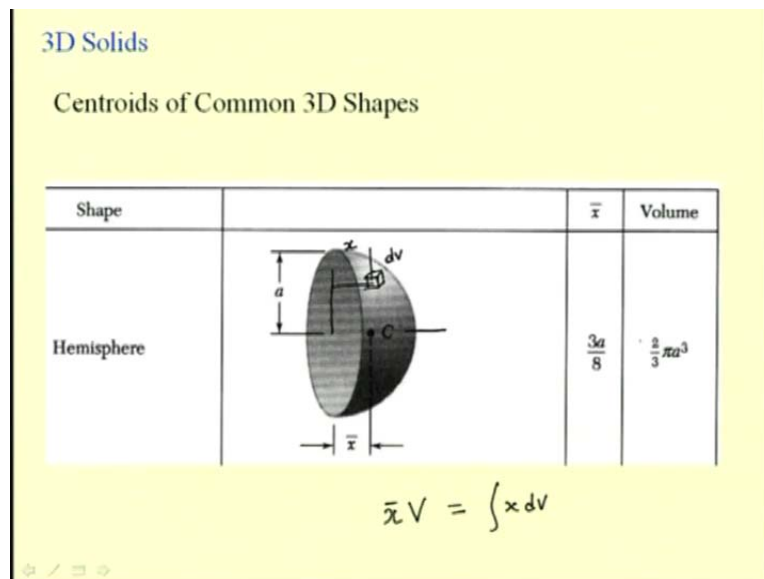


Engineering Mechanics
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Module 6 Lecture 14
Centroids and Area of Moments

Today, we will see some more topics on centroids and additionally we will see area moments. For your reference, this is module 6 lecture 14, of the engineering mechanics course. In the last lecture, we saw how to determine centroids by integration, as well as by methods of simple decomposition to determine centroids for plates. Today we will extend the discussion to volumes and then we will move on to discuss the second moment of the area.

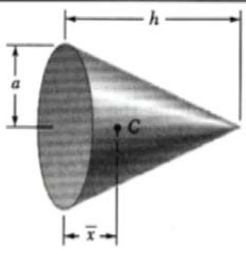
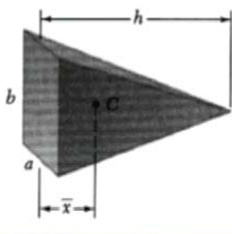
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For 3D solids, the centroid can be determined by the integration and for common shapes these values are known. Again, for 3D solids, we determine the centroids by finding the first moment of the differential weights of the small elements that we take and then, we integrate it to find the location of the centroid.

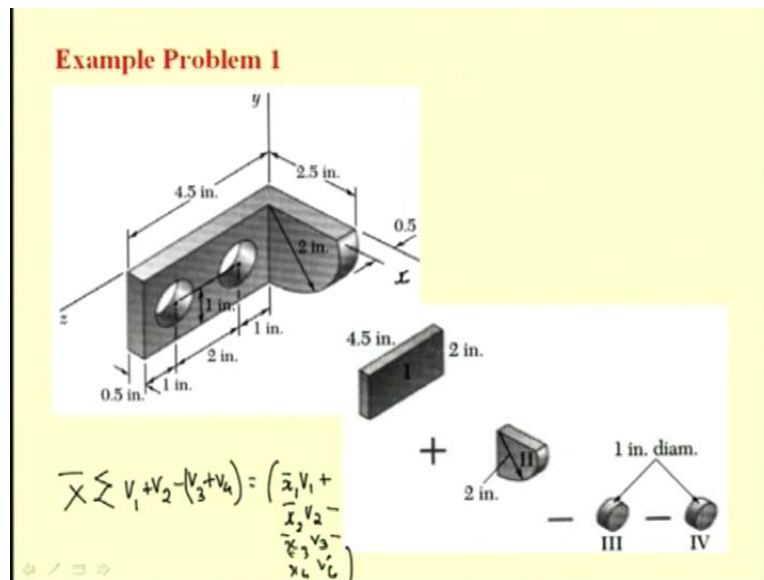
Here, you see one hemisphere and the centroid of the same is located at this point C. Since we have this symmetry along this diametrical plane, the centroid lies in that and it lies at a distance of \bar{x} from this diametrical plane. That distance is $3a/8$, where a is the radius of the hemisphere. Volume, again by integration can be found. For finding this, we take a small differential element with respect to this axis, this is dv and its location is x . We find this location \bar{x} , by writing $\bar{x} V$ is equal to integral $x dv$ and for the hemisphere, this V is the integral dv , which is $2/3 \pi a^3$. Once we know this V and this integral value, we can determine this \bar{x} .

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Cone		$\frac{h}{4}$	$\frac{1}{3} \pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3} abh$

In the same way for other simple solids, like cone and pyramid, the location of the centroids can be found. In case of cone, again we have symmetry. It lies along this axis and lies at a distance of $h/4$ from the face of the cone. For this pyramid, we have the centroid located at $h/4$ from this base.

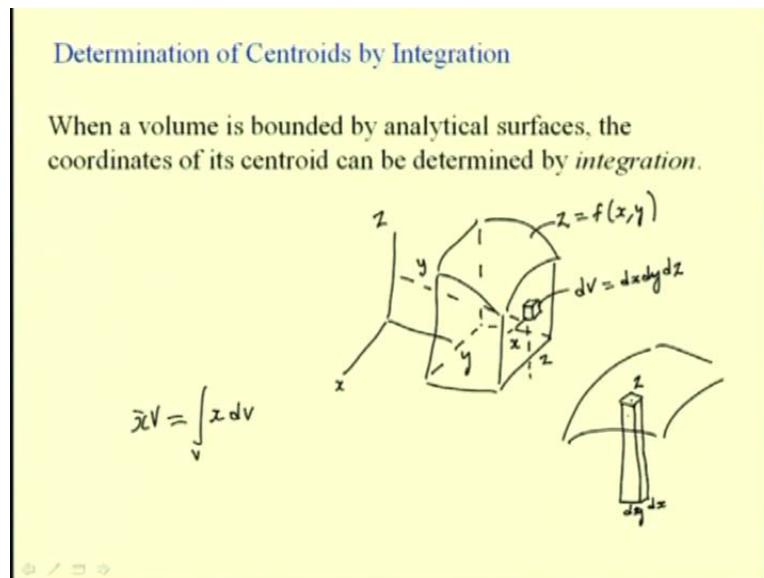
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Here we have an example, where we find the centroid of a complex shape by decomposing it into simpler shapes. For these simpler shapes, we know the location of the centroid and thus, we can find the location of the centroid of a complex object. Here, you see an object constituting of several features like we have 2 through holes and some circular filleted region. This component can be split into a rectangular plate of 0.5 inches thickness, from which we have to remove these circular holes; so negation of these volumes, that is III and IV, and then we have to add the volume corresponding to this feature, that is volume II and for these shapes the location of the centroid is known.

The location of the centroid for this component can be written with respect to the zy plane that is the location of the x direction. The location of \bar{x} , for the sum of the volumes V_1 plus V_2 minus V_3 plus V_4 is equal to sum of the location of \bar{x}_1 bar V_1 plus \bar{x}_2 bar V_2 minus \bar{x}_3 bar V_3 minus \bar{x}_4 bar V_4 . For these shapes, we know these values and the volumes are also known; from this, we can find this quantity. Same way, we can find the location of the centroid with respect to the other principle planes.

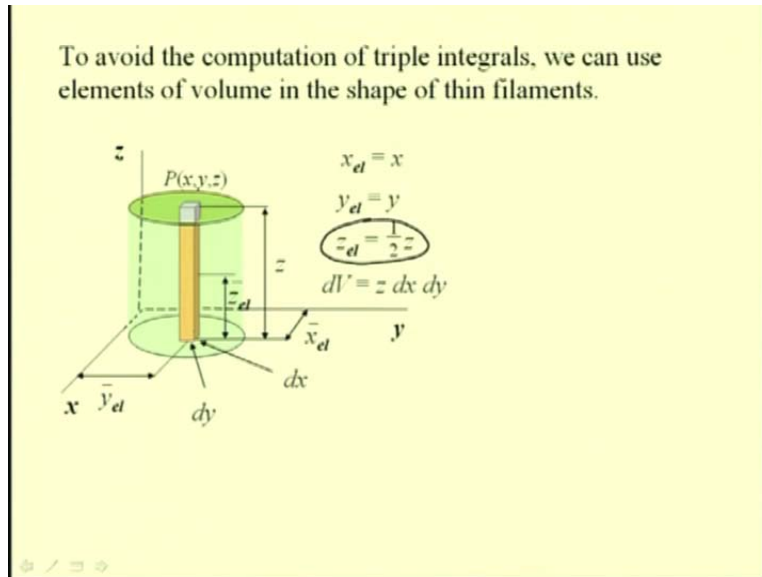
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Centroids can be found by the method of direct integration for analytical surfaces. If we take a surface whose equation is of the form, Z equal to function of x, y ; the surface is an analytical surface. Since we have the definition of the surface in terms of these coordinates, we find the centroid of the volume under this surface, by considering a small element whose volume is dv , which is equal to $dx dy dz$ and which is located at distances, say y, z and x .

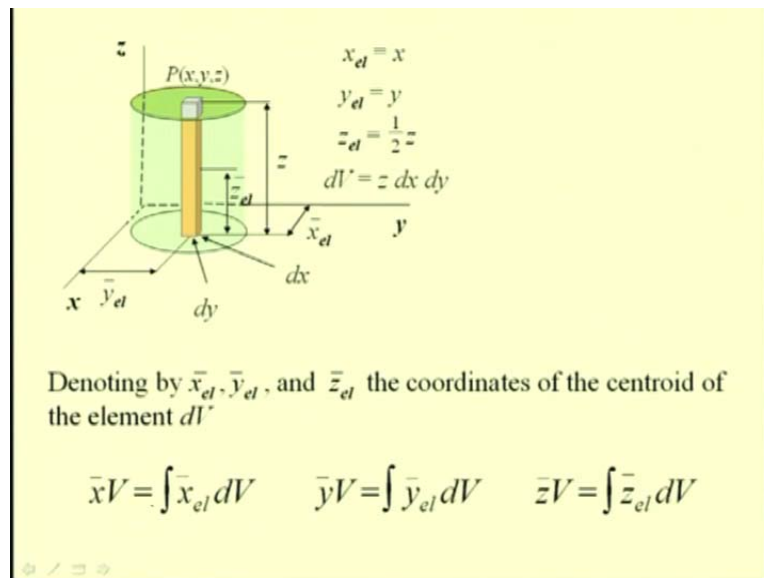
We can find the location of the centroid for this volume with respect to this plane as $x dv$ integration over the volume. This integral is a triple integral, that means, we have to integrate along x, y and z -axis. If you choose proper elements like instead of choosing this kind of an element, we can choose, for this analytical surface, a thin column of volume which is having dimensions dy and dx and along z the length of this column is equal to the z coordinate of the surface. We can now integrate this only along say x and y and we do not need to integrate along the z -axis. So, by properly choosing the elements, we can convert this triple integral into double integral and in certain cases it can be converted into a single integral also.

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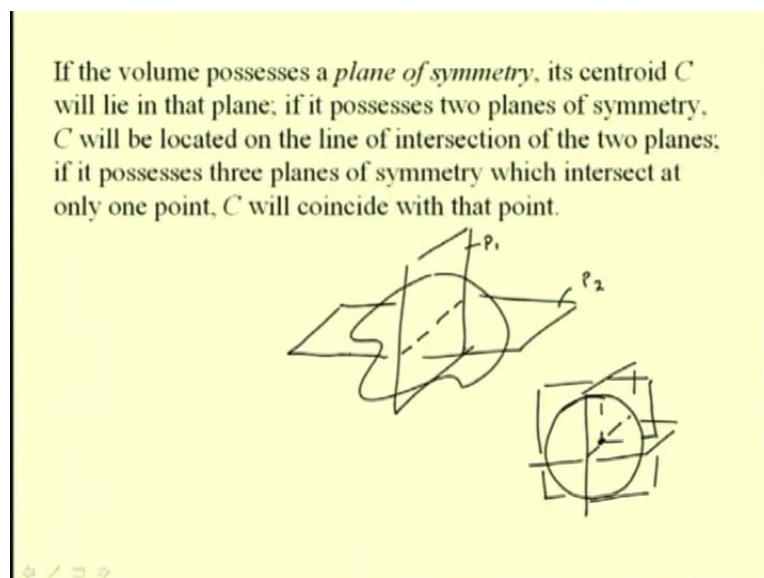
This example clearly illustrates that if we take a thin filament whose area, cross sectional area, is dy times dx and the length of this element is z , which is equal to the height of this cylindrical volume for which we are interested to find the location of the centroid. For this element, z element bar is the location of the centroid which is nothing but z by 2. The location of centroid in the x and y directions are, x and y itself. Because, we are considering it as a very small element, where dx and dy tends to 0. So the volume is z times dx dy .

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We can find these locations of the centroid \bar{x} , \bar{y} and \bar{z} by these integrals; that is $\bar{x}V$, where V is the volume of this cylinder is equal to integral $\bar{x}_{el} dV$, and same way, for the location of y and z coordinates of the centroids can be found.

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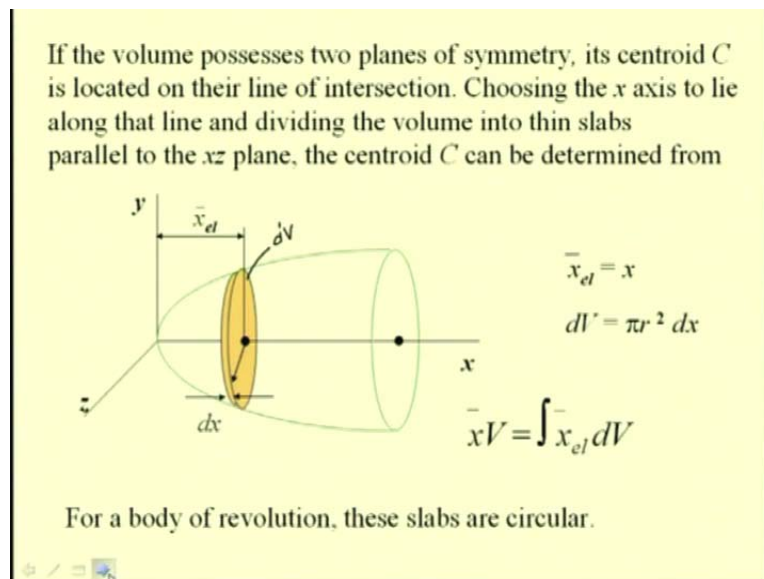


We have seen in 2D, that, if we have a line of symmetry for a plate, then, the centroid lies along that line of symmetry. Because, the moments, the first moment of the differential elements are

symmetrical and that is they are having positive sign on one side and negative side on the other and they cancel out. So, the centroid lies along the line of symmetry. In the same way, we can extend this discussion to the volume. For volumes, the first moments have equal magnitude and opposite signs with respect to the plane of symmetry, the centroid lies on the plane of symmetry.

If we have two planes of symmetry, then the centroid lies, let us take an object which is having one plane of symmetry, then the centroid lies in this plane. If it has another plane of symmetry then the centroid lies along the line, that is an intersection of these two symmetrical planes, say P_1 and P_2 . If it had a third symmetrical plane, like in case of spheres, we have three planes of symmetry say 1, 2 and then we have the third plane of symmetry. Thus, we have the location of the centroid, which is nothing but the center of the sphere itself. If it is possible to find the planes of symmetry, in case of three dimensional objects, then, determination of centroids becomes easier and we can do only the required integration along the non symmetrical direction.

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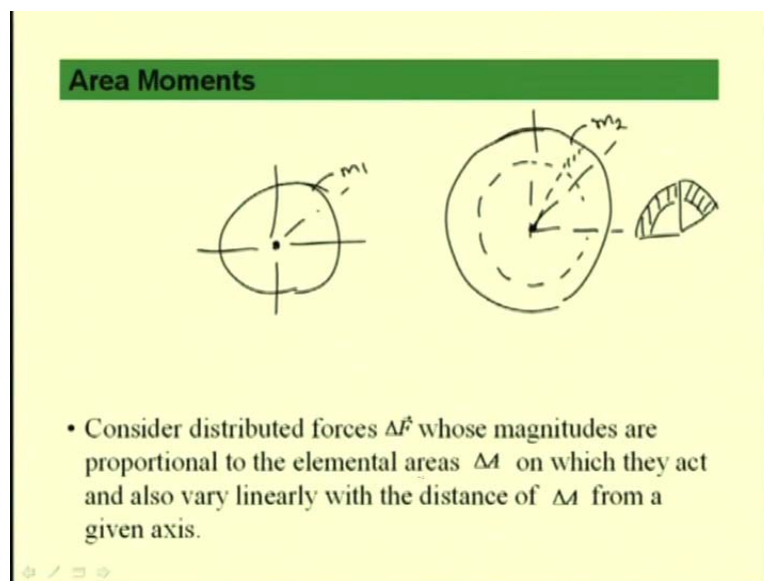
Here, you see an example, where we have two planes of symmetry, that is, this object is symmetrical about this plane yx , as well as it is symmetrical about this plane zx . The centroid lies along the line of intersection of these two planes, that is, the x -axis. We are interested to find the location of the centroid, along this x -axis. For that, we consider this thin slab which is

parallel to this plane yz or parallel to the plane, perpendicular to the axis, along which the centroid lies.

Now, we take these as the differential elements and integrate it. The location of the centroid for this thin slab is the x coordinate of this element, that is \bar{x} element, is the x coordinate of this element. The volume of this element dv can be determined from the geometry and we can integrate it to find this. For objects, which are symmetrical about this axis, these slabs become circles; that means these become thin discs. We have the volume as $\pi r^2 dx$ and now we can integrate this integral $\bar{x} dv$, to find the first moment of the volume. From this, we get this \bar{x} or the location of the centroid along the x -axis.

So, we saw how to determine these first moments. The first moments help in determining the location of centroids or the point through which we can assume the weight of the distributed body to act as a single concentrated weight. But we cannot differentiate two objects having distribution of the weight if we only know the centroid or the first moment.

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So, let us consider a solid sphere, which is having the centroid at its center and another sphere which is hollow from inside; that means, when you will cut it, you will have something like this. The inside is hollow for this solid. Also, we will see that the centroid lies in the same location. If

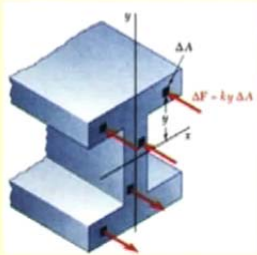
the mass of this sphere and the mass of this sphere are the same, and also, we have seen that the centroid is same, then, we have an equivalent weight acting through this point, both in magnitude as well as at the point of location of the load.

But the behavior of these two solids will be different when they are subjected to various forces and moments. That is because of the distribution of this mass. Here, it is solid and the distribution is uniform in any radial direction. But here, we see that we have the mass distribution which is not uniform. We do not have any mass up to certain point along any radial line and then we have the mass. It could be other way round like solids which are having different densities, which are porous inside and could be solid in the exterior. Their behavior is different when subjected to the forces and moments.

So, in order to quantify this, we have to take the second moments, which will take care of this distribution of the mass. In this context we will study determining the area moments and later on we can extend this discussion, to determine the second moments for mass, or the moment of inertia of mass. In order to study this, let us take a distributed force ΔF , whose magnitude are proportional to the elements ΔA , on which they act and also, vary linearly with the distance of ΔA from a given axis.

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• Example: Consider a beam subjected to pure bending. Internal forces vary linearly with distance from the neutral axis which passes through the section centroid.

$$\Delta \vec{F} = ky\Delta A$$


$$R = k \int y dA = 0 \quad \int y dA = Q_x = \text{first moment}$$

$$M = k \int y^2 dA \quad \int y^2 dA = \text{second moment}$$

This kind of a situation happens in pure bending of the beams. Here, in this picture, you see a cross section of a beam, here we have taken an I section beam and it is subjected to pure bending. That means, there are no resultant axial forces on this element. Sum of all these forces will be zero. But here, we see that these forces are compressive in this section, which is the top section above this x-axis and they are of the opposite sense. That is, here they are tensile in nature. Their sum becomes zero when we compute it and these forces have a magnitude, which is proportional to the distance from the axis. We have this δF equal to k , some constant of proportionality times, y times of δA . So, the force in these elements is proportional to the distance - that is y .

If you want to find the resultant, if you sum all these things that is $\int \delta F$, which is k being a constant taken out, $\int y \, dA$ and since, this geometry is symmetrical about this axis, this first moment $\int y \, dA$ is zero and thereby the resultant becomes zero. But if we take the moments of all these forces, it will be equal to y times the force.

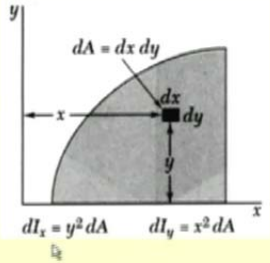
The momentum is y times the force, which is $ky \, \delta A$ and the total resultant moment of all these forces, that is M is equal to $k \int y^2 \, dA$ and k being a constant, taken out of the integration. We have, $\int y^2 \, dA$. This quantity, that is $\int y^2 \, dA$ is nothing but the second moment of the differential area dA . This is helpful to determine the resultant moment.

So, in the context of the beams, we can say that for the same bending moments that is subjected, for the same bending moment, two beams having different cross section or different cross sectional shapes will develop different distribution of these forces.

It is interesting from the point of design to choose cross sections that will resist this bending moment better. In the sense, the distribution of the internal forces will be safe. The design of these beams and choosing the cross section depends on the cross sectional shape and the distribution takes place. Let us see, how we determine this quantity, that is the second moment, for a given area, that is $\int y^2 \, dA$.

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Moment of Inertia of an Area by Integration



- *Second moments or moments of inertia of an area with respect to the x and y axes,*

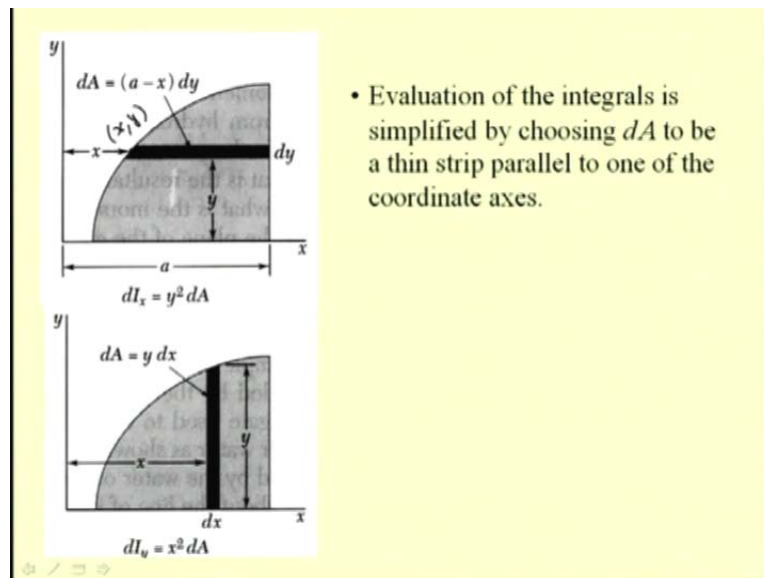
$$I_x = \int y^2 dA \quad I_y = \int x^2 dA$$

We can determine the moments of inertia of an area with respect to x and y-axis. We define it has I_x , the second moment of the area as integral $y^2 dA$. If we take this picture, we are interested to find the second moment of this total area. We take a small differential element $dx dy$, which is situated at a distance of x from the y-axis and at the distance of y from the x-axis.

If you are interested to find the second moment of this elemental area, with respect to the x-axis, it is equal to the distance of this area from the axis that is $y^2 dA$. If we integrate it, we get the moment of or the second moment of this area or the moment of inertia of the area, with respect to this axis.

Same way, with respect to y, we have it has integral $x^2 dA$. For these differential areas the differential moments of inertia dI_x and dI_y are defined as $y^2 dA$ and $x^2 dA$ respectively. We can simplify this integration by choosing suitable elements; that is, we can take thin strips of rectangular shape or triangular shapes, depending upon the geometry of the area to simplify the integration.

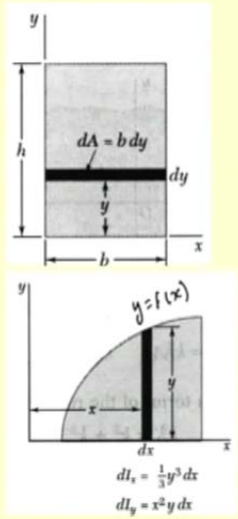
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Here for the same example, we take a thin strip which is horizontal or parallel to the x-axis. In order to determine the second moment of this area, with respect to this axis, that is x-axis, we define the second moment of this thin strip dI_x , with respect to this x-axis as y square, the distance square from the axis times dA , the area of this element.

What is the area of this element? It is nothing **but...** if this distance is A , then A minus x this point on this curve has the coordinate x comma y times the thickness of this element, that is dy . Same way, you can do the integration by taking a vertical strip also. In this case, this will be helpful to determine the moment of inertia of this area with respect to the y-axis. The differential second moment of this area, dI_y , is defined as x square. The square of the distance of this thin strip from the y-axis times dA , where the area of this element is y times dx , dx being the width of the strip and y the height or the length of the strip. By choosing these elements, the integration becomes simpler; that means the double integration has been simplified to single integration.

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The diagram consists of two parts. The top part shows a rectangular area in the first quadrant of a Cartesian coordinate system, with width b and height h . A horizontal differential strip of thickness dy is shown at a distance y from the x-axis. The area of this strip is labeled $dA = b dy$. The bottom part shows a curved area under a curve $y = f(x)$ in the first quadrant. A vertical differential strip of width dx is shown at a distance x from the y-axis and height y from the x-axis. Below this strip, the differential second moments are given as $dI_x = \frac{1}{3} y^3 dx$ and $dI_y = x^2 y dx$.

- For a rectangular area,

$$I_x = \int y^2 dA = \int_0^h y^2 b dy = \frac{1}{3} b h^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

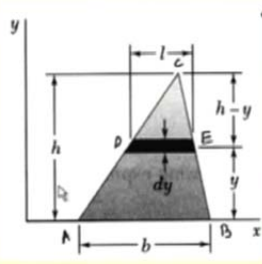
$$dI_x = \frac{1}{3} y^3 dx \quad dI_y = x^2 dA = x^2 y dx$$

Let us consider for a rectangular plate. For this we consider this horizontal element dA . If b is the width of this rectangular area and h the height of the rectangular area, then area of this thin strip is b times dy and it is located at a distance of y from the x -axis. If you find the second moment of this area with respect to the x -axis, then we have I_x equal to integral y square dA , where dA is b times dy and we have the quantity y square $b dy$, which has to be integrated between the limits that is 0 to h , where y varies from 0 to h . If we integrate this, we have the value as one-third $b h$ cube. Since, we are going to consider thin rectangular strips for the integration for any other kind of a area, these results can be used.

Let us say that we have a general curve. This equation is defined as y is some function of x . So, we define this thin vertical strip; the second moment of this strip with respect to the x -axis is one-third width of this strip, which is b , which is nothing but dx , in this case times height cube, in this case height is y . We have one-third y cube dx is the second moment of this thin strip with respect to this x -axis and the second moment of this thin strip with respect to the y -axis is x square times the area which is y times dx . So, we have dI_y as x square $y dx$. If we integrate this between the limits we have the results.

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Example Problem 2
Determine the moment of inertia of a triangle with respect to its base.



- A differential strip parallel to the x axis is chosen for dA .

$$dI_x = y^2 dA \quad dA = l dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \quad l = b \frac{h-y}{h}$$

$$dA = b \frac{h-y}{h} dy$$

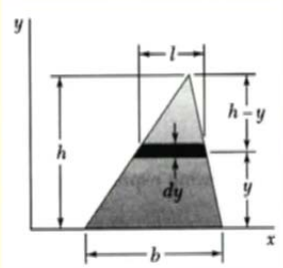
Let us take this example problem. Here, you see a triangle for which we are interested to find the second moment of this area with respect to its base; that is in this case, with respect to the x-axis because the base is located on the x-axis. In order to do that, we define a thin horizontal strip of width dy , which is located at a distance of y , from the x-axis.

If h is the height of the triangle, then we have these dimensions that the vertical height of the strip from the base is y . So, the remaining distance is h minus y . If we have b as the base of the triangle then we have to determine what will be the length of this strip at this height of y . For this we can use these two triangles $A B C D E$; then, we can use these triangles $A C B$ and $D C E$, which are symmetric triangles and we can find what will be this l at this height of y . So these are the values. dI_x is the second moment of this strip with respect to the x-axis, which is y square dA and where dA is l times dy . We have to determine this l . From these similar triangles, we have l divided by b equal to h minus y by h or h minus y divided by the total height h . From this we have the length as b times h minus y by h , which gives this value of dA as, b times h minus y by h times dy .

We can use this area dA in the expression and integrate it to get the location of the centroid. In this case, we are interested in the second moment. So, we can find the second moment with respect to the axis.

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• Integrating dI_x from $y = 0$ to $y = h$,



$$I_x = \int y^2 dA = \int_0^h y^2 b \frac{h-y}{h} dy$$

$$= \frac{b}{h} \int_0^h (hy^2 - y^3) dy$$

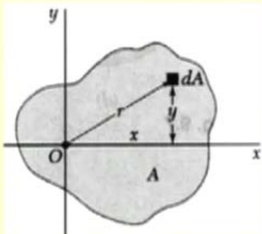
$$= \frac{b}{h} \left[h \frac{y^3}{3} - \frac{y^4}{4} \right]_0^h$$

$$I_x = \frac{bh^3}{12}$$

Integrating this quantity dI_x between 0 and h , we have I_x as integral dI_x , which is y square dA , which is equal to the limits of integration is y from 0 to h y square. This dA , is nothing but b times h minus y divided by h times dy . If we simplify this term, we have this as b by h ; these two being the constants, we take it out; hy square minus y cube dy and if you integrate it, we have this as, hy cube by 3 minus y to the power 4 by 4 in the limit of 0 to h . When we put these limits, we have I_x as bh cube by 12. This example illustrated for a simple geometry of a triangle how we use the method of integration, and we use elements which reduce the double integration to single integration or we use thin strips instead of differential small elements, which vary both in x and y direction.

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Polar Moment of Inertia



- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

$$J_0 = \int r^2 dA$$

- The polar moment of inertia is related to the rectangular moments of inertia,

$$J_0 = \int r^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA = I_y + I_x$$

We will discuss the concept of polar moment of inertia of an area. Till now we saw, determining the second moment or moment of inertia of the area with respect to x and y-axis or the axis in the plane of the area. The polar moment of inertia is the second moment of the area with respect to the axis perpendicular to the plane of the area. This is particularly of interest in the problems concerning the torsion of the cylindrical shafts where we are interested to find the angular twist caused in a shaft, because of a given twisting moment.

We saw that the area moments are helpful in determining the forces and that occur in the beams. In the same way, these polar moments are helpful in determining the required quantity. You know the quantities when solving torsion problems. Also this quantities help in problems of rotation of slabs, where this polar moment of inertia, helps in determining the response of this slab for the rotation when subjected to some rotational moments. Let us take this area dA , which is having the distance y from the x-axis and x from the y-axis. The second moment of this area with respect to y, we have already seen is $y^2 dA$,

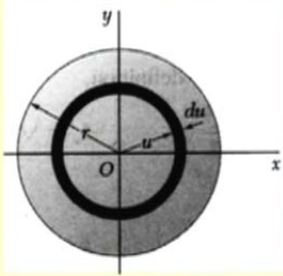
In the same way, the second moment of this area with respect to the axis, that is z-axis, passing through this point o is nothing but the square of the distance that is $r^2 dA$. If we integrate this quantity, we have the polar moment of inertia, which we designate it as J , with respect to o,

as integral $r^2 dA$. From geometry, we know that, this r and x and y are nothing but sides of right angle triangle.

We have r^2 as, x^2 plus y^2 . So, the polar moment of inertia can be related to the rectangular moments of inertia by substituting this r^2 as x^2 plus y^2 in the integration. We find that this is equal to integral $x^2 dA$ plus integral $y^2 dA$, which is nothing but the second moment of the area with respect to y and second moment of the area with respect to x .

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Example Problem 3
Determine the moment of inertia of a circular area with respect to a diameter.



- An annular differential area element is chosen,

$$dJ_O = u^2 dA \quad dA = 2\pi u du$$

$$J_O = \int dJ_O = \int_0^r u^2 (2\pi u du) = 2\pi \int_0^r u^3 du$$

$$J_O = \frac{\pi}{2} r^4$$

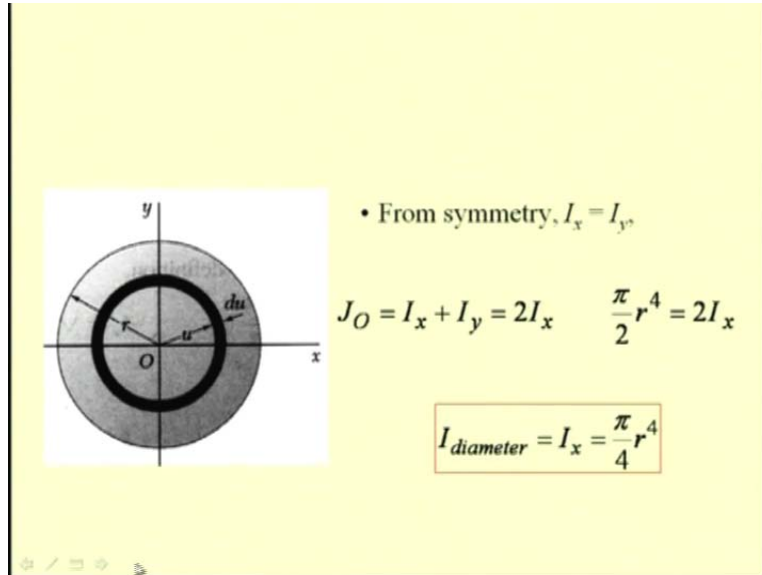
Let us take one example. Here, we see a circular area and we are interested to find the moment of inertia of this circular area, with respect to this diametrical line, say, with respect to this x -axis.

This problem we solve by first finding the polar moment of inertia with respect to the z -axis and then, we determine the rectangular moment of inertia that is I_x . You will just see why this is convenient; because, in this case, we can take an annular differential element, which is located at a distance of u from the z -axis and whose thickness is du .

The polar moment of inertia can be found by considering the polar moment or the moment with respect to the z -axis of this element, that is, u^2 the distance of this element square times dA , the area of this element, which is $2\pi r$. In this case, the radius is u times du , the change in

the radius. This is the area of this differential element and when we integrate this, we get the polar moment of inertia as $2\pi \int_0^r u^3 du$, which is nothing but, πr^4 . We have already seen that the sum of the rectangular components or the rectangular moments of inertia I_x and I_y is equal to J_O or the polar moment of inertia and we use this to find I_x .

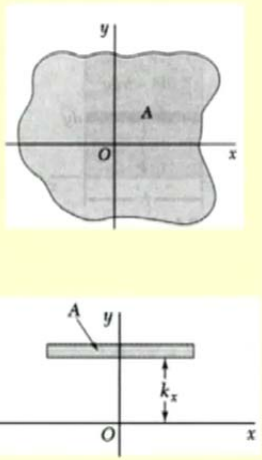
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In the case of this circular lamina, the moment of inertia with respect to x and the moment of inertia of the area with respect to y are the same, by symmetry. So, we have I_x is equal to I_y . The rectangular second moments are the same. So we write, J_O is equal to sum of the rectangular second moments, which is equal to $2I_x$ or $2I_y$ both being same. Since we have now determined J_O , we can determine this I_x , which has been found as $\pi r^4 / 4$ and this is the same with respect to the other diametrical line, that is, y -axis also.

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Radius of Gyration of an Area



- Consider area A with moment of inertia I_x . Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x .

$$I_x = k_x^2 A \quad k_x = \sqrt{\frac{I_x}{A}}$$

k_x = radius of gyration with respect to the x axis

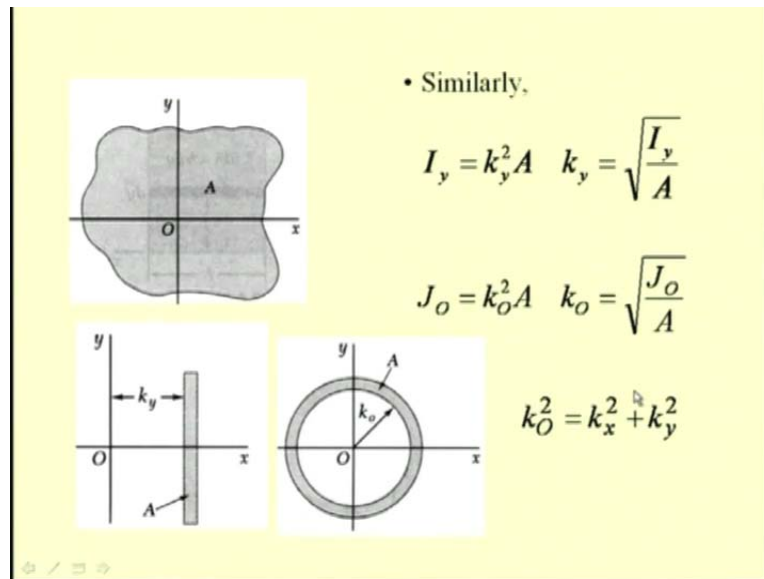
We define this radius of gyration of the area. For defining this for any element or any area element, if we assume that, this element is of a form of a thin strip and if we want to equate the second moment of this thin strip to the second moment of the area then the distance at which this thin strip will be situated is known as the radius of gyration.

So, what are we doing? We are finding an equivalent thin strip, which has same second moment with respect to the concerned axis. If we have this area A and we are interested to find their second moments, with respect to this x and y -axis passing through O , then, for finding this second moment with respect to x , we imagine that a thin concentrated area exists whose second moment with respect to this x -axis is same as the second moment of this area with respect to the x -axis.

If such an imaginary thin strip exists whose area is same as the area of the area of interest, then we are interested to know the distance at which this thin strip will be located with respect to the axis, that is ox . We equate their second moments. So, we know that, I_x is equal to k_x square times A for this thin strip because k_x is the distance of this thin strip and we have k_x as root of I_x by A , where I_x is the second moment for this area, with respect to the x -axis.

This is a unique quantity with respect to a given axis. So, for various areas that are of engineering importance like beams of cross sections, like I section or L sections are the typical cross sections, so we take this radius of gyration as tabulated values, to proceed with our engineering calculation. This is the radius of gyration with respect to x-axis.

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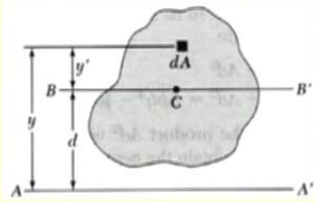


In the same way, we can define the radius of gyration with respect to the y-axis, by considering an imaginary area having the same area of this element and situated at the distance of k_y . Then k_y is equal to root of I_y , the second moment of this area with respect to y-axis divided by A. If we know these two quantities k_x and k_y , it is possible to define the radius of gyration with respect to the z-axis or the polar axis. We assume that this area is concentrated as a thin annular strip of radius k_o then the second moment of this area with respect to the z-axis, which is J_o is equal to k_o square times of A and thus k_o is J_o by A.

We know the relation between J_o , I_x and I_y . From that we can easily find that, k_o square is equal to k_x square plus k_y square or square of the radius of gyration with respect to x-axis, plus square of the radius of gyration with respect to y-axis is equal to the radius of gyration with respect to the polar axis square.

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Parallel Axis Theorem



- Consider moment of inertia I of an area A with respect to the axis AA'

$$I = \int y^2 dA$$

- The axis BB' passes through the area centroid and is called a *centroidal axis*.

$$I = \int y^2 dA = \int (y' + d)^2 dA$$

$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

$$I = \bar{I} + Ad^2$$

If we determine the second moment with respect to a given coordinate system then we can determine the second moment with respect to a parallel axis. Because, if in a particular engineering calculation, the area is now transformed with respect to a given axis, its second moment changes, but we are not interested to compute every time the second moment by let us say a method of integration.

If we relate the second moment of a given area with respect to two parallel axes, then it will reduce the amount of computation required to compute these values. Let us consider, in this case, this axis BB' , which is a centroidal axis that means it passes through the centroid C of a given area; that means, the first moment of the area with respect to this axis BB' is zero. We have this x -axis or this AA' axis and the distance of any given element differential element dA as y from this axis AA' and if d is the distance between these two parallel axes, then the distance of this element dA with respect to this axis, is nothing but, y minus d or this y' distance.

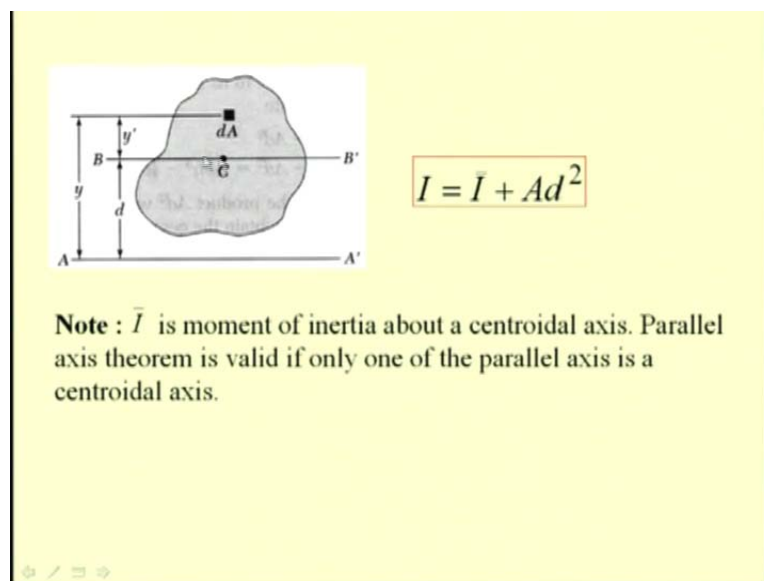
We have this second moment with respect to this axis AA' as integral $y^2 dA$, this y is nothing but y' plus d . So, we write this integration as y' plus d square times dA . If we expand this we have integral $y'^2 dA$ plus two d , d being constant, it has been taken out of the integration, integral $y' dA$ plus d^2 integral dA .

This quantity is nothing but the first moment of this elemental area dA with respect to this axis BB' , which is nothing but a centroidal axis. We know that the first moment of the summation of the first moment of the area is zero with respect to a centroidal axis and this quantity now becomes zero.

We have this integration as, this quantity which is integral y prime square dA , which we designate it as, I bar plus integral dA is nothing but the area of this element A times d square. This quantity I bar is nothing but, the second moment of the area, with respect to this axis BB' prime or the centroidal axis. This quantity I bar is nothing but the second moment of this area with respect to this centroidal axis.

We have a relation between the second moment of the area, with the respect to the centroidal axis and any other axis which is parallel to the centroidal axis and located at a distance of d . You should note that, this relation between the two parallel axes is valid only if one of the axes passes through the centroid of the given area.

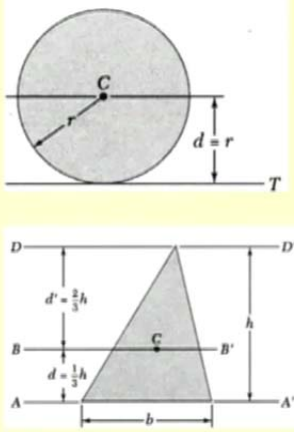
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This point you should note that this relation is valid only if one of the axes is a centroidal axis.

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Example Problem 4



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \frac{1}{4} \pi r^4 + (\pi r^2) r^2 = \frac{5}{4} \pi r^4$$
- Moment of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$

$$I_{BB'} = I_{AA'} - Ad^2 = \frac{1}{12} bh^3 - \frac{1}{2} bh \left(\frac{1}{3} h \right)^2 = \frac{1}{36} bh^3$$

Let us take one example to illustrate this. We have this circular lamina of radius r and we are interested to find the moment of this area or the second moment of this area with respect to tangential axis T which is situated at a distance of r or the radius of the disc from the centroidal axis. We have this I_T or the second moment of this area, with respect to this tangential axis as the centroidal moment or the moment of inertia of this area with respect to the centroidal axis \bar{I} plus A times d square.

These values are available for our engineering computation. For known shapes like circles, triangles and rectangles, it is possible to find the second moment by integration and this can be made as a table and used for computation. The value of \bar{I} is $\frac{1}{4} \pi r^4$ which we have already seen in our earlier examples. We have found the moment of inertia with respect to one of the diametrical axis, for a circular lamina, as $\frac{1}{4} \pi r^4$.

We know this distance is r and from this we get the moment of inertia, with respect to this axis as $\frac{5}{4} \pi r^4$. We see that we can advantageously use this parallel axis theorem to determine the second moments, with respect to the axis parallel to a centroidal axis.

We can take one more example. Here, you see a triangular lamina and BB prime is a centroidal axis. We know the centroid for this triangle is situated at one-third of h from the base and if you know the moment of inertia of this area or the second moment of the area with respect to this axis then we can find the second moment with respect to this AA prime axis. We have $I_{AA \text{ prime}}$ is equal to the second moment with respect to the centroidal axis plus Ad square. If we know the second moment with respect to this AA prime, which we have just computed in one of our earlier examples where we determine the second moment with respect to the base of the same triangle we determine that as $\frac{1}{12} b h^3$.

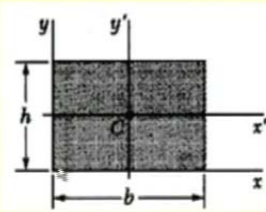
We can use this to determine the second moment, with respect to the centroidal axis and knowing that, this d is one-third of h, we can determine the second moment as $\frac{1}{36} b h^3$ with respect to the centroidal axis. So these examples illustrate how we can use the parallel axis theorem to determine the second moments.

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Moments of Inertia of Composite Areas

The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, \dots , with respect to the same axis.

\bar{I} the moment of inertia about the centroidal axis for common shapes are available as table for reference

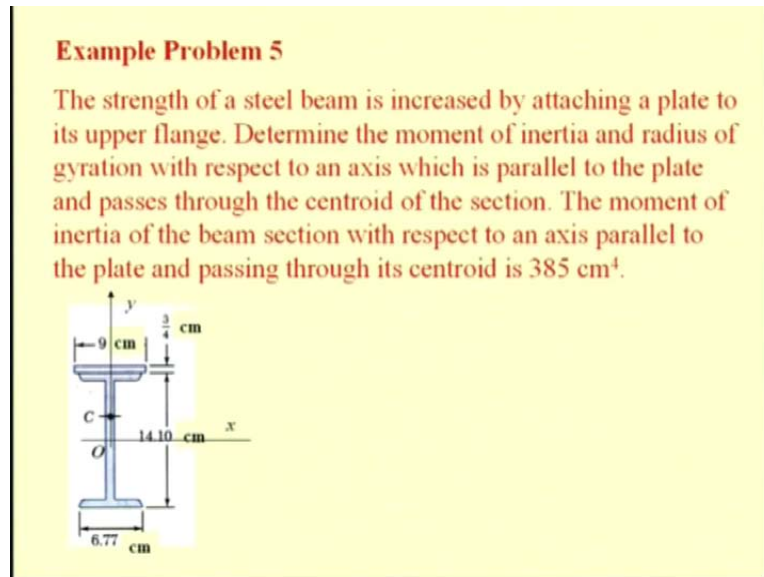


$\bar{I}_{x'} = \frac{1}{12} b h^3$
$\bar{I}_{y'} = \frac{1}{12} b^3 h$
$I_x = \frac{1}{3} b h^3$
$I_y = \frac{1}{3} b^3 h$
$J_C = \frac{1}{12} b h (b^2 + h^2)$

If we have composite areas, then we can determine the second moments by considering the second moments of the individual areas and summing up in order to determine the second moment for the component or the compound area. For simpler shapes these are known. For this rectangular area we have these various quantities like the second moment with respect to the centroidal axis, say x x prime or the centroidal axis y prime and the polar moment of inertia of

this area with respect to the centroid. For many areas of standard shapes like triangles, rectangles these quantities are known and an area which is composed of such simpler areas, we can determine the second moment, by considering this individual component areas.

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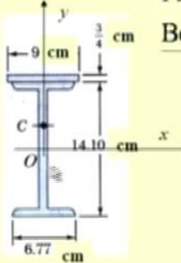


Let us see one example, here you see a typical I section, cross section for a beam. In order to strengthen this, we are actually welding a thin strip or a plate to the top of this I section beam. We are interested to find the moment of inertia and the radius of gyration, with respect to an axis which is parallel to this plate and which passes through the centroid of this section.

It is given that the moment of inertia of only the beam section for an axis which is parallel to the plate and passing through the centroid, the centroid of this beam section is this O itself, because we see that, it has two axis of symmetry and it passes through O and so with respect to this axis, it is given has 385 centimeter to the power of 4. In order to determine the moment of inertia and radius of gyration, we decompose into the areas, for which we know these quantities.

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To find the location of centroid of the section sum the first moment of the area about O



Section	A, cm^2	$\bar{y}, \text{cm.}$	$\bar{y}A, \text{cm}^3$
Plate	6.75	7.425	50.12
BeamSection	11.20	0	0
	$\sum A = 17.95$		$\sum \bar{y}A = 50.12$

$$\bar{Y} \sum A = \sum \bar{y}A$$

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{50.12 \text{ cm}^3}{17.95 \text{ cm}^2}$$

$$= 2.792 \text{ cm}$$

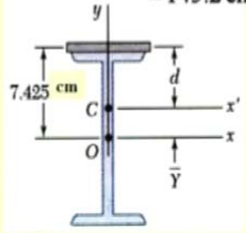
We decompose this into two areas, that is one in the I section and other, the plate. We determine the centroid of the section first. In order to determine that we have the areas of individual sections, that is, the beam section, for which the value is 11.20 centimeter square and for plate which is 9 centimeter by 3 meters, the area is 6.75 centimeter square. The location of the centroid for the beam section is 0 with respect to this x-axis, because you know it is symmetrical about this x-axis, \bar{y} is 0. For this thin plate also, the first moment is 50.12 for the plate and 7.425 for this plate.

This quantity can be found by parallel axis. We have the summation of this area as 17.95 and $\sum \bar{y}A$ as 50.12. From this we can determine the centroidal location of this composite section, with respect to this axis Ox , which is, $\bar{Y} \sum A = \sum \bar{y}A$ or the location of the centroid of component areas. From this, we determine the location as 2.792 centimeters.

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Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.

$$I_{x', \text{beam section}} = \bar{I}_x + A\bar{Y}^2 = 385 + (11.20)(2.792)^2 = 472.3 \text{ cm}^4$$

$$I_{x', \text{plate}} = \bar{I}_x + Ad^2 = \frac{1}{12}(9)\left(\frac{3}{4}\right)^3 + (6.75)(7.425 - 2.792)^2 = 145.2 \text{ cm}^4$$


$$I_{x'} = I_{x', \text{beam section}} + I_{x', \text{plate}} = 472.3 + 145.2$$

$$I_{x'} = 618 \text{ cm}^4$$

We can determine the second moment, with respect to the axis passing through this centroid C for this composite section. We have this I_x for the beam section as I_x with respect to its centroidal axis plus the area of the cross section of the beam section times y bar square. It has been determined as 472.3 centimeter to the power of 4; for the plate section again by using the parallel axis theorem we have $I_{x'}$ prime plate is equal to I_x bar plus Ad square, where A is the area of the plate and d is the distance of this plate, with respect to this axis. We have this quantity as 145.2 centimeter to the power of 4. We can determine for the complete section as sum of these moments, which is equal to 618 centimeter to the power 4.

So these problems illustrated how we can use this parallel axis theorem, for computing the second moments of the area.