Engineering Mechanics

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Module 8 Lecture 18
Principle of virtual work

Today, we will continue our lectures on the engineering mechanics in statics where we will see a new method of solving equilibrium problems - what we call as the virtual work method and energy methods. These methods are particularly useful when solving equilibrium of large structures with many interconnected rigid bodies. When such a structure or a machine has to be analyzed by the traditional way of breaking them or dismembering them into individual rigid bodies and trying to analyze the equilibrium of the individual bodies, the procedure often becomes too lengthy.

This method of virtual work and energy is particularly helpful in these cases where we have large number of interconnected rigid bodies. We are interested in analyzing the equilibrium of such a body or we are interested to find some unknown forces etc. Before going to the discussion, we will see how to define work done by a force and then what we mean by the virtual work. For your reference, this is module 8, lecture number 18 of the engineering mechanics course.

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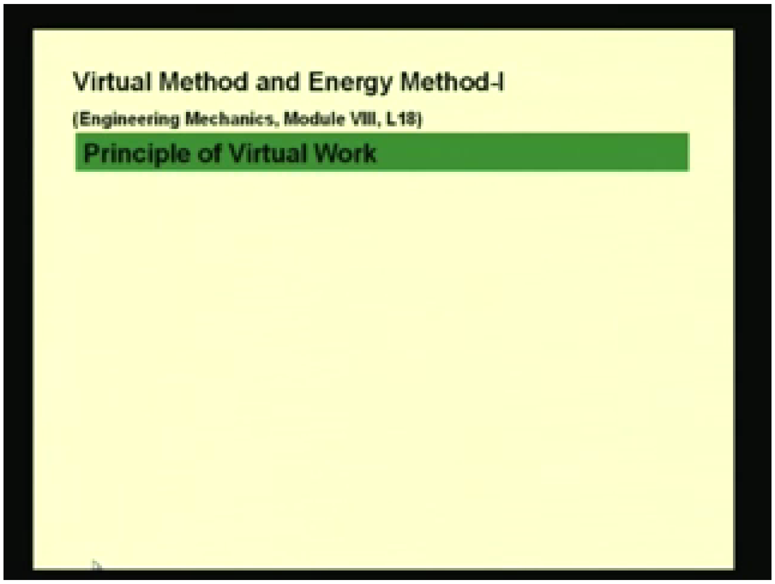

Let us first define what a virtual work is. Before defining that, let us first define the work done by a force.

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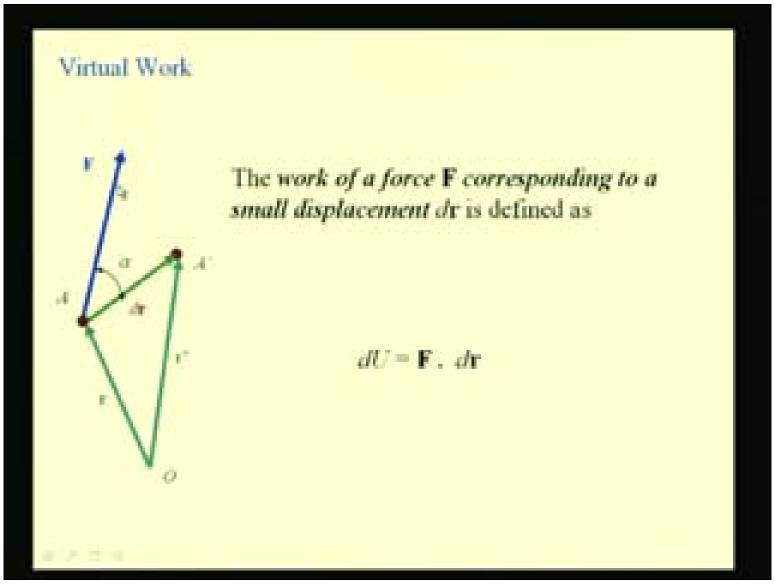

Here, you see a force F acting on a particle A whose position is defined with respect to an origin O by this position vector r. After a small interval of time dt the position of this particle A moves to a new position A prime and the new position vector is r prime and the vector dr gives the displacement vector for this small interval of time dt. Let alpha be the angle between this force vector and the displacement vector, dr. We define work done by this force in this interval of time in displacing this particle from A to A prime as the dot product of these two vectors; that is, the force vector and the displacement vector dr. In other words, we find the component of this force along this displacement vector dr and multiply them to get the magnitude of the work done. It could be other way round that we find the component of this displacement along this force vector and the work done by this force is the product of the force and the component of this displacement vector along this force.

We define this work done during this small interval of time dt; we designate it by dU, the incremental work done by this force F in displacing the particle from A to A prime; we define this dU as F dot dr.

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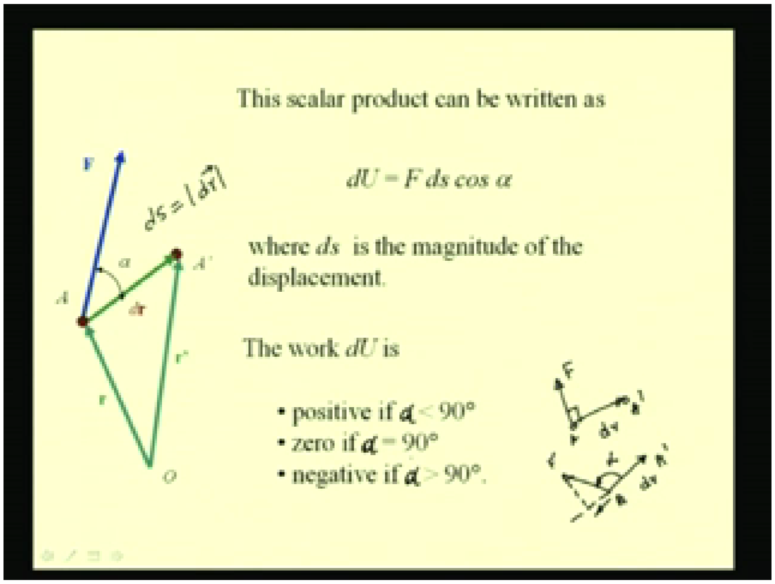

In scalar form, we have dU as Fds where ds is nothing but the magnitude of this vector dr. We define this work dU as F the magnitude of the force vector times ds the magnitude of the displacement vector times cos alpha the cosine of the angle between the two vectors F and dr. This comes from our definition of the dot product.

We can see few specific cases. What happens to this work dU if this angle is less than 90 degrees? The projection of this along dr is positive and so, the work done or this quantity Fds cos alpha is also positive. We say that the force does a positive work in displacing the particle from A to A prime. What happens if this alpha? Here this is alpha. If this alpha is 90 degrees, we have the case where the force vector and the displacement vector dr are 90 degrees. In that case cos 90 is 0. So, this work becomes 0, or the work done by this force when the particle is moved from A to A prime where the displacement is perpendicular to the direction of this force then the work done is 0. Then we have the work done as negative quantity if alpha is greater than 90 degrees. That means, the force vector is like this and we have the displacement vector dr. Particle is moved from A to A prime. This angle is alpha which is greater than 90 degrees.

The component of the force along the vector dr is in the negative direction. So, the work done by this force in moving the particle from A to A prime is negative. This way we have three cases where the work done by a force can be positive or 0 or negative depending upon the direction in which the force vector acts with respective to the direction of the actual displacement of the particle.

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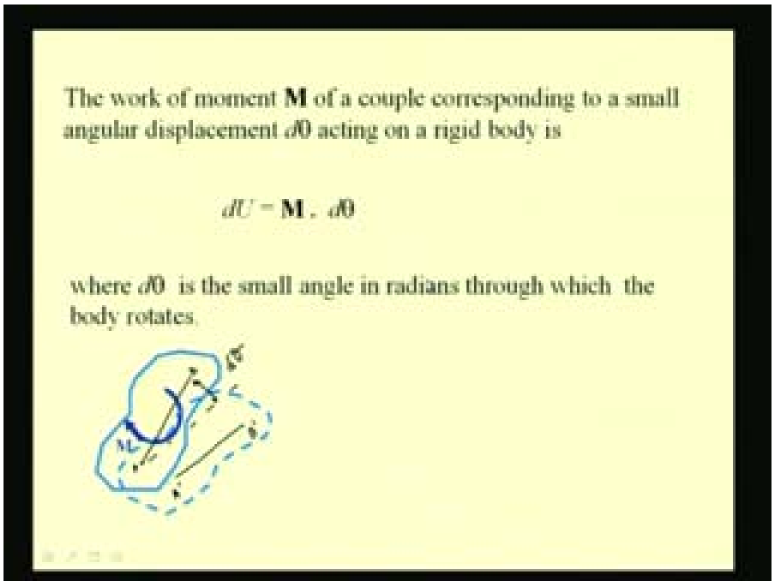

Let us see the work done by a moment. We have seen the work done by a force. So let us see the work done by a moment. The effect of the force is to displace the particle or linearly translate it.

The effect of a moment is to rotate the object. In case of the planar motion, if we have this moment acting on this rigid body then this object tends to rotate because of this application of this moment and it takes a new position. The work done by this moment can be computed. For an incremental moment d theta, say we have a vector on this body, a and b are particles and we have in the transformed position A prime B prime. The angle between this position is d theta. A small angular displacement has taken place because of this application of the moment. In that case we say the work done is M dot d theta.

For small rotations the d theta or the incremental angular displacement is a vector quantity. Finite moments are like an angular displacement of 40 degrees or so; it is not a vector quantity and it will not follow the laws of vector addition. But an incremental quantity d theta is a vector quantity because it will follow all the laws of vector addition. We can define dU as M dot d theta. Here, we define this d theta in radians.

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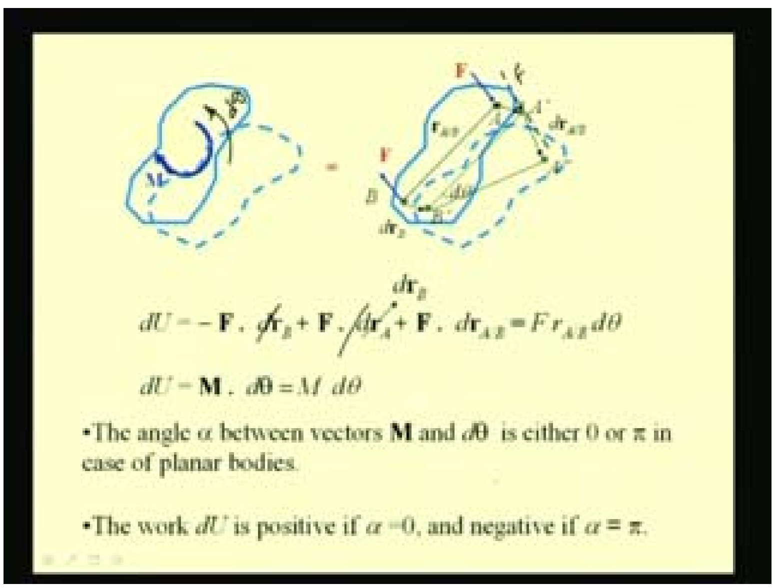

Let us try to derive this quantity based on our understanding that any rigid body motion can be decomposed into a translation and a rotation. Because of the application of this moment M the body has moved from this position to this dotted position. This total moment can be decomposed into a translation and a rotation. Let us take two points on the body A and B and the final position that means on this dotted position we have the positions as B prime and A double prime.

The moment of A to A double prime has been decomposed into a pure translation where the translation of this point A to the new position A prime is equal to the translation of this point B to B prime. This vector dr_B which defines the small incremental displacement of this point B to the new position B prime is same as this vector from A to A prime. Then this point A prime moves to the new position that is A double prime by pure rotation about this point B. This point B or the translated position that is B prime does not move and with respect to this new position B prime the body undergoes a pure rotation; because of that, the position changes from A prime to A double prime. We call it as $dr_{A/B.}$ That means the change in position of point A with respect to B. This can be defined by a pure rotation. Thus, by this angular displacement d theta if we know this relative position of A with respect to B that is $r_{A/B}$ then we can define this $dr_{A/B}$ with respect to this angular displacement, d theta. This moment M which causes a rotation of this body can be represented by a couple or these two equal and opposite forces F acting at B and A such that the moment M is equal to the couple of these two forces that is F times r_{AB}.

We can define the work done dU as the work done by this force F in displacing the point B to B prime. So, we have this as minus F dot $dr_{B.}$ Why? Because, we know that these two vectors are in the opposite directions. Thus, the component or the magnitude of this work will have a negative sign.

Let us see the work done by this force F on this particle A in moving it from A to A prime. It is equal to F dot dr_A. Since the direction cosine of this vector along A A prime is in the positive sense we have this work done as a positive quantity. This point A prime is moved to this new position A double prime. So, some work is done by the force again to move the particle from A prime to A double prime. That is given by the same force F that is acting here. I can just represent that the same force F is acting here, F dot the displacement vector that is dr_{AB}. Again, they have same sense or the direction cosine of this vector is along the A, A prime vector. So we have this work also as positive quantity. This is equal to $Fr_{A/B}$ d theta. Why? Because, these two quantities are the same because dr_B or the translation of this B to B prime is same as the translation of this point A to A prime.

These two quantities are the same and they cancel out each other and what we have is only F dot $dr_{A/B.}$ In other words, this dot product can be written as the magnitude of this force times the magnitude of this vector times the angle between these two vectors that is d theta because we have this $dr_{A/B}$ as $r_{A/B}$ d theta. In other words, we can say that this is equal to M dot d theta and this M d theta can be replaced by $Fr_{A/B}$ d theta. The angle alpha between the vectors M and d theta is 0 or pi in case of planar bodies. That means this vector d theta is perpendicular to the plane of the paper. Thus, it could be either be pointing upwards or it can be pointing downwards; that means, the vector can go into the plane of the paper or it can emerge out of the plane of the paper. That means the angular displacement can be in the same sense of the moment.

In this case, the moment is counter clockwise. So d theta can also be counter clockwise in which case the angle between these two vectors is 0. When the angular displacement is other way round that means if d theta is in the counter clockwise direction and M is in the clockwise direction, then they are in the opposite sense and we have the angle between the vectors as pi. Thus, the work done is positive if the angle is 0. This quantity M d theta is a positive quantity if the angle between these two vectors is 0. So, cos 0 is 1. If it is pi it is minus 1, cos pi is minus 1; we have the work as negative. That means if both the angular displacement and the moment the applied moment are in the same direction. That means if they both are in the clockwise direction or both in the counter clockwise direction the work is positive. If the sense is opposite that means the applied moment is counter clockwise and displacement is clockwise then we have the work as negative work.

Now that we have seen how to define the work done by a force and the work done by the moment, let us define what we mean by the virtual work. We define first the concept of virtual displacement.

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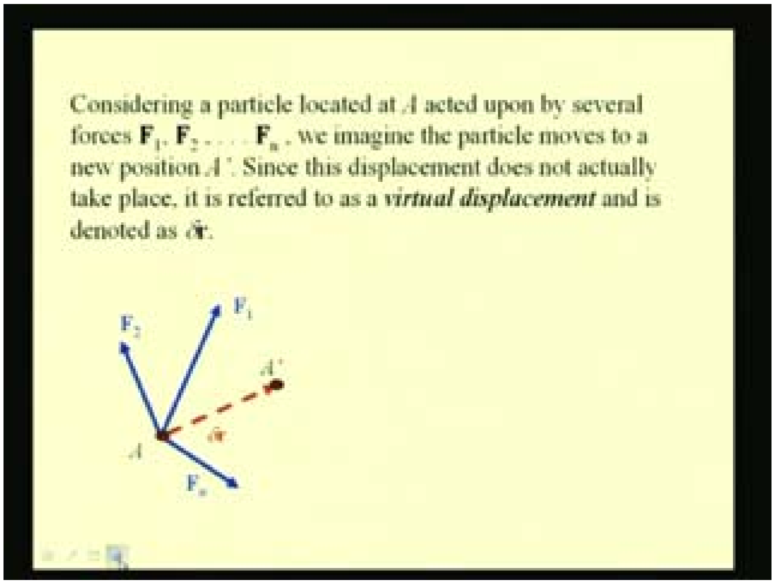

Let us consider a particle A as shown in this picture and which is subjected to these various forces say F_1, F_2 and so forth; n number of such forces are acting on this particle A. We imagine that this particle undergoes a displacement and moves to this new position A prime under the action of these forces. This is just an imagination and actually the displacement is not taking place. We are assuming that the particle is moving from this position A to A prime which is virtual; there is actual no real displacement taking place. Let this displacement which is virtual be delta r. The symbol delta we use for the virtual displacement and this displacement which is virtual is termed as the virtual displacement. These forces are not causing the movement of this particle A. There is no real displacement, but we imagine that such a displacement is taking place. In this way, we define the concept of virtual displacement.

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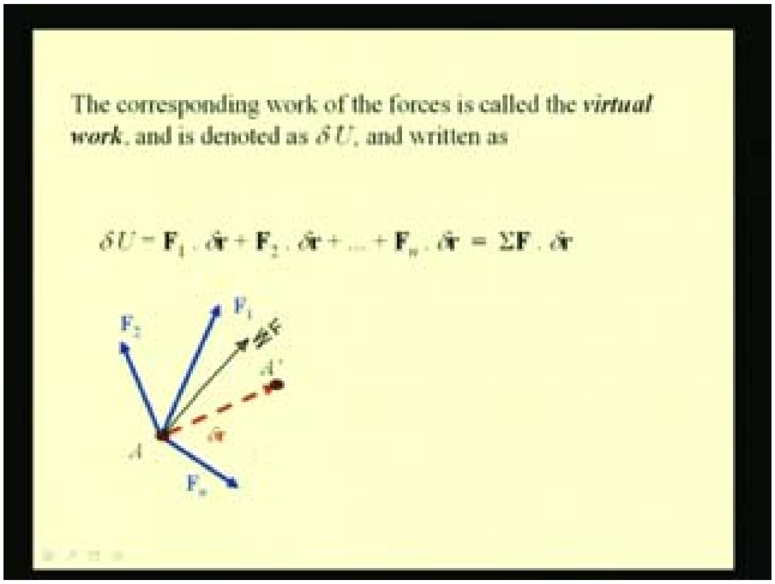

Let us see what will be the work done because of this virtual displacement. If we define such a work or the work done because of the virtual displacement such a work is a virtual work of the forces. Let us designate that as delta U, the virtual work. We can write that as the work done by these individual forces. For this force F_1, the work done due to this virtual displacement delta r is nothing but F_1 dot delta r. Same way, for other forces we can write the virtual work done by these forces as F_2 dot delta r and so forth. If we have n number of forces, we have n number of terms giving the total virtual work done by these set of forces. Since we have this delta r as the common term, we can write this as sum of all these forces that is sigma F which is nothing but the resultant of all these forces dot delta r.

I can say that if I can represent the resultant of all these forces by a vector say sigma F which is the resultant of all these forces, then the work done by these forces in doing this virtual displacement is equal to sigma F dot virtual displacement vector delta r. Based on this concept of virtual work done by the forces on the particle, let us try to derive the equilibrium equations of particle based on this virtual work.

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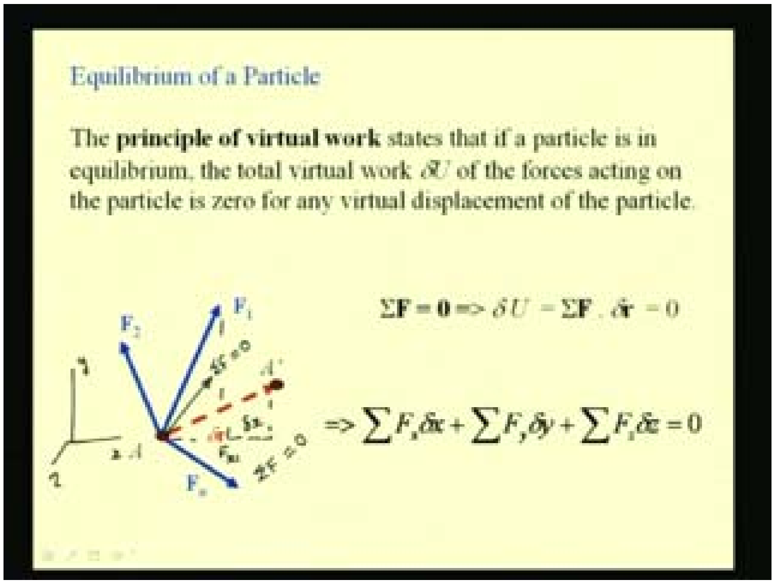

We define the equilibrium of the particle to exist when a set of forces act on the same if the resultant of all these forces is equal to 0 from our earlier discussion. Let us see how we define the equilibrium of the particle by this virtual work concept. Here again you see this particle A subjected to a number of forces and undergoing a virtual displacement of delta r where the position moves from A to A prime. Let us analyze the condition when these forces sum to 0 that is sigma F; in this case, sigma F is equal to 0 for the equilibrium of this particle. If this condition is satisfied then let us see what happens to the virtual work.

The principle of virtual work states that if the particle is in equilibrium the total virtual work delta U of the forces acting on the particle is 0 for any virtual displacement of the particle. Since we have already defined that this particle A is in equilibrium because of the action of these various forces sigma F is 0. Irrespective of this vector, sigma F is equal to 0. The dot product of this vector to any virtual displacement vector delta r, it could be of any magnitude in any direction, that is 0 because this vector sigma F is 0. Any virtual displacement undergone by this particle the work corresponding to that or the virtual work because of that virtual displacement is 0. We state that the equilibrium equation of the particle which is sigma F equal to 0 can be rewritten as the virtual work delta U equal to 0 because sigma F dot delta r is 0. This equation which is the equation of equilibrium for the particle can be replaced by the virtual work equation that is delta U is equal to 0 because we know that this delta r is not 0. So, sigma F is 0.

If you would like to solve the other way round that means first we defined the virtual displacement and then we find the virtual work done by the various forces and then equate that virtual work to 0. For equilibrium to exist, we know that this delta r is not 0 because we are taking any arbitrary virtual displacement which is not 0, this sigma F has to be 0. Let us write them in the component form. So, we have F_x delta x is the virtual work done along the x direction. Sum of all such virtual works is sigma F_x delta x or if we define these coordinate systems then we are taking the component of this virtual displacement delta r along the x direction which we define as delta x and the component of these forces is F_x say 1, 2 and so forth.

Sum of all such components times this virtual displacement delta x will give the virtual work done by these forces along the x direction plus the virtual work done for the Y component of the virtual displacement and the virtual work done because of the virtual displacement along the Z direction. This summation has to be 0 and this gives you the three equations which can be used to find the components of sum of the unknown forces.

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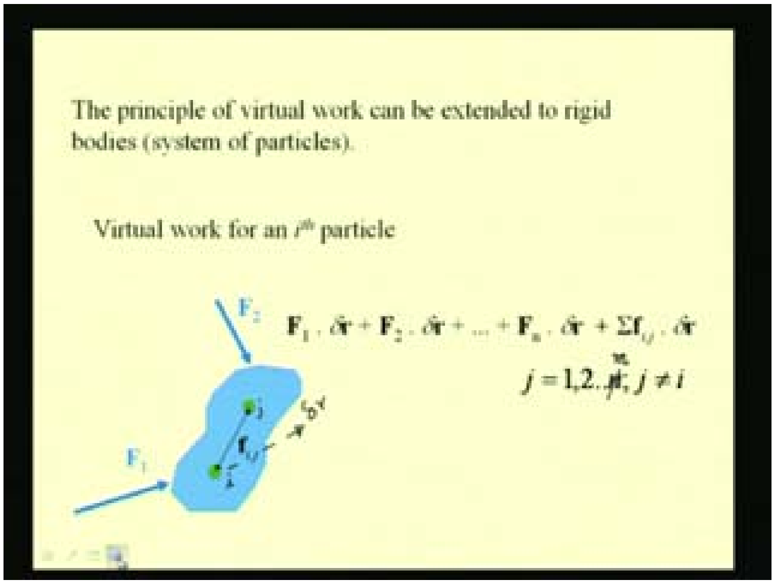

Let us extend this discussion for a system of particles or two rigid bodies. A rigid body is a system of particle where the relative distance between two particles does not change under the action of the forces. Here you see a system of particle or a rigid body and here we are just depicting two particles say a particle i and a particle j. There is some n number of forces acting on this rigid body.

Let us try to write the work done by these forces when this body undergoes a virtual displacement. When these external forces are acting in order to keep the integrity of this body, there exists a force between the various particles of the rigid body in order that the rigid body is stable and it could maintain its relative orientation and configuration. Intermolecular or interatomic forces develop which hold together the various parts of the rigid body. Under the application of the forces, the body is prevented from disintegration because of this intermolecular or interatomic forces. These forces, I have marked as F_{ij} or the force of this j th particle on the i. It is an equal and opposite force. That means on this particle i this force is acting in this direction, on this particle j the same force F_{ij} is acting in the other direction, equal and opposite direction. They are equal in magnitude but opposite in the direction.

If this rigid body undergoes some virtual displacement say delta r, let us try to find the work done by these forces when this rigid body undergoes a virtual displacement. For this i th particle we are writing For the single particle we are writing this equation, delta r is the virtual displacement. First we write the work done by these external forces say F_1 dot delta r plus F_2 dot delta r etc., say n numbers of forces are acting and then we write the work done by the internal forces. Here, we have this force F_{ij} which is the force of this j th particle on this i and we write this work done by all such forces as sigma F_{ij} dot delta r where this j moves from 1 to say some quantity m, where there are m such particles constituting the rigid body.

We take all those forces. This term sigma F_{ij} represents the sum of all the internal forces acting on that particle i the resultant of all such forces. The resultant of all the internal forces times the virtual displacement delta r, that is what this j not equal to i states that it takes care of all the internal forces except for F_{ii} because the particle does not put a force on itself because of its existence.

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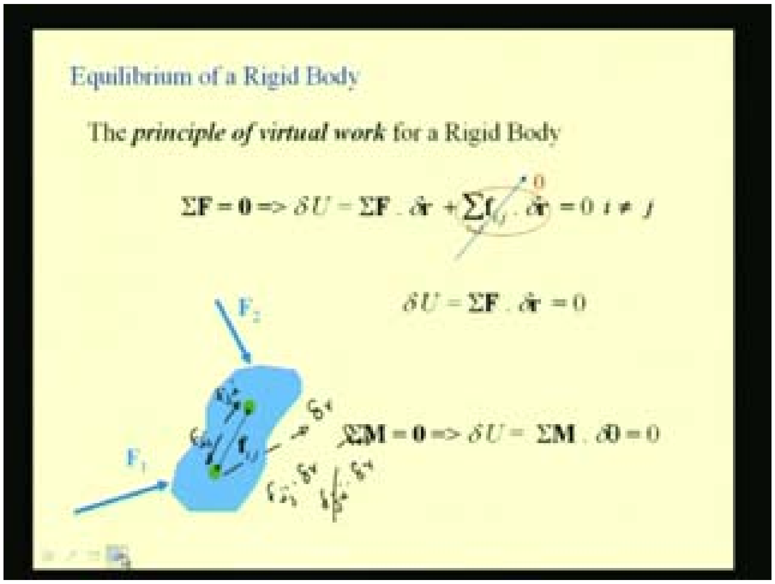

Let us define the equilibrium of the rigid body based on the virtual work that we have defined for a single particle. For this rigid body to be in equilibrium the sum of all the virtual work done by all these forces has to be 0. Let us write that equation. We have this equation sigma F is equal to 0 coming from our equilibrium definition. That means for any consistent virtual displacement the virtual work has to be 0. Let us write this virtual work delta U and equate it to 0.

What is a virtual work? The virtual work is the work done by the external forces. So, this quantity sigma F is nothing but the resultant of all these externally applied forces F_1, F_2 etc. Sigma F dot delta r is nothing but the virtual work done by the external forces when this rigid body undergoes a virtual displacement of delta r. This quantity is the total virtual work by the external forces. Let us sum up the work done by all these internal forces. We have sigma_{ij} F_{ij} dot delta r equal to 0 where i is not equal to j, that means this term accounts for the virtual work done by all the internal forces. When we take let say this i th particle and j th particle, we have this force say F_{ij} and this force say F_{ji}.

The work done by this force F_{ij} when this particle or this rigid body undergoes this virtual displacement delta r is F_{ij} dot delta r and the work done by this force is F_{ji} dot delta r. But we know that F_{ji} is nothing but this quantity is equal to minus F_{ij} because these two forces are equal and opposite. We see that the work done F_{ij} dot delta r minus F_{ij} dot delta r becomes 0. So, sum of all such virtual displacement when we take the various particle combinations becomes 0 and this quantity is 0 when we take for the complete rigid body. This quantity becomes 0 and we have the equation delta U the virtual work equal to 0 becomes sigma F dot delta r equal to 0. If equilibrium exists, that means sigma F is 0 for any virtual displacement delta r, this virtual work delta U is 0. This is how we define an alternative equation for equilibrium. We can state that sigma F equal to 0 is the equilibrium equation for a rigid body to be in equilibrium. In terms of virtual work, the virtual work done by these external forces for any consistent virtual displacement should be also 0.

The other equation - that is sigma M equal to 0 because for particle we only have sigma F equal to 0 defining the equilibrium state for the rigid body. The sum of all these forces has to be 0 and as well as the moment of all these external forces has to be 0; that is this condition sigma M equal to 0. We can write the virtual work done by the moment of the constituent forces. We can write this equation that is delta U where delta U is the virtual work done by the moments of these forces for causing a virtual angular displacement delta theta.

We have this delta U as sigma M dot delta theta which has to be also 0. So, the equilibrium equations for the rigid body can be written in view of these two equations, that is sigma F equal to 0 and sigma M equal to 0, we can write sigma F dot delta r equal to 0 and sigma M dot delta theta equal to 0 and we can solve some problems.

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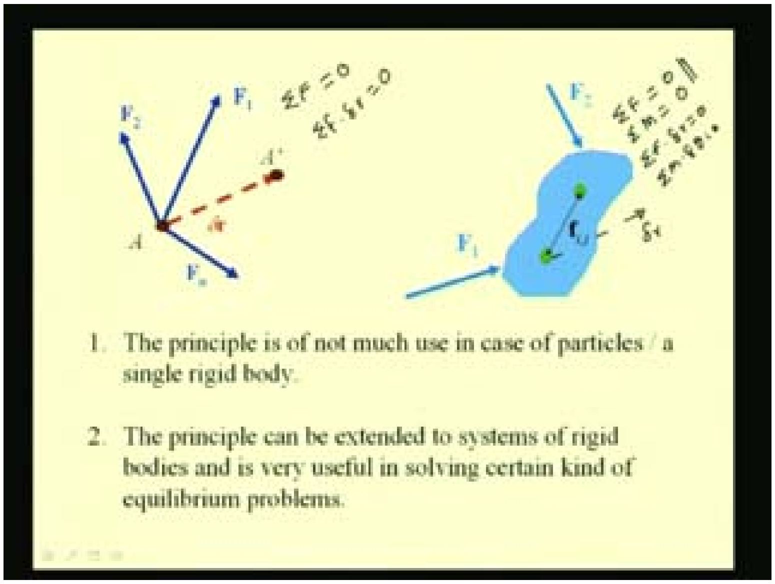

We have seen how we define the equilibrium equations for a particle and a rigid body based on this virtual displacement concept. The principle as we would have seen is not of much use in case of particles and single rigid body equation because if we would have written the equilibrium equation that is sigma F equal to 0, again we would have got the three scalar form equations and we could found the three unknowns. What we have done in view of this, we just write that sigma F dot delta r is equal to 0; again, this provides the three equations to solve, let us say three unknowns. We do not see much use in writing the equations in the alternative form. Same is the case in case of the rigid body, where we have written in lieu of these two equations. We are writing sigma F dot delta r equal to 0 and sigma M dot delta theta equal to 0. We are just replacing these two equations by the other two equations. But we will see that in case of connected rigid bodies, if we extend this principle of virtual work it becomes very useful to solve certain class of equilibrium problems. We will see some problems so that we realize the usefulness of this method for interconnected rigid bodies. Let us first define or extend this principle of virtual work to the system of connected rigid bodies.

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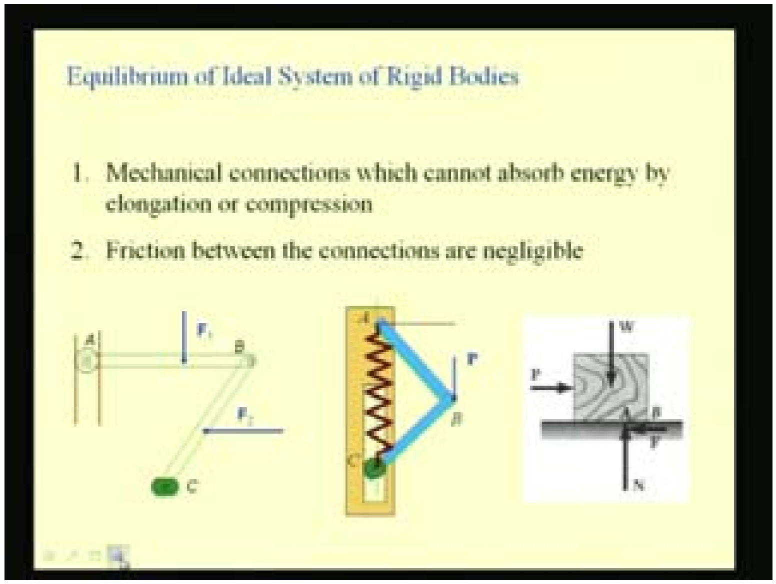

Before we do that, we have to have certain definition. First, we would like to define what we mean by an ideal system of rigid bodies. Let us first define what we mean by the ideal system of rigid bodies. We define a system of rigid body or a connected rigid body if the mechanical connections do not absorb energy by elongation or compression. That means that system should not have elastic members like springs or other kinds of elastic members which can absorb energy when some forces are applied to that. Because we are going to apply the method of virtual work, the work done should not be absorbed by any of the elastic members in the structure or the system under consideration. The work has to be conserved, in the sense whatever work is input to the system it should be output from the system and the system should not absorb any energy by elongation or compression. This is the first assumption in case we would like to idealize a system of rigid bodies.

The second assumption is that there is no friction between the connections and if all it is there it is very negligible quantity because we will see a little later that friction does negative work. Whenever friction is present, the work done by the friction is not available. It is normally dissipated as heat or other forms energy which is not recoverable. So again, we assume that there is no loss of energy because of the friction in the system. If we could take these two assumptions then we can idealize a real system and say that it is an ideal system of rigid bodies. Here, let us see some examples like this system of connected rigid bodies where we have a link AB and CB connected at this pin joint connected by a smooth roller at A and pinned at C. We assume that these are rigid, that is AB BC are rigid, thereby they do not elongate or deflect and absorb energy. If we assume as well that the contact between the roller and these guide ways is very smooth or it is lubricated as well as this pin connection at C as well as at B are well lubricated such that the friction that is existing is negligible then this system can be idealized and can be stated that this is an ideal system of connected rigid bodies.

Let us see other examples. Here you have a system of connected bodies where we have two links say AB and CB and there is a spring element that connects this pin at A and the roller at C. Here if this element is taken as a constituent member of the system that is this member AC which is a spring is taken as a component of the system then when some external forces are applied, the spring elongates or compresses thereby absorbing energy. This system cannot be idealized if we take the spring as a constituent member of the system.

Later on we will see that, we can actually solve this problem by removing this from the system and equivalently representing the force because of this element on the remaining system and we can then solve the problem. We have other systems like here this force P is trying to move this block and there is a frictional contact between this block and the surface where we have the friction which does a negative work. Thereby, this is also not an ideal system if this force F is considerable and it leads to the dissipation of the work done by heat. If such assumptions can be made like we can remove the spring elements or the elements that can absorb energy, we can also assume that such connections have negligible friction, then we can idealize the system and perform the virtual work method to analyze the equilibrium of such connected rigid bodies.

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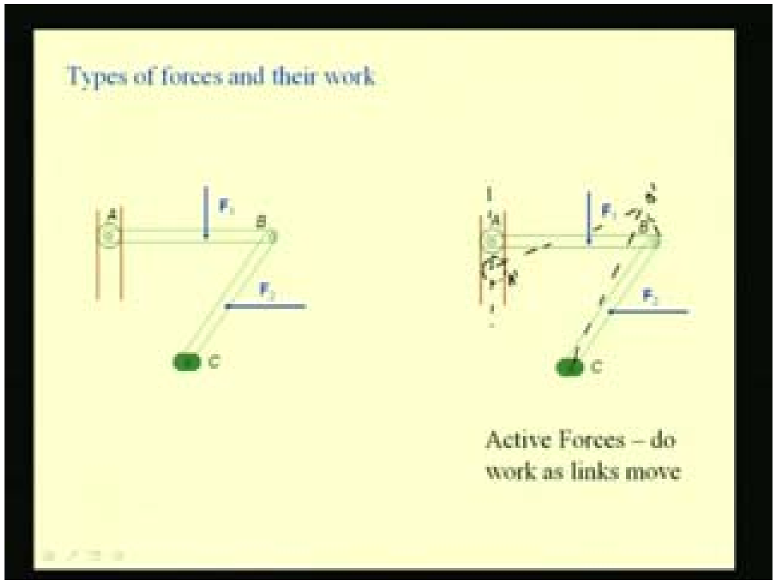

Before we go into the actual method, let us see the kinds of work that occur in the system of connected bodies and let us try to analyze the work done by these various forces. Let us take the same example of two links AB and CB connected at B and having a roller at A and pin connected at C and some external forces F_1, F_2 are acting on these members. Even though by our first analysis this member seems to be a two force member because it is connected at A and B, but since this force F is acting not at the extremities both the slender members that is AB and CB are not two force members. Let us try to see the various kinds of forces that are occurring.

First, let us see the applied forces say F_1 and F_2. Because of the application of these forces, the system tries to undergo a displacement consistent to the constraint. That means the displacement should be consistent with the constraint that at C the position of this link cannot move or this link CB can only rotate about this point C and at A, this point can only track along this dotted line. That means this roller is confined to the slide or the slots that is available. Consistent with these constraints if we assume that the link takes a new position this is your B prime this your A prime and C is same as C prime, but this link has moved to a new position, consistent to the fact that the length CB prime remains as CB and so forth. That is this is a pure rotation; the movement from B to B prime is a pure rotation about this point C.

If consistent to these constraints, this displacement has taken place then these forces that is F_1 and $F_{2,}$ would have done some work causing this displacement to occur. Thus, these works are to be considered when we analyze the problem of equilibrium or let us say when we are interested in finding some unknown forces, we have to consider the work done by these forces which we define as active forces because these forces do work when a displacement consistent to the constraint takes place in the system of connected rigid bodies. Neither we see that the displacement is perpendicular to these forces, nor do we see that the displacement is 0. That means there is some displacement which is not 0 and which is also not perpendicular to the direction of these forces that is F_1 and F_2. The work done by these force is not 0. Thus, we say that these are active forces or the forces which actually do the work and thus we term them as active forces.

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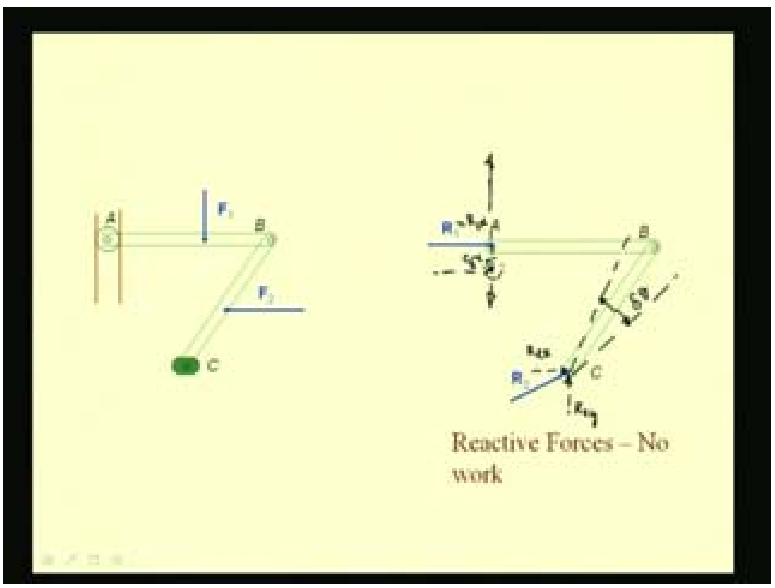

Let us see the other kinds of force that is occurring; that is the reaction forces. So, at A it is supported by a roller and so we have the reaction perpendicular to this slot along which this point A can move. At C since it is pin connected, the reaction can have both the horizontal component as well as the vertical component say R_2y R_2x and here it is only along the x direction. Consistent with this constraint, if we see any virtual displacement can happen, this point A can move to another point only along this line and this force R_1 is perpendicular to the direction in which any virtual displacement delta r can take place. Delta r can be either in this direction or it can be in this direction. For any such displacement, we see that since R_1 is perpendicular to this virtual displacement always the work done by this force that is R_1 the reaction at this point A is 0.

Let us see the work done by this force R_2 and because this point C is fixed, the link can rotate with respect to say this could be another virtual displacement or this could be delta theta. We see that since this point C cannot move and it is fixed, whatever may be the virtual displacement this point is not displaced and so the work done by this force R_2 is 0 again. We see that these reactions R_1 and R_2, they do no work when some virtual displacement takes place in the system.

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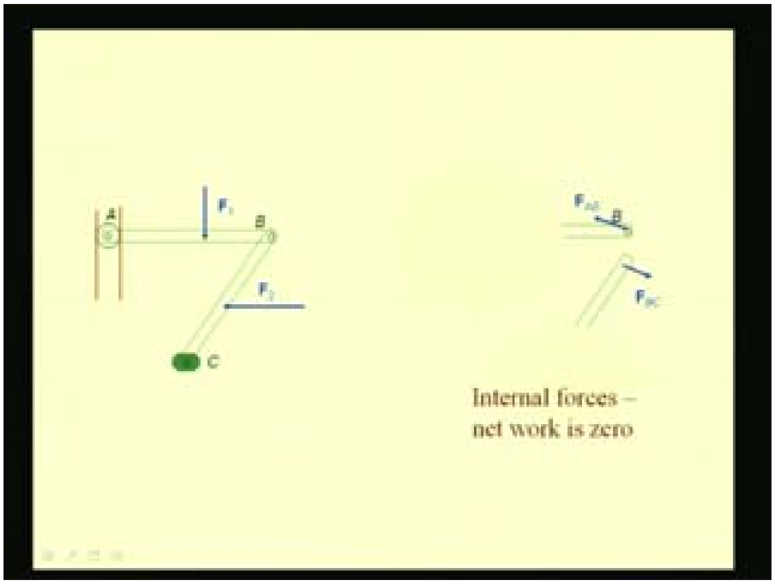

Let us see the other kind of force that is occurring that is the forces in the joints. If we dismember this, we will have a force in this multi force member AB say F_{AB} is the force in the multi force member. We again see that it has both components, along the member as well as perpendicular to the member and at the other link that is BC, we have an equal and opposite force F_{BC} because equilibrium should exist at this point B. So, the sum of these two forces should be 0. If this point B undergoes a virtual displacement delta r consistent with these constraints that means point A can move only in the vertical direction in the slot and point C cannot move and thus CB has to purely rotate about this point C. If consistent to these constraints, if we can define this delta r then the work done by this force F_{AB} and the work done by this force F_{BC} will cancel out each other and thus the net work becomes 0. We see that the work done only by the external forces have to be counted in a virtual work equation.

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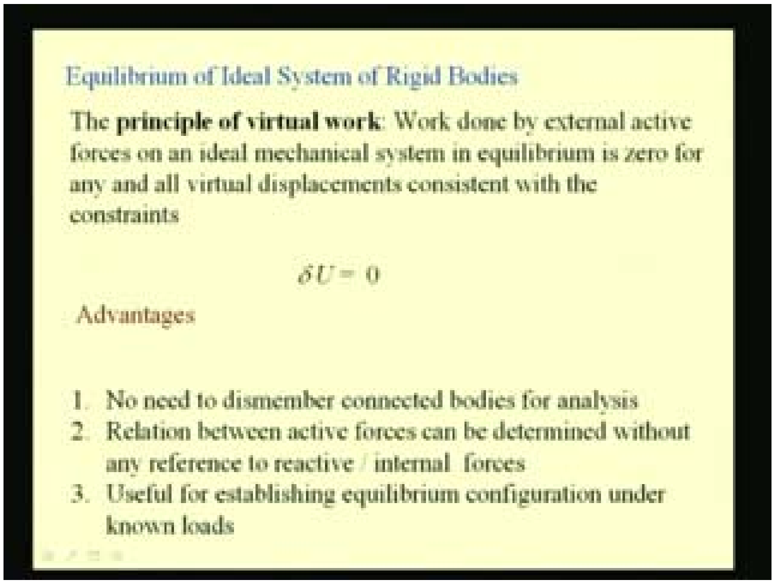

When we apply this equation to the ideal system of rigid bodies, the principle of virtual work becomes the work done by the external active forces on this ideal mechanical system in equilibrium is 0 for any and all virtual displacement consistent with the constraints. We can write this delta U equal to 0. So, the advantage is that we do not need to dismember because we have already seen that the internal forces constitute no work. So, for the analysis we do not need to dismember.

The second thing is that the relation between the active forces can be determined without any reference to reactive or internal forces, because we see that we do not have to find any internal forces and finally it is useful for establishing equilibrium configurations under known loads.

We will continue in the next lecture and we will discuss some concepts of degree of freedom and then we will solve some problems on this ideal system before taking up real systems.