Theory and Practice of Rotor Dynamics
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Module - 2
Simple Rotors
Lecture - 6
Variant of Jeffcott Rotor Model

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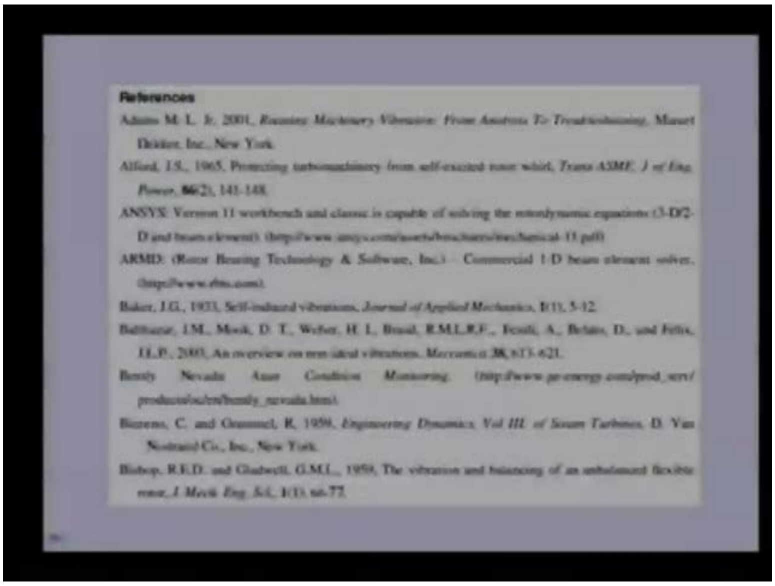

I will be showing some of the references, which is useful for whatever I have covered in this particular lecture and in the subsequent lecture. Actually this is a exhaustive list of reference of various researchers and even textbook I have referred in this.

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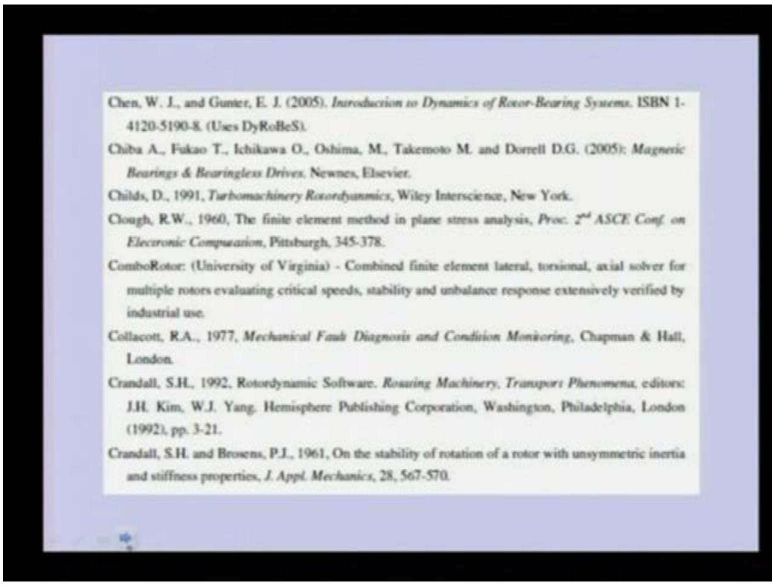

So, these are all references, which can be preferred for more detail of these lectures.

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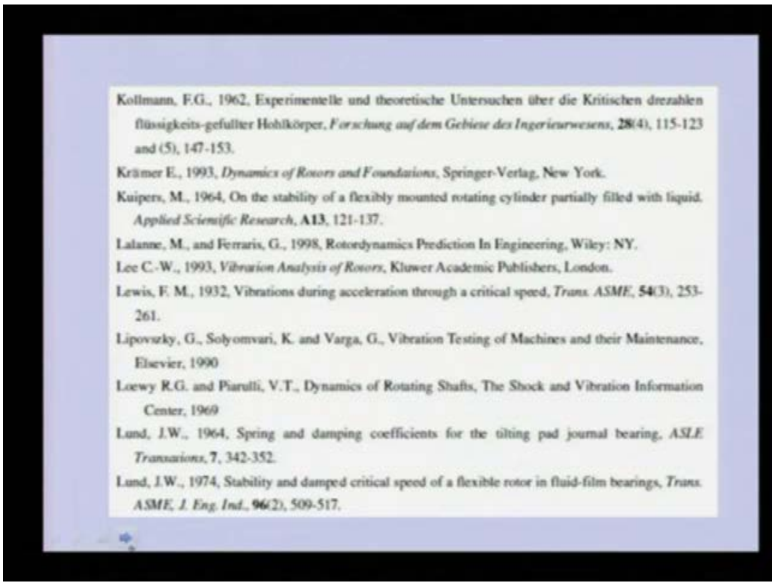

This several text book reference are also.

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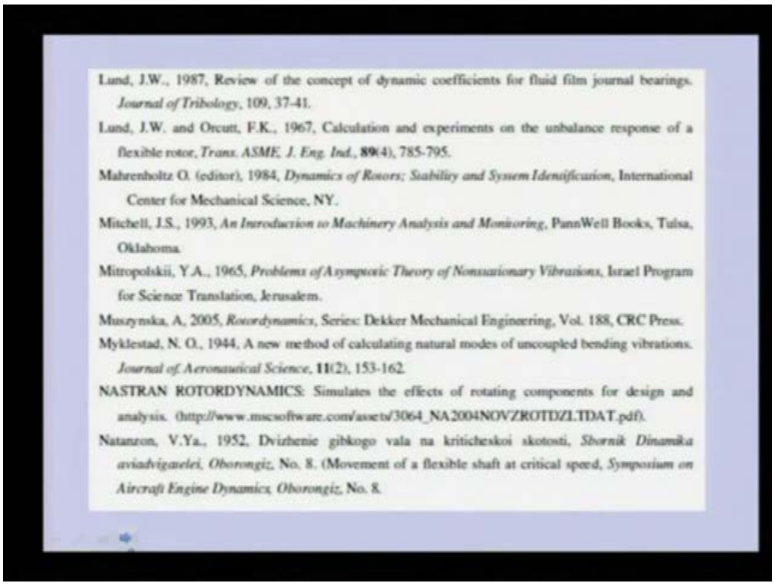

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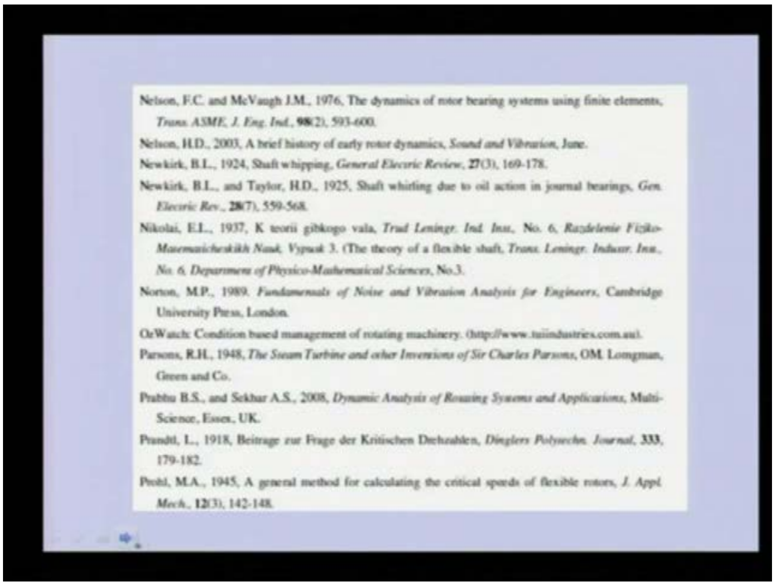

There apart from even some softwares reference, which are developed especially for rotor dynamics purposes.

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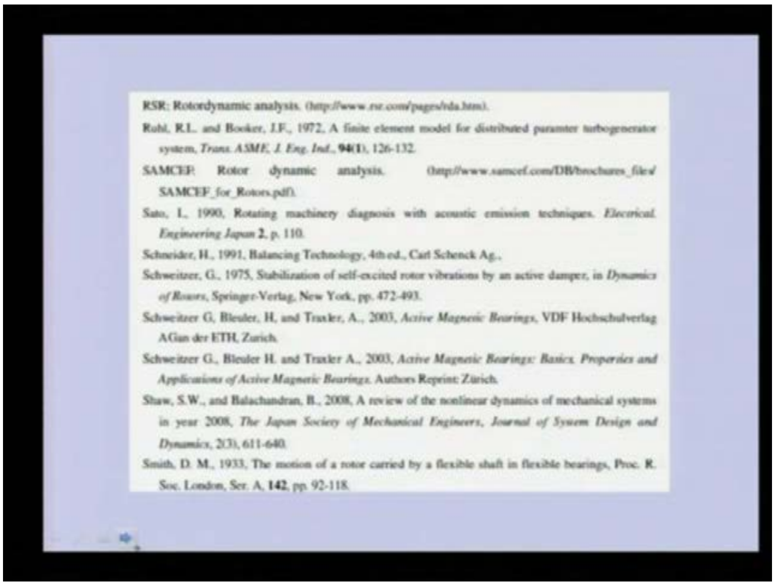

These are very exhaustive list.

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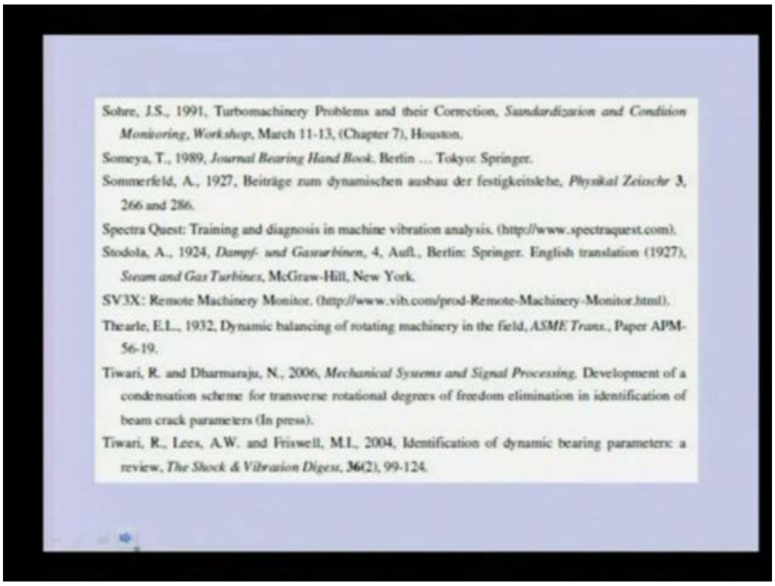

Which is very useful for overall understanding of basic concepts of the rotor dynamics in the previous lecture, we seen the Jeffcott rotor modal analysis and in that particular model, that was a flexible shaft that was mass less and there was a rigid disk at the centre and the bearings where rigid.

Why is this particular disk is at the shaft centre line the slope at the centre is zero during the whirling and because of that the tilting or the wobbling of the disk does not take place. Today we will consider another case in which, we will be taking the jeffcott rotor model variant in which we will be keeping the disk with slightly offset from the mid span. So, that during whirling, not only the disk will behave in the translatory motion also it will tilt about its diameter in two planes.

So, in this particular case we will see that we will be having some assumptions in this will not take the gyroscopic effect in this particular analysis for time being for simplification, in this particular case we will be having a four degree of freedom system, as compared to the jeffcott rotor model in which we had only two degree of freedom system.

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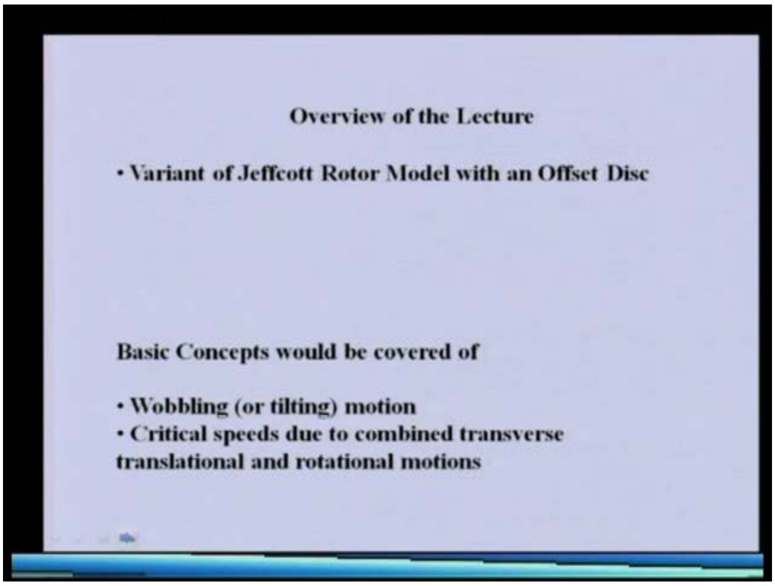

This is the overview of the lecture in which, we will be taking the variant of the Jeffcott rotor model, in which the disk will be offset some of the concept, which we will be seeing is the wobbling motion or the tilting motion of the disk along with the translatory motion and then we will be studying the critical speed due to the combined transverse and rotational motion of the disk. So, in this particular case let us consider.

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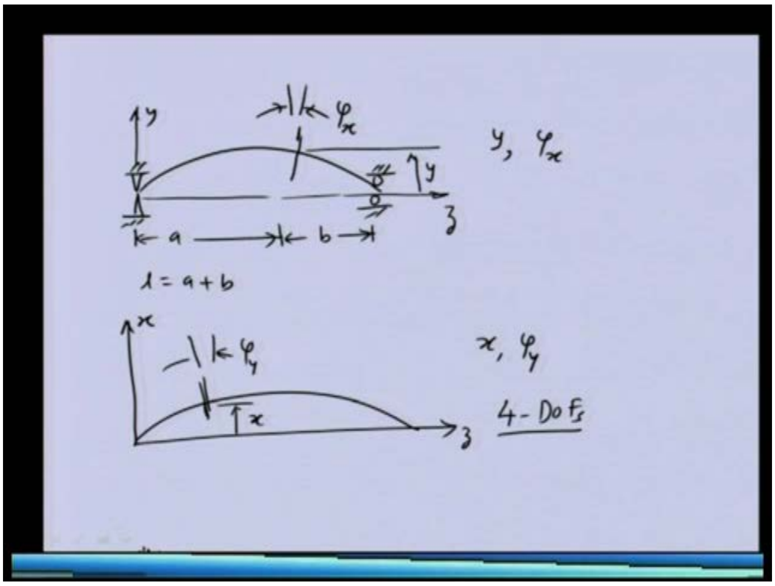

A flexible shaft which is supported at ends with friction less bearing let us say this is simply supported at ends. And there, is a disk which is offset from the centre this is the disk, this is the bearing axis. So, let us say this is z axis let us say vertical axis is y, and there is another axis which is perpendicular to the plane of screw, we will consider that subsequently let us say we are considering the motion in this two plane at present.

So, we can able to see that let us say define the verification of the disk from one end is a from other end is b, total length is a plus b of the shaft, we have linear displacement of the disk along the y axis that is y and there is a tilting of the disk with vertical axis, this can be defined by this angle that is phi x and in this particular case now in this in this plane, we have two coordinates to define the position of the disk, one is linear or translatory displacement y and it is the angle displacement phi x, similarly when we will consider the motion in other plane that is in z x plane of the shaft we will be having one is linear displacement.Let us say disk is this one and it is tilting on vertical that is phi y.

So, in this particular case we will be having another two displacements to define the position of the disk. So, now, we have total 4 degree of freedom system and we will be getting 4 equation of motion corresponding to these directions. So, now, let us take the free body diagram of such a rotor system.

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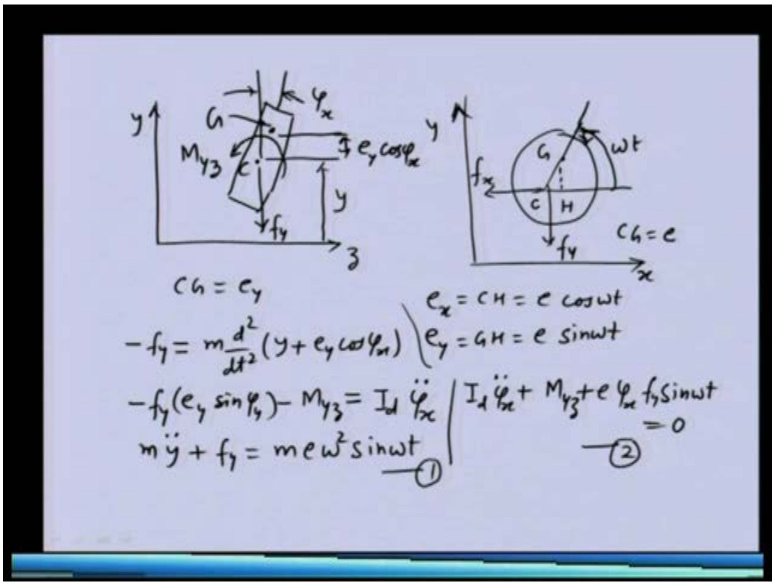

So, if we take let us say free body diagram of the disk in z y plane, z is the bearing axis. So, let us say I am taking the disk free body diagram separately. So, this is a disk is having vertical displacement y and is having tilting with respect to the vertical axis that is phi x and because we have removed the shaft, there will be elastic force which will be coming from the shaft on to the disk apart from that, we will be having moment that is the reaction moment on this plane, that is M y z which will be acting opposite to the tilt because tilt is in this particular plane is in the clock wise direction. So, this moment will be opposite to that in the anti clock wise direction similarly displacement is in the vertical direction.

So, force will be downward direction, this is the reactive force from the shaft on to the bearing and if, we see the disk in x y plane will be having the elastic force in y opposite to the y direction, similarly in x direction there will be f x, which is coming from the shaft let us say unbalance position is this centre of gravity G is here, this is the centre of rotation. So, this particular angle will define as for constant angular speed of the rotor as omega t, let us say the disk is rotating in the counter clock wise direction in the x y plane.

So, now, we can able to see that the eccentricity, which is defined as C G is having two component, one is in the x direction, if i drop a perpendicular up to here. Let us say I am referring that as H then the component of the eccentricity in the x direction is C H. So, that is given as e cos omega t and component in the y direction is G H, that is e sin omega t, x axis is here now coming to here in the z y plane, we will see that centre of gravity is somewhere here, this is the G centre of gravity and centre of rotation is here.

So, this particular distance. So, this particular distance is nothing but basically in this particular plane y z plane, where C G is component of the eccentricity in the y direction that is e y. So, this particular distance looking into the tilt of the disk will be e y cos phi x. Now we can able to write the equation of motion of the disk in y direction and even the moment we can able to balance.

So, let us say first the force balance. So, we have f y that is elastic force, which is at t then inertia force. So, location of the centre of gravity is y that is this distance and then e y cos y x. So, right hand side term is the inertia term and left hand side term is the reaction from the shaft, that is the elastic force, which is coming from the shaft.

Apart from this we will be having moment. So, if you are taking moment let us say about the centre of gravity then we can have moment due to this force f y, the moment on will be e y sin phi y, that will be you can able to see this force is acting here and the moment on about G in the horizontal direction will be why because vertical direction is e y cos phi.

So, horizontal direction will be e y sin phi and the moment is also acting this particular moment and this will be equal to the inertia rotor inertia of the disk, now we can able to simplify this equation and we can able to write this as first equation, we can able to write it as M y double dot plus f y is equal to M e omega square sin omega t and the moment equation, we can able to write certain equation is I d y double dot x plus M y z is equal to even you can able to write this term, this side plus e y x f y sin omega t is equal to 0. So, this is 1 equation of motion in the y direction and this is the another equation of motion in the x direction. We obtain a equation of motion in y z plane on the same line, we can able to obtain the equation of motion in other plane, that is z x plane. So, let us see that derivation.

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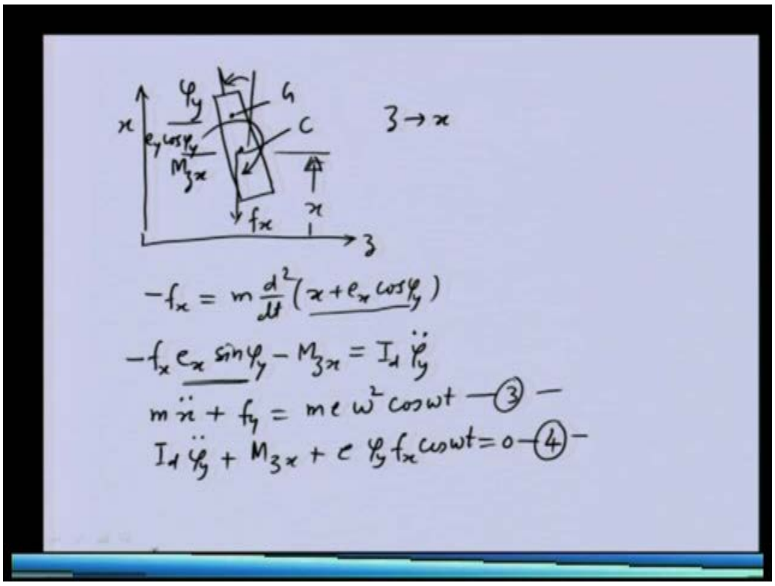

So, now we will obtain the equation of motion in z x plane let us say z is the bearing axis direction x is the direction in the horizontal plane, if we draw the free body diagram of the disk on it, I am taking the tilting the positive direction, I will show this. So, this is the x displacement translatory displacement and there is a tilting, which I am taking as positive direction is z 2 x, I am calling that as phi z. So, in the right hand this particular angle, I have taken in the positive direction that is z to x the tilting, I have taken in the positive direction also the linear displacement translatory displacement in the positive direction.

Now, because we removed the shaft, there will be elastic force in opposite to the x direction f x apart from this, there will be moment which will oppose the tilting motion, this will be in the clock wise direction, this will be n z x, now if we see this is the centre of rotation if centre of gravity is here G then this vertical distance is nothing but the e y cos phi x then vertical distance between the G and C is e y cos phi x and now we can able to write the equation of motion in x direction an in phi z phi y direction, this particular angle is phi y.

So, the equation of motion in x direction is phi x is equal to inertia force, that is x plus e x cos phi y, this is the position of the centre of gravity. So, we can able to see this is x and then from C to G is e y cos phi x, this is also phi y because angle is phi y. This angle is phi y, yes then now we can able to do the force moment balance also. So, that f y this is opposite to the f y is acting again opposite to the tilting because this f y is acting opposite to the tilting because we are taking moment about G. So, this particular effects will give a clock wise moment.

So, this is e x and sin phi y this is the vertical distance from C and G, that is this particular is horizontal distance sorry and vertical distance was e y cos phi, that is from C to G and this will be the horizontal distance e x sin phi y, this is the momentum of the f y with respect to the centre of gravity G and moment is acting that is z x and z x and then it should be equal to the inertia rotor inertia of the disk. This two equations, now we can able to simplify and we can able to write the first equation as M x double dot plus f y is equal to M e omega square cos omega t and the second moment equation can be written as I d phi double dot y plus m z x plus e phi y f x.

This is also f x cos omega t is equal to 0. So, this the third equation of motion and this is the fourth equation of motion, these equation of motion now we can able write in one place in a matrix form, now these four equation of motion will be writing in one place as a matrix form. So, that it is more convenient to handle the four equations in very compact form.

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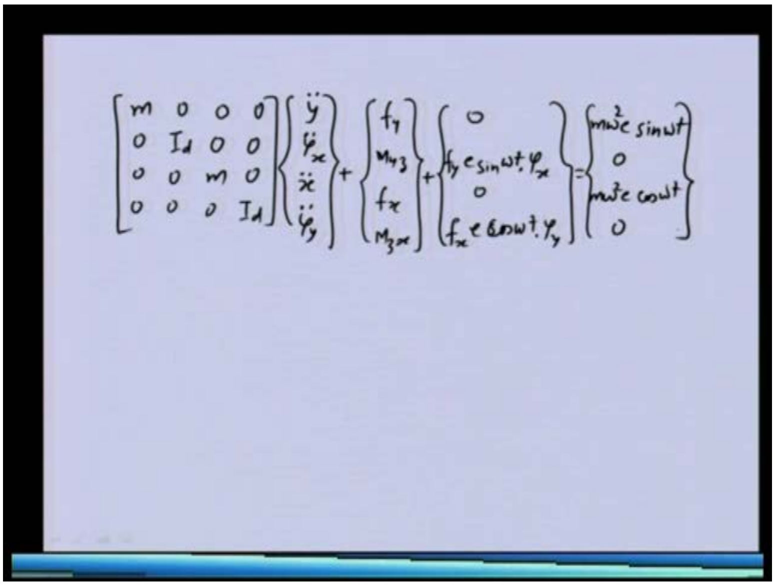

So, this particular four equation, we can able to write like this. So, this is the equation of motion, this is the mass matrix which will be coming from the inertia term of the four equations. So, you can able to see that we have arranged the various variables like y double dot phi double dot x, then x and then phi double dot y because y and phi x are in one plane and x and phi y in another plane. So, we arrange these inertia terms like this.

Then we have elastic forces f y and y z f x and z x these will determine the equation of motion, then we had some terms that was related to the moment equation f y e cos omega t phi x, then corresponding to the force equation it was 0 and then for moment in z x plane, it was this was sin and the second was cos omega t and then we had phi y, this is phi x there outside and then we had the centrifugal force, which I am keeping in the extreme right hand side and omega square e sin omega t no there was no moment, then n omega square e cos omega t and there was no moment, this is the equation of motion i have written these four equation of motion.

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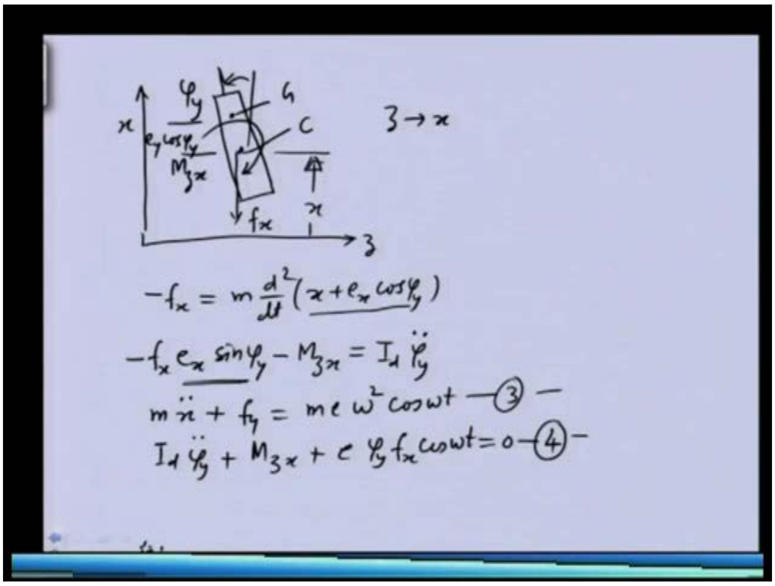

So, if we again will see.

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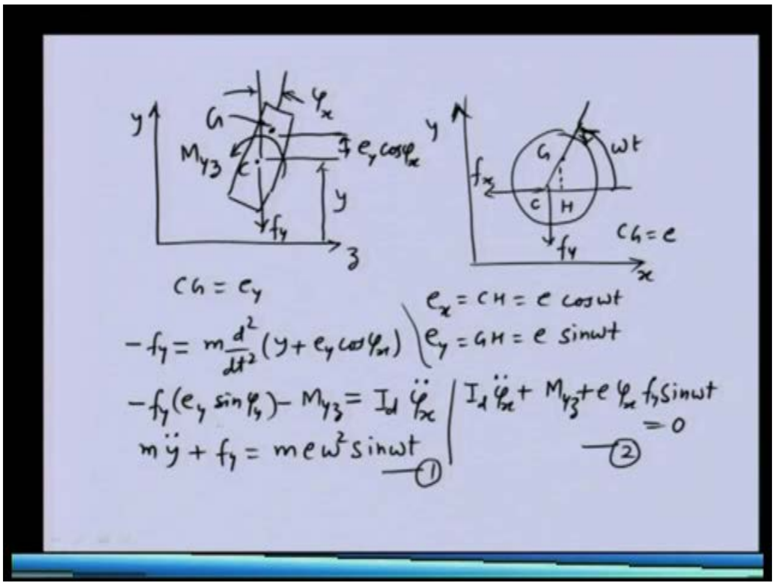

The this is the first equation. So, this term will give to the mass matrix this was tagged separately and this is the force unbalance force, similarly we have the moment equation, there was no moment external moment. So, that term was 0 in that, so, these two equations.

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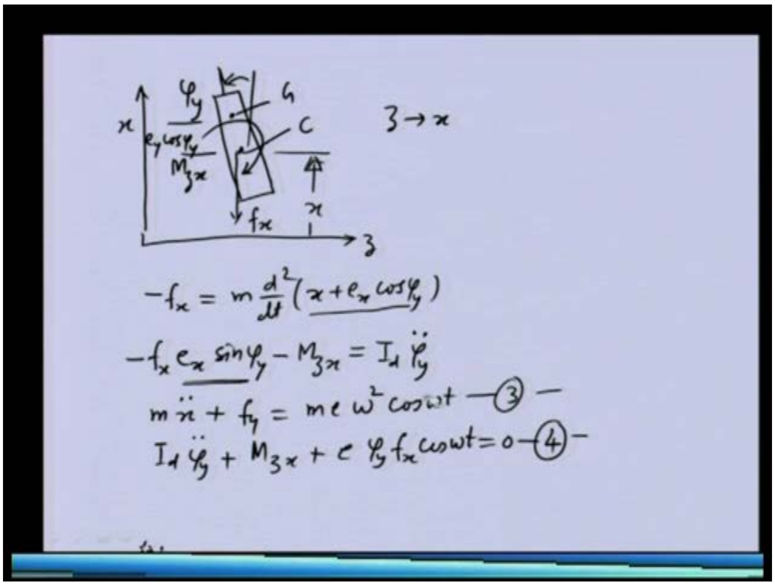

And then the third and fourth equation, I have kept in a matrix form.

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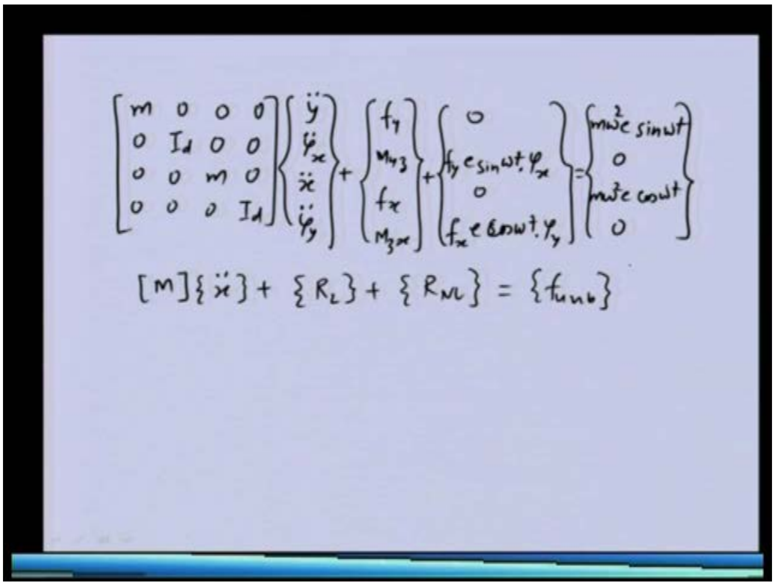

Now once we have these equations in matrix form, but still this can be written as, plus let us say I am representing this reaction, a linear plus reaction non-linear term, we will see how the these terms are non-linear subsequently, once we obtain f x and M y z in terms of the linear and angular displacements and is equal to the unbalance force. Now, we will relate the elastic force and moment, which is coming from the shaft onto the disk to the linear and angular displacement with the help of influence coefficients. So, let us see this particular.

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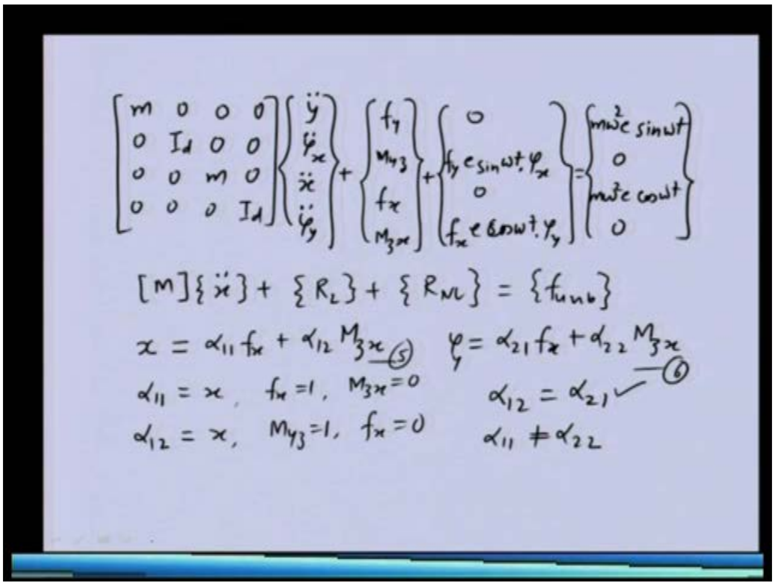

Relations we generally study in the strength of material that once we have the displacement, we can able to write in terms of the influence coefficient multiplied by the force in that particular plane and the moment in that particular plane. So, we have this relation. So, the linear displacement because in beam the linear displacement would take place not only because of force but also because of the moment and these are the elastic constant, this is called influence coefficient and you can able to see the definition the influence coefficient, this is the if, we want to define the alpha 1 1.

So, basically this is a linear displacement for a unit force keeping all other moments and forces to 0. So that means this will be the x when we have f x is equal to 1 and all other moment and forces like this are 0, so it can be defined like this or if you want to define the 1 2. So, this is the linear displacement corresponding to a unit moment keeping all other force and moment equal to 0. So, this is the definition of the influence coefficient, similarly we can able to extend the same thing for the angular displacement.

Let us say this is alpha 2 1, f x plus alpha 2 2, M z x in this particular case we will see that generally, these two influence coefficients are symmetric and they may not be equal. So, this cross coupled terms will be equal now with the help of this 2 equation let us say this is the fifth equation and this is the sixth equation they can be combined in the following way.

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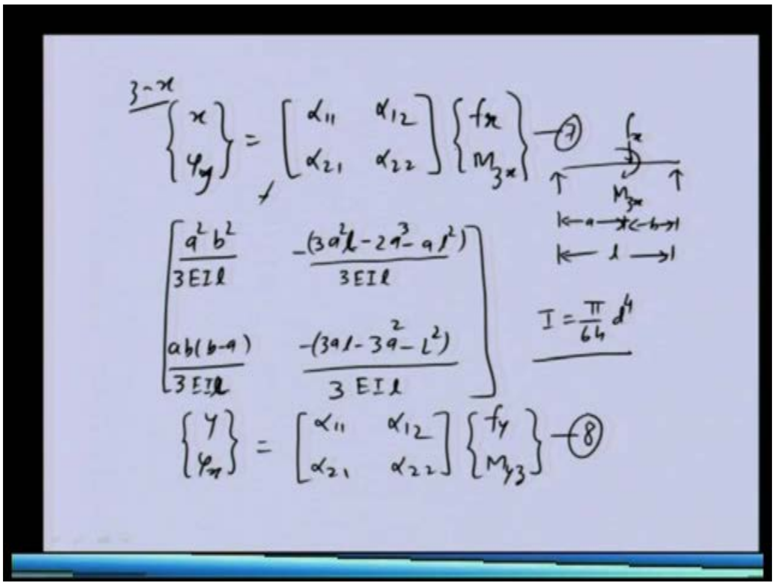

So, I am writing those two equations in a one matrix form. So, this will combine the 2 equation in 1. So, if I simply supported beam generally, this influence coefficient matrix take the following form just for illustration, I am giving the if, let us say we have simply supported beam these are the support.

If we are applying a force f y and moment z f x and M z x then the displacement linear and angular will be related let us say this distance is a this is b, total length is l of the beam, then this influence coefficient matrix would take this particular form a l square divided by 3 E I l and a b minus a divided by 3 E I and the alpha 2 2 is 3 a l minus 3 a cube l square divided by this a square divided by 3 E I l.

So, this is the typical influence coefficient for a simply supported beam, but for other kind of support condition, this influence coefficient will change here, you can able to see this is the young's modulus, I is the second moment of area that is 6 pi by 64 d 4 for circular disk circular shaft. So, this is the influence coefficient and using this, we have related the linear and angular displacement with the beam force and momentum, now these relations will be valid for other plane also, this was in the z x plane the similar expression we can able to write in the y z plane and because shaft is symmetric.

So, we will be having similar influence coefficient in that plane also only thing is the forces or moment are now different. So, this equation and this equation they further can be combined in a bigger matrix form. These equations 7 and 8 can be expressed in terms of the stiffness terms by inverting the this influence coefficient. So, these equations can be written

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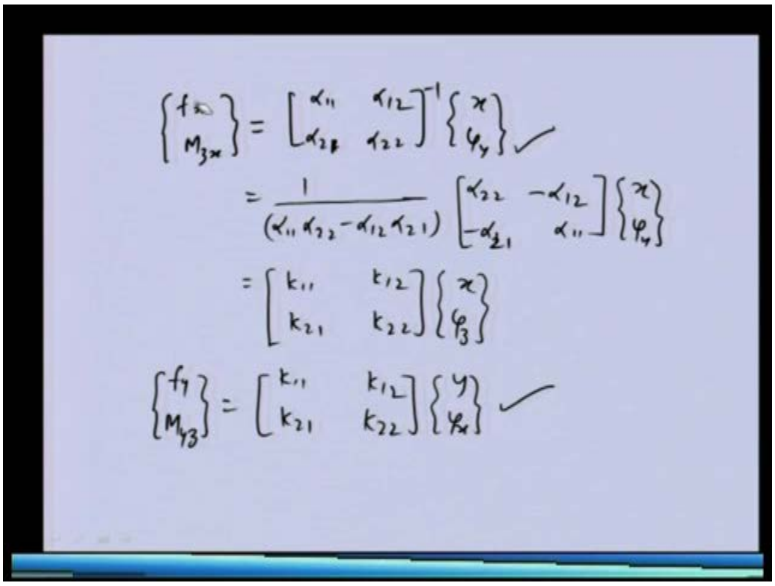

In the stiffness coefficient form like this f x M z x, let us say we are inverting this particular matrix. So, alpha 1 1 alpha 1 2 alpha 2 1 alpha 2 2 inverse into x phi y, now we have a simple relation of the inversion of a matrix, this particular matrix we can able to invert the inversion is alpha 1 1 alpha 2 2 minus alpha 1 2 alpha 2 1 that is a determinant of this particular matrix and we need to change the position of the influence coefficient or here this 2 2 was below, now I have kept here and we need to change the sign and the cos coupled terms.

So, this is the inversion of the 2 by 2 matrix. So, now, this we can able to write it as k 1 1 k 1 2 k 2 1 k 2 2 and this is. So, you can able to see the k 1 1 is nothing but alpha 2 2 divided by the determinant of the influence coefficient matrix, similarly other terms can be defined. So, once we have this in the stiffness coefficient term. We can able to write the other also in the same form and the form of the coefficients will be same as the other plane because this is symmetric shaft, now this equation and this equation can be combined to get the reaction force and moment.

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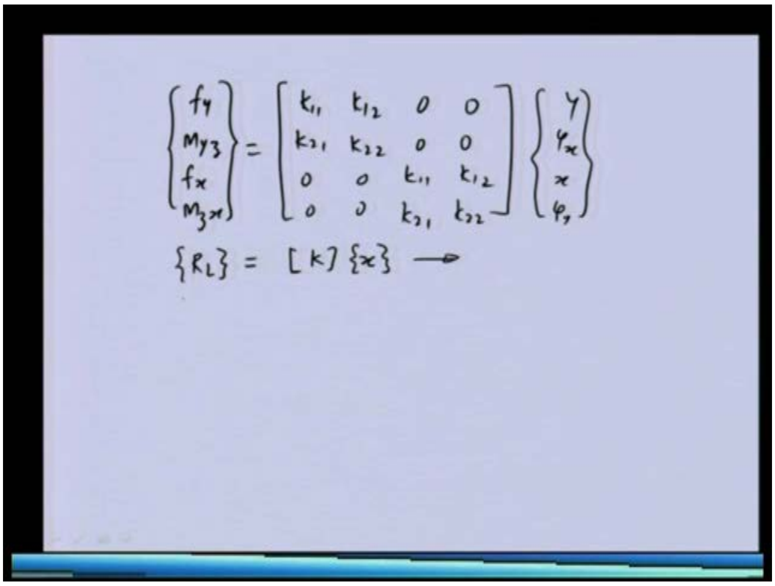

So, we can able to write the moment and forces in this form I am combining the 2 in 1 and here we have displacements. So, here we will be having k 1 1 k 1 2 0 0 k 2 1 k 2 2 0 0 0 0 k 1 2 k 2 1 k 2 2. So, this the relation of the linear elastic force, which we had earlier, this we can able to write it as let us say k matrix and this is the displacement vector. So, the equation of motion which we derived we can able to use this 0 equation there and we can able to write now the equation of motion which we had earlier.

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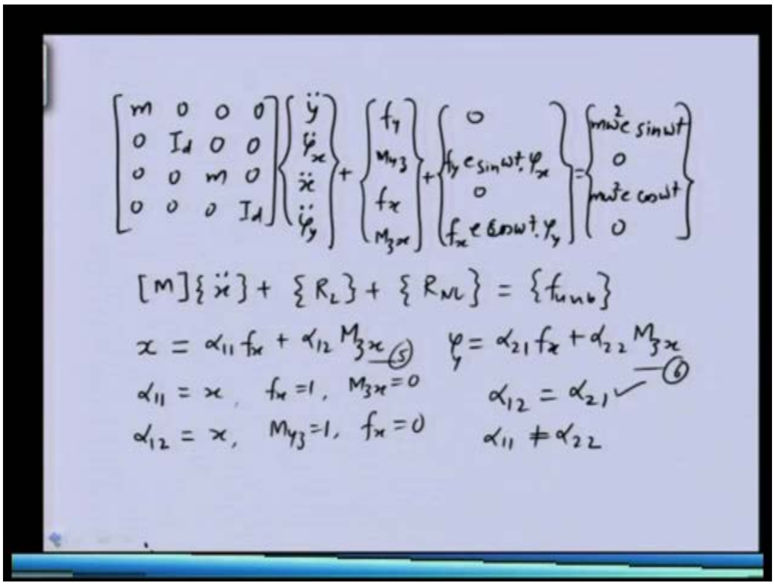

This equation of motion we can able to replace this term.

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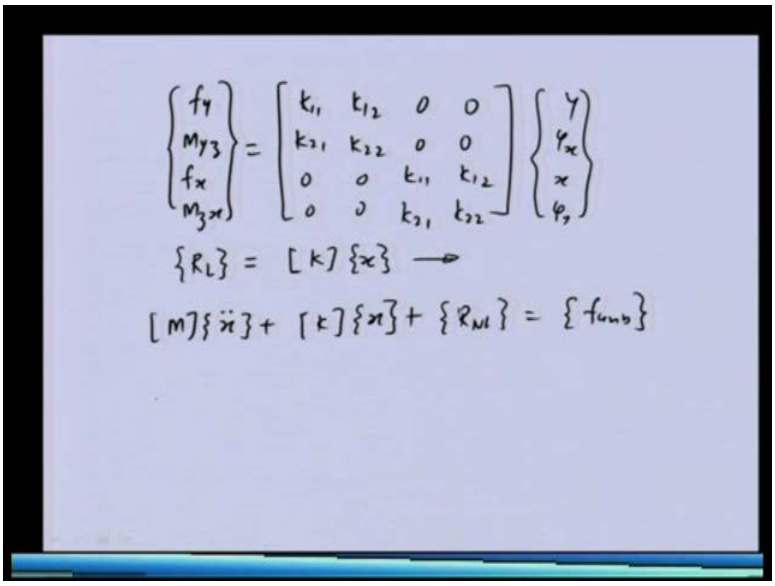

And we can able to write. M x double dot plus k into x and we have still the non-linear term equal to the unbalance force, the non-linear term we had I am taking one of the term that was I will show that expression.

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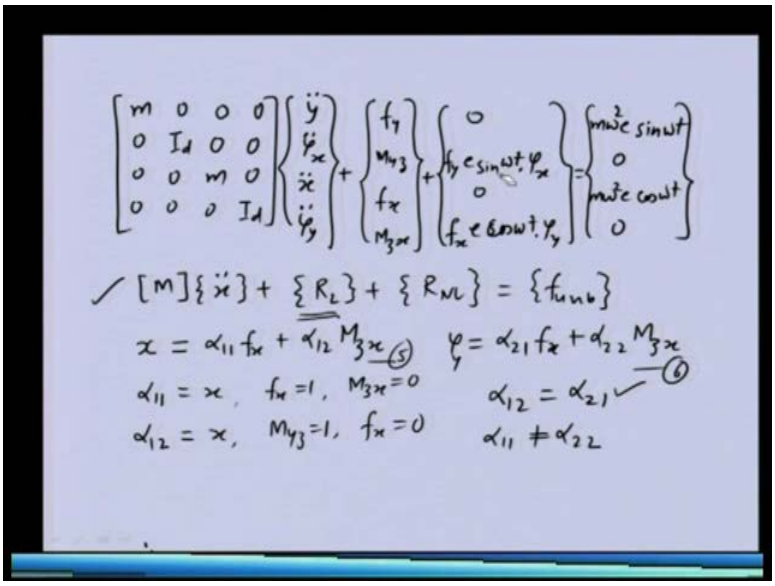

A non linear term. So, we had this one and this one. So, we will now put the f y f x and f y here and we will see that will be giving non-linear terms.

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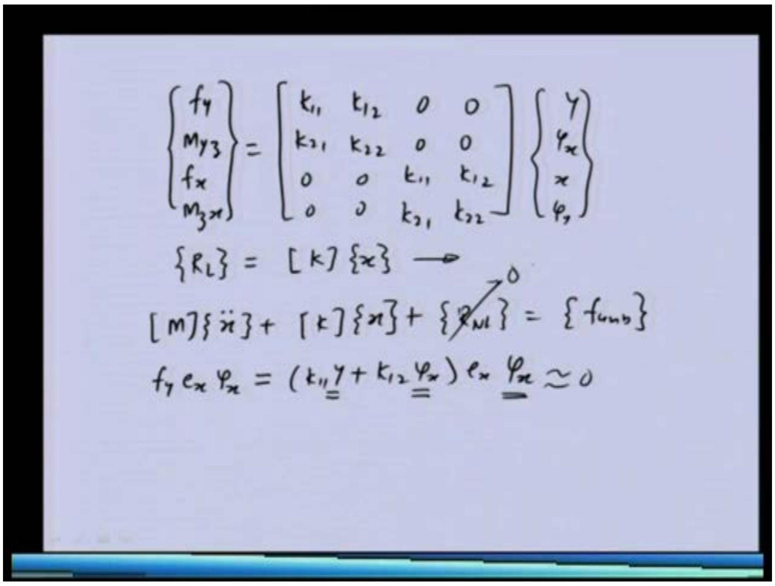

So, one of the term is f y e x and then phi x is equal to f y from above equation, we can able to see f y is k 1 1 into y plus k 1 2 into phi x e x phi x. So, now, you can able to see this is a angular displacement, this is a linear displacement. So, product of these will give us the non-linear term similarly the other expression will also give the non-linear term we are dealing with the linear systems. So, we will drop these terms. So, we can neglect these terms because we are dealing with the linear system. Now, we could able to get the equation of motion in a matrix form of the Jeffcott rotor with it is disk at offset position, we observed that the size of the matrices are 4 by 4 matrices and now let us see how we can able to get the force response from these equation of motion.

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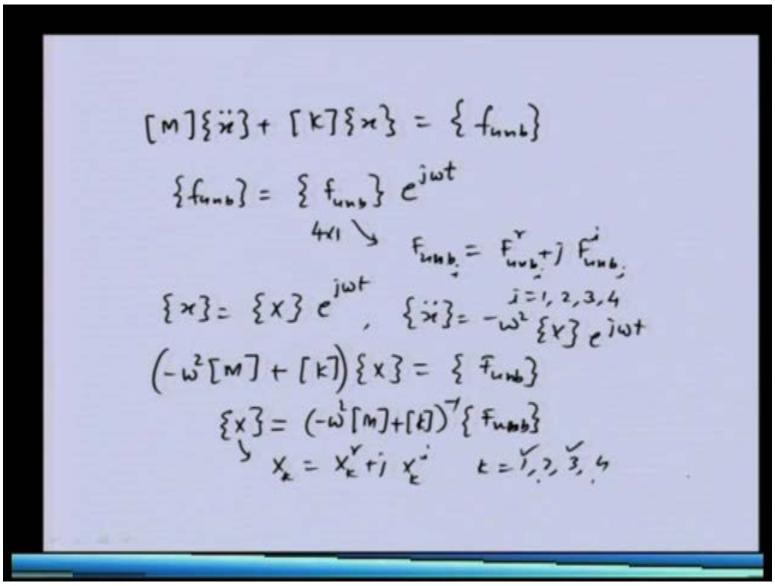

So, we have equation of motion of this form, the linear system we are considering this particular force, we have seen is coming from the unbalance force. So, we can able to write this as amplitude, that is complex amplitude and frequency component, I am taking here as j omega t in this particular case, this force will be having this form that is a real part of unbalance plus imaginary part.

And because the size of the matrix is 2 by 1 vector is 2 by 1. So, that many components will be there in this particular case, if you has I here the subscript. So, I will be from 1 to up to 4 3 4. Once, we have chosen the force like this is a response x can also be assumed in a this form in which capital X is the complex displacement and this gives us the double derivative as minus omega square capital X e j omega t, this can be substituted in the equation of motion.

So, we will get from equation of motion capital x is equal to unbalance force magnitude that is a complex amplitude time dependent term will vanish, now x can be that is a response of the system can be you can able to get by inverting this particular matrix because now it is 4 by 4 matrix. So, we need to do some kind of computer programming to do this inversion and we can able to get the response.

So, response will be having 4 components, corresponding to the linear displacement that is 2 linear displacement and 2 angular displacement and there will be having this particular form, let us say a real part plus imaginary part and k will be 3.

So, first M 3 as we have or a linear displacement and the second and fourth will be the angular displacement. So, once we have obtained the response in the form of the complex form, we can able to get it is magnitude and phase from the complex displacements.

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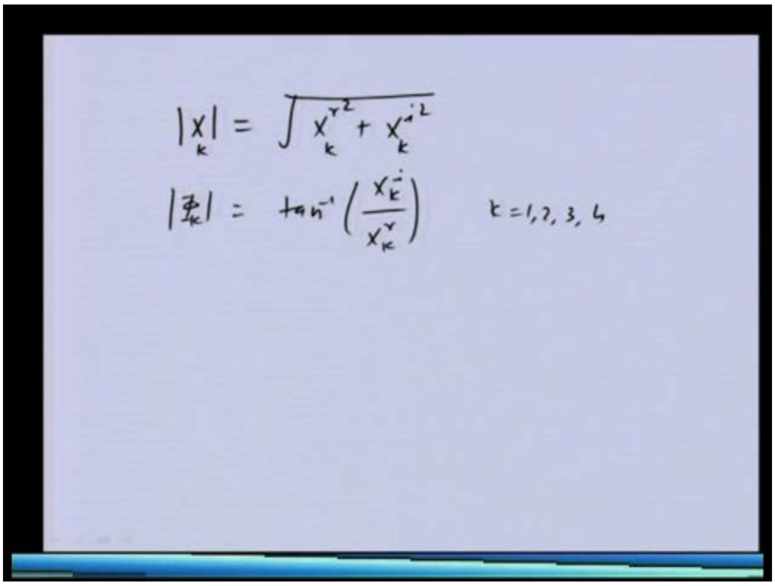

So, if we want the amplitude of the displacement one corresponding the displacement obviously, we need to square the real part and imaginary part and sum up. So, k if I am writing here. So, 4 displacement will be getting 2 will be translator motion and 2 will be rotory motion displacements. And in this particular case the phase we can able to get by tan inverse x k imaginary divided by x real because anyway k is again 1 2 3 4. So, this particular phase is there for the linear displacement or the translatory displacement as well as for the rotational displacement, this particular phase is the phase between the response and the force.

So, we should not confuse this with the rotational displacement, rotational displacements can have a phase with respect to the force and we are obtaining those phase for each of the displacements either it is a translatory displacement or rotationally rotational displacement, now we will take up some objective type of questions just to brush up our fundamentals on the basic vibration and also in the rotor dynamics which till, now we have studied. So, first question you can able to see.

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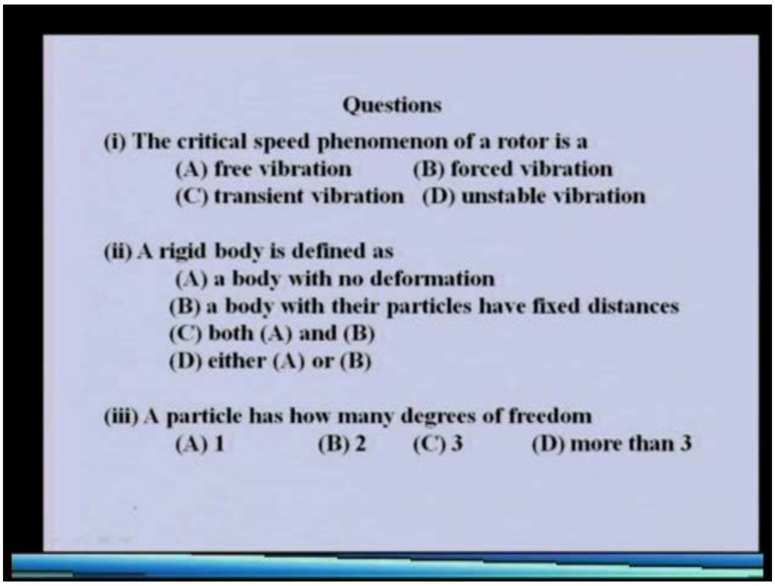

The critical speed phenomena of a rotor is a as we know the critical speed is the speed at which, the natural frequency is equal to the spin speed and in this particular case speed of the rotor is important and this particular phenomena is gone by the unbalance force and because of this critical speed concept, which is a resonance phenomena is a force vibration phenomena, this is not a free vibration even not a transient vibration or unstable vibration.

This is basically a forced vibration phenomena, now let us see a rigid body, how we define a rigid body basically rigid body. So, I have given two option a body with no deformation. So, that is is one of the correct answer then even a body with a particle have fix distance, if we consider let us say two particles in a body, if they are not the distance between them is not changing during motion that means, we can able to consider that body is a rigid.

So, we can able to see that both A and B options are correct, but the last option is not correct, apart from this now let us see the particle was a rigid body and particle is having as such no dimension. So, how many degree of a particle can have. So, particle, because this is having no physical dimension. So, it will be having only the translatory or the linear motions in the x y z direction. There will not be any tilting of the particle, we cannot able to represent that. So, we will be having degree of freedom of a particles 3.

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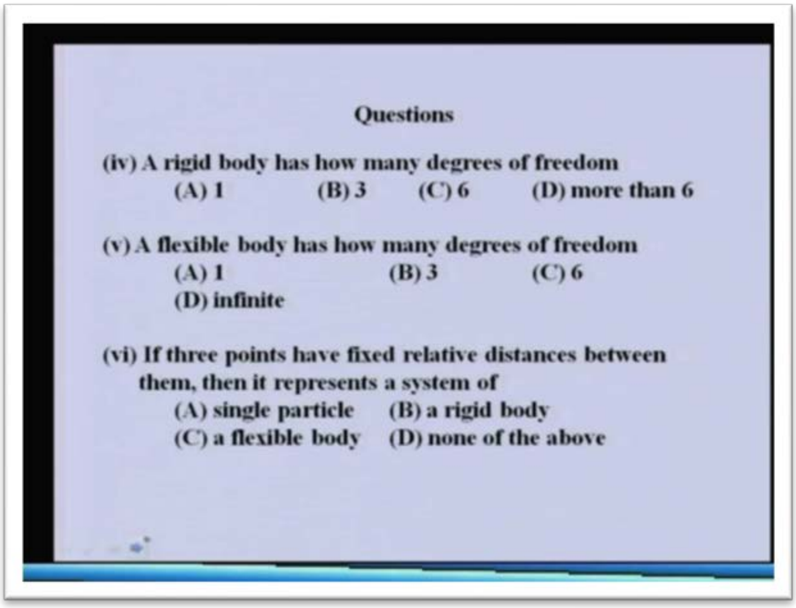

Then if we are coming to the rigid body we need to define it is position not only by x y z coordinate, but also with the 3 rotational angle or the tilting angle. So, there will be 3 linear displacement or and 3 rotational coordinates will be required to define the position of a rigid body. So, it will be having 6 degree of freedom as compared to the particle in which it is only 3 degree of freedom, now when we are coming to the flexible body generally like shafts, we have flexible body in which relative displacement between them between various particles they take place. So, in that particular case a flexible body you can able to see that we can able to define there are, infinite number of points on this and each point is having relative displacement. So, you can able to see that because this is infinite degree of infinite particles are there.

And each particle requires three degree of freedom. So, we will be having an the degree of freedom of a flexible body is infinite if, we consider let us say the 5 question, if 3 points have fixed relative distance between them. Then it represent a system, if 3 points are fixed relative to each other so obviously that cannot be a particle and it cannot be flexible body and it must be a rigid body because the rigid body cannot have relative motion between its particles.

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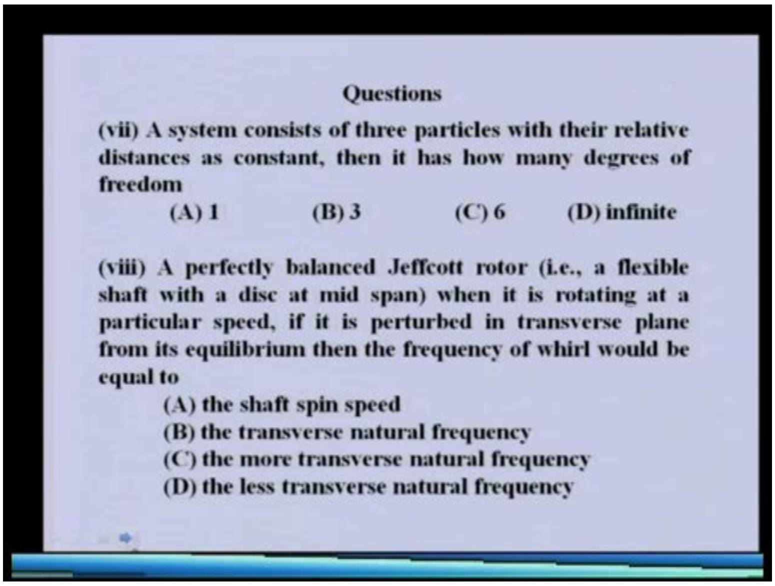

Now if we have a system consist of three particles we already seen that three particle, if they are having no relative motion that represent a rigid body. So, three particle with their relative distances constant then it has how many degree of freedom so obviously, a rigid body we are referring to and it has 6 degree of freedom system, now a perfectly balanced Jeffcott rotor.

Jeffcott rotor we already know this is a flexible shaft with a disk at mid span and simply supported conditions are there at the end of the shaft when it is rotating at a particular speed. It is and if it is perturbed in a transverse plane from it is equilibrium then the frequency of whirling would be.

So, in this particular case what we are considering, a Jeffcott rotor is perfectly balanced, there is no unbalance in the rotor and is spinning at very high speed, now if you are giving some kind of disturbance to this particular rotor, what will be the whirl frequency of that during the disturbance once we disturb it we are giving some initial conditions to the rotor by giving some tap. So, what will be the frequency or whirl.

So, obviously because there is no unbalance, so there will not be any force to excite at the spin speed. So, the this particular Jeffcott rotor will whirl at it is natural frequency because this is a perfectly free vibration phenomena, we are giving a initial disturbance to the system there is no external excitation to the system and because of that it will be whirling at the natural frequency of the shaft.

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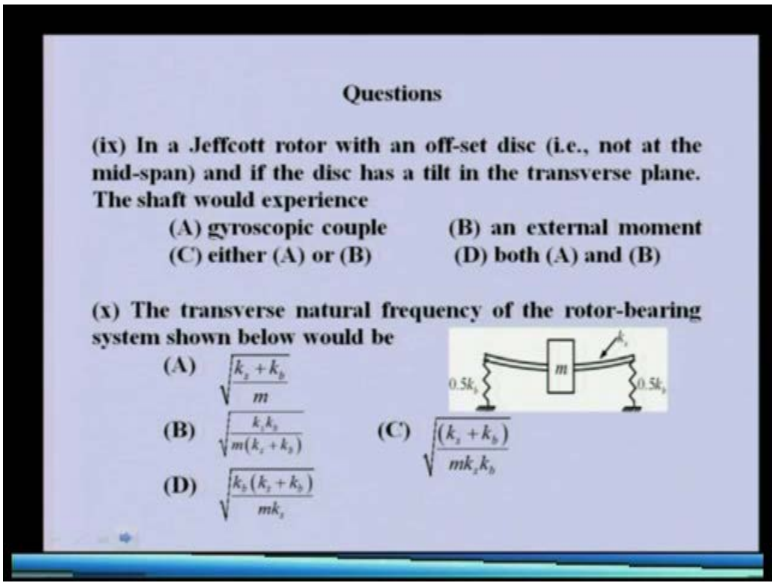

Now coming to the next question, we have already seen a jeffcott rotor with an off-set disk. So, if it is not at mid span obviously, we have seen that the tilting of the disk take place in the transverse plane and this would not only additional external moment, but also it will give a gyroscopic couple, which will be seeing in subsequent lectures, that this particular disk is spinning and is having precision about it is diameter. So, gyroscopic couple would also be acting.

In this particular case last question, we have a mass supported on flexible shaft is having stiffness k s and the support bearings are having this stiffness. So, we can able to calculate because now we can able to see the shaft and the bearing are in series. So, we can able to obtain the effective stiffness of this and that can be effective stiffness of k b, that is in series with k s.

So, we can able to see that this particular solution will be valid, today we have seen the analysis of a Jeffcott rotor with an disk is offset from the centre of the shaft or the mid span of the shaft and in this particular case we derived the equation of motion also we have seen the procedure to obtain the unbalance response, now because the degree of freedom of the system is 4 because we have 4 equation of motion when we obtain the response with varying speed, if we vary the speed and solve the unbalance response for various speeds.

We will find that wherever there will be coincident of the speed with the natural frequency of the system, there will be critical speed and these can be plotted that means, the amplitude of the response and the speed can be plotted and we can able to see that we will observed 4 critical speed in this particular system, as a guess in the previous case for the Jeffcott rotor only one was there and because the rotor is symmetric in two plane. So, it is one, but here we will find that now the, it will be having four critical speeds.