Theory and Practice of Rotor Dynamics
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Module - 2
Simple Rotors
Lecture - 5
Jeffcott Rotor Model

In the last lecture, we analyze a rotor with a single mass system, and we will consider single degree of freedom system and in which the difficulty is that we can able to get the response, but as we know that in a rotor the motion take place into two planes. In two orthogonal planes, the whirling motion which takes place we cannot able to represent with this single mass rotor system. So, today we will see another model that is called Jeffcott rotor model in which we will be considering two degree of freedom system by which we can able to analyze the orbit of the shaft, also in two transfers direction. Subsequently, we will be analyzing some other variant of the Jeffcott rotor in which we will be considering four degree of freedom system.

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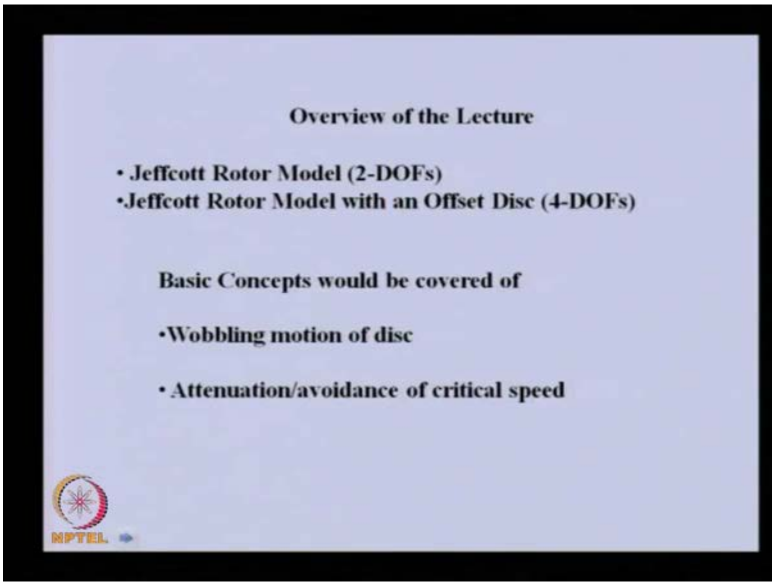

So, let us see the overview of the lecture the Jeffcott rotor model with two degree of freedom, then Jeffcott rotor model with an offset disc that will be having four degree of freedom system. In the previous first case, will be considering only the linear that is translationary motion of the disc.

In second case, not only the translation motion, but also the rotational motion, that is the wobbling motion will be considering of the disc when it is especially offset in this simply supported case. So, some of the concepts like wobbling of the motion and this avoidance of the critical speed, these concepts will look into this particular lecture, now is Jeffcott rotor model. In the previous lectures, we have come across regarding this particular model in which there is a elastic mass less shaft and the disc is there at the middle and bearings are rigid.

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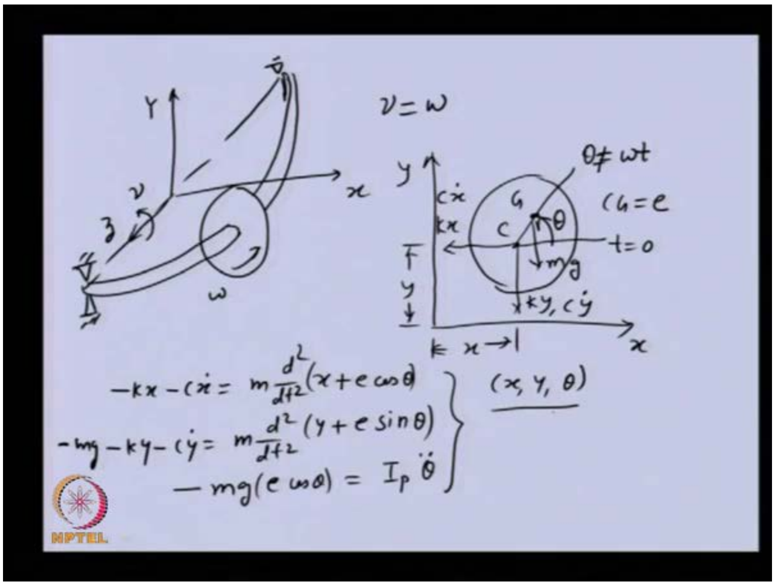

So, if we look at this particular rotor model this is the bearing axis and we have a disc which is attached with a shaft and the disc at the middle of the shaft span and the bearings are rigid. They are allowing the rotor to have rotary motion that no transverse displacement takes place. If you see axis, this is x axis horizontal direction, vertical axis y direction and accordingly we have z axis and this direction. According to the positive convection, the rotor is having spins p omega and this whirling about its bearing axis there deflected shaft at a new frequency, but we are consider the synchronous whirl condition. This means we are considering the whirling frequency is equal to this spin speed, because main force which is causing here is the unbalance.

Now, if we see the free body diagram of the disc in x y plane this is the disc, the shafts will exert plastic force, let us say k y k x that these are the displacements x and y and this is the centre of rotation of the disc that is the centre of gravity is offset. So, we have if about time reference is x axis then this particular angle I am representing that as the theta which is function of time. Theta is equal to omega t for constant speed, but at present we are not considering that as a constant, so let this is not equal to omega t. So, basically in this particular case we have x y and theta as the generalized coordinate to define the position of the disc.

Now, we can able to obtain the equation of motion by equating the force balance in x direction y direction and theta direction. In this particular case let us see the position of the centre of gravity which is here C G, let us take it as e. So, you can able to see that position of the disc, the position of this centre of gravity of the disc will be in x direction. As x plus e cos theta and y direction it is position will be y plus e sin theta, so this is the position of the centre of gravity. If we derivate this twice that will give acceleration of the disc in x and y directions and if you multiply by mass these are the inertia forces. Now, we need to equate this with the external forces, if we have damping also in the in the system will be having damping forces also along the x and y direction like this.

So, if we want to balance the forces in x direction will be having elastic force and damping force similarly, in y direction will be having elastic force and damping force and right hand side is the inertia forces. I perform this; we will see that because of centre of gravity mass m g is acting, so that will also come in the force in the y direction that is in this particular case it will be minus m g. Not only it will give force, it will give moment about the centre of rotation.

So, if we take the force moment balance in the theta direction that is, m g will give us a moment m g into e cos theta that is horizontal component of the eccentricity. There is a momentum and it is acting opposite to the motion that is motion in the counter clock wise direction, theta direction. This couple is opposite direction clock wise, so it will be negative should be equal to the inertia of the disc mass moment of inertia of the disc and the angular acceleration.

So, these are the three equation of motion corresponding to three generalize coordinates. In this particular case of Jeffcott rotor model initially we started with three degree of freedom system that is the two translational motion x and y and one theta that is the torsional oscillation. Generally, this torsional oscillation and the transverse vibration coupling, we neglect in this particular analysis also we will see will be neglecting this by assuming the speed of the rotor is uniform so that we can able to write the theta equal to omega t for uniform velocity of the spins rotor speed. Now, these equations can be simplified by taking the derivative of the terms here and here and let us see what it takes the form so in this particular case.

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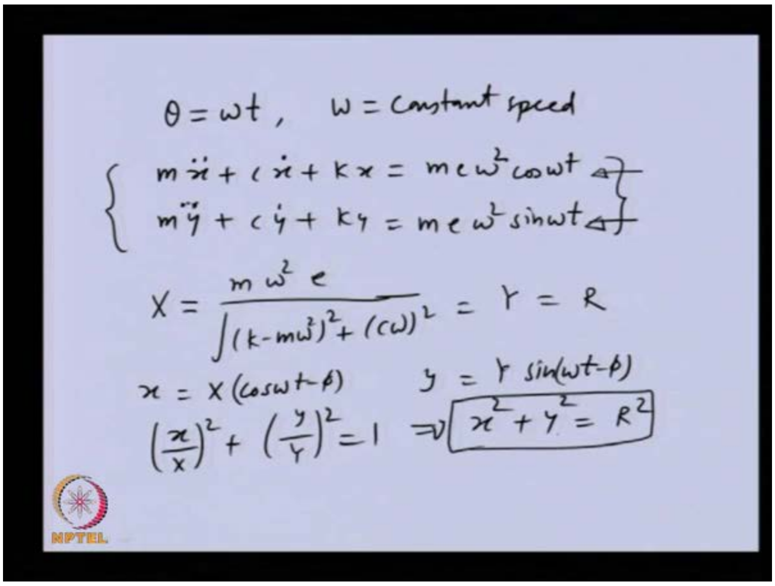

The first thing I am taking is omega is equal to omega t so that we have this as constant speed of the rotor. So, the third equation is having no meaning, now the first two equations will give us equation of motion of this form and in y direction from y double dot plus c y dot plus k y is equal to m e omega square sin omega t. So, we can able to see these equations or representing transverse vibration in x and y direction as such. These two equations are independent of each other; there are not coupled equation, so we can able to solve the equation separately.

If you recall in the previous lecture for single degree of freedom system the second equation, we solved it and those solutions are valid here also. For second equation also, because we can able to see rotor is symmetric, their properties are same, so from the previous analysis we can able to write even the x displacement as m omega square e, that is the amplitude in the x direction. It will be same as y it will be same as y because rotor is symmetric is equal to y. So, the response which we write in the form of like this there will be some phase also, damping you will see that these can be written as, and since x and y are same in this particular case. So, basically it is representing an equation of a circle because these two are same equal to let us r.

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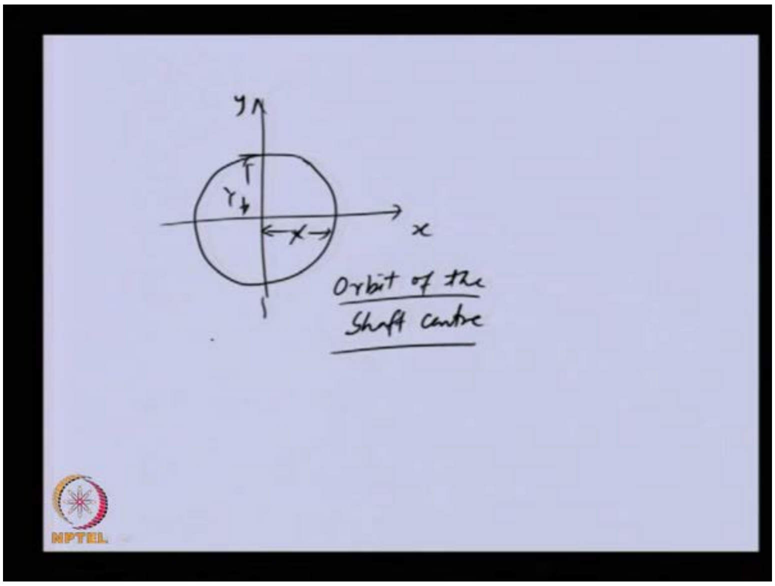

So, this will represent the orbit of the shaft which is circular, so if we plot the orbit of the shaft let us say x in direction there is these will be circular, where this amplitude in the x direction and y direction. Basically, they are same equal to r, so this is the orbit of the shaft centre these two equations we can able to combine in the rotor. Again, I am repeating, the equations of motion in which we have obtained in x and y direction, they can be combined and they can be solved in a complex form and this particular approach is quite popular in rotor dynamics. So, let us to understand that approach, we solve these two equations simultaneously.

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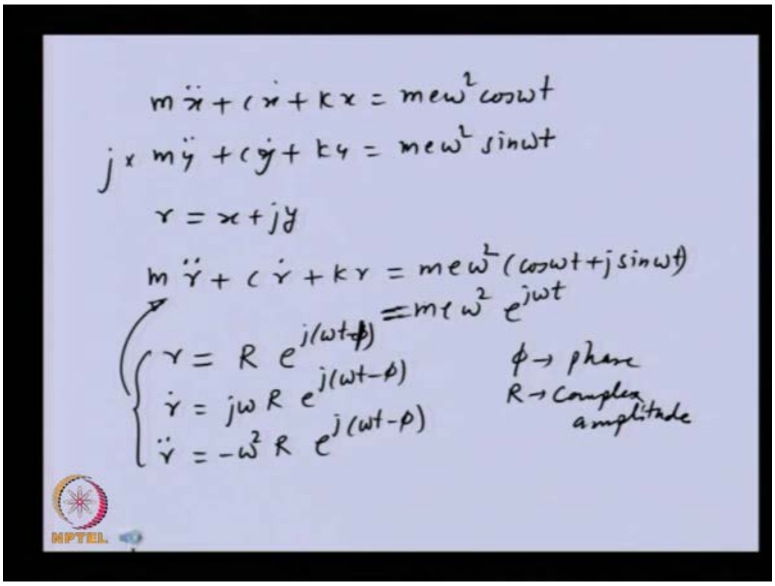

So, in this particular case we have two equations again I am writing of this form and y direction of this form. Now, we defined a complex vector R which is x plus j y and this can this particular vector, complex vector we can use in the two equations by first multiplying second equation by j and adding it to the first one, so we would get these a combined equation in complex domain. Here will get cos omega t plus j sin omega t and they this can be simplified as e j omega t.

Now, the response can be assumed as an amplitude and is having some frequency that is coming here and the phase also phi is the phase and R is the complexion amplitude. If we take derivative of this, we will get j omega R j omega t minus phi t another derivative because in the equation of motion we have two derivatives. We get omega square minus omega square R j omega t minus phi, these assume solution and its derivative we can able to substitute in the equation of motion.

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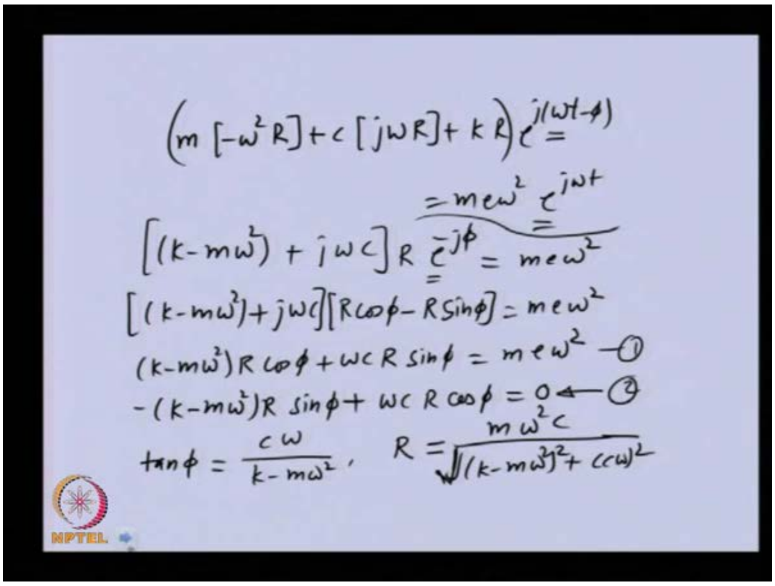

If we substitute we will get equation of this form, that is m then omega square R, I am taking some of the terms common plus c j omega R plus k R. I have taken j omega t minus phi as common and there is equal to right hand side, we have we obtain earlier m e omega square e j omega t. So, this was from the unbalance force, I can able to see that e omega t both side will vanish and will left with this particular equation, I can able to simplify this.

Also, I can combine the real part in one place and the imaginary part in other place and R is common j minus phi is equal to m e omega square. Time dependent terms will vanish now, so this is separate, now we can able to expand this so that we can able to separate it of the real part and imaginary part. So, this will give us this is expansion of the exponential raise to j, phi is equal to m e omega square. Now, if we equate the real part and imaginary part, we will get expression that is in the from the real part equating the real part represent c.

I am equating their real part right hand side we have m e omega square then imaginary part am equating both sides of the equation and in the right hand side there is no real imaginary part so it will be 0. We can able to see that second equation this is first, this is second, second equation can be solved to get the phase, tan phi can be obtained from second equation. So, this is c w by k m omega square and this we can able to substitute in first to eliminate the phi terms from, where we can get the r, then because r and phi are unknown. This can written as if you recall similar expression we obtain when we did the analysis with single degree of freedom system, let us see these equation more carefully we can able to non dimensionalize these terms.

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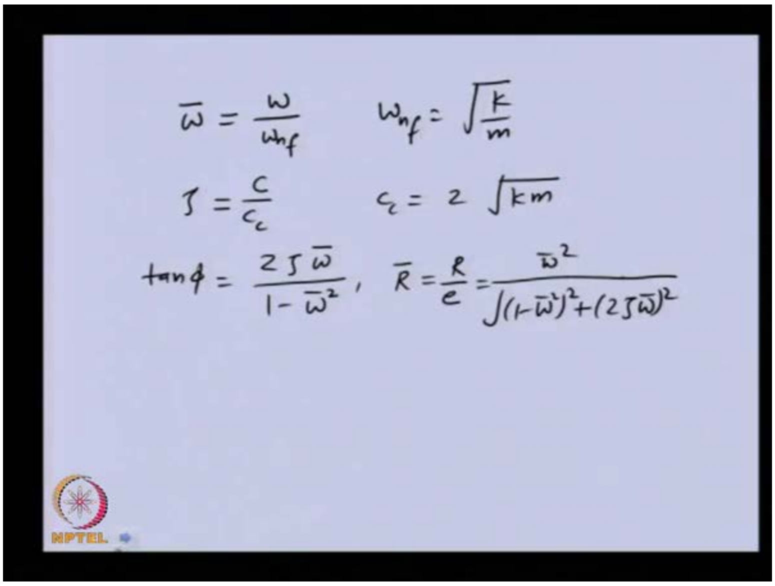

Let us say, I am defining a frequency ratio as spin speed divided by natural frequency, un damped natural frequency, where un damped natural frequency is root k by m. I am defining the damping ratio, which is damping factor divided by the critical damping; critical damping itself is defined as 2 root k into m. So, using this we can able to write the tan phi which was in the terms of the various parameter like k n c it can be written as a non dimensional form and similarly, the response ratio I am defining as the actual complex amplitude divided by eccentricity. This can be written as frequency ratio square divided by this expression, these expressions, now we can able to plot then now this non-dimensional complex amplitude and the phase will be plotting and this is the plot of those particular responses.

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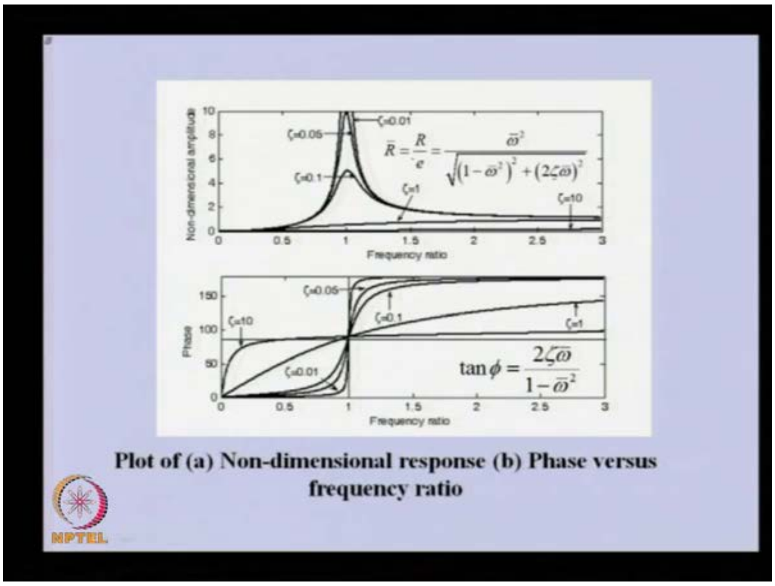

Plot of (a) Non-dimensional response (b) Phase versus frequency ratio

You can able to see the horizontal axis is the frequency ratio of the first plot and non dimensional amplitude, complex amplitude is in the second and you can able to see it has been plotted for various damping. At zero damping it will be at very high amplitude will occur and the frequency ratio 1, but as the damping increases this peak across that will less than this frequency ratio, because this frequency ratio has been defined with respect to the undamped natural frequency.

With damping the damp natural frequency will be slightly less than the undamping one, so that is why these peaks are before the ratio 1 especially when the dampings are high. We can able to see that as we are increasing the frequency this amplitudes are going towards the value 1, because this for that particular case this our particular amplitude becomes equal to the eccentricity. The rotor tries to rotate about its centre of gravity instead of centre of rotation c and this is the phase plot in which various damping ratios having chosen.

You can able to see that they all meet at 90 degree phase where there is a frequency ratio 1 even at zero damping or very high damping case as the damping increases the asymptotically. So, there is a change in the phase that particular speed we can able to see the of the order of 180 degree the meaning of that is the response and response changes. It is phase with respect to the force, and look there is a change of 180 degrees that means the unbalance force with respect to the force and there is a change of 180 degree. This means the unbalance force, which was outside it becomes inside toward the centre of rotation of the shaft.

And because of that it tries to pull the rotor towards the bearing axes and because of that we have smaller displacements as we are going towards the high frequencies. This particular Jeffcott rotor model which we derived for two degree of freedom system we analyze them using complex variable approach. Now, the same problem will be analyzing using a matrix method because we will see that as we as the degree of freedom of the rotor increases the matrix form is more convenient to handle. So, let us see in a very simple form how this matrix method is useful for handling these kinds of equations.

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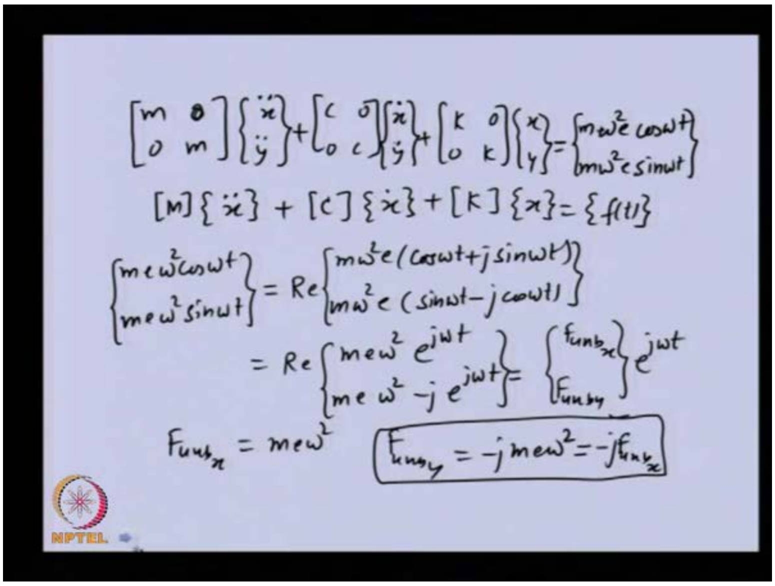

So, these equations we can able to write in matrix form we can able to combine them which we derived earlier like this because we had two equations. So, this matrices are two by two, this is velocity vector, then we have stiffness term as such these equations are in coupled. But to show the matrix method more detail I am showing this for a this particular simple case, but this analysis will be valid for more complex cases also. So, this is the unbalance force, this also we have kept in a matrix form, so basically this particular matrix we can able to represent in more complex form like this as a force vector.

So, first one is the mass matrix, then we have damping matrix, and then we have stiffness matrix, and these are the inertia velocity displacement and this is the external force.

Now, this particular force vector let us simplify them, so I am writing this particular vector separately. Now, this can be written as let us say real part of m e omega square e cos omega t plus j sin omega t, so that means we are interested in the real part only, the imaginary part just we are adding for our convenience.

Then second is sin omega t and here we are writing j cos omega t, now here I am writing minus we will see what is the advantage of this, but basically we are interested in the real part of this. We can add the imaginary part re value or here we are looking into the real part we are not considering the imaginary part. So, this can be written then as real part and m omega square first term can be written as j omega t m, the term, which is below.

We can able to take out the minus j outside and this can be simplified as minus j e j omega t, so if we take minus j out from this will become cos omega t plus j sin omega t. So, we can able to write this particular expression, so if we take minus j from here we will get this expression and now you can able to see that we writing this as a real part. Now, let us say F unbalance in x direction and F unbalance in y direction and j omega t is common, so I can take outside now we know that we are interested in the real part only, because in left hand side of the expression all are real quantity.

So, even we can drop this term from this and it will not make any difference and where this F unbalance in x direction is m e omega square and f unbalance in y direction is minus j m e omega square. Now, you can able to see that this is nothing but, minus j equal to minus j into F unbalance in x direction.

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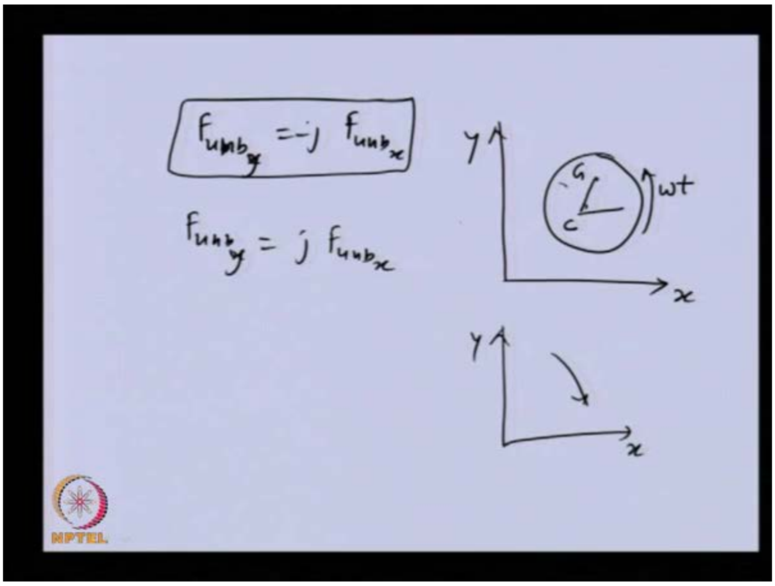

So, we have this relation for this particular case and this is valid only for one particular direction of rotation of the relation which we obtain in the previous slide that is in y direction is equal to minus j F unbalance in x direction. This is valid for a particular direction of the axis and the direction of the rotation, so in this particular case this is x in y direction, rotor is having whirling in counter clockwise direction. So, this is centre of rotation, this is centre of gravity, this is the reference axis, so in this particular case omega t is in this direction.

So, you can see in this particular case y axis is lagging behind x axis for this particular direction of whirling so this expression is valid. If our whirling direction is opposite, that is clockwise direction then we will be having this relation valid. In this particular case our x axis will be when in this particular case whirling direction is let us say in this direction, so x axis will lag the and the y axis by 90 degree which is taken care by the j and minus j is the for the first case.

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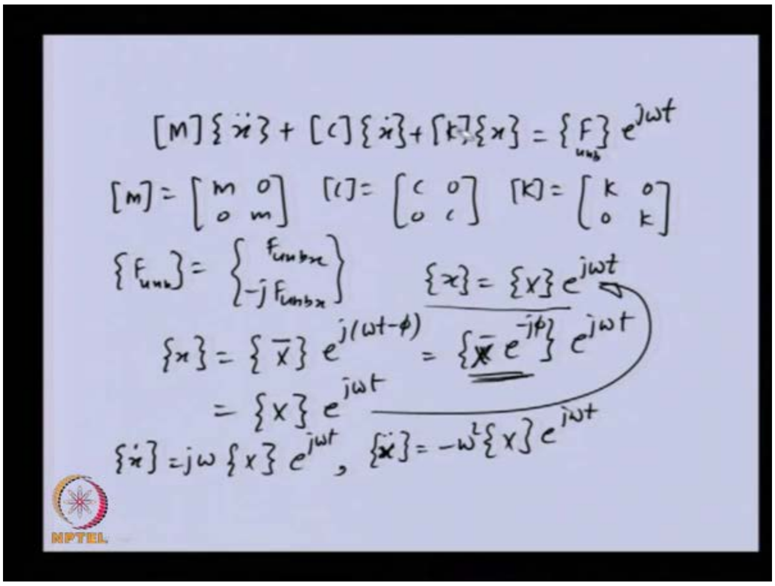

Now, with previous slide analysis we can able to write the equation of motion in matrix form like this and F amplitude and j omega t. The time dependent term is outside, where we have a mass matrix earlier defined as like this stiffness matrix or damping matrix as this and stiffness matrix k diagonal terms are there in all the unbalance force. This particular unbalance force we have as unbalance in x direction and minus j unbalance in x direction because this is taken care of the 90 degree phase.

Now, we able to be assume the response as complex amplitude and the same frequency as the force. In this particular case the phase is taken care inside the capital x, if we are considering let us say phase separately, let us say x R is the real quantity and if we are taking the phase inside here, then this can be written as minus j phi this can be brought inside and j omega t. So, you can see now x bar is real, but this quantity is complex, so they can be combined and written as a complex displacement and e j omega t.

This particular thing we have written here where x capital x is the complex displacement and this is the frequency of excitation, which is coming from the unbalance force now. Once we have chosen the solution of this form, now we can able to take the derivative, first derivative. So, we will get this and for second derivative we will get this and these solutions and it derivatives we can able to put in the equation of motion which is there in the matrix form.

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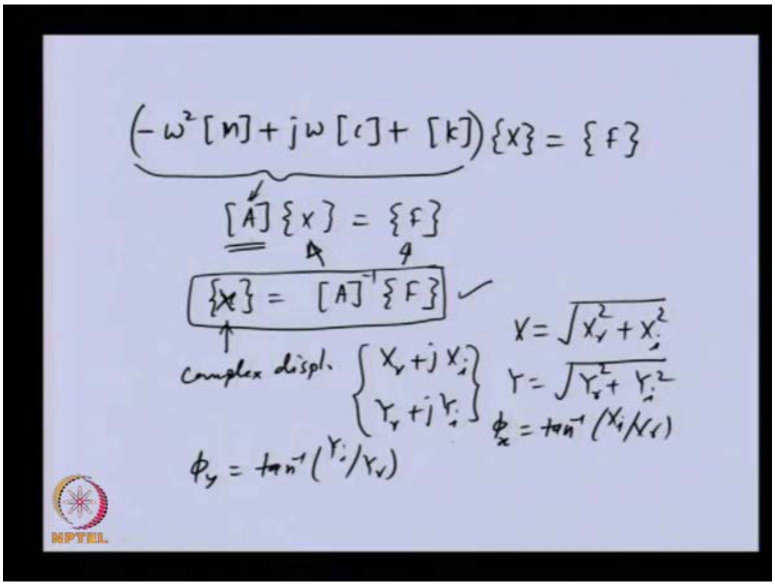

So, we will get summation like this, we can able to take the common the complex displacement vector and the force the time dependent terms e j omega t will vanish from both sides. If we are defining this as let us say a matrix then we can able to write this as simple equation, like this our aim is to obtain the response of the system for a known unbalance force. So, this can be obtained by inverting this particular matrix to get the response, so you can able to see that we are getting the response using this equation which is in matrix form here, sorry this is capital, this is a complex displacement.

These particular displacements will be having this form x real part plus x imaginary part and y component real part and its imaginary part. So, when we solve this equation we will get two quantities that is first one corresponding to the x displacement second corresponding to the y displacement and amplitude in x direction. We can able to get from these components of real part and imaginary part by squaring it, adding it; I am taking this square root.

Similarly, at the y we can able to take, we can able to obtain from this complex quantity, so these will be the displacement amplitude in x and y direction their phase can be obtained, let us say phase of x displacement tan inverse x i divided by x r. Similarly, phase of the y direction displacement can be obtained as tan inverse y i by y r. So, we have obtained not only the displacement amplitudes also the using matrix method we have seen that it is very convenient specially if we are using some kind of a matlab which can handle the complex matrices, we can able to solve the amplitude of vibration and its phase easily. Let us see because at present we have illustrated the method using simple two by two matrixes, so I will show in more detail how these can be interpreted, if you are trying to solve by hand calculation using the matrix using the matrix method.

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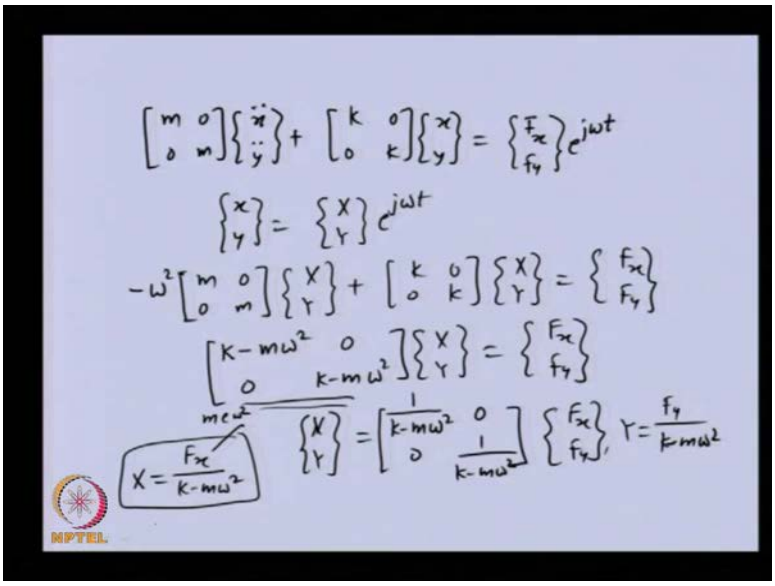

So, for illustration I am taking the equation of motion in more simpler form that is, I am taking the mass matrix and I am neglecting the damping at present and this is the forcing and the these are the complex force amplitudes which take cares of the phase also. Excitation frequency is omega, which is corresponding to the spin speed, now we can able to solve this. If we assume the solution I am writing, the expanded form if we are assuming the solution in this form where x and y are complex displacement.

We can able to write the equation of motion with this assumed solution of this form and so your time dependent terms are getting vanished, they are getting cancelled out. Here, we can able to write this as we can able to combine these two terms the left hand inside and this will be something like this equal to the forcing in x and y direction. Now, we want to inverse this particular matrix to get the x and y, if we want to inverse this matrix and multiply with the F x and F y inverse, inversion is simple just we need to put this in this form.

So, this is the inverse of the matrix, this particular matrix. Now you can able to see that we have x is equal to F x divided by K m omega square and y also we can able to write here F y divided by K m omega square, so this kind of expressions where we obtained in which effects was m e omega square. So, we can able to see that how the matrix method works very conveniently, only thing is when damping is there, these matrices will be complex and need to be inverted in complex form.

We have seen the simple Jeffcott rotor model that is the two degree of freedom model, in which we neglected the torsional and the transverse vibration coupling. In this particular case as we see from in this particular model this the shaft is very flexible, the bearing are rigid there at the two ends and the disc is at the mid span of the shaft. So, when the whirling is taking place in vertical and two planes, this particular disc always remains vertical because this is at the mid span and the slope of the shaft elastic line is always 0, that means no mid span.

So, there will not be any tilting or the wobbling of the disc plane take place, but if same disc is at the upside it is not at the centre now, you see that when it is at the mid span, when it is at the equilibrium the disc will be vertical. When the shaft is wobbling the disc tilts, we can able to see during whirling it tilts about its diameter and this particular motion is taking place in two plane, so continuous wobbling this kind of wobbling will take place of this particular disc.

In the previous case when we considered the disc at middle, we could able to define the position of the disc by just x and y position that is the translation motion. But when it is at the offset position, now we need to define the orientation of the disc also because of its wobbling. So, not only we need to define the two linear or the two translational displacement x and y also we needs to define the tilting of the disc about two axes that is x and y axes.

So, here we will be having four degree of freedom or four variables to define the position of the disc and this is the variant of the Jeffcott rotor in which disc is offset from the centre. In the subsequent topics, we will see this particular kind of wobbling not only introduces additional degree of freedom, but also it introduce it introduces the gyroscopic coupler. Also, because this particular disc is spinning about its own axis not because of this it starts wobbling, so there is a precision of the disc also along with the spinning and because of that we find that the gyroscopic couple will also be there.

In subsequent lecture we will try to derive the equation of motion of this kind of disc without considering the gyroscopic couple. Initially, we will take the four degree of freedom system and we will obtain the corresponding four equation of motion that is two in the translational direction and two tilting. This particular tilting is the different which we considered initially in the case of the Jeffcott rotor and that was the torsional displacement theta we considered.

So, in the present lecture we have analyze the Jeffcott rotor model in detail even it is a solution procedure by a complex method and matrix method, we have seen so that we can get more insight into the method how it works. Apart from this, I defined the wobbling motion when the disc is upside from the centre, how additional angular displacements introduces in the model and this particular model will be analyzing in a subsequent lecture.

Thank you.