Engineering Mechanics

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Module 6 Lecture 13
Centroids and Center of Mass

In all our previous lectures, in engineering mechanics, we primarily saw how to compute the various forces and the moments for concentrated loads. When we have distributed loads in a system, then we have to find their equivalent, in order to evaluate the moments due to those distributed forces and their effects on the equilibrium of the body. So, we will see certain procedures to compute the points of concentrated load that can be said to be equivalent to the distributed loads.

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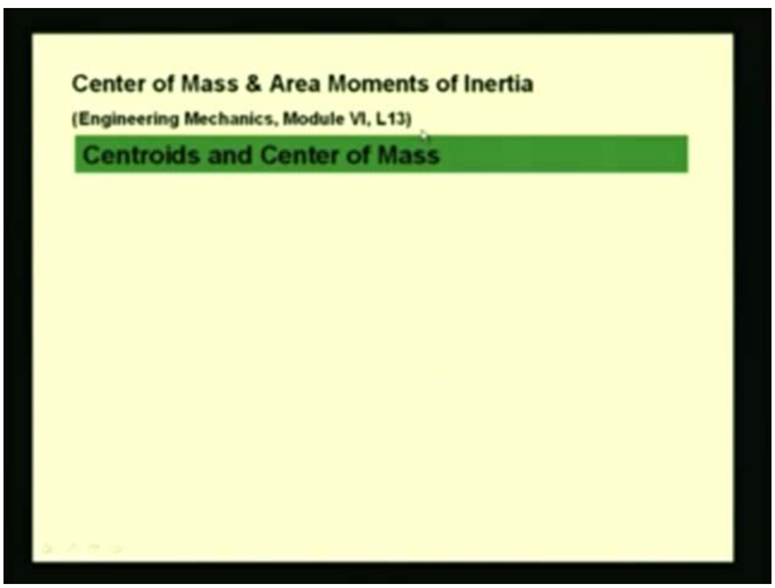

One such measure is the centroid or the center of mass. We will discuss about them and how to compute these values for various shapes, in this lecture. So, for your reference, this is module 6, lecture 13, of Engineering Mechanics course.

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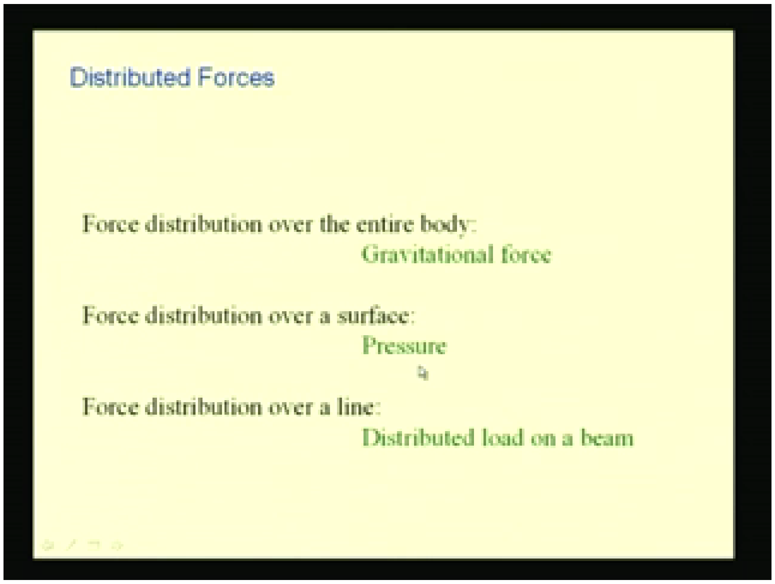

Let us see what kinds of distributed forces occur. We very well know that gravity acts on all the constituent parts of matter. Thus, it is in fact a distributed force. So, weight of a body, is in fact a distributed force. Here, the distribution of the force is throughout the volume of the body. Every part of the body experiences the gravity. Thus, the weight of the body is in fact a distributed force over the volume.

Next, we have some examples of force distributed on surfaces, like the water pressure on the walls of the dams or the walls of the pressure vessels, is in fact a distributed force on surface. So pressure is an example of force distribution on a surface. Then, we have also force distribution on line or on curves. We have examples like, the loading of the beams that we have already seen. We see the dimension like, here it is 3 dimension, here it is 2-D and here it is 1-D and we have the force distribution and examples for all these cases.

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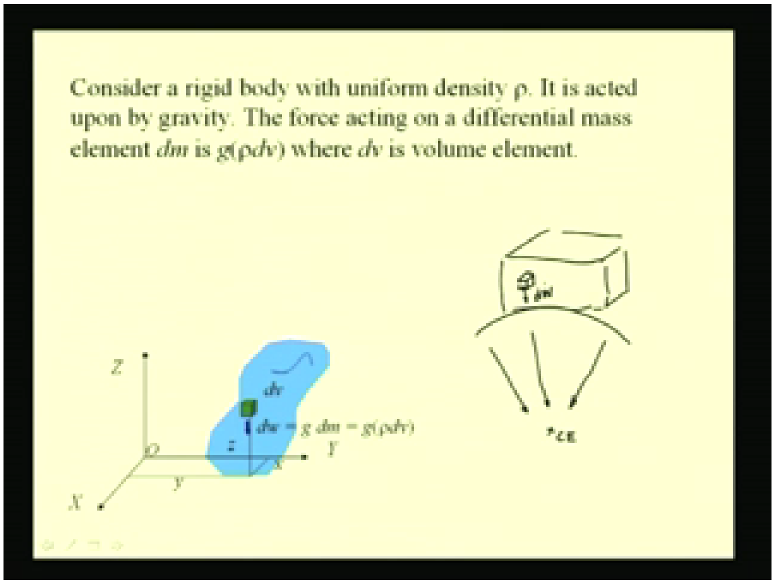

Let us see how to compute this mass center or equivalent points, through which we can assume the mass to be acting, in order to compute the weight of the system or the moment of the weight of the body, with respect to some coordinate frame. Let us consider a rigid body which is having uniform density, rho. Here in this picture, you see one such body and it is acted upon by gravity; this body is acted upon by gravity. Let us fix some reference frame, OXYZ. If we consider a differential mass element of this body, say this small q that you see over here, the mass of this element is rho dv, if dv is the volume of the element. The weight is gravity times the differential mass, which is equal to g times rho dv. This vector shows both magnitude and direction of this weight of this differential element, that is dw.

If we consider the object which has dimensions comparable to say the size of the earth, then, if we consider a differential element in this body and if say this is the center of the earth, then, we know that gravity at any point acts radially towards this center. This vector which is dw, is in fact acting towards the center of the earth, say, CE. If we consider the size of this body to be small, then we can assume that all these vectors are in fact parallel. In this case, or in reality these vectors are not parallel, but if the size of the object can be considered to be small, then these vectors can be assumed to be parallel.

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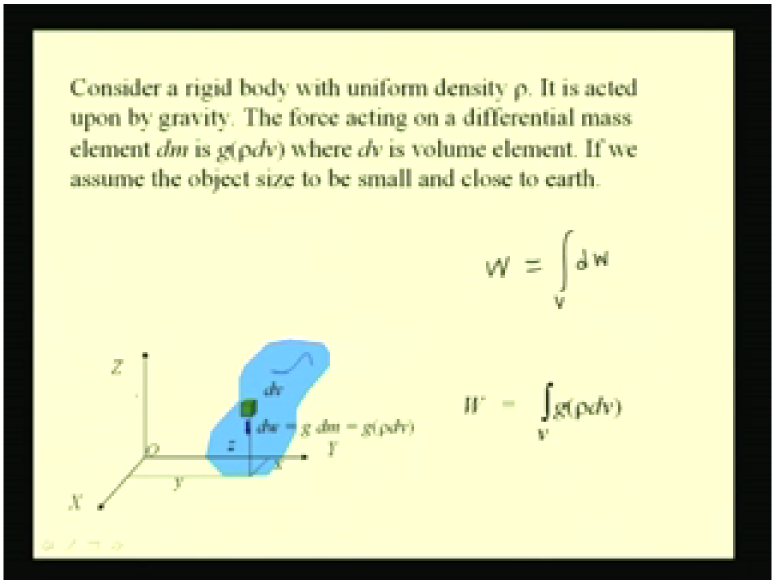

That is the assumption that we make that if we assume the object size to be small and is close to the Earth's surface, then we can assume these directions to be vertical and all these differential elements have this weight dw, which is parallel to the Z-axis. It is equal to g times rho dv, where g is the acceleration due to gravity, and rho is the density and dv is the volume. If we assume the size of the object to be small and close to Earth, then the assumption that all these vectors dw are parallel is valid. With this assumption, let us see, how to find the resultant of all these forces, that is dw. The resultant of these forces if we say is w is equal to summing up all these differential weights over the volume v of this object, which is equal to integral over the volume of g rho dv. This expression gives the resultant of all these differential weights. But now, we have only equated the forces to find the resultant. Now we have to find if the moments are also equal, so that we can find the point of action of this equivalent force W.

Let us consider the location of this differential element dv at x, y and z with respect to this reference that is OXYZ.

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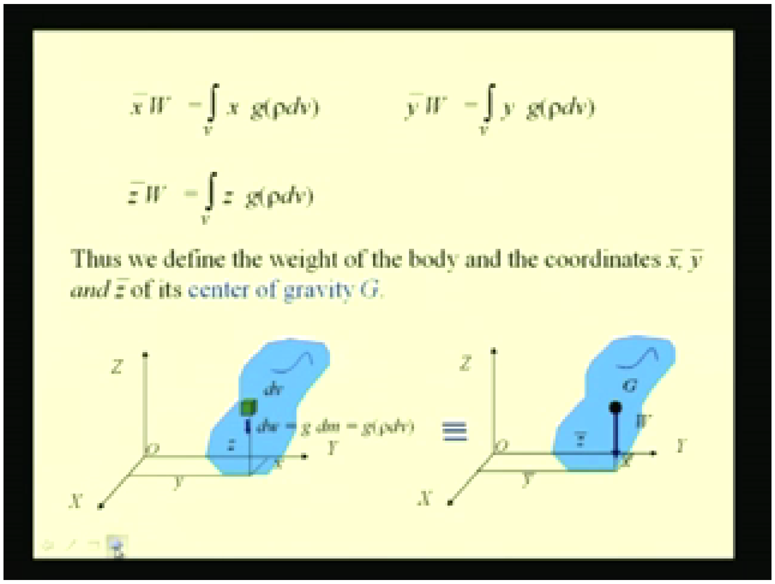

If we are interested to find the moment of these differential weights with respect to this axis OX, and equate it to the moment of this concentrated load W, which, let us say, acts through a point G. Let the coordinate of this point G be x bar, y bar and z bar with respect to this frame of reference. If this concentrated load W has to be equivalent to all these differential weights dw, then, sum of the moments of these forces about this axis OX, has to be same as the moment of this vector w, with respect to this axis OX. We write that, if we take the axis OY, then we can write x bar, which is the momentum for this vector w, with respect to this axis' OY times w is the moment, should be equal to the summation of the moments of all these vectors dw, over the entire volume. So integral vx, which is the momentum for this vector dw, which is g times rho dv.

From this expression, we can find what is x bar or the location of the coordinate for the concentrated load, with respect to the axis OY. Similarly, we can sum the moments with respect to the other axes, say with respect to OX, in which case the momentum is y bar. If we take the axis for this OZ, then we have this momentum of z bar. So, we can write, these additional equations, that is, y bar W equal to integral v y times g rho dv.

From this expression, we can find what this y bar or the location of this vector, with respect this plane OXZ. In the same way, z bar W equal to integral v z g rho dv, which gives this value z bar or the location of this point with respect to this plane OXY. From these expressions, we can now locate the point x bar, y bar, z bar, which can be considered as the point through which the equivalent load or the equivalent weight W, acts. This point is known as the center of gravity for this body. Weight of the body is also defined by this integral dw.

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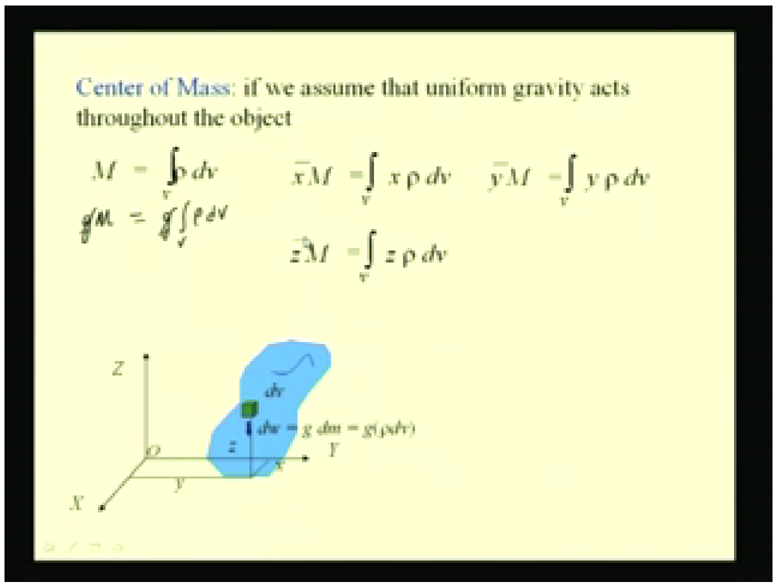

Let us see how we can extend this discussion to find other centers, that we call as center of mass or centroids. To find this quantity center of mass, which is nothing but a point through which all the differential mass elements act as a single quantity, that is, the mass of the object. It can be said to be the equivalent point through which the complete mass of the body is assumed to be concentrated. Let us assume that uniform gravity acts throughout the object. In that case, this quantity g can be taken out from the integral. So, we get this g times of M as the weight is equal to g times of integral v rho dv. So, the quantity has been defined in this way, from which we remove this common terms and we get this expression for M as integral v rho dv. It is nothing but the integration over the volume of dM, which is nothing but rho dv. Now, we can take moments of this mass and equate it to the moment of these differential mass elements, that is dM.

We have x bar M is equal to integral v xdm, which is rho dv. From this expression we get the coordinate x bar or the location of this point through which it is assumed that the complete mass M acts with respect to say the plane OZY. In the same way, the other two coordinates, that is y bar and z bar can be computed by taking the corresponding first moment of this mass. This coordinate x bar, y bar, z bar defines the center of mass. If the gravity is acting throughout and it is same, then we have the mass center and the center of gravity as the same location.

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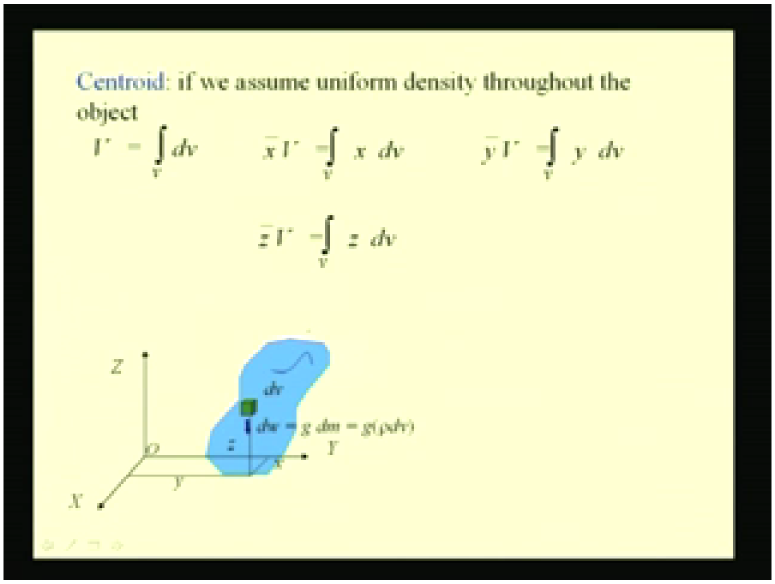

We can extend this discussion, to define this centroid or the point, where we can assume that the complete volume is concentrated. If we assume that the body is constituted with uniform density material, then, this rho can be taken out of this integration and can be cancelled. Thus, we have the volume of the body as integral over the volume dv, the differential volume of these elements. The coordinates of the first moment are x bar v is equal to integral over v x dv. In the same way, the other coordinates that is y bar and z bar are defined. These coordinates x bar, y bar and z bar are the centroids of this volume.

From these discussions, we see that, if we assume the density to be uniform and the gravity to be the same throughout the volume of the body, then, the center of gravity, the center of mass and the centroid of the volume are all located in the same place.

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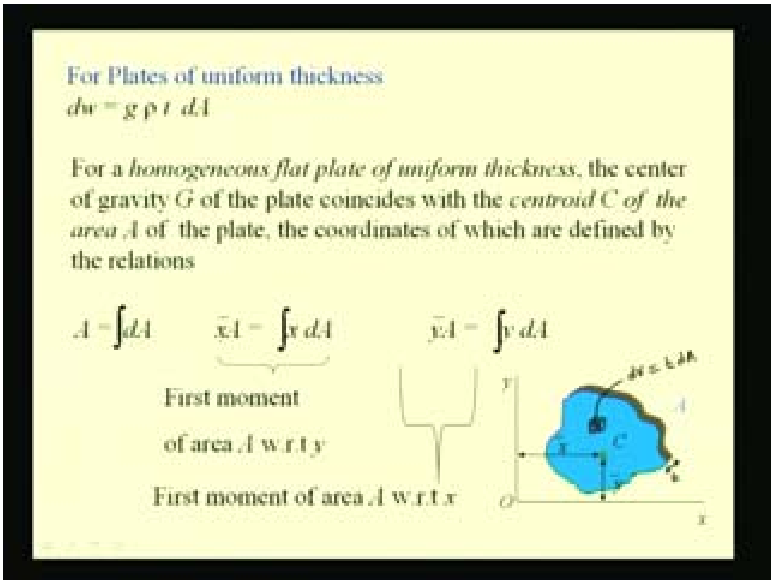

Let us discuss and extend these derived quantities for thin plates. For thin plates, the thickness can be assumed to be uniform and equal to t. So the differential element dv is equal to t times the differential element dA. If I consider an element whose thickness is t, then if dv is the volume of this differential element, it is equal to t times or the thickness of this body times dA, which is the differential area of this element. So, we write dw, or the weight of this element as g times rho times dv, which is t times of dA. If we compute those quantities, we will find that if other things remain constant, that is, it is a homogeneous material, so rho is constant and gravity is also acting constant and it is a thin plate or a uniform plate, then t is constant. Then, we will see that, the centroid C of this area coincides with the center of gravity and center of mass of this plate. So, we have the area of the plate as the integral of this dA. The first moments are x bar A equal to integral xdA and this quantity integral xdA, is known as the first moment of this area with respect to the y-axis. Because we are taking this momentum with respect to this y-axis, this quantity is known as the first moment of the area with respect to y-axis.

Same way, the coordinate y bar is defined and is known as the first moment with respect to the x-axis. So these quantities that is x bar and y bar can be found for various shapes, [….] tabulated and engineers generally use these values for their computations. That means, the location of the mass center or the center of gravity is same as the location of the centroid and this is useful for various computations.

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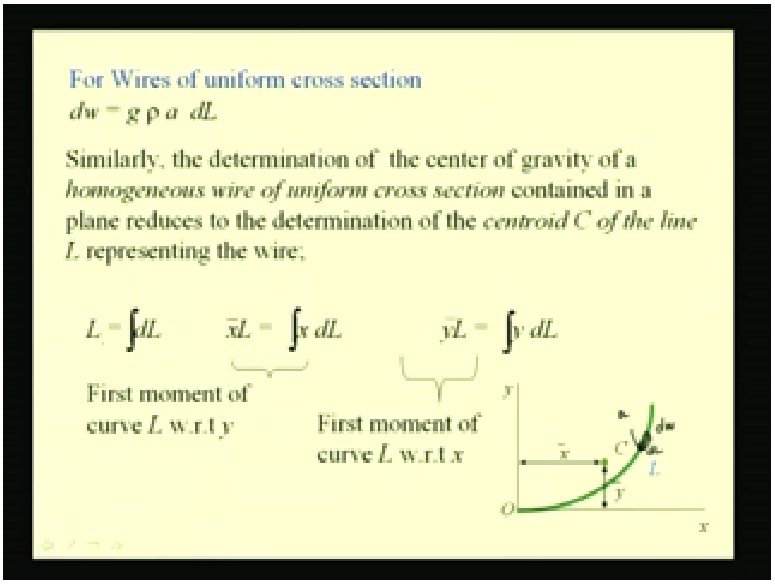

Let us further extend this discussion to uniform wires. If you consider a small element in this wire, the weight of this element is dw, which is equal to g times rho times a. If a is the cross section area, let us assume that, this is the differential element and it is having uniform cross section throughout this wire. Let us say, a, is the area of cross section of this element, which is constant throughout this wire times dL or the differential length that we have taken along this wire. We have dw as g times rho a dL. If we integrate this, we find the quantities, that is, if we take out these constants like g and rho and a, from the integration, because we assume homogeneous material and small objects; so g is also constant. We are additionally assuming a wire of uniform cross section; so, this a is constant. So the integration simplifies to L, which is the length of the wire equal to integral dL. We have the first moments as x bar L equal to integral x dL, which is the first moment of this curve with respect to the y-axis. We have this quantity y bar L equal to integral y dL or the first moment of this curve, with respect to the x-axis. So, we have seen, in general, how to compute the coordinates of the centroids or the mass center and the center of gravity.

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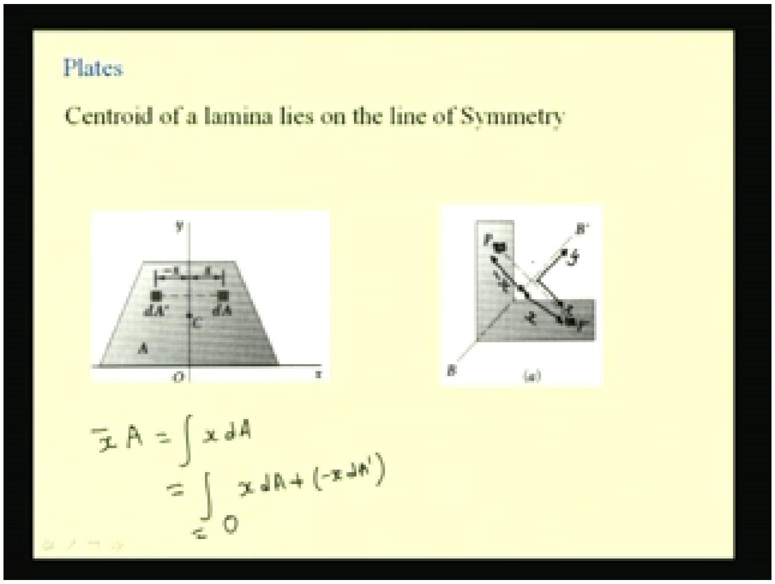

We will discuss little more the discussions on finding the centroids and mass centers for the case of plates. For lamina or plates, the centroid lies on the line of symmetry. This can be found from this example, where we have taken this plate or lamina, which is having symmetry with respect to this axis say Oy. We have already seen that the centroid is defined based on the first moment of the area. So if we take any differential element say dA, the first moment of this element with respect to this axis Oy is nothing but xdA. We defined the centroid x bar and if A is the total area as the sum of this value xdA, for the entire region, we see that if symmetry exists for this object, then for every element dA which is at a distance of x from this Oy, we have a corresponding element dA prime, which is at a distance of minus x.

This quantity can now be grouped as integral xdA plus minus xdA prime and for every element that we find we have a complementary element dA prime, which is having the same momentum. These two quantities cancel out and this becomes zero. So obviously area is not zero, so this quantity x bar has to be zero. That means, the moment or the first moment of this area with respect to this Oy is zero; that means it lies on the same axis. It can be proved that the centroid lies on the line of symmetry, by considering this first moment.

Here, you see another example. We see that this geometry or the shape of the plate has a symmetry with respect to this axis b b prime. If we shift our axes, let us say our y and x and compute the moments, then we can find an element at p and p prime, having the same moments say x and minus x. So if we sum up the first moment of these differential elements, they sum to zero and we find that the coordinate x bar has to be zero that means it has to lie along this line. So this is valid for any shape, which is having an axis of symmetry. If we have two axes of symmetries, then the centroid lies at the location where these lines of symmetries intersect for the lamina.

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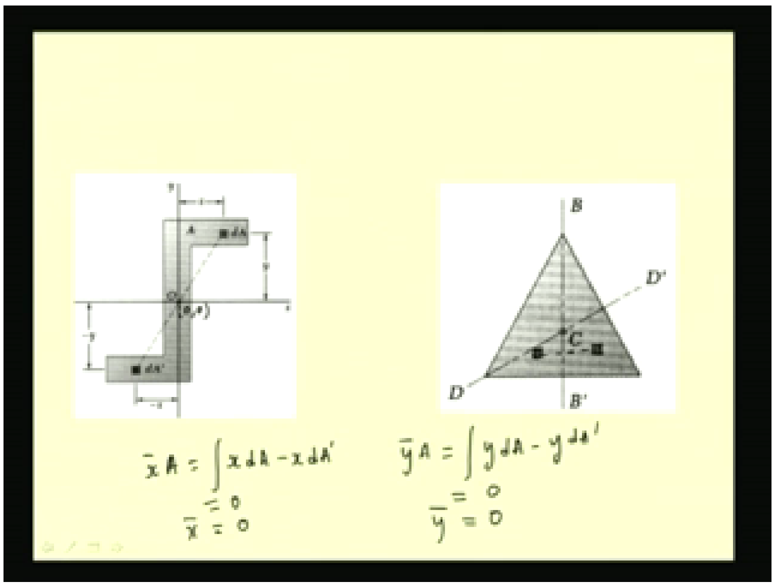

Here, you see an example where this shape has symmetry, with respect Oy or the y-axis as well as with respect to Ox. So, if we try to find the first moment of this area with respect to this axis Oy, we have x bar A equal to integral xdA. We see that these elements have a corresponding element which is having the same area. The integral xdA can be said to be constituting of integral xdA minus the complementary elements, that is dA prime and this quantity being zero, we have x bar as zero. Similarly, if we take the other integral that is y bar A equal to integral ydA, we again see that for each of this elements we have the complementary element dA prime and this quantity is zero and thus we find y bar is also zero. We see that, the centroid is located at x bar y bar which is zero comma zero, which is nothing but this origin and we see that, this origin is nothing but the intersection of the lines of symmetry and this is valid for any shape.

Let us consider this triangle; it has three axes of symmetries. We consider these two that is D, D prime and B, B prime, and the intersection of these two lines of symmetry is at C. This discussion, whatever we have done, can be extended to this, by considering any element, any differential element. It is possible to find a corresponding element and thus the sum of the moments of these elements is zero, with respect to this axis. Same way, it is possible to find elements which are having the same moment with respect to D D prime. Thus, if we sum them it becomes zero and thus they lie along this axis.

If the mass or center or the centroid has to lie along this line, as well as along this line, we see that, the possibility exists only at the point of intersection that is C. So, this discussion holds good for any shape. If we have the lines of symmetries, then we can advantageously use it and we can only find the other coordinate that has to be found by the integral.

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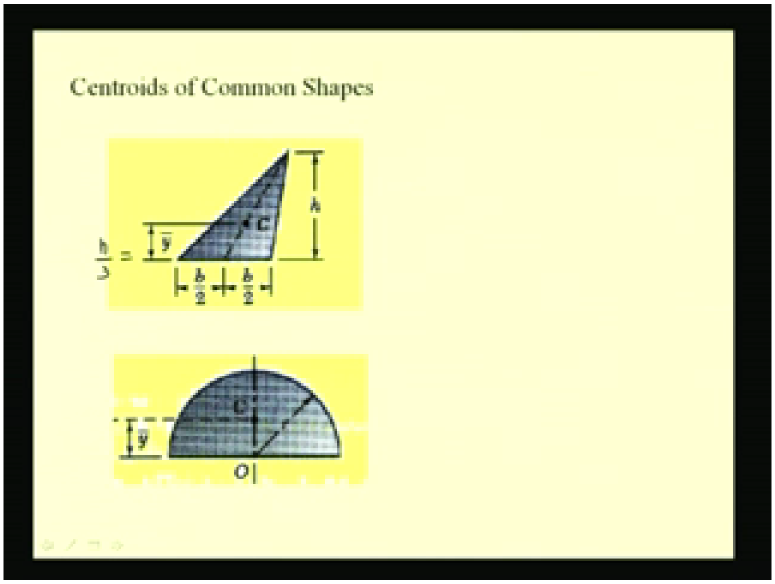

For shapes which are very common for engineering application, we have the centroids defined. Like for this picture, we have the centroid located along the axis of the angle bisector and only we have to determine this y bar, which is, in this case, if you find the integral, it will be h by 3. In the same way, let us consider another example, like a semicircular lamina and since we have one axis of symmetry, we can say that C or the centroid, will lie along this line and we can only find the location of this C, with respect to this diametrical line; if r is the radius and if we integrate it, we can find this quantity y bar as 4 r by 3 pi. So for common shapes, these values, that is, the centroids x bar and y bar are tabulated and they are available for use for various computations.

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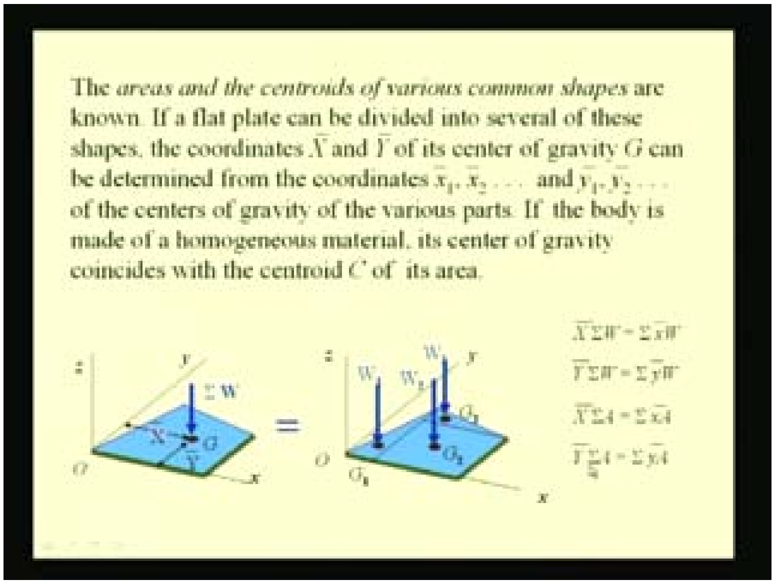

Generally, the shapes that we use are composite shapes, which constitute a combination of these simpler shapes like the circular lamina, the rectangular lamina, or the triangular lamina. If you are interested to find the mass center or the centroids or center of gravity for these complex shapes and if it is possible to divide them into simpler shapes, for which the centroids are defined, then the problem of finding the centroids is made easy. Let us see this; the areas of the centroids of various shapes are known. If a flat plate can be divided into several of these shapes, then the coordinates X bar and Y bar of the center of gravity can be determined from the coordinates, that is x_1 bar, x_2 bar etc of these various parts.

Let us see this example. We have this shape, complex shape and we are interested to find the location of its centroid or mass center, that is, these quantities X bar and capital Y bar. We find that this shape can be decompressed into, let us say, one rectangle and two triangles. For these simple shapes, the location of the mass center is known, that is, let us say, G_1, G_2 and G_3 are the location of these mass centers. If W_1, W_2, W_3 are the weights of these plates, then, the first moment of all these quantities, with respect to this axis Oy, the sum of all these moments is equal to the moment of this force, single force, which is nothing but sum of W_1, W_2, W_3 into the momentum. So that is what we have, X bar sigma W is equal to sigma x bar, which is nothing but the location of the centroid, mass center and center of gravity, if we assume homogeneous materials, and same way Y bar sigma W is equal to sigma y bar W. From this, we have X bar sigma A is equal to sigma x bar A and Y bar is equal to Y bar sigma A is equal to sigma y bar A. In this way, it is possible to find the location of the centroids of plates, if they can be decomposed into simpler shapes.

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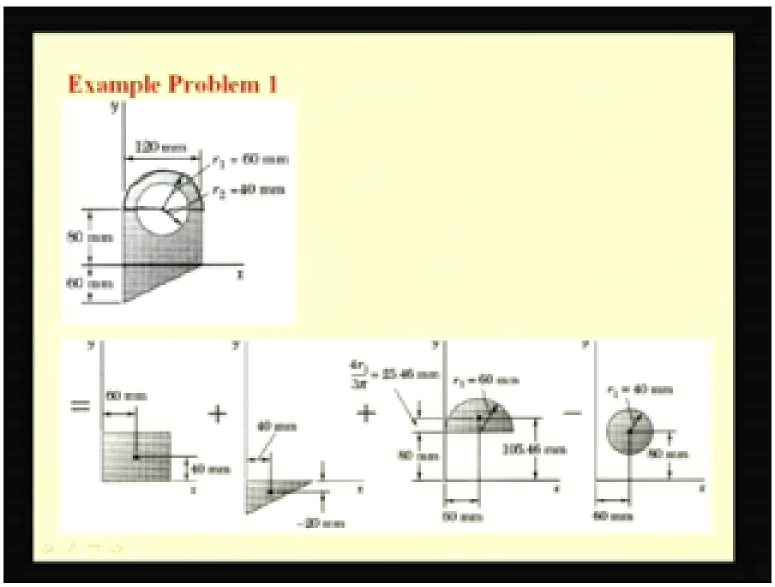

Let us take one example to find the centroid of a plate, which can be decomposed into simpler shapes. Here, you see a lamina which is composed of certain annular hole region and composed of a triangle and a rectangle. This shape, from our understanding, can be decomposed into a rectangle, a triangle and a semicircular lamina. We have to remove from the sum of these areas, this area corresponding to this circle. We see that this complex shape can be decomposed into simpler shapes, for which the centroid is known. We have this rectangle, whose centroid is located. This rectangle has two axis of symmetry. So, it is located at 40 mm from x-axis and 60 mm from y-axis.

Let us take this triangular lamina. We know that the centroid is located at a distance one-third height, so in this case the height is 60 mm. So it is located 20 mm below the x-axis. If we consider this semicircular lamina, it has an axis of symmetry. So, its centroid lies at 60 mm from y-axis, which is equal to the radius of this semicircular lamina. It is located from its diametrical line at a distance of 4r by 3 pi, which is in this case 25.46 mm. For this circle, the centroid lies at the center of the circle, which is 80 mm from the x-axis and 60 mm from the y-axis.

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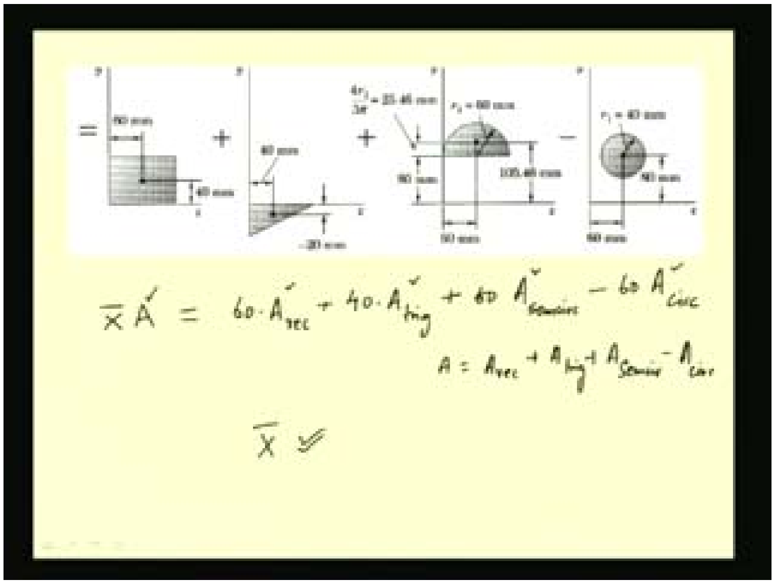

Knowing these quantities, we can find the first moment about the required axes, let us say x and y-axis, and from which, we can find the first moment of the sum of all these areas. For x-axis, I can write x bar A is equal to, if I say this as the area of the rectangle and I have the location of the centroid with respect to the y-axis as 60 mm, so 60 times the area of the rectangle, which is 40 times 60 plus 40 times area of the triangular lamina plus 60 times the area of the semicircular lamina minus 60 times the area of the circular lamina. Since we know all these quantities can be computed from the geometry and we also know the total area A, which is area of the rectangle, plus area of the triangle, plus area of the semicircular lamina, minus area of the circular lamina, we find this quantity X bar can be determined.

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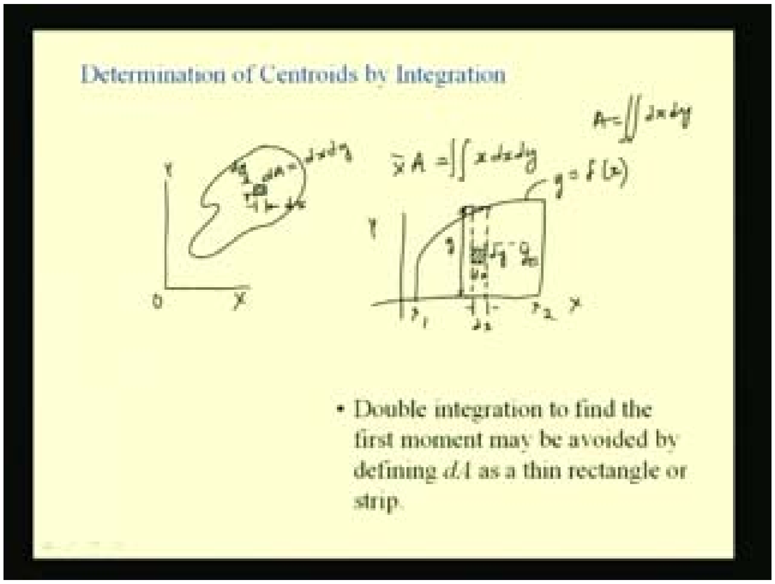

Let us move on to discuss how to find these centroids by the method of integration. We have already considered that, if I have an area and I am interested to find the centroid of this lamina, or the plate, then I consider a small element, say dA, which is equal to dx times dy, where these are the dimension of these elements. So I find the equivalent x bar A as integral x dx dy, where A is equal to integral dx dy and this is a double integral, because you have to integrate for the limits in x and y. So for curves, if I consider this curve and I am interested to find the centroid of the area, under this curve between the limits say x_1 and x_2, then I consider this differential element dx dy and I can find this quantity.

I can avoid this double integral, if I consider, instead of an element whose dimensions are dx dy, I consider this rectangular strip, whose dimensions are dx and y. For a small element, I can say that this is equal to a rectangular strip whose dimension is y. So if I have the equation of this curve defined as y equal to some function of x, then I can define a series of such rectangular strips and integrate them.

So the double integral has been converted to a single integral, where if I consider these elements, I know that, the centroid of this rectangular strip will lie here, which I may call it as y bar of the element. I can sum these quantities to find the centroid of the area under this curve. By taking suitable elements, one can avoid this multiple integrals. Let us see some examples. So you can either consider thin rectangles as shown in this example or you can consider other forms of thin strips also.

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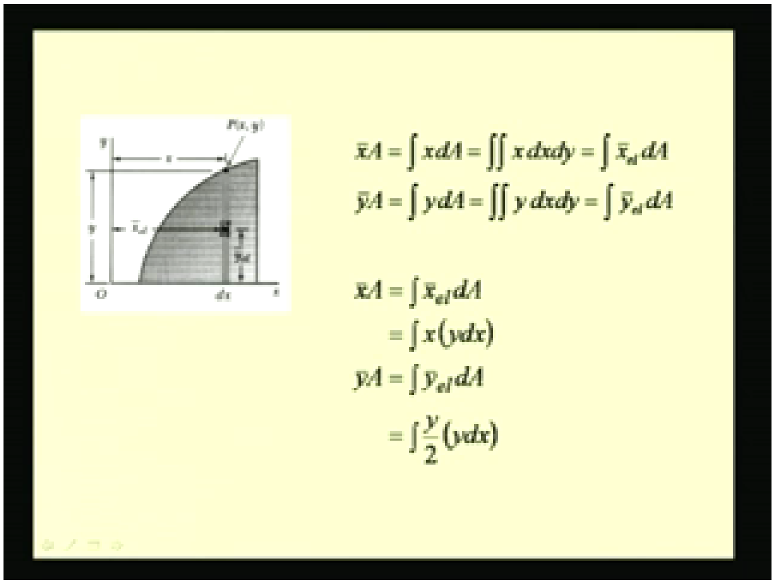

Here, you see one example where we have this curve and whose equation is known and we are interested to find the centroid of this area, that is, the area under this curve. Let us take this thin element which is a vertical strip, whose centroid is located at y bar element and at x bar element for axis the Oy and Ox. So, we can write x bar A as integral x dA, which is the double integral, if we take a small differential element of this form. But since we have taken this rectangular element, it becomes a single integral with x bar of this rectangular element times dA, where dA is the area of this thin strip. In the same way, we can define this y bar.

We know that for this thin strip the area is ydx, so we have this quantity as integral x ydx, where y is a function of x again. So now, it becomes a single integral in x. In the same way, here we have, for determining y bar, it is integral y by 2 which is this distance times y dx, which is the area of this element. Again, this is a function of x, because we have y as a function of x and so it is again a single integral.

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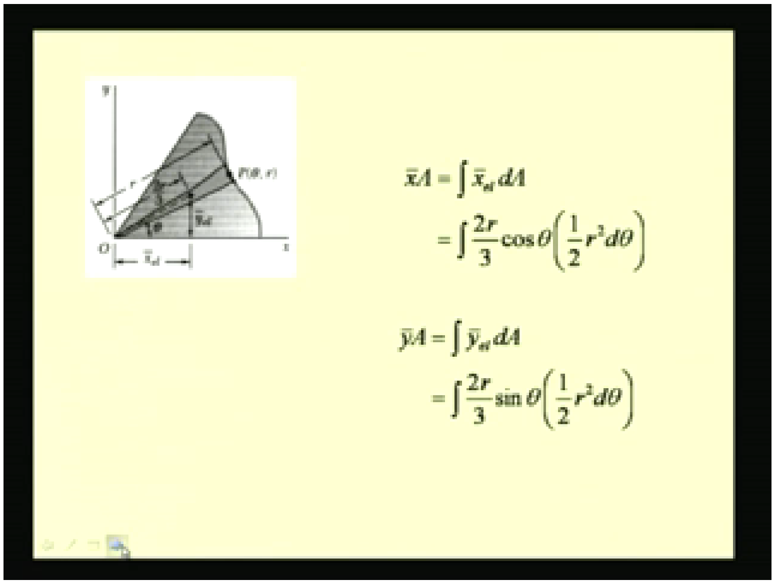

We can also take other forms of elements, like in this example, we have a thin triangular strip which can be swept in order to approximate this area. So, we can define the centroids by taking these elements, where dA is the area of this element. In same way, y bar can be defined as integral 2r by 3, which is the location of the centroid with respect to this coordinate. We have this area as 1 by 2r square d theta, where d theta is the angle swept by this small triangular element.

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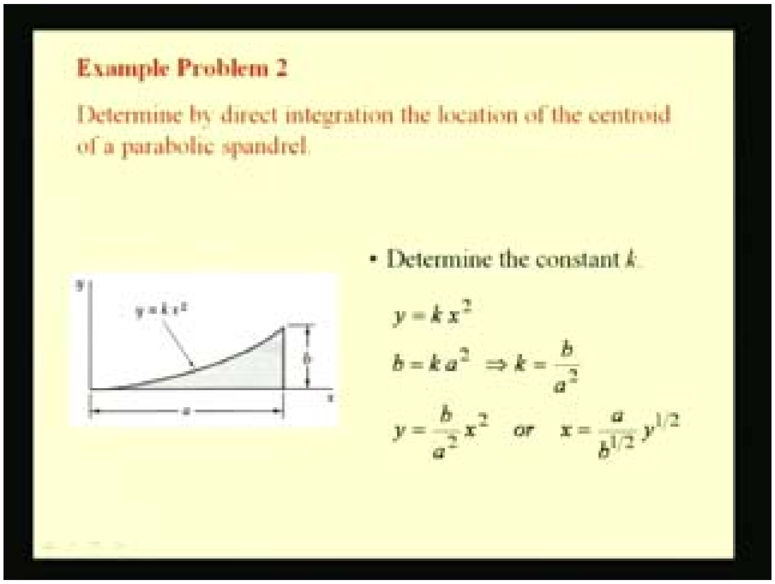

Let us consider one example to completely illustrate this procedure. We have this curve, y equal to kx square, which is a parabola and we are interested to find the location of the centroid. To proceed with this problem, first we have to determine this value of k. We find that by substituting the end condition that y is equal to b, when x is equal to a and from that, we find this value of k. Thus, we write the equation of this parabola as, y equal to b by a square x square.

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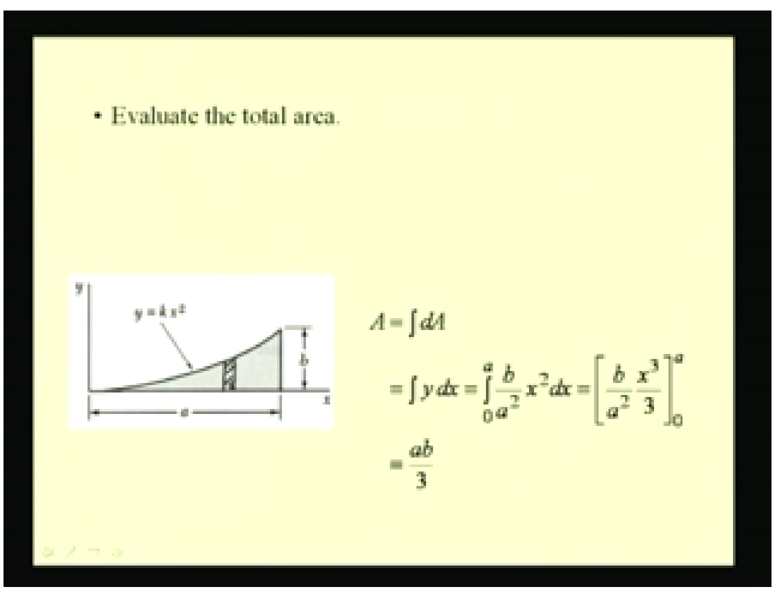

Now we find the total area. We have this differential element dA, which is considered as a thin vertical strip and we integrate this between 0 to A and we have the area as ab by 3.

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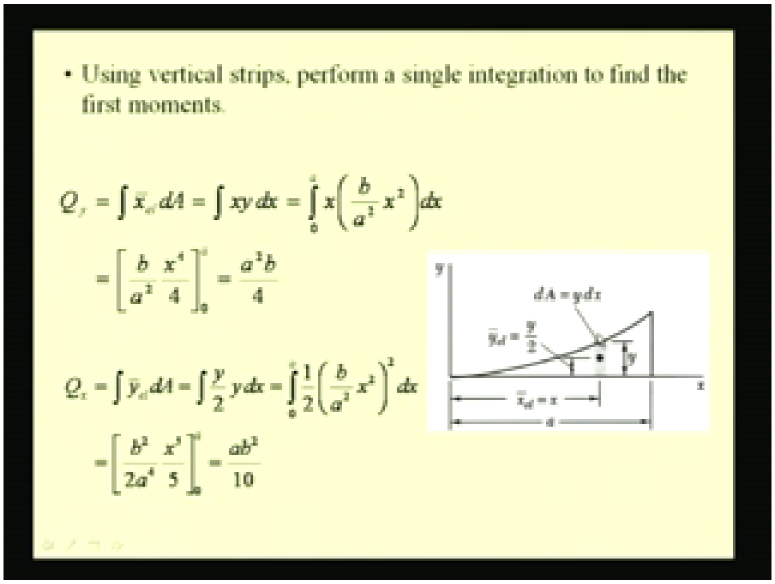

In order to find the first moments, we consider this element, which has the location of the centroid as y element, which is y by 2, and x element, which is nothing but x itself. Q_y, which defines the first moment of this element and the sum of all these first moments is the first moment of this area with respect to y-axis and it is found as a square b by 4.

Here, you will see that xy dx is the first moment of the small elemental area and it is integrated between the elements 0 to A. Similarly, we can find the other moment, that is the moment with respect to x-axis. Once we know these moments, we also know the area and we can find the coordinates that is x bar and y bar.

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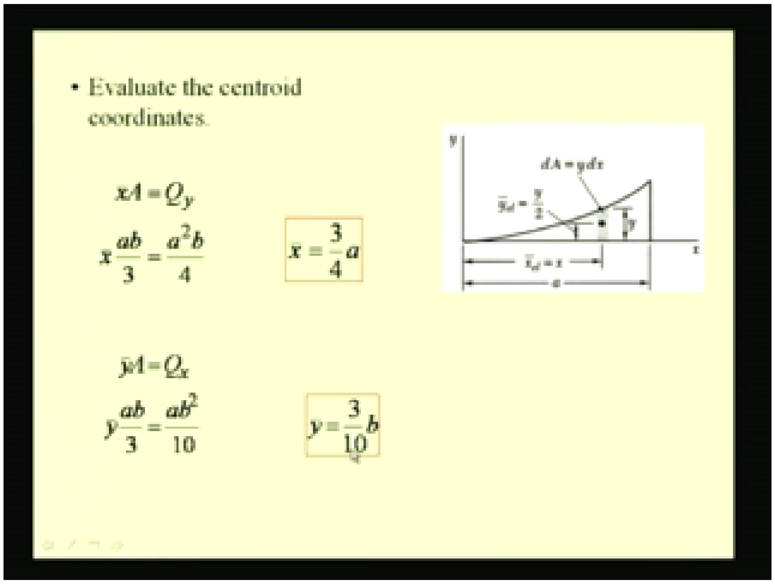

We have area as well as Q_y found and so we can find this x bar and it is found as 3 by 4a. Similarly, we can find y bar as 3 by 10 times of b.

This example illustrated the method of finding the centroids by integration.