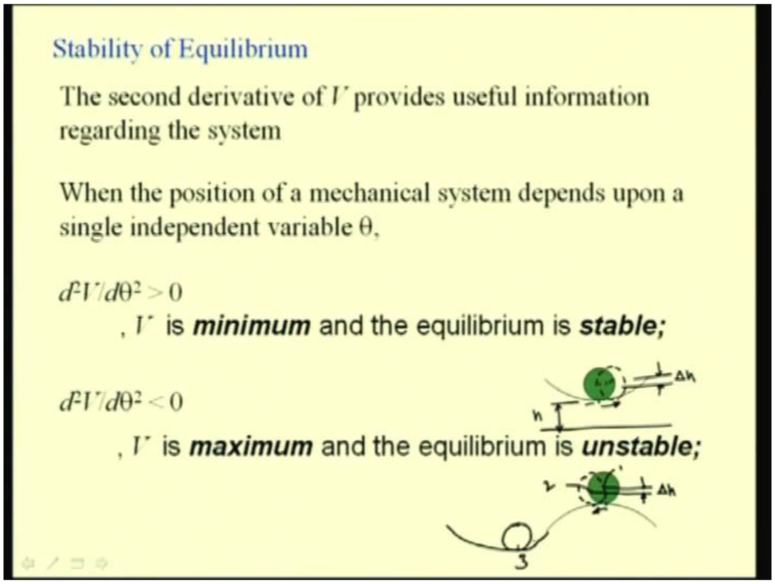

The second derivative of this potential energy function, in case if it is a single degree of freedom system having a coordinate theta then, V is a function of theta. The second derivative of this function provides useful information regarding the stability of the equilibrium. If the mechanical system depends upon the independent variable theta, the configuration of the system depends on the independent variable theta then the second derivative is d square V by d theta square and the value of this particularly the sign of this quantity is useful in analyzing the stability of equilibrium. If it is greater than 0 then the potential energy is minimum and the equilibrium is stable.

This can be further explained using this analogy of a sphere trying to roll in a surface. If the sphere is slightly displaced to a new position say it is displaced from this point to this point the potential energy of the sphere increases. If we have this as the datum, originally the mass center or the center of gravity lies here. So, now it moves to a new position. Thereby, there is a change in the height of this center of gravity which increases the potential energy associated with this sphere. This has a tendency to bring it back to its original position. So, this equilibrium position is stable; in the sense if we displace this sphere by small amounts it comes back to its equilibrium position. which is defined by this point which is the… If this height is h, then this gives the least potential energy for the system. So, the system always comes back to occupy that position where the potential energy is minimum. The same is the case if we displace in the other direction also.

If we have this condition that d square V by d theta square is a positive quantity then the potential energy is minimum and the equilibrium is a stable equilibrium. In the sense, at this position the sphere is in equilibrium and also for small perturbations the sphere comes back to the same equilibrium position. If this sign of this quantity d square V by d theta square is negative that it is less than 0 then V is maximum and the equilibrium is unstable.

This is depicted by a sphere situated on a surface but for this position the sphere is in equilibrium. But if it is displaced by small amount say it is moved from this position to this there is a small change. Here, in the earlier case we saw that the CG rises; so potential energy rises when there is a small disturbance and the system always tries to come back to the least potential energy configuration. So, again it came to the original position. So, we said that the equilibrium is stable. But we see in this case, for the displaced position given by these dotted lines the change in its mass center reduces the potential energy for this new configuration. Let us say this is state 1 and this is state 2; for this position 2, the sphere has a lower potential energy. Thus, the sphere further tries to move down so that its potential energy keeps reducing. So, it does not have a tendency to come back to its original position.

We see that the sphere is in equilibrium for this position 1 but its equilibrium is unstable; that means for small perturbances the sphere starts to roll out and it does not come back to the same equilibrium position. It tries to go and settle in another equilibrium position which could be something like this where the sphere will go and take a new equilibrium position say 3 where again the sphere is in equilibrium and here it is in stable equilibrium. This quantity that is d square V by d theta square, the second derivative of the potential energy provides this useful information about analyzing the stability of equilibrium of a mechanical system.

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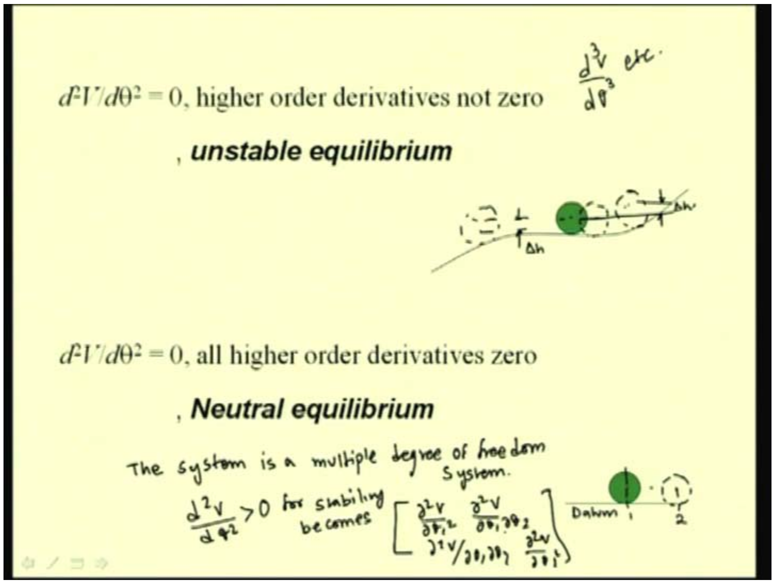

Let us see few more cases. If this quantity d square V by d theta square is 0 and higher order derivatives are not 0; that means the quantity d cube V by d theta cube etc., are not 0. In that case, the system is in unstable equilibrium, what we call as an unstable equilibrium where you have the sphere situated in a surface which has a a local plateau. If this is the potential energy function then for small regions the potential energy of the sphere remains the same. We see that for perturbances which are on this side, for certain virtual displacement, say over here the potential energy increases; whereas for other kinds of virtual displacements the potential energy decreases. Though there is a stability of this equilibrium for small vicinity around the current equilibrium position, this equilibrium is still an unstable equilibrium because for large displacement the system does not regain back the original equilibrium position and it starts to roll down to occupy a new equilibrium position.

We can have another kind of a situation where the second derivative that is d square V by d theta square is equal to 0 and also all higher order derivative terms are also 0. In that case, we call that it is in neutral equilibrium. That means it is like a sphere which is kept on a flat surface which is flat infinitely. So, for all displacements of the sphere to any new configuration, we see that the potential energy of this, if this is the datum, does not change. This position again is an equilibrium position. If this position is 1 and this position is 2, the position 1 as well as position 2 has same potential energy and both the positions are equilibrium positions. In fact, any position occupied by this sphere is an equilibrium position. The sphere does not have again a tendency to come back to any of the equilibrium position if we disturb the sphere. So, such an equilibrium configuration is known as neutral equilibrium.

The equilibrium state of a system can be analyzed whether it is stable or it is unstable or it has a neutral stability by considering this second derivative that is d square V by d theta square and by analyzing the sign of this. This we have seen for a single degree of freedom system. For multiple degree of freedom system, we can formulate this as if the system is a multiple degree of freedom system then the condition that d square V by d theta square should be greater than zero for stability becomes …. the matrix for a two degree of freedom system, dow square V by dow $theta_2$ square, this matrix should be a positive definite matrix. It is possible to write a similar kind of a matrix for higher degrees of freedom system and derive the stability equations as discussed for the two degree of freedom system. Let us first see a single degree of freedom system example and analyze the stability of the equilibrium.

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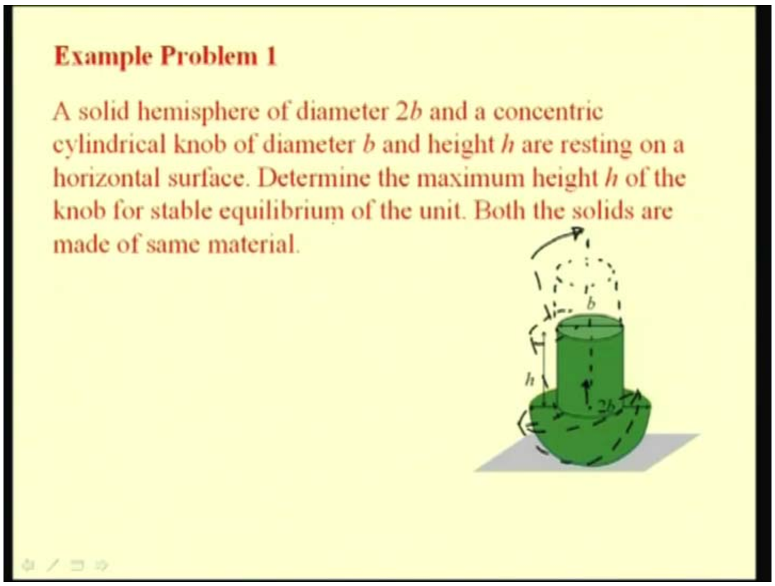

Here you have a solid which is composed of a cylindrical knob and a hemispherical base. The diameter of the hemispherical base is 2b which is twice the diameter of the cylindrical portion. The height of the cylindrical knob is h. We are interested to determine the maximum height h of the knob for stable equilibrium of the unit. Let us assume that both the solids are made of same material.

If this height h keeps on increasing that is if the height of this knob keeps on increasing, the CG of this entire unit which is originally situated somewhere here along the axis; obviously, it will be lying along the axis because both these objects are symmetrical about this axis, so the CG will lie along this axis. This CG will keep rising and for small disturbances like if this entire unit is displaced like this, then it may not come back to its original equilibrium position and it may just fall off. We are just interested to find the maximum height h of the knob that will keep this entire unit in equilibrium position.

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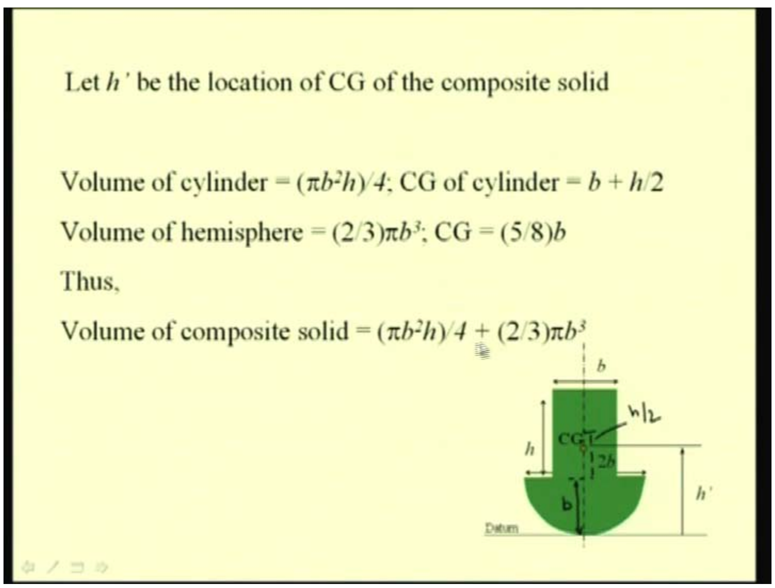

Let us try to first find the expression for the center of gravity for this composite solid. Let us define a datum as described in this picture. Let h prime be the location of the CG of this composite solid composed of the cylindrical portion and the hemispherical portion. The various dimensions are given; like h being the height of the cylindrical portion, b the diameter of the cylindrical portion and 2b the diameter of the hemispherical portion. Let us first compute or get the expression for this h prime. This we obtained by decomposing this object into two simple shapes that is one the cylinder and the other the hemisphere for which the location of the CG and the volume is known. The volume of the cylinder is pi b square h divided by 4 and the CG of the cylinder is b which is the radius of the hemispherical portion plus h by 2. So, up to this, this is b and we have h by 2. If this is the location of CG of the cylindrical portion then this is h by 2 and we have the CG of the cylinder located at b plus h by 2. The volume of hemisphere is two-third pi b cube and the CG is located at five by eighth of b.

So, somewhere here from the datum in order to compute the CG of the composite solid, we first compute the volume of the composite solid which is equal to sum of these two volumes that is the volume of the hemisphere and the volume of the cylinder, which is equal to pi b square h by 4 plus 2 by 3 pi b cube.

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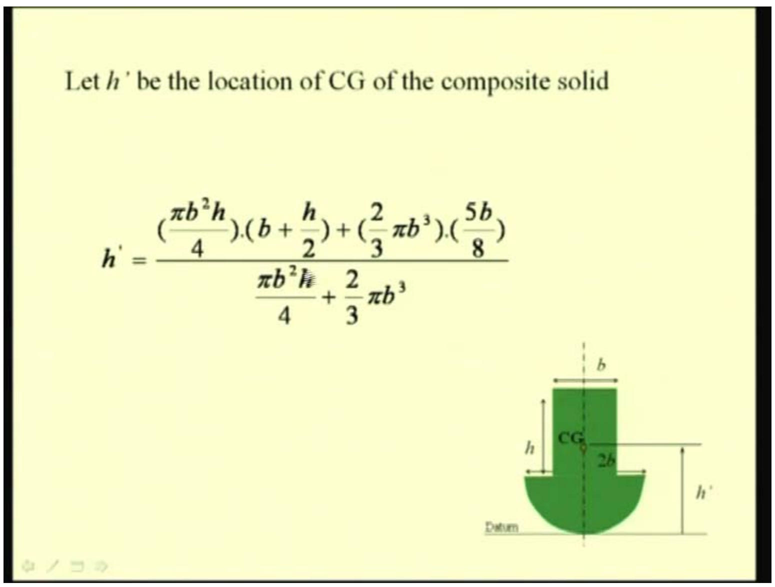

The location of the CG h prime is equal to the sum of these two quantities that is the volume of the cylinder into its location of the CG which gives the first moment plus the volume of the sphere times its location of the CG divided by the volume of the composite solid. From this expression, we get the expression for h prime. If we simplify this, we will get the required expression.

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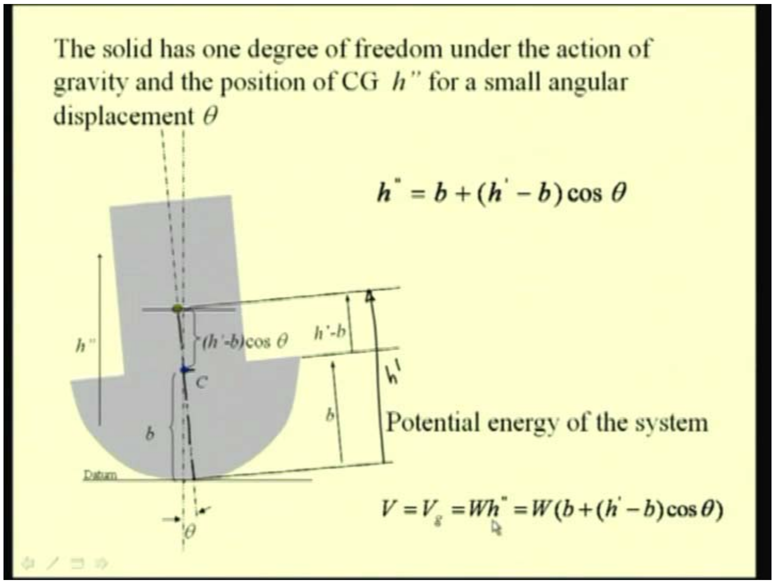

Let us consider a slightly displaced position of this composite solid from its equilibrium position and analyze how its CG moves because the potential energy of this system is the gravitational potential energy. It depends only on the location of the CG of the composite solid. In this picture, you see the same knob in a slightly displaced position; that means the point of contact which was originally here has been displaced to a new position because of a small angular displacement, theta. If we say that c is the point on the diametrical plane of this hemisphere then this point is at a distance of b because the object when it is tilted by this angle theta will roll with this as its instantaneous center. This distance will be the radius of the hemispherical portion that is b.

Let us see what will be the position of the CG of this object for this small angular displacement theta. This portion we have seen that it is b; again the point corresponding to the original equilibrium position. This portion is nothing but h prime minus b because this is the location of the CG and this axis represents the location of the CG in the undisplaced position that is when this knob was in the vertical position. This total height is h prime and so this distance is h prime minus b. If this angular displacement that we are considering is theta then this distance that is marked on the present vertical axis passing though the point of contact between the composite object and the datum surface is h prime minus b times cos theta. The location of the CG with respect to the datum if we term it as h double prime is equal to this distance that is b plus h prime minus b cos theta. This expression gives the location of the CG of the composite solid with respect to the datum surface for any angular displacement, theta.

We can write the potential energy function in terms of this h prime. V the potential energy of the system equal to the gravitational potential energy of the system that is V_g is equal to the weight of the composite body times the location of its CG that is h double prime which is equal to W times b plus h prime minus b cos theta. This gives the expression for the potential energy of the system.

Let us first find the equilibrium position and then try to analyze the stability of this equilibrium for this equilibrium position.

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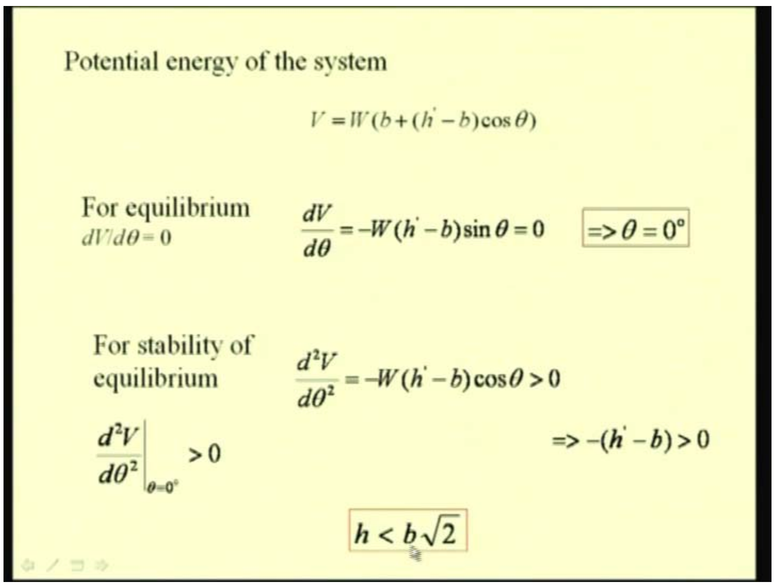

We have this, the potential energy function that is V equal to W times b plus h prime minus b cos theta. To find the equilibrium position, we put dV by d theta equal to 0. First, we differentiate this function and equate it to 0. The differentiation of this is minus W times h prime minus b times sin theta because this term is a constant term and what we have is the differentiation of the term W times h prime minus b cos theta. If you equate this to 0, these terms being constant that is W, h prime and b this term becomes 0 only if this term becomes 0 that is sin theta becomes 0, which gives that theta equal to 0 is the equilibrium position; that means the vertical position is the equilibrium position which also comes from our first visualization of the knob and its placement.

Let us see the stability of equilibrium which can be analyzed by seeing the sign of this quantity that is d square V by d theta square. At theta equal to 0 that is this equilibrium position, we have to see if the sign of this is greater than 0 for the equilibrium to be a stable equilibrium, or if you want to derive the condition, we have to put the condition that the second derivative has to be a positive quantity. The second derivative is the differentiation of this again, which is minus W h prime minus b times cos theta and this has to be greater than 0. This implies that h prime minus b has to be greater than 0, because at theta equal to 0 cos theta is 1 and W does not have any effect on the sign of this quantity. So, this quantity that is minus h prime minus b has to be greater than 0. If you substitute the value of h prime in terms of the height of the cylinder then we have height of the cylinder should be less than b times root 2.

This gives the expression for the height of the cylinder in terms of the radius of the hemisphere that is b such that the composite solid is in equilibrium and its equilibrium is also a stable equilibrium. Let us take another example in which we have both the potential energy as well the elastic potential energy term.

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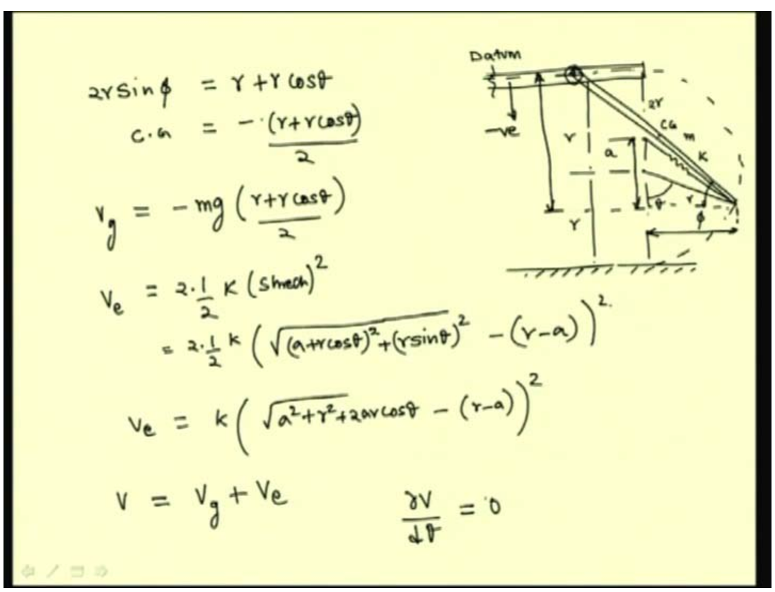

In this picture, you see a cross section of a car garage. This is the door of mass m which is attached by two springs whose spring constants are k to this point. It is having one link of length r in order to guide the motion of this car garage door. It is supported by a roller and so it can be opened. These dotted lines show the …… by this end of the door as the door is lifted. This is the completely opened position and this represents the closed position of the garage door. We have two sets of such springs, one on each side to support the door. We are interested to know the stability of the equilibrium of this door in the closed position. That is, the spring constant of this spring if not properly chosen may lead to instability at this point; that means, for a small displacement the door may swing open and this is not a desirable effect.

We would like to make sure that the spring constant is chosen such that this door when it is kept in the closed position that means for this angle theta to be 0. As this door is opened this link will have theta equal to 0 or it will be vertical. As the door is completely opened, this link will be in the vertical position. We are interested to know that when this theta is equal to 0 or for this door in the completely closed position the door is having a stable equilibrium. Let us fix this as our datum that is the top slot in which this roller slides as our datum and measure the CG of this door with respect to the datum.

For any given position theta if this angle is phi, the angle between the garage door and the horizontal if this is phi, then the location of the CG with respect to this datum point is half times 2r times sin phi that is the vertical projection of this door on the vertical direction. Let us write that equation. We have 2r sin phi which is equal to r plus rcos theta because this distance which is rcos phi is equal to r plus this distance which is equal to r times cos theta. So, the CG is located at minus r plus rcos theta divided by 2, half of this distance; because this is the datum this is in the negative direction. So, this CG is located at a distance of minus r plus rcos theta by 2. We can write the potential energy in terms of this quantity. V_g is equal to minus mg, the weight times the location of the CG. This gives the potential energy associated with the car garage door for any given position theta.

Same way, let us write the potential energy associated with the spring. It is known that for the completely open position that is when this door is in this position, the spring is in the un-stretched position that means the un-stretched length is r minus a. For any given position, we should find the stretch of the spring in order to compute the elastic potential energy associated with the stretch of the spring. For that, we compute it as 2 times; because we have two such springs 2 times half k that stretch square, which is equal to 2 times half k the stretch is root of a plus rcos theta square plus rsin theta square. This is the distance rsin theta, the horizontal distance of this point. This distance is rsin theta and this distance is a plus rcos theta. This is rcos theta and this is the distance a. So, this length the square is equal to this length square plus this length square that is what is written. So, the stretch or the length of this spring is a plus rcos theta square plus rsin theta square minus r minus a which is the original length of the spring that is r minus a the whole square. So, now we have got the expression for the potential energy as well as the potential energy associated with gravity as well as the elastic potential energy.

We can try to simplify this and write it as k times root of a square plus r square plus 2 a rcos theta minus r minus a whole square. This is the elastic potential energy. The total potential energy of the system is V_g plus V_e. So now, this is a function of theta. We see that V_g is a function of theta as well as V_e is a function of theta. So, in order to find the equilibrium position we have to find this dV by d theta and equate it to 0.

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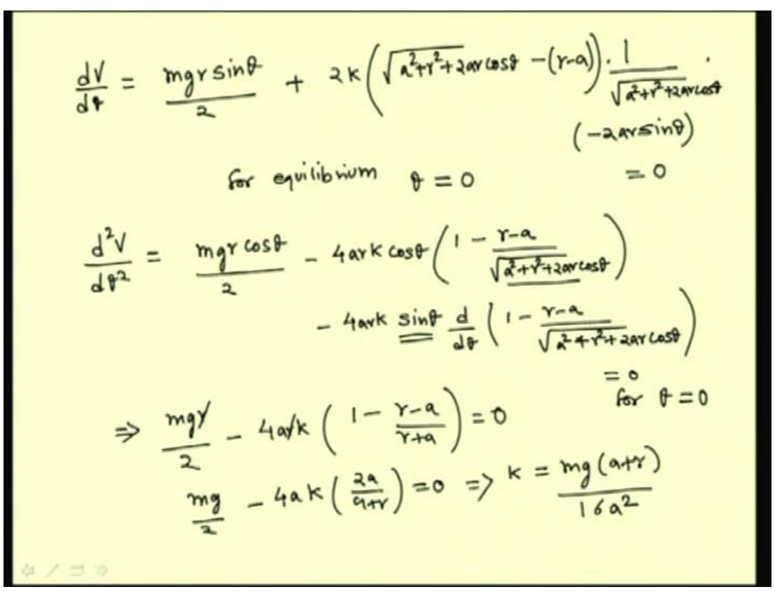

If you will differentiate this dV by d theta is equal to mgr sin theta by 2, the differentiation of the gravitational potential energy term plus 2k times root of a square plus r square plus 2 a rcos theta minus r minus a times the differentiation of this quantity which is 1 over root of this term that is a square plus r square plus 2ar cos theta times minus 2ar sin theta. This is the value for dV by d theta. This we have to equate to 0 for the equilibrium position.

We see that, this is equal to 0 if this term becomes 0 that is sin theta has to be 0 or in other words theta has to be 0. This term also we see that, the product of this times the product of this times the product of this term, here also we have the sin theta term. So we clearly see that, for equilibrium the condition is theta has to be 0. So theta equal to 0 is nothing but the closed position of the door and this is clearly an equilibrium position. Let us see whether this equilibrium is stable or under what condition this equilibrium is stable because that is our interest. We are interested to find the stability of the equilibrium for the closed position of the door. Let us find this quantity that is d square V by d theta square the second derivative.

Let us differentiate the terms that we have got for dV by d theta. This is equal to mgr cos theta by 2 minus, let us try to differentiate this term. If we will try to simplify this term and then differentiate it we will have this as 4ark cos theta times 1 minus r minus a divided by root of a square plus r square plus 2a rcos theta minus 4a r k sin theta then d by d theta of this term that is 1 minus r minus a divided by root of a square plus r square plus 2a rcos theta. This term, anyway we are not differentiating. For the stability, here we are interested that the door has to be insensitive to any disturbances; that means the second derivative has to be also 0, if we equate this to 0, for the position theta equal to 0. We know that for the position theta equal to 0 this term becomes 0 because we have this sin theta term and this term, since we have this cos theta these terms become 1 and this entire term becomes mgr divided by 2, this first term because cos theta for theta equal to zero is 1 minus 4ark cos theta is again 1. Here in this case cos theta is 1. We have this a square plus r square plus 2ar which is a plus r square root of that is nothing but a plus r. If we simplify this we have theta as 1 minus r minus a divided by r plus a should be 0.

Let us try to rearrange these terms mg by 2 minus 4ak times 2a by a plus r equal to 0, the common terms that is this r has been cancelled and this term has been simplified. Thus, we have this as r plus a minus of r minus a divided by r plus a, which is 2a by a plus r. If we further simplify this, we have k is equal to mg times a plus r divided by 16 a square; that means the spring constant of each of these springs has to be equal to the mass of the door times g times a plus r, the various dimensions associated with the garage door divided by 16 a square.

This problem illustrated how we can use the concept of the stability of equilibrium and analyze the same using the potential energy function and the derivatives of this potential energy function with respect to the equilibrium positions. Let us take another example and see a system which has two degrees of freedom and how to analyze the stability of the equilibrium of multi degree of freedom system.

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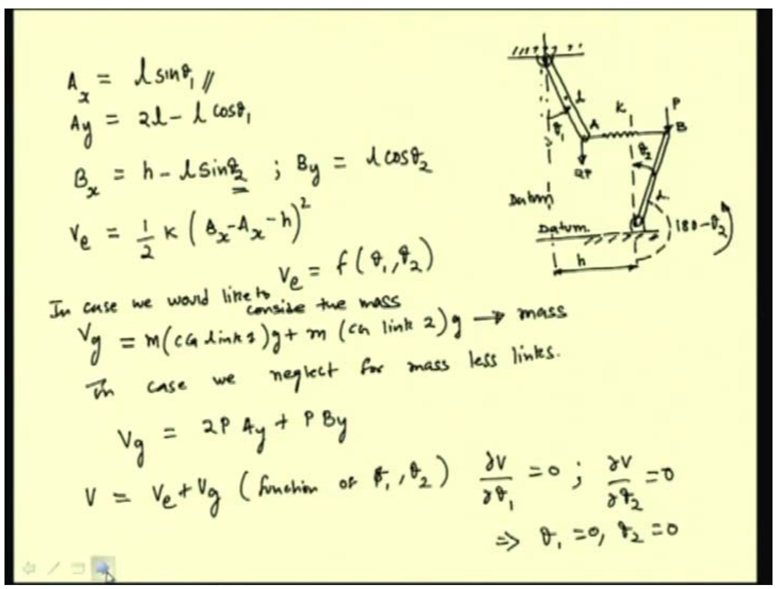

Here we are taking up a system with a two degree of freedom. You see a mechanical system constituting of two links of length l which are connected by the spring. They are pinned at these two positions. So, the vertical difference between the points of connection of these two links is equal to 2l and the horizontal separation is h. It is also known that the spring is unstretched for the vertical position that is when the links are vertical that is, this link and this link are both vertical then the spring is in unstretched condition. Then we see that, the particular configuration of this system is defined by both these angles theta$_1$ and theta$_2$ which have to be individually specified to specify the configuration of the system completely.

This is a two degree of freedom system with $theta_1$ and $theta_2$ as the coordinates that have to be specified. The potential energy of this system, if we neglect this mass and if we only consider the elastic potential energy then, the stretch of the spring for any position is governed by this motion for this $theta_1$ as well as for $theta_2$. Let us also additionally have the assumption that for small disturbances the spring is stretched only in the horizontal direction; that is there is no misalignment of this spring because of this small perturbation of $theta_1$ and $theta_{2.}$ The spring is aligned in the horizontal direction itself. In order to derive the quantity for the stretch of the spring, we should know the coordinates of this a and the point b. Let us try to write with respect to some datum. We have taken this as the datum for measuring the horizontal displacement and this as the datum for measuring the vertical displacement. We can write A_x the horizontal distance at which this point A is located as l sin $theta_{1.}$ This is the length of the link and $theta_1$ is the angular displacement of this link. The horizontal displacement of this point A is l times sin $theta_1$ and the vertical position of this point A from this datum is equal to 2l which is this total length that is from this point to the datum; that is the point where this link is hinged that is at the distance of 2l minus l times cos $theta_1$ that is cos of this link length l. That will give the vertical position of this point A for any displacement $theta_1$.

Same way, we can write B_x as h minus l sin $theta_2$ where h is the horizontal position of this pin connection from the datum that we have chosen. So, this is the datum for measuring the x- axis minus l sin $theta_2$ because this is measured in the counter clockwise direction it has this negative sign. Otherwise, we can measure this from the datum and so this will be 180 minus $theta_{2.}$ We have this quantity as minus l sin $theta_2$ because position of these links has to be measured from the datum in the positive sense which is 180 minus $theta_2$ h plus l times sin 180 minus $theta_2$ which is nothing but minus l sin $theta_2$. We can write the potential energy. Same way, we can write the vertical position as l cos $theta_2$.

The potential energy is the potential energy associated with the mass and with the elastic spring, but here we are assuming mass less links. So, the potential energy is only the potential energy due to the elastic force that is the spring force k. The stretch of the spring is nothing but B_x minus A_x the horizontal position of B and A minus this h which is the original length of the spring. We can write this as half k times stretch square which is B_x minus A_x minus h whole Square.

We see that, this is a function of both $theta_1$ and $theta_2$ because B_x is written in terms of $theta_2$ and A_x is written in terms of $theta_1$. We have this V_e as a function of $theta_1$ as well as $theta_2$. If you would like to include the potential energy associated with the mass then let us assume that the masses are acting through its CG of these links. Already we know these positions A_y and B_y. We can also write V_g. In case we would like to consider the mass then we can write this V_g as the mass of this link times this position the CG of link 1 plus times gravity and CG of link 2 times g.

If we see that these two forces that is the force P at B and the force 2P at A both are vertical forces and thus they are conservative forces. In case we neglect this above term that is this term due to the mass for mass less links, then we have this V_g as only the force 2P times A_y plus P times B_y. The energy associated with these two conservative forces that is the force 2P at A and P at B. The total potential energy V is written as V_e plus V_g which is a function of $theta_1$ and $theta_2$. So, for equilibrium this has to be 0 and this has to be 0 which implies that $theta_1$ is 0 and $theta_2$ is 0.

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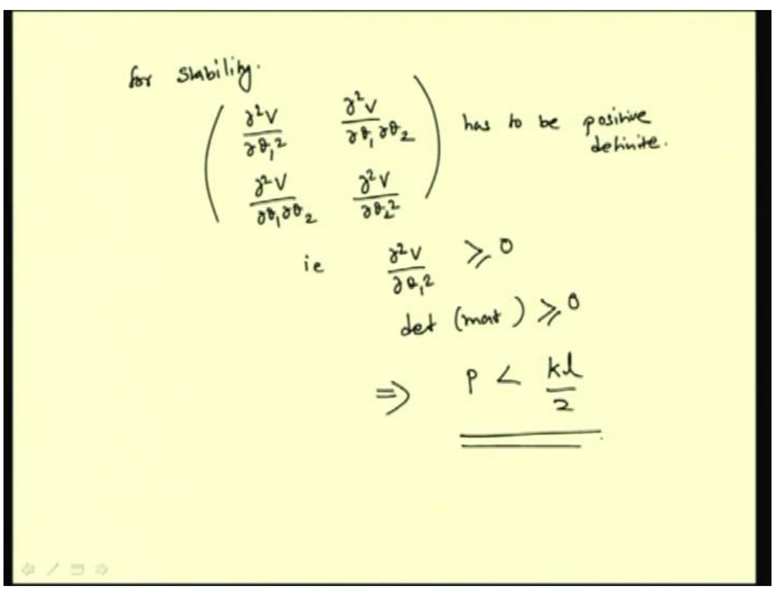

For stability, we have this quantity that is dow square V by dow $theta_1$ square dow square V by dow theta 1 dow $theta_2$ dow square V by dow $theta_1$ and dow $theta_2$ and dow square V by dow $theta_2$ square has to be positive definite that is dow square V by dow $theta_1$ square is greater than 0 and the determinant of this matrix has to be greater than 0. This implies if we solve that P has to be less than k l by 2. So, this gives the condition for the stability of these two links. In this lecture, we saw how to analyze the stability of equilibrium of structures using the potential energy functions.