Engineering Mechanics

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Module - 9
Virtual method and energy method -2
Lecture - 21
Potential energy

In today's topic on engineering mechanics, we will discuss a new method of analysis which is known as the method of potential energy. So, the discussion will be extended from our understanding of this virtual work principle and the way we have used this virtual work to analyze equilibrium or to analyze for the efficiency of a real mechanical system.

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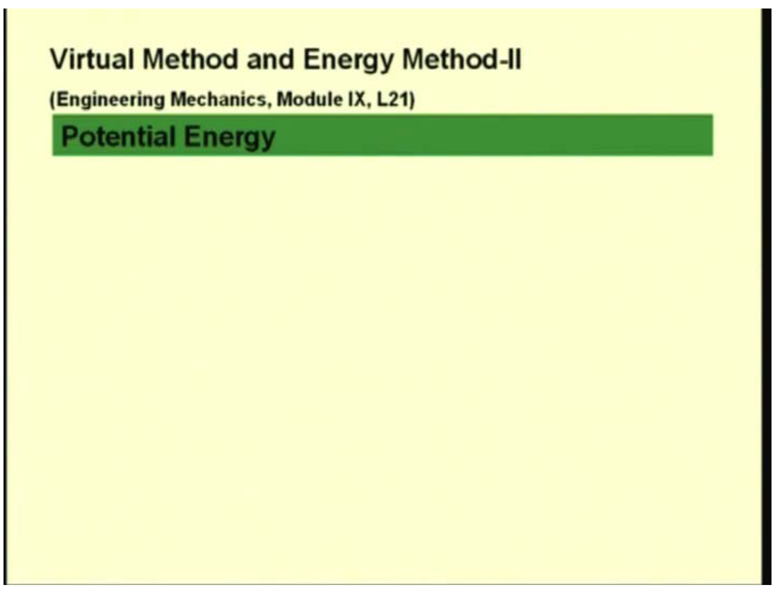

For your reference, this is module 9, lecture number 21 of the engineering mechanics course. Before we have an understanding of the method based on this potential energy, we have to associate the potential energy to certain class of forces what we call as conservative forces. We have to understand the energy associated with the work done by the conservative forces. Before we go to the discussion, we will first define the energy or the work done for a finite displacement by a force.

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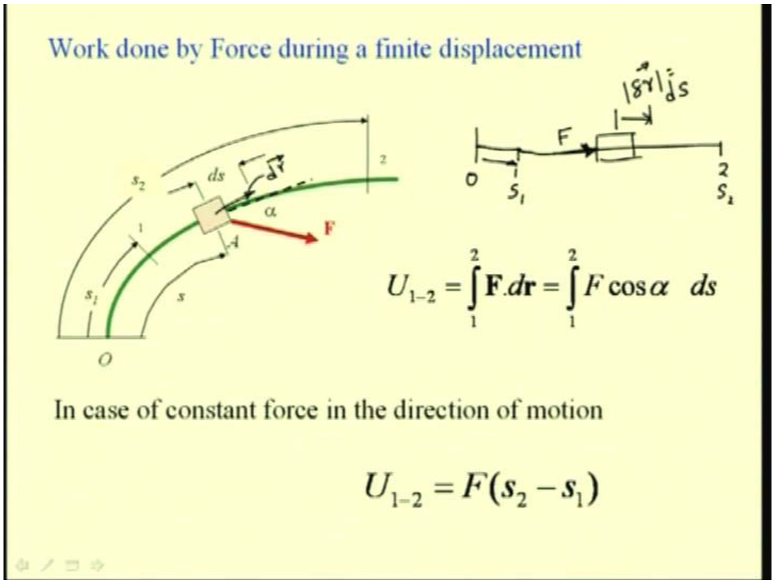

Let us consider an object or a particle that is being acted by this force F and the body or the particle moves along this path when this force F is being applied. So, we define an origin O. The position of this body along this path is defined by its path coordinates, S. So, S is the coordinate that specifies the location of this body along this path; any two positions say 1 and 2 are defined by these coordinates S_1 and S_2 which are the coordinates of this path. Let this force F act on this body at an angle of alpha to this direction which is the instantaneous tangent to this path; that is this dotted line is the tangent to this path at this location or the coordinate S.

If we have a displacement, ds then, the work done by this force is F dot dr where dr is the vector along the tangent to the path at this location A. The total work done between these two positions 1 and 2 is nothing but the integration of this value. We have the total work done between the position 1 and 2 as integral 1 to 2 F dot dr. This we can write as F ds cos alpha, where alpha is the angle between these two vectors, that is dr and the force vector. ds is the magnitude of this vector dr or the incremental displacement along the path coordinate because for small displacement, the distance travelled along the curve and along the tangent are the same.

With that assumption, we can write that integral 1 to 2 F dot dr is equal to integral 1 to 2 F cos alpha ds. This gives the work done by this force from 1 to 2 or the work done by this force in moving this body or the particle from this position 1 to 2. In case this force is constant in the direction of the motion then we can say the work done between 1 to 2 is F times S_2 minus S_1.

This is applicable for cases like this. We consider a path coordinate and we have this position 1 and position 2. We have a particle and a constant force F is acting and we have the incremental displacement delta r whose magnitude is ds. We can now integrate it and it will be nothing but F times S_2 minus S_1 where the coordinates are defined from an origin. This way, it is possible to define the work of this constant force in displacing this object from position 1 to 2. We see that the work done is dependent on the final positions that is the position 2 and the initial position S_1.

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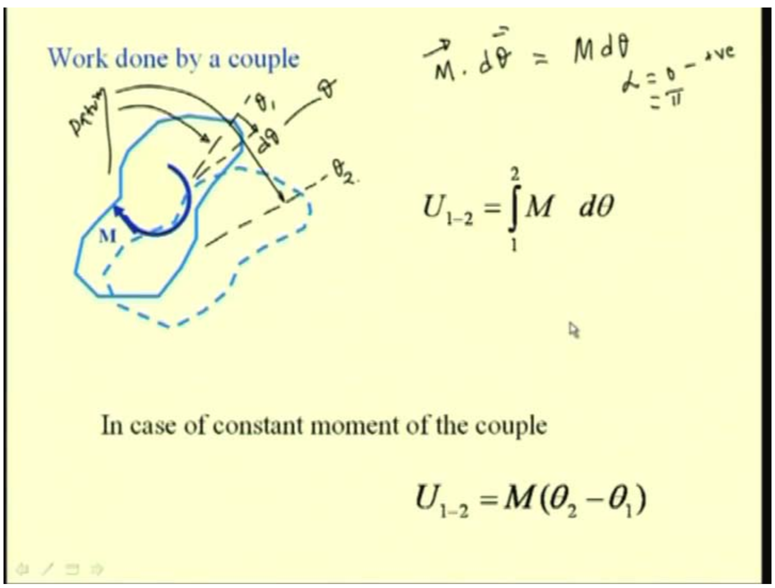

Let us see the work done by a moment. We have just seen the work done for finite displacement for a force; let us see the work done by the moment for a finite angular displacement. If you consider that this is the position theta 1 and this body rotates and this position is marked by this coordinate theta 2 where these angles are measured from some data, say these angles, where this angular displacement theta is measured from a given data. The work done between this position 1 and position 2 is the integral of M d theta because this is the work done for incremental displacements. This is the incremental displacement d theta. Then from our earlier discussion, we know that the work done for this incremental angular displacement d theta is M dot d theta where both of these are vectors. For the planar case, we have seen that this is either positive or negative depending upon whether both the moment and the angular displacement are in the same direction or in the opposite direction. In case there are in the opposite direction then this work done is negative.

So, it is equal in magnitude to this value M d theta. The sign of this depends upon whether this angle between these two vectors is either 0 or pi. In this case, the vector d theta is perpendicular to the plane of paper and it is towards us. Same way, this moment vector M which is also clockwise the vector is perpendicular to the plane and it is towards us. So, the angle between these two vectors, alpha is 0. So, this value is positive. We have this value M d theta as positive. So, now if we integrate this between this position 1 and 2, we have the work done as integral M d theta between these two positions.

In case this moment is constant like a constant couple is applied then this integration becomes M times $theta_2$ minus $theta_1$ where $theta_2$ is this final position and $theta_1$ is the initial position. Again, we see that for a constant moment the total work done depends on this position that is the initial position $theta_1$ and the final position $theta_2$. From our understanding of the work done by a force for an infinitesimal displacement, we have found the work done by the force and the moment for finite displacement. This work done can be associated with the potential energy particularly for conservative forces.

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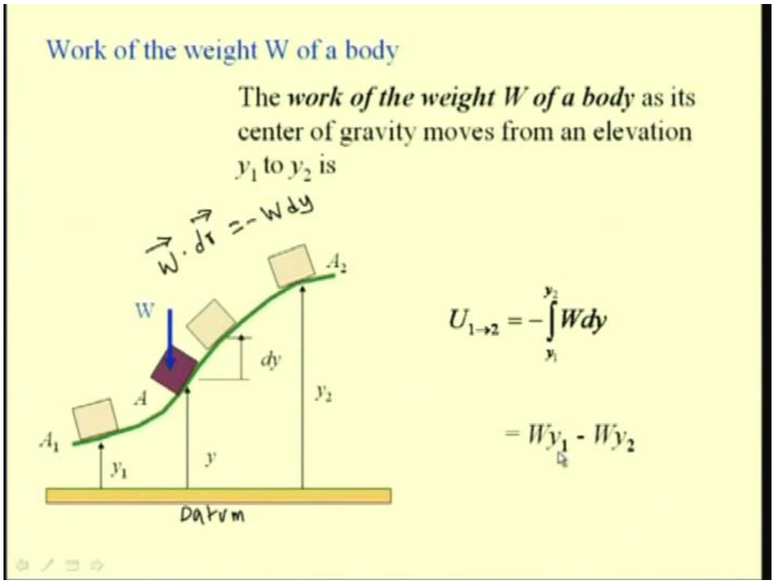

Let us first see two kinds of conservative forces and the work done by these forces. One is the weight of the body. We will just say why we call these forces as conservative forces after this discussion. Let us compute the work done by the weight of the body in raising the object. So here, we see one path along which this body is being raised. The body is being raised from the position A_1 to A_2 and at any position the vertical displacement from the datum that we have considered; this is our datum and this is the positive displacement, which is given by this value y. So, at A_1 we have the value that is the coordinate as y_1 and at A_2 we have the vertical displacement as y_2. This W is the weight of the body and it always acts in the vertical direction. For this position A, we have this coordinate as y.

Let us assume a small vertical displacement dy and find the work done by this force W for this displacement dy and then find the total work done by this force W for the displacement from the position A_1 to A_2. The work done of the weight as the center of gravity moves from the elevation y_1 to y_2 is equal to minus integral y_1 to y_2 W dy. This negative sign comes from the fact that this force W is in the negative direction of the displacement that we are considering. So, the work done which is W dot dr is nothing but the work done which is W vector dot dr, which is minus W dy because these two vectors are in the opposite direction. W is in the downward direction and the vector, dr the displacement vector or in this case dy is in the upward direction. We have this negative sign because of the opposite sense of these two vectors. So, we have this work done between the position 1 and 2 as the integral of this value that is minus W y dy, which is equal to W which is the constant is taken out and this integration is equal to y_1 minus y_2. So, we have W y_1 minus W y_2 as the work done by this force W when the body is being raised from the position A_1 to A_2.

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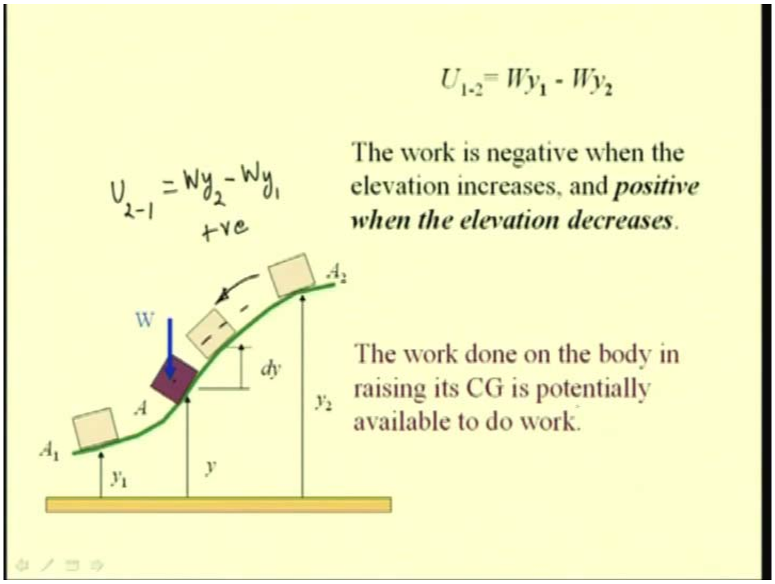

This work done is negative when the elevation rises because we know this value Wy_1 is smaller than this value Wy_2 because this coordinate y_2 is greater than the coordinate y_1. So this value is negative and we see that this work done between this position 1 to 2 is negative when the CG of this object is being raised along this path. So, the CG of this object is being raised and so we find that this work is a negative work. For a motion which is other way that is if this body moves from A_2 to A_1 where the elevation drops, the same work will be positive because then the work done will be U_{2-1} will be Wy_2 - Wy_1 which will be a positive quantity. That means if we raise this weight then the work is negative. That means one has to supply work in order to raise the weight. The work is positive when the elevation decreases means that work is available to do some work.

The weight of the body when the body lowers gives a work which can be used and when the body is raised one has to do work against this weight. That is what we say that the work done on the body in raising in its CG is potentially available to do work. Whatever work was used to raise this body from this location A_1 to A_2 was the negative work. That means it is supplied externally to this system, is potentially available to do work. When the body drops from this position 2 to 1, this is a positive work or the work is available to do work. That is why we call that it is potentially available to do the work.

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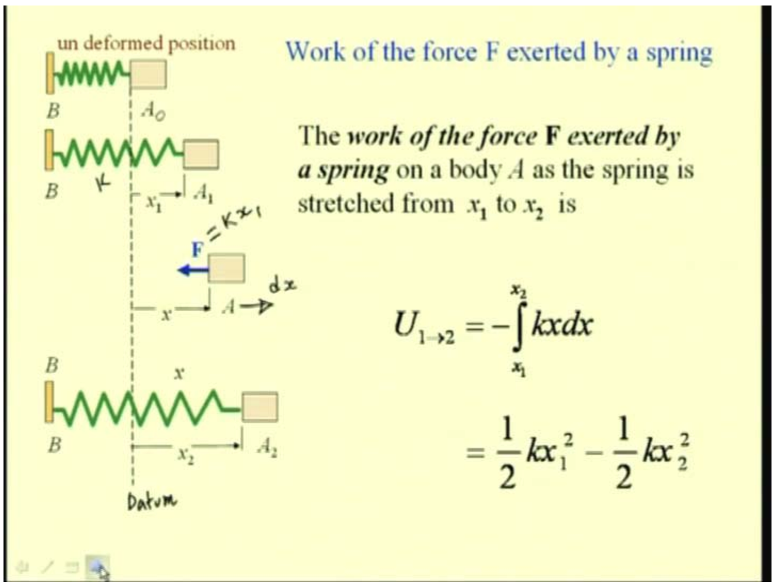

Let us see another kind of a force that is exerted by the elastics springs. Here, in this picture you see a spring in an un-deformed position attached to this body. Let us define this dotted line as our datum. This position A_0 is the mean position where the spring is neither elongated nor compressed. So no force acts on this body. Consider the position A_1 where the spring has been elongated by a distance of x_1. So this is the positive displacement of this body along this direction. Since this spring has been elongated, it applies a force of F on this body. If you consider the free body diagram of this block then this F represents the spring force. If we have K as the spring constant then this F is equal to k times x_1 or the displacement. So this is the spring force that acts on this body for any position say x. So let us take two positions A_1 and A_2 whose coordinates are given by this value x_1 and x_2. For any intermediate position the coordinate is x. So, let us try to see the work done by this force F when the body is being displaced from this position A_1 to the position A_2.

The work of the force F that is exerted by the spring on this body as the spring is stretched from this position x_1 to x_2 is given by integral minus kxdx between the limits x_1 x_2. Why this is negative is because the positive displacement that we are considering that is dx is along this direction and this force F is in the opposite direction of the finite or the infinitesimally small displacement that we are considering. We have the work done between this position 1and 2 as minus integral x_1 to x_2 kx dx. If we do the integration and put the limits, we have this as half kx_1 square minus half kx_2 square.

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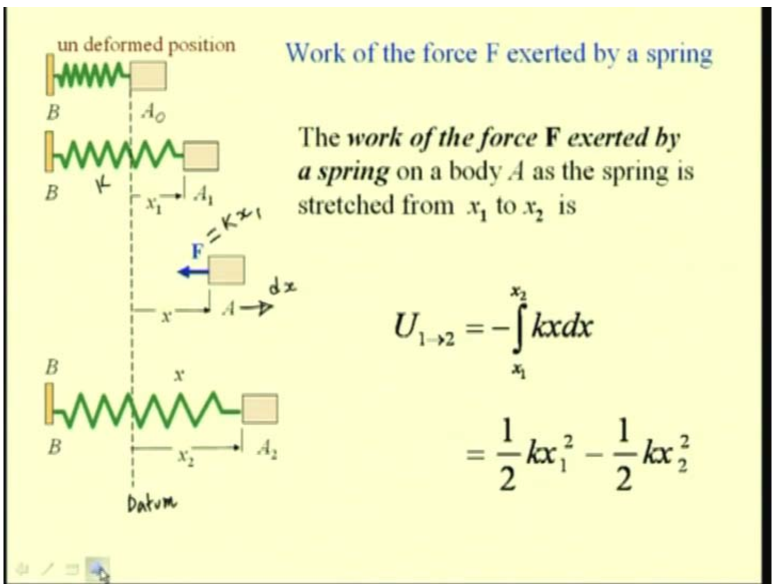

If we again see this term, this work is positive when the spring is returning to its un-deformed position. That means when the spring moves from A_2 to A_1, when the body moves from A_2 to A_1 this work is positive or the work is done by the spring on the body; but when we move from this position A_1 to A_2 an external force equal and opposite to the force exerted by the spring has to be applied in order to do the work. When the body is moved from this position 1 to 2, an external force equal to and opposite to the spring force does the work. So this work is a negative work. But when the body is returning to its mean position that is from A_2 to A_1 then we have the work as positive or the spring force is doing the work. So that means that work is potentially available to do the work. Again we say that the spring in compression or elongation whatever work is done is potentially available to do work. We have seen that for these two kinds of forces that is the elastic force and the gravitational force, there is an associated energy that is potentially available to do work for any given position and such forces where the work done by the forces is independent of the path followed between these points are known as conservative forces.

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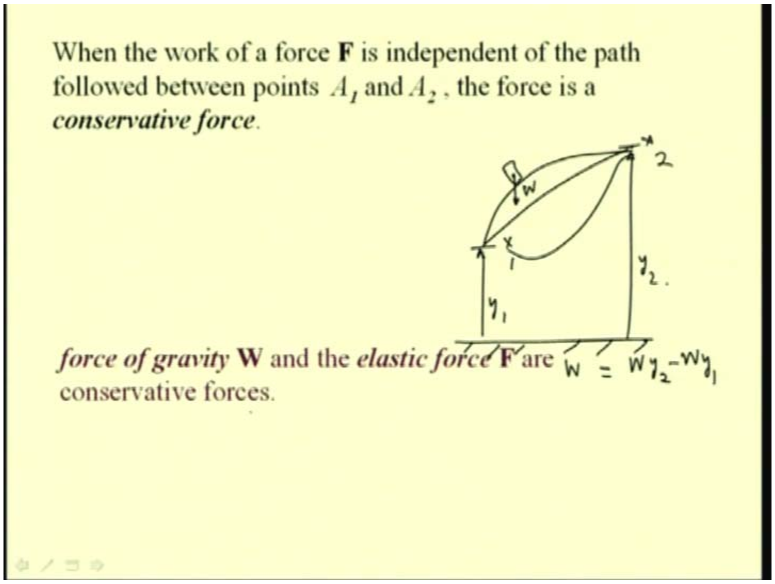

If this is the datum and these are the two positions 1 and 2 at a height of say y_1 and a height of say y_2, the work done in raising a block from this position to this position does not depend whether the block is being raised along these various paths but rather it only depends on this final position 2 and the initial position because we have seen that the work done will be equal to, if W is the weight of the body then, $Wy_2 - Wy_1$.

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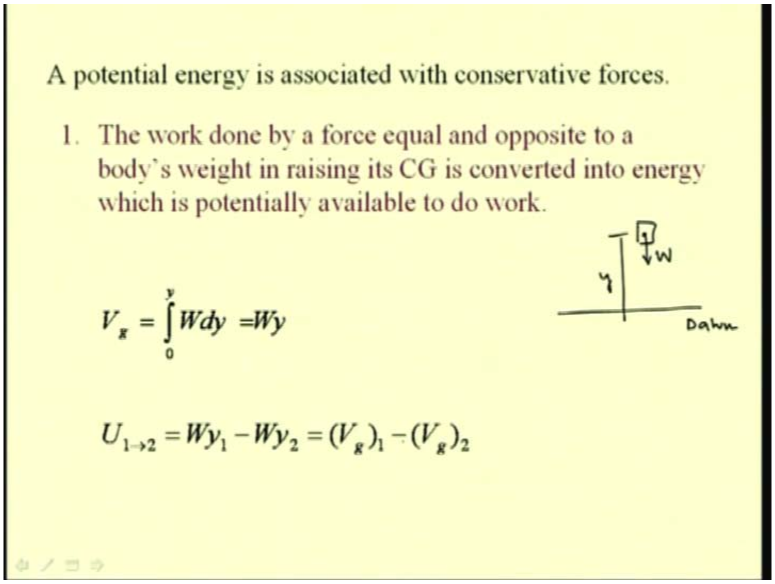

The force of gravity and the elastic forces are conservative forces and for such conservative forces, one can associate the potential energy. Let us first consider the weight or the gravitational force. So the work done by this force equal and opposite to the body's weight in raising its CG is converted into energy which is potentially available to do work. That is what we saw in the example.

So let us rewrite that. I define this term V_g, the potential energy because of gravity, is equal to integral 0 to y Wdy. That is, this is the work done in order to raise the weight from the datum that is this 0 to y. This is the work done to raise a given body from its datum to the given position y this is the weight W. Wy is the potential energy associated with this position. For any two positions the work done is equal to the difference of this potential energy that is Wy_1 minus Wy_2. From our derivation of the work done for the weight of the object, we have found that it is same as this Wy_1 minus Wy_2 which is equal to the potential energy at 1 minus the potential energy at 2. So, the work done from 1 to 2 is equal to the potential energy at 1 minus the potential energy at 2.

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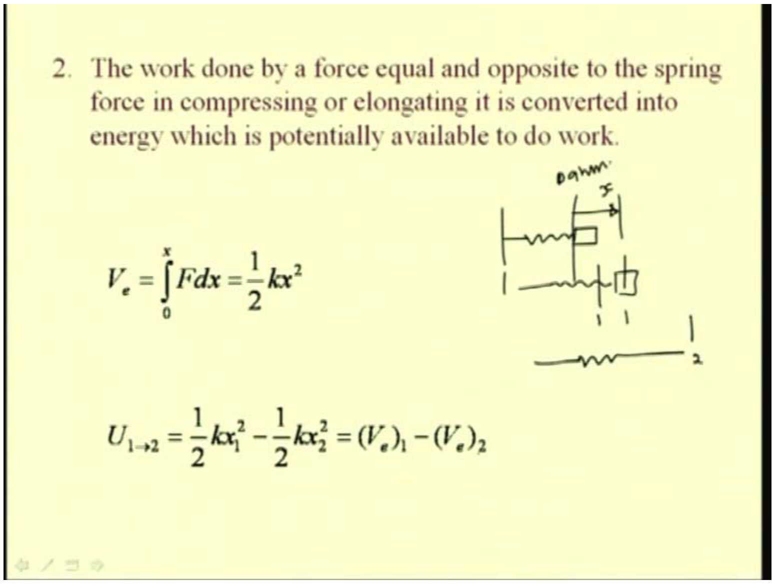

In the same way we can define for the spring force, the work done by the force equal and opposite to the spring force in compressing or elongating the same is converted into an energy which is potentially available to do work. Again, we write this potential energy V_e as the total work done by this force F in displacing. So, Fdx is the work done from the un-deformed position which corresponds to this 0 and to any particular position x, which is equal to half kx square. This is the datum position and the block here and it is being elongated to a new position which is x from the datum and this is available as the elastic potential energy for any given position.

The work done between two positions 1 and 2, for these two positions is equal to half kx_1 square minus half kx_2 square which is equal to the potential energy because of this elastic force for the position 1 minus the potential energy of this elastic force for this position 2.

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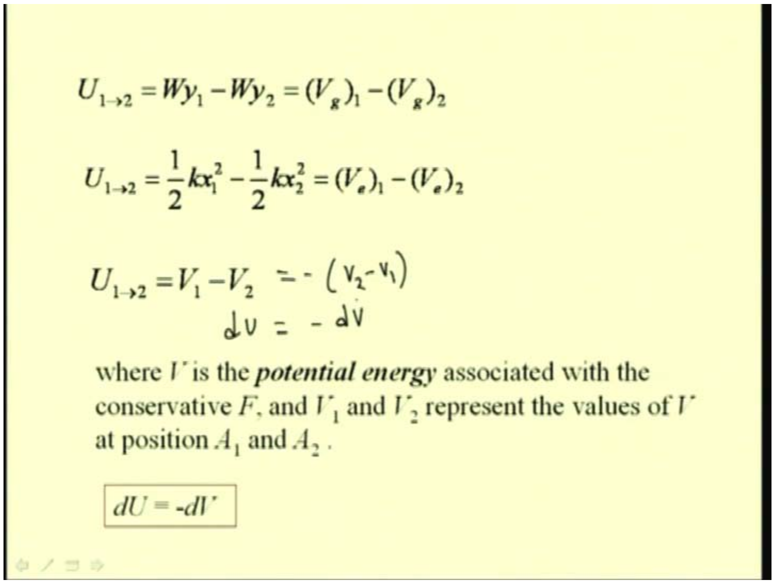

These two works done that is the work done by the weight and the work done by the spring force can be written as the difference of the potential energy of the corresponding states. This V is the potential energy associated with the conservative force. V_1 and V_2 represent the values of this potential energy for the positions A_1 and A_2.This is true for both the conservative force that is weight and the conservative elastic force.

So, we can write dU or the change in the work is equal to the negative of dV because we see that the work done from 1 to 2 is equal to V_1 minus V_2 or this is equal to minus V_2 minus $V_{1.}$ For a small change in dU it is equal to minus small change in the potential energy. This differential form can now be advantageously used to analyze the systems where conservative forces are involved.

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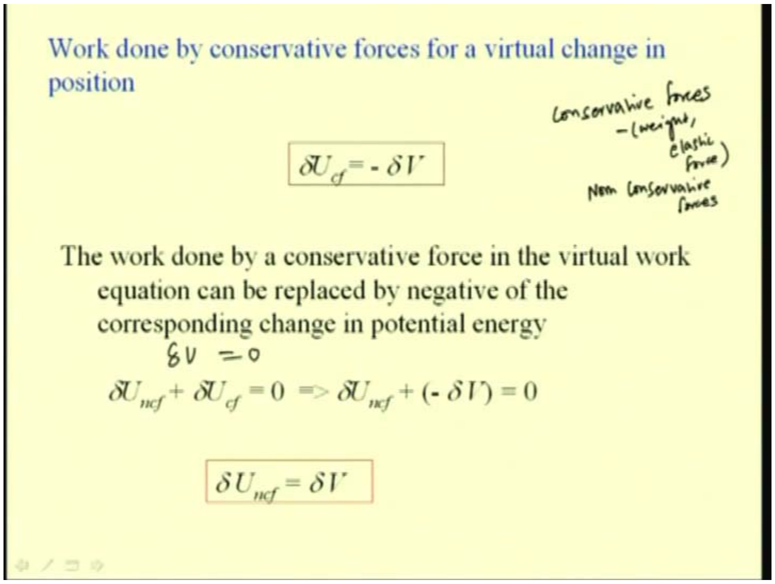

The work done by the conservative forces for a virtual change in the position is delta U_{cf} is equal to minus delta V. In the method of virtual work where we have the work done by various forces, we can classify the forces as conservative forces which are the weight and the elastic force and the non-conservative forces. In the active force diagram, we can remove these conservative forces that is the weight and the elastic force and only consider the non-conservative forces to compute the virtual work. The work done by these conservative forces can be replaced by the negative of the change in potential energy of the system for that particular virtual displacement. We say that the work done by the conservative force in the virtual work equation can be replaced by the negative of the corresponding change in the potential energy.

This equation is nothing but delta U is equal to 0; the total virtual work has to be 0 for the system to be in equilibrium. That work is divided as two components; the work done by the conservative force and the work done by the non-conservative force. We have this delta U non-conservative force plus delta U conservative force equal to 0 and this work done by the conservative force is replaced by the negative of the potential energy of the system that is minus delta V. This is nothing but the potential energy change for the given virtual displacement. In other words, delta U non-conservative force is equal to delta V or the virtual work done by the non-conservative forces is equal to the change in potential energy for the given virtual displacement. The equation of equilibrium by the method of virtual work can be replaced by this equation where we equate the work done by the non-conservative forces to the change in the potential energy of the system.

We can solve equilibrium problems by using this equation where we can avoid this weight and the elastic force in the active force diagram and one can compute the equilibrium.

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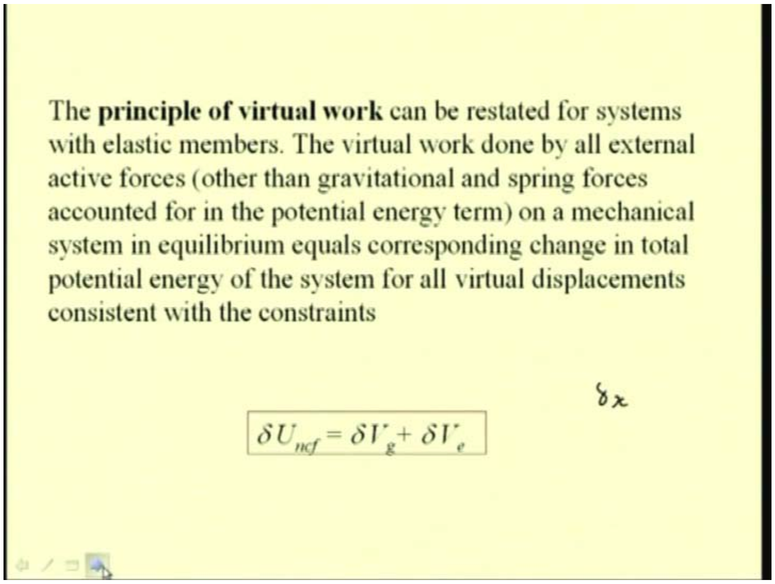

The principle of virtual work is restated for systems with elastic members and systems where mass has to be considered. The virtual work done by all external active forces other than gravitational and spring forces accounted for in the potential energy term on a mechanical system in equilibrium equals corresponding change in the total potential energy of the system for all virtual displacement consistent with the constraint.

This is the significance of the term that we have written that delta U non-conservative force is equal to the change in the potential energy of the system which is equal to the change in the gravitational potential energy plus the change in the elastic potential energy for the given virtual displacement delta x.

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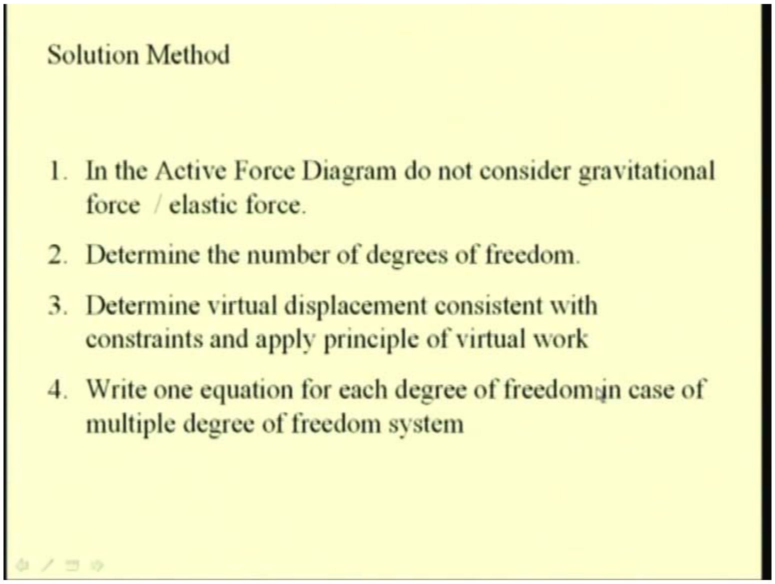

Let us see the solution method. In the active force diagram, we do not consider the gravitational forces or the elastic forces because these are taken care of by the potential energy change term that we are going to use. Then determine the number of degrees of freedom as we do. Then determine the virtual displacement consistent with the constraint and apply the principle of virtual work and write one equation for each degree of freedom. In case of multiple degree of freedom system, we have for an n degree of freedom system we have n equations.

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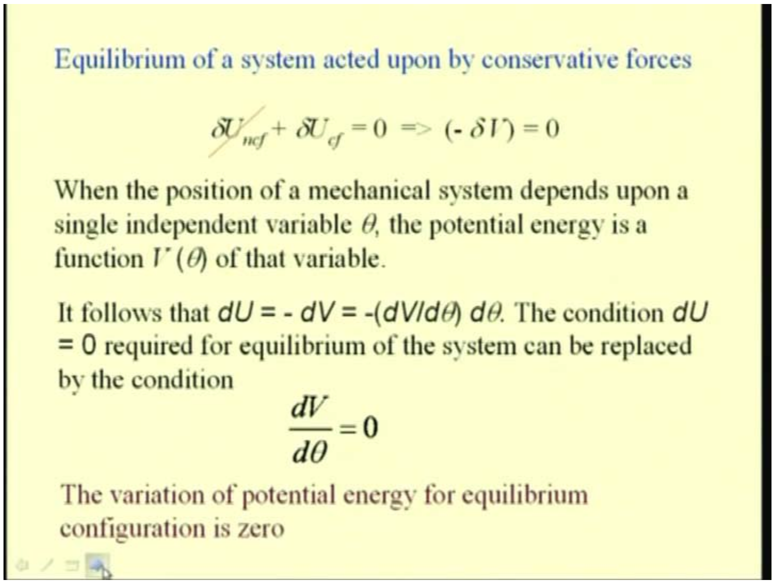

Let us see the equilibrium of systems acted upon by the conservative forces. We have this term delta U non-conservative force plus delta U conservative force is equal to 0. If a system has only conservative forces then this term becomes 0 that is delta U non-conservative force it is 0. So, we have delta U conservative force is equal to 0. The work done by the conservative force is equal to the negative of potential energy or in other words minus delta V is equal to 0. If we have systems which are having only conservative forces that is they constitute masses and only elastic members then, the virtual work equation can be replaced by the change in the potential energy is 0 for equilibrium. That is delta V which is nothing but the change in the potential energy corresponding to the virtual displacement is equal to 0.

So we state that when the position of the mechanical system depends upon a single degree or a single independent variable theta then the potential energy is in terms of that variable theta. Then we can redefine the equilibrium as dU which is the change in the work is equal to minus dV which is minus dV by d theta times d theta. Since this has to be 0 dV by d theta is equal to 0.

The condition that the change or the virtual work is to be 0 that is required for the equilibrium of the system can be replaced by the condition dV by d theta equal to 0. The principle of virtual work when applied to systems with conservative forces, the equilibrium equation that is delta U equal to 0 can be replaced by this equation dV by d theta equal to 0. That means the change in the potential energy of the system is 0 for the equilibrium configuration.

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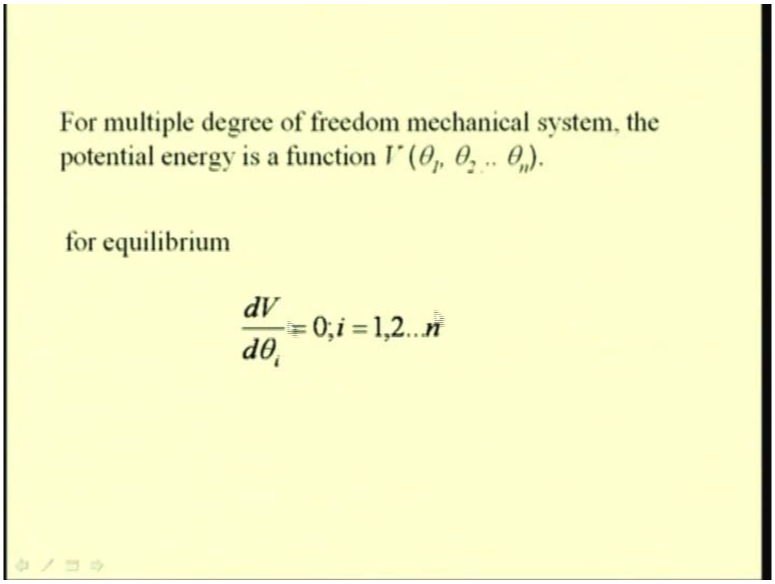

For the n degree of freedom system, the potential energy is a function of those n variables theta 1, theta 2 to theta n. For equilibrium each of these values that is dV by d theta should be 0 for i equal to 1, 2, 3 etc., up to n. This again gives n equations from which you know n unknowns can be determined. Let us take one example problem.

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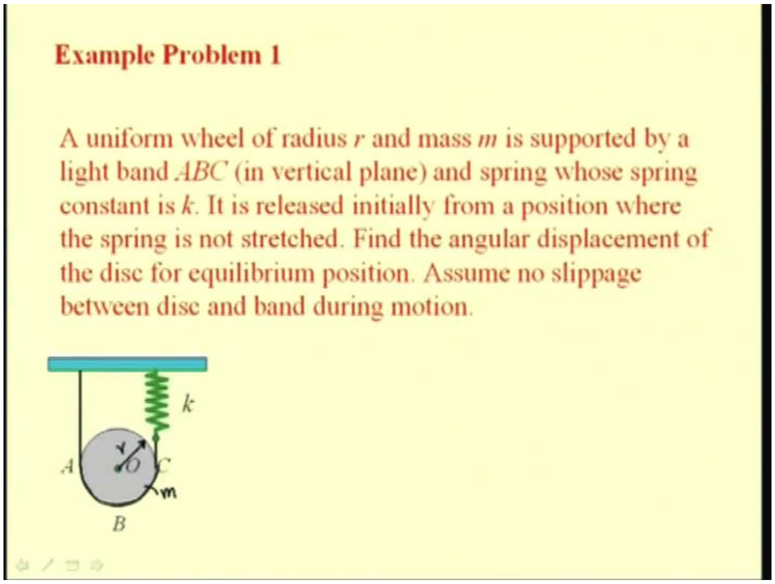

Here you see a uniform wheel of radius r. Let the radius of this wheel be r and which is having a mass m supported by a light band ABC. This band passes over the wheel and it is connected by a spring. It is released initially from the given position where the spring is not stretched. For the given position, this spring whose spring constraint is k is in the un-stretched position. We are interested to find the angular displacement of the disk for equilibrium position. As soon as this disk is released, the disk will descend and then it will reach a position where the disk will be in equilibrium. That means the spring force and the weight of this disk will be balanced and that will be the equilibrium position of this disk. We are interested to find that position. We are interested to find the angular displacement of this disk for reaching this equilibrium position. We assume that no slippage occurs between the disk and the band during the motion.

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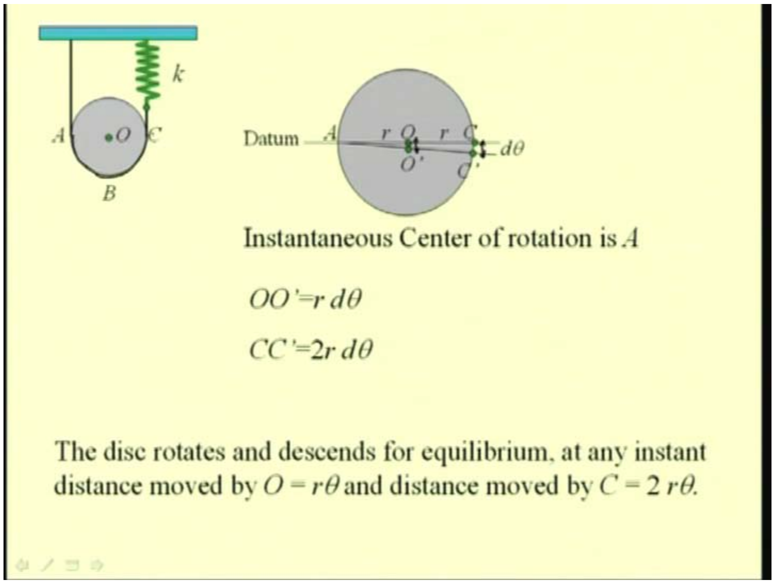

Let us take the disk. Since we assume that there is no slippage occurring between the band and the disk, we can assume that the disk instantaneously rotates about this point A. So, we can say that this position AOC which is the datum and this disk instantaneously rotates about this position A because there is no slippage. So, these displacements that is the displacement of the mass center of this disk and the displacement of this point C which is connected to the spring can be derived in terms of this angular displacement d theta. These distances that is O O prime is nothing but r times d theta or this distance. This distance between CC prime is equal to 2 r or the diameter of the disk times d theta.

These values give the corresponding displacement of the points O and C for this angular displacement d theta of the disk. So, the disk rotates and descends for equilibrium. At any instance, the point O descends by a value r theta because we integrate this value for position 0 that is the datum position, to a given position where the angular displacement that has taken place is theta. So, the point O would have descended by a distance r theta. The distance moved by this point C will be 2 r theta. These are the distances which are of interest to us because the point O corresponds to the mass center where the weight of the disk acts and the point C corresponds to the point where this spring force acts.

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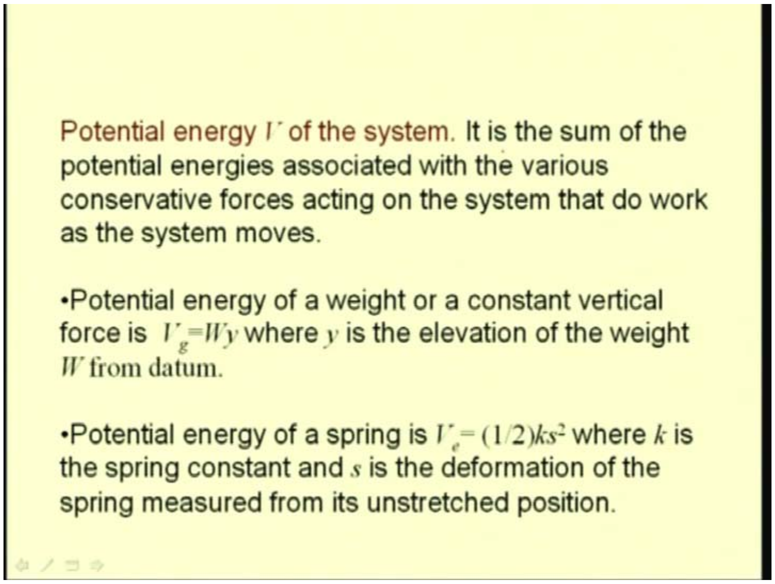

We will apply this potential energy principle where this is the sum of the potential energies associated with the various conservative forces. Here, the two conservative forces are the weight and the spring force and the potential energy associated with the weight is W times y this value and the potential energy of the spring is half k s square.

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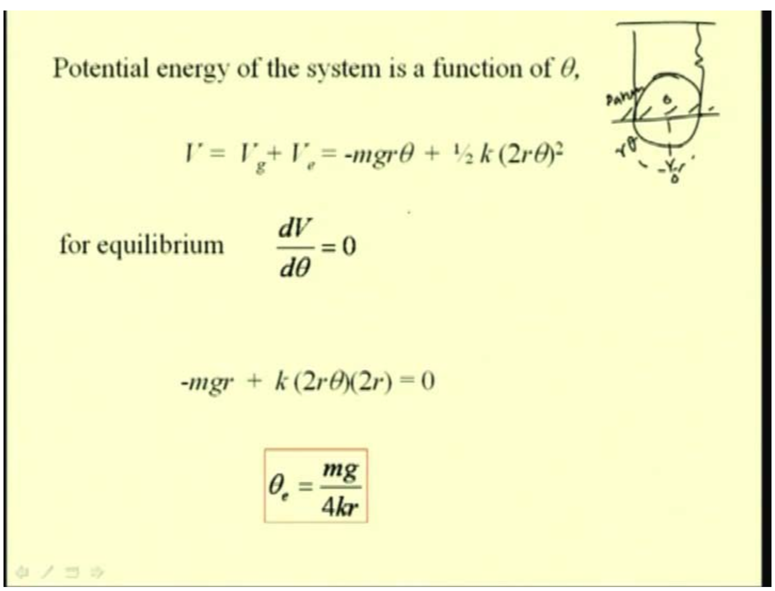

Let us determine the potential energy of the system as a function of this angular displacement theta. We write the potential energy of the system as the potential energy due to the weight or the gravitational potential energy plus the elastic potential energy V_e which is equal to minus mgr theta because the weight is mg and the displacement that has taken place is r theta in the negative direction. Because we have seen that, this is the datum that we have taken where the disk is originally placed and the spring is attached. The disk has now descended from O to O prime. Since we have taken this as datum, the displacement is in the negative direction or below datum. So the value is negative. We have mgr theta where this displacement between O to O prime is r theta. For this displacement, this point C would have moved by 2 r theta. So, the energy associated with that position is half k the stretch that is 2r theta square. This gives the potential energy of the disk for any angular displacement theta.

For equilibrium the change in the potential energy for a small virtual displacement should be 0. The virtual work equation for the conservative systems with only conservative forces is given as dV by d theta equal to 0. So, we will first differentiate this equation with respect to theta and then equate it to 0. The differentiation of this is mgr d theta and the differentiation of this is k times 2r theta times 2r divided by d theta. So, that term gets cancelled and we equate this to 0. From this we get that equilibrium position theta e as mg divided by 4 kr. This example illustrated how we can replace the virtual work equation by the equation that is the change in potential energy for the virtual displacement has to be zero and the problem can be solved for systems where we have only conservative forces.

Let us see one more example on the application of this potential energy method to determine the equilibrium position.

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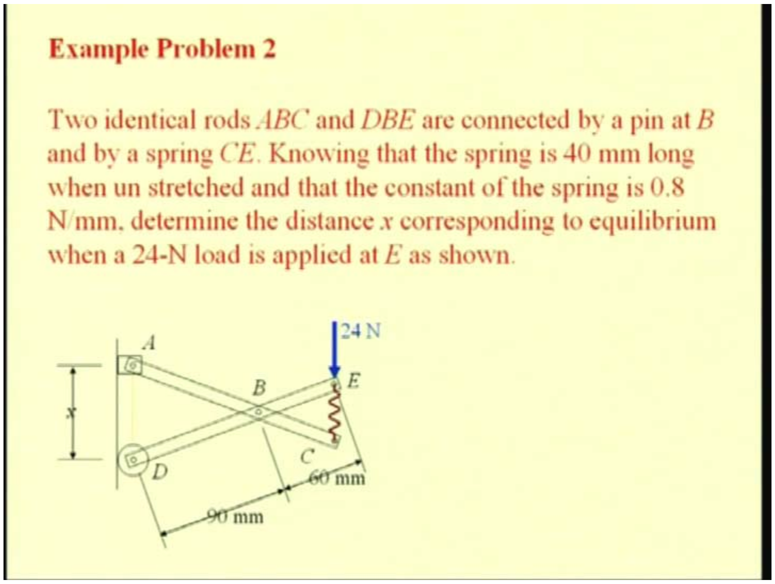

In this example you see two identical rods ABC and DBE which are pinned at B and the points E and C of these two links are connected by a spring. The spring constant is 0.8 Newton per mm and it is known that the spring is 40 mm long when it is un-stretched. That is the free length position of the spring is 40 mm long.

We are interested to determine the distance x that is the distance the roller that is connected to this end D of this rod DBE moves for the equilibrium position when this force of 24 Newtons is applied at this point E of this link BBE. So, we assume that the friction is negligible. So, the roller moves down and attains this equilibrium position x and there is no loss of energy because of any friction that is involved.

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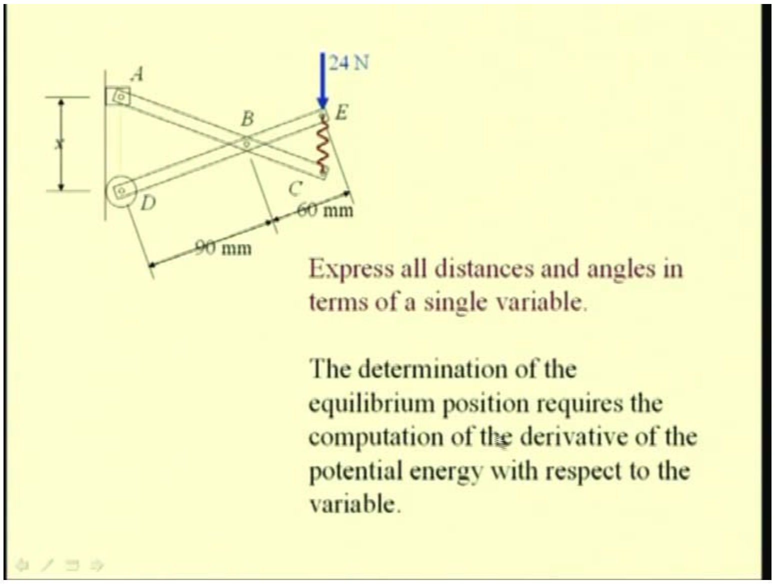

If we are interested to apply the method of potential energy, first we have to determine the various distances that is the distance of these points E where this force 24 Newton force is being applied with respect to a datum that we have chosen to compute the potential energy associated with this system. For that position, we have to also know the length of the spring that is the distance between these two points E and C. We can see that, this system is a single degree of freedom system which can be defined by the position x that is reached by the pin D or this roller. So, let us express all the other distances that are of interest to us that is, the position of this 24 kilo Newton force on this link DBE and the length this spring CE for a particular displacement of x of this roller. We determine the equilibrium position by computing the derivative of the potential energy.

First, we write the potential energy term and then we differentiate it to find the equilibrium position.

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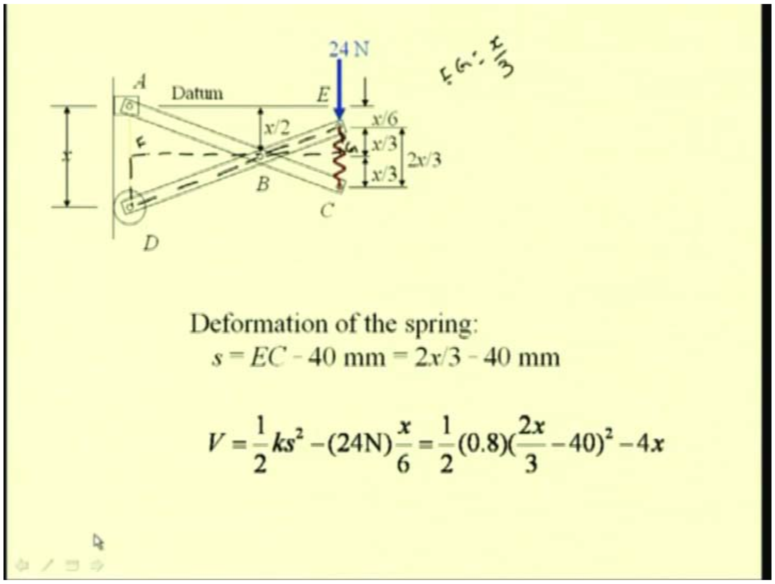

First let us draw the active force diagram and mark all the distances. We have chosen the horizontal line passing through this pin at A as the datum. At any position for the mechanical system the position of the roller is x units below this datum line. For this position, let us try to see how the other positions are related to this displacement x. The pin at B is located at a distance of x by 2. This comes from the symmetry of this geometry because we know that the length of these two links that is ABC and DBE are the same. They are pinned at the same locations along the length of the links. So this point B is located at a distance of x by 2 from the datum. The position of E below datum is x by 6. This comes from the symmetric triangles, from the triangles DBF and the triangle BEG knowing these distances that is DB and DE. Also, these distances that is FD which is x by 2 and this distance can be determined; that is EG can be determined. Once this distance EG has been determined, it is possible to determine the distance of this point E from the datum.

First, we determine this EG as x by 3 from the symmetry and then we can determine the location of this E with respect to datum as x by 6. Now, we see that we have determined all the required dimensions to compute the various potential energy terms. So, the deformation of the spring is equal to EC minus 40 mm because 40 mm is the original length and EC is the current length of the spring for a given position. We have this as 2x by 3 minus 40 mm. Because this distance that is EC is 2x by 3 for any given displacement of x of this roller at D. So we have the stretch of the spring for any given displacement x.

The potential energy associated with the spring is given by half k into stretch of the spring square minus the potential energy or the work done by this conservative force 24 Newton force. Because this force 24 Newton force is always a vertical force, this can be considered as a conservative force. We can associate a potential energy with this force which is also given by force times the location of the force with respect to the datum which is x by 6. Since this point E has moved down from the datum, the term is a negative term.

We have this as half k s square which is the potential energy of this spring minus the potential energy of this conservative force 24 Newton force which is 24 Newton times x by 6, which is equal to half the spring constant is 0.8 times the stretch which is determined as 2x by 3 minus 40. We write that 2x minus 3 minus 40 square minus this term which when simplified it is 4 times x. Now we have the potential energy term. For equilibrium we have to differentiate this with respect to the coordinate that is x the coordinate that specifies the configuration of the system.

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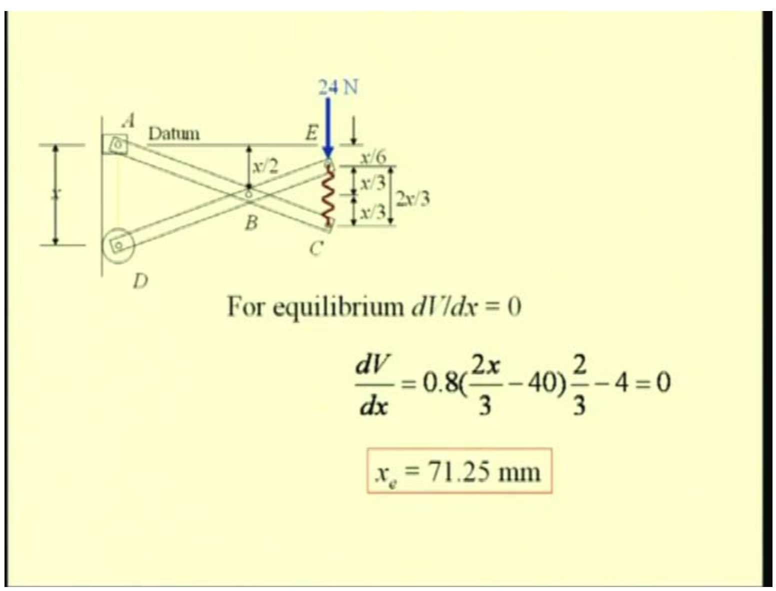

We differentiate it and equate it to 0. So we have dV by dx equal to 0. dV by dx is 0.8 times 2x by 3 minus 40 times the differentiation of this term that is 2 by 3 minus the differentiation of the term 4x is 4 which is equal to 0. When we simplify this, we have x as 71.25 mm. The equilibrium position is 71.25 mm that is when the roller reaches a distance of 71.25 mm from the original datum position. This problem illustrated how we can apply this potential energy method to analyze the equilibrium of a mechanical system with conservative forces, the weight of the system or the various elastic forces that may be involved in the system or there could be other kinds of conservative forces that may be applied in a system.