Theory and Practice of Rotor Dynamics
Prof. Dr. Rajiv Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 2
Simple Rotors
Lecture - 4
Simple Rotor Models with Rigid Bearings

Till now we have studied introduction to the rotor dynamics, and even in the previous lectures we have seen lot of information regarding the rotor dynamics history, state of the art. So, during the previous three lectures you will find that I have given a overall overview of the this particular subject. And you may now aware of various kinds of words or formulas which are being used in rotor dynamics. Now from present lecture we will start very simple models of a rotors to analyze some of the phenomena of the rotor dynamics. So, present module is on simple rotor models with rigid bearing.

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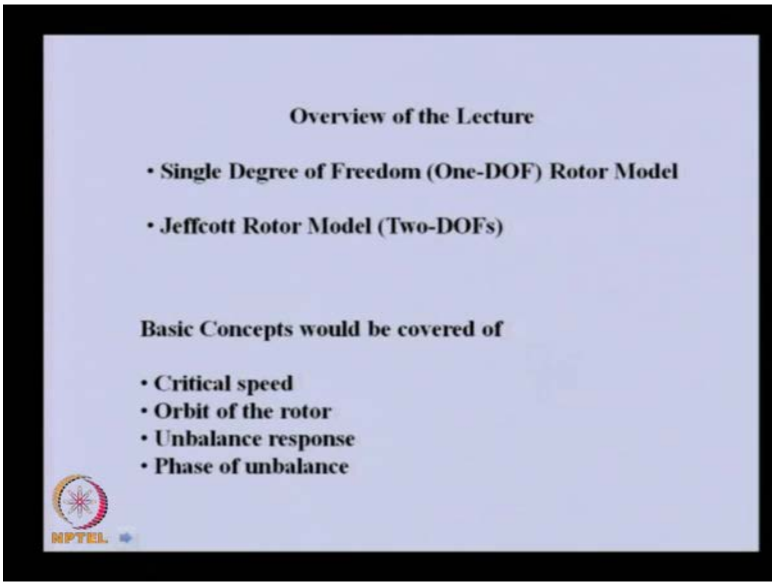

So, this is the overall overview of the lecture I will be covering. So, I will be taking initially the single degree of freedom of rotor model and then I will move to two degree of freedom Jeffcott rotor model which is very important model to understand various rotor dynamics phenomena's. And overall in this lecture we will see various concepts like critical speed, orbit of the rotor, unbalance response, phase of the unbalance and with this let us take one single degree of freedom rotor model.

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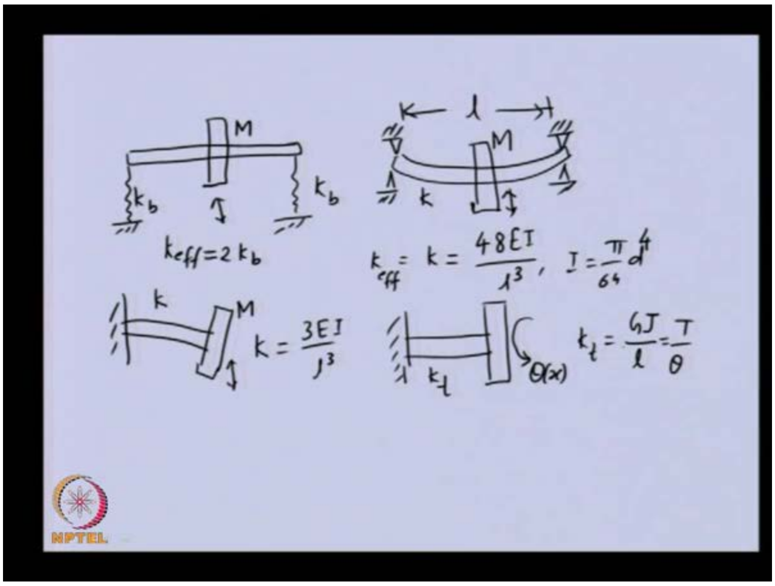

In which, let us say there is a rigid shaft and there is a heavy disc on that. And this particular rotor is supported on some flexible bearing which is having stiffness property, let us say K b. The rotor mass is M. This is one particular model in which you can able to see that these bearings they are parallel. So, the effective stiffness with this particular rotor is experiencing is 2 K b. If we take another model in which we take the bearings as rigid. If we take another model in which, let us say the shaft is flexible which is at mid span we are mounting one rotor having mass M. The mass of the shaft is, let us say negligible as compare to the mass of the disc at the centre and bearings are rigid.

And the stiffness, the transfer stiffness of this particular shaft from the strength of material concerned we can able to write it as 48 E I by l cube. Where l is the length of the shaft, E is the young's modulus and I is the areas certain moment of area. And for a circular shaft we know I is pi by 64, diameter raise to four. So, the stiffness for this particular case whatever the effective stiffness of the, which this particular disc is experiencing from the shaft is this much.

We can have other systems like a flexible shaft with a cantilever, a beam model in which the stiffness is of the shaft is here, mass less shaft. But disc is having some mass. In this particular case the stiffness is given as 3 E I by l cube. In this particular case we are trying to see. In all these three cases we are trying to see up and down motion of these masses that we call it transverse vibration. Another case we can have a cantilever shaft with a disc.

But let us say we are trying to analyze the torsional oscillation of this. That means twisting of the disc about its own axis. In this particular case the stiffness K t will be given by a different expression from strength of material I can able to write like G J by l, which is nothing but T by theta is the stiffness. In this particular you can able to see that these are all single mass model.

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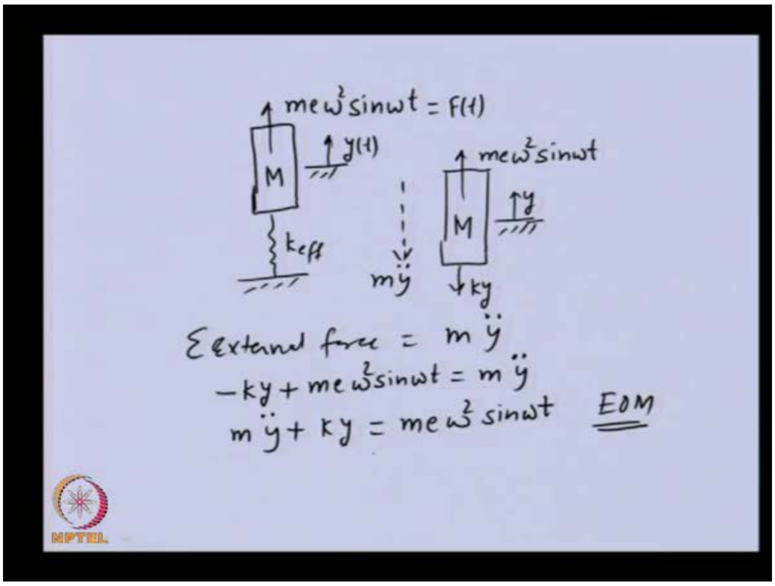

And all these models we can able to represent by a mass and a effective stiffness of the support or of the shaft. Mass is having value M and we are having some kind of displacement in vertical direction which will be time dependent. If you consider the previous model, let us say the first one or second one. We can have some kind of centrifugal force due to the unbalance of the mass and that we can able to apply on this particular mass like m e omega square sin omega t.

Where m is the mass of the rotor, e is the eccentricity, omega is the spin speed of the shaft and t is the time because this particular unbalance force changes with harmonic. So, that is why we can able to write as sin omega t. This is the external force which the mass is experiencing from the unbalance. Now, this particular single degree of freedom model we can able to obtain the equation of motion of this by drawing the free body diagram of the mass. Let us say this is the mass or in which elastic force will be acting downward which will be opposite to the motion.

If motion is upward, this will act downward. Let us say centrifugal, this external force is acting here. Gravity effect we have neglected here. We are assuming that whatever the reference we are choosing is from static equilibrium position. Apart from this there will be inertia force which will be acting. Now, we can able to write the force balance sum of external force is equal to mass into acceleration. External forces are the elastic force which is opposite to the motion.

So, it will be minus, plus unbalance force which is in upward direction. So, it will be positive is equal to the inertia force. And this can be written in very standard form like this. This is the equation of motion generally we write as equation of motion, short form like this. So, we have seen that several single mass rotor systems can be modeled by spring and a mass, single mass system. Like I can give example of practical example of a engine. Engine, which is having reciprocity parts that give some kind of harmonic motion, excitation.

And if that engine is mounted on some kind of flexible support, that particular flexible support can have some kind of effective stiffness. The whole engine we can able to consider as a single mass and the model which we were discussed can be used to analyze that kind of engine system which is supported on flexible shaft. That is a very simple model. But it will give some kind of information's like we can able to analyze the natural frequency of the system. Also we can able to analyze the resonance condition of the system as we will be discussing in this particular lecture.

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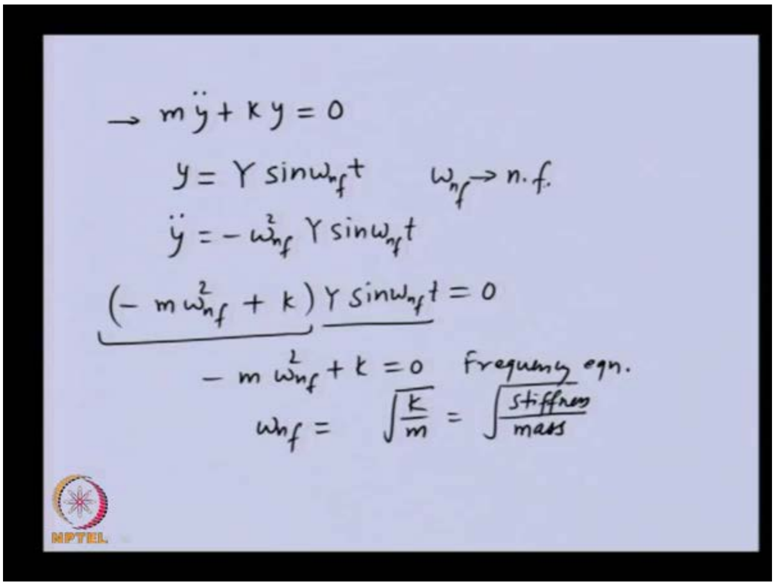

So this particular equation of motion which we obtain, we can able to solve first may be the free vibration part of that, that is by putting the external force is equal to zero. In this particular case, because this is a free vibration we expect the motion should be some kind of simple harmonic motion with frequency equal to the natural frequency of the system. And this capital Y is the amplitude of the free vibration. Omega nu t, omega n f is the natural frequency. t is the time, where this is natural frequency.

This is not the spin speed of the shaft there was at present you are neglected the unbalanced force. So, frequency is corresponding to the natural frequency. If we take double derivative of this, we will get this expression. Then we can able to substitute this in the equation of motion. So, we will get sin omega t will be outside, sin omega n f t will be outside. Because this term cannot be 0 for all value of time. So, the term within the bracket we should equate it to 0 and y is common.

So, we will be having m terms will be there and replaces is equal to 0. y is common so it will like amplitude cannot be 0. So, we should have this term 0. This is called frequency equation. I will repeat, I will repeat some part. These two expressions can be written in the equation of motion. To get this expression this sin omega n f t and y is common. Now we can able to see that this term cannot be 0. So, we need to put the terms within the bracket equal to 0.

So, that will give us, this is called frequency equation and from here we can able to get the natural frequency of the system as root k by m. And we can able to see that frequency cannot be negative. So, the negative value is having no meaning, so only positive value to be retained. This is the natural frequency of the system which is nothing but square root of stiffness, if x changes stiffness of the system divided by the mass of the rotor.

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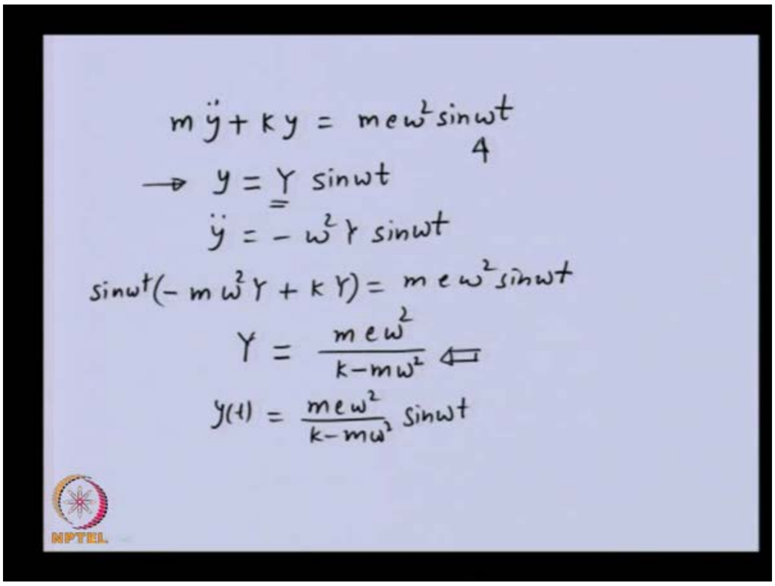

Now we will analyze the force vibration due to the unbalance which we neglected in the previous case. In this particular case when we choose the solution, the frequency will be spin speed because now unbalance is giving us the spin speed as the forcing frequency. So, the response should also, should have the same frequency. Here if we take double derivative of this, will get this expression. This can be substituted in the equation of motion.

Now our aim is to obtain the amplitude of vibration. So, we can able to solve this equation form y, capital Y was here sin omega t will be there that we cancel this particular term, because this amplitude is not time dependant. So, here we will be having this expression. In this particular case if you see, because there is no damping in the system. So, we will not consider the phase difference between the response which is Y and the force. Force is having sin omega t term, response is also having sin omega t term.

If damping is there then we need to take consider the damping phase in the response. So, in that particular case the response will be sin omega t minus phi. Now this particular amplitude will give us the actual response with time using this expression. This particular amplitude is having importance so let us see this particular amplitude how it varies with frequency.

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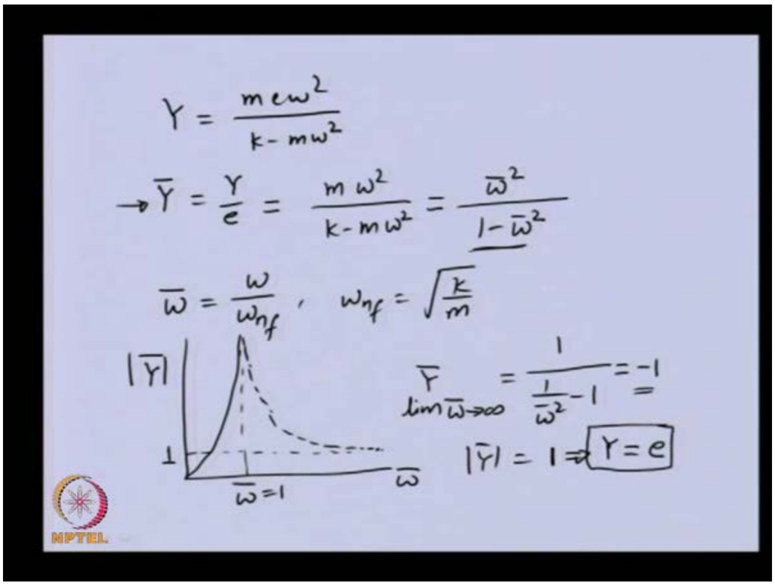

So, I am again writing that expression. If we divide by omega square then numerator and denominator we will get. Also I am dividing by mass. So, this can be written as, I will repeat this slide again. This particular amplitude we can able to write, I am writing that expression again here. So, we can able to define a non dimensional amplitude as Y by eccentricity which is given by this. Now we can define a expression frequency ratio is omega by omega n f, where omega is the spin speed, omega n f is the natural frequency which is given as root K by m.

With this we can able to write this particular expression as omega bar square divided by one minus omega bar square by rearrangement of the equation. Now if we want to plot this particular response that is with respect to frequency ratio, the amplitude ratio. So, if we plot this we will get very high amplitude at omega bar is equal to 1. Because in this expression this denominator will become 0 and we expect very high amplitude of vibration and if you increase frequency further this will go asymptotically to equal to value 1.

So, this particular amplitude ratio will be 1. Why it is 1? Because this Y bar when we are putting limit omega bar is equal to infinity, we can able to see from the above expression that can be written as 1 by 1 by omega bar square minus 1. That is equal to minus 1. So, basically I have drawn the more mod of that, so the value is mod of that is positive 1. So, we can able to see what is the meaning of this? Here if you put the Y bar as Y by e it will give Y is equal to eccentricity. So, you will see that at a very high speed for the single mass rotor system the amplitude becomes equal to the eccentricity and this the sign is minus. So, let us see what is the meaning of this?

So, from the, this analysis we have seen that when we are increasing the speed of the rotor from 0 speed at the frequency ratio 1, the resonance condition take place. That is very dangerous condition generally we avoid the operating, generally we have avoid the operating speed to fall at the natural frequency, because there is a resonance condition where high level of amplitude take place. Because we did not consider the damping on the system, so the amplitude at the resonance was infinity.

But in factors always damping will be there to restrict the amplitude to some finite value. And also we have seen that once we were increasing the speed to very high speed, theoretically infinite speed, practically at above the critical speed may be three to four times. We will see that the rotor vibrations will be very less and that will be equal to the whatever the vibrations will be taking will be equal to the eccentricity of the rotor. So, let us see this particular thing how it happens.

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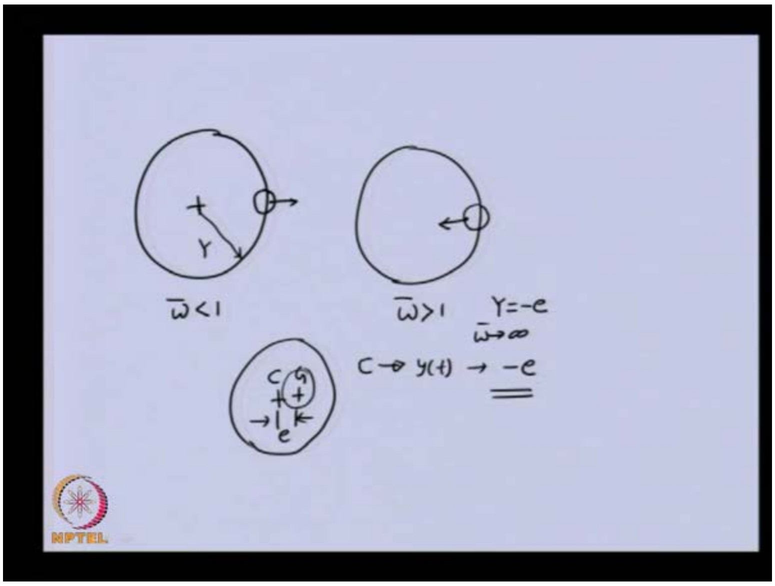

So, earlier we have seen that the volume of the shaft about its varying axis and when we are operating below the critical speed, that is frequency ratio 1. The unbalance force is towards the outward direction and this is the amplitude Y which we are talking about in the previous analysis. But the same thing when we were cross the critical sped, this particular unbalance force comes towards the negative direction. We have seen that the Y becomes negative at very high value of this.

So, in this particular case the rotor if you see the rotor, rotor is having center of rotation this one, center of gravity this one, this is the eccentricity. So, C when it is displacing, that is the displacement of the rotor, if C is displaced by minus e. What is the meaning of that? That means the center of the gravity will try to come at the rotational axis of the system. So, it will the disc will rotate about the center of gravity and that is why we are having this particular less vibration once we are crossing the critical speed. Now let us extend this particular analysis which we in which we have taken the mass and the spring. Now we will consider a system in which the damping is also present.

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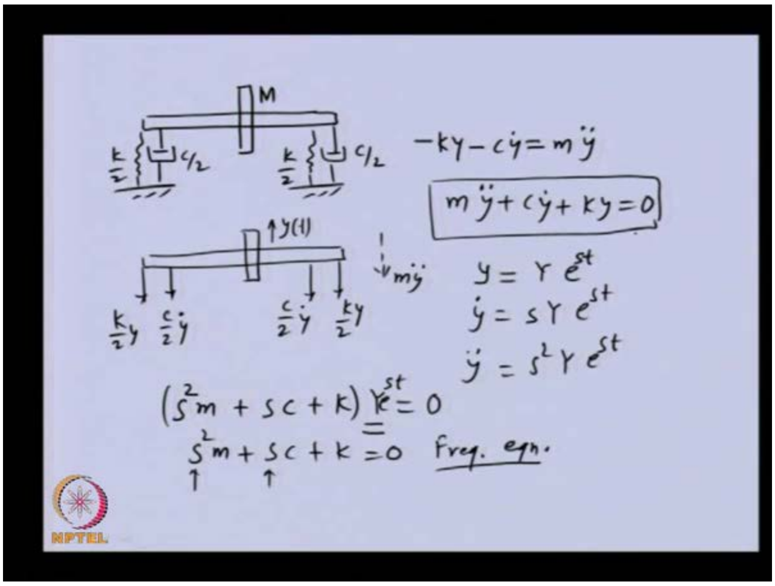

So, in this particular case let us say we are considering a simple rotor system which is rigid shaft and a mass. This is supported on a spring and mass spring and damper system. Let us say the damping is viscous damping and the coefficient is c, stiffness is half, k by half, k by 2. Here c by 2, k by 2. This is M. So, in this particular case you can able to see from free body diagram of the rotor system will be having, we have displacement in this direction will be having k by 2 y and c by 2 y dot.

Because viscous damping is gives us a force which is proportional to the velocity. Here also because we have considered the bearing as symmetric. An inertia force will be opposite to this. So, the equation of motion if we balance on the forces and moment we will see that this will be minus k y that is from both the bearing minus c y dot on both the bearing. There is no unbalance let us say. So, external force due to unbalance is 0 should be equal to inertia term. So, basically this equation is like this. So, this equation of motion is very popular equation of motion of a spring mass damper system.

And to solve this for a free vibration firstly we need to assume the solution. Let us say we are assuming as s t, if we take derivatives twice. Once we will get this, another derivative we get s square Y e s t. And if we substitute this in the equation of motion you will get this expression. Where y is common is equal to 0. e s t is also common, these expressions cannot be 0, that was a no-solution there were not be any motion. So, we need to put terms with them in the bracket equal to 0 and this is called frequency equation or characteristic equation. From this we can get the frequency of vibration. So, let us solve this equation.

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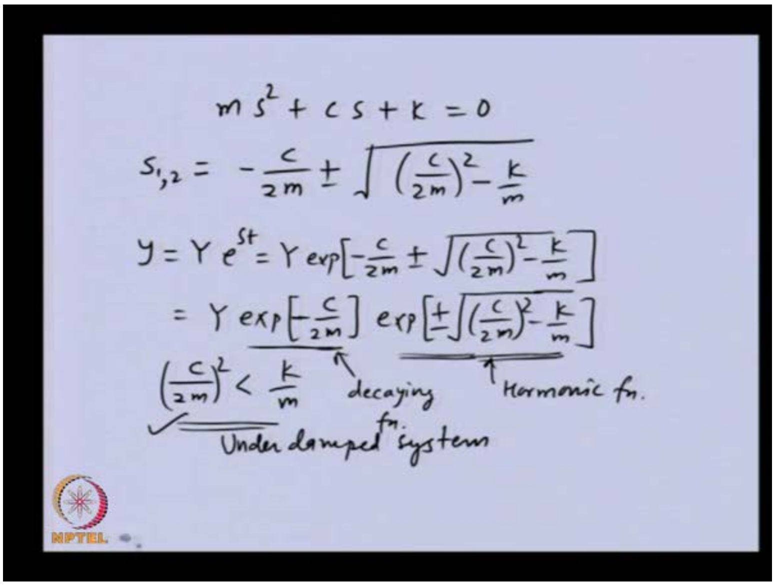

So this is the equation of frequency equation we got form the spring mass damper system. We can able to solve for s because this is a quadratic equation. So, we will get two roots and the solution will be of this form, and if we substitute this in the assumed solution. Let us say I am writing this as exponential and within the bracket these terms. Now, this can be written as we can able to separate it out some of the terms like this. And we can take this square root term separately.

Now, we will see that when we have this condition which is inside the bracket this term. This is negative then we will be having the terms from this as sinusoidal terms because once it is negative we will be having complex to exponential raise to complex. That will give us the sin and cosine terms. The first term, this one is I can able to see this is having minus sign all other c and M are positive. So, this expression gives us a decaying it is a decaying function. And for this condition, this a harmonic function. And for this case we this particular case we call it as under damped system which is generally the case.

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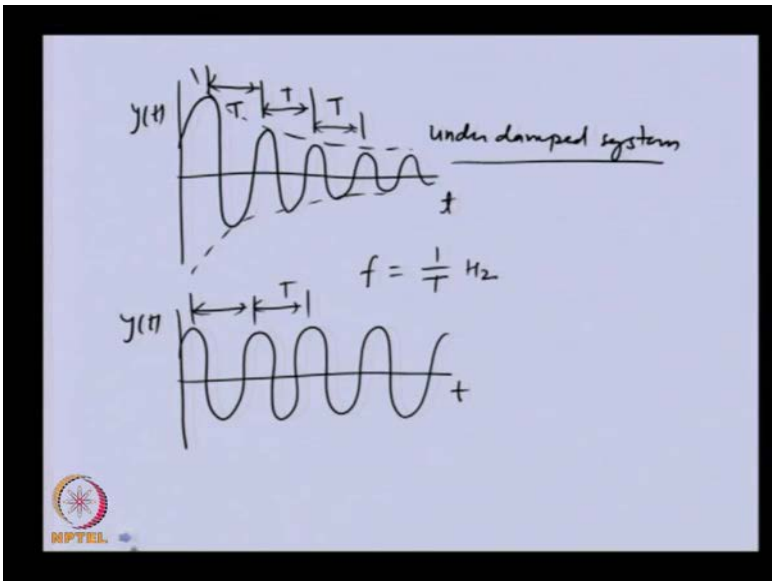

And for this particular case let us see how the y will vary. For under damped system if we want to plot the y t with respect to time the decaying function which is the first term will be of this form. And depending upon the initial condition we will be having the free vibration like this, after sometime it will decay. Now you can able to see that this particular plot, this is the time period and that remain same. It does not change with time, always this time period will be defined with respect to this.

So, when you are working in the linear system when response is small always this will be the time period. And the frequency will be one of by the time period. This will be in the

Hertz. And this particular response as I mentioned this is called under damped system. In the previous case when there was no damping we expect the response plot will be simple harmonic. It will not decay this is time period always this time period will be same which is related with the natural frequency of the system.

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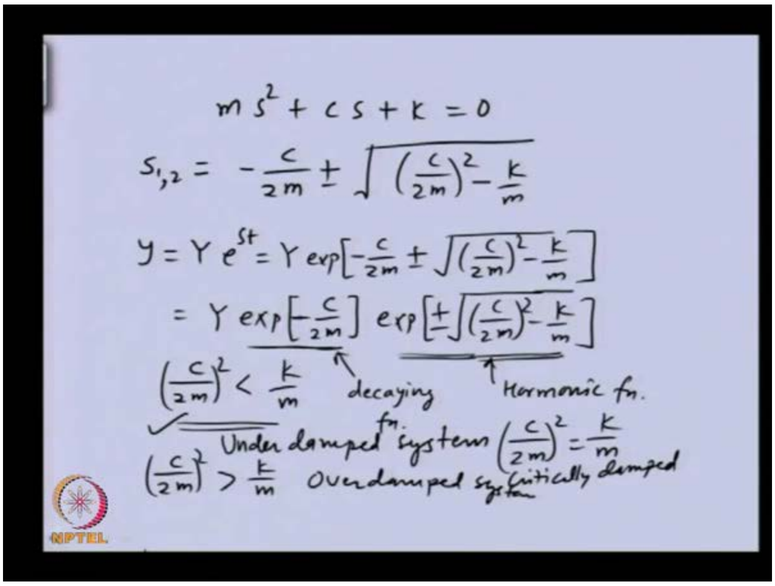

Okay in the previous case if this particular condition is equal. That means, we have c by 2 m square is equal to k by m. That is we call it as critically damped system. Damped system and the another case in which c by 2 m square is more than k by m, that is called over damped system. And in this particular case let us see how the response will be for over damped system. In this particular case the first term, I will again go back to the expression.

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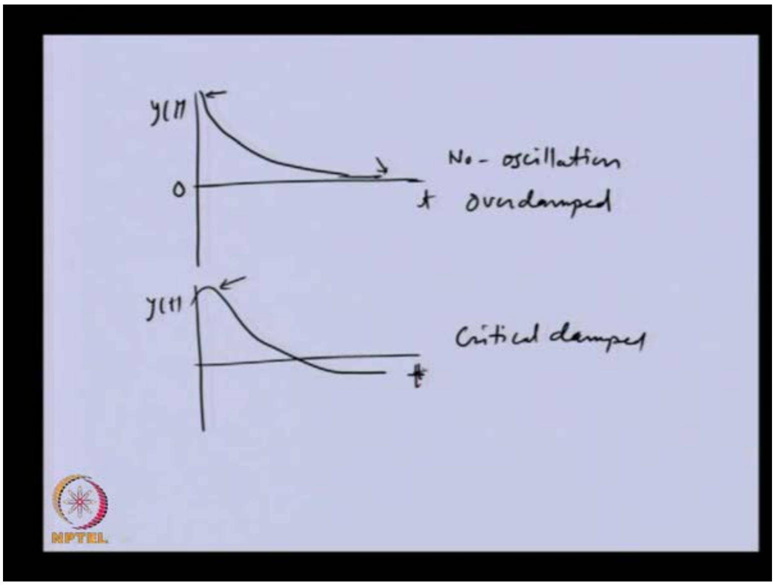

Here you can able to see when over damped system is there, we or let us cut this one. For over damped system we will be having a response like this. There will not be any crossing of the response from the 0 line. So, after sometime gradually if we are giving some disturbance to the system from here resonate will come to the rest. It will not cross, there will not be any no oscillation will take place. This is for over damped system.

For critically damped system also similar behavior is there in which the resonate comes to the rest and in and for very special initial condition sometime for critically damped system it may cross one solid. So, this is a special velocity initial condition we need to give by which it can able to cross once. ((Refer Time: 36.04)) Now we will solve the force vibration of a single mass, spring mass damper system. And for that let us take the equation of motion.

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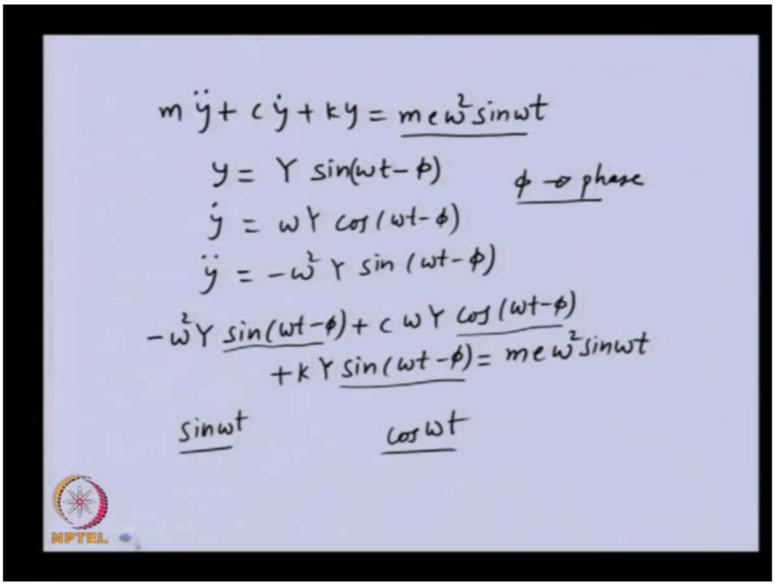

So, I will write the equation of motion. Earlier we did not considered the forcing but due to unbalance this kind of forcing can come into the system. So, for this particular case the response can be assumed as sin omega t. But now this time we need to consider some phase because damping is there in the system. Earlier this particular phase was not there because there was no damping in the system, because when we are applying some force the response will lag behind by the force by this phase angle.

Now, we can take the derivatives, this will be positive if you taken double derivative then it will become negative. Now, we can able to substitute this in the equation of motion. So, we can able to write in this omega t minus phi and then the last term corresponding to the stiffness in the left hand side. This should be equal to the unbalance force. Now, because now the expressions are in these forms we need to expand them. These expressions we need to expand them and then we need to separate the sin omega term, then cos omega term. So, let us do that particular expansion of these terms in the next slide.

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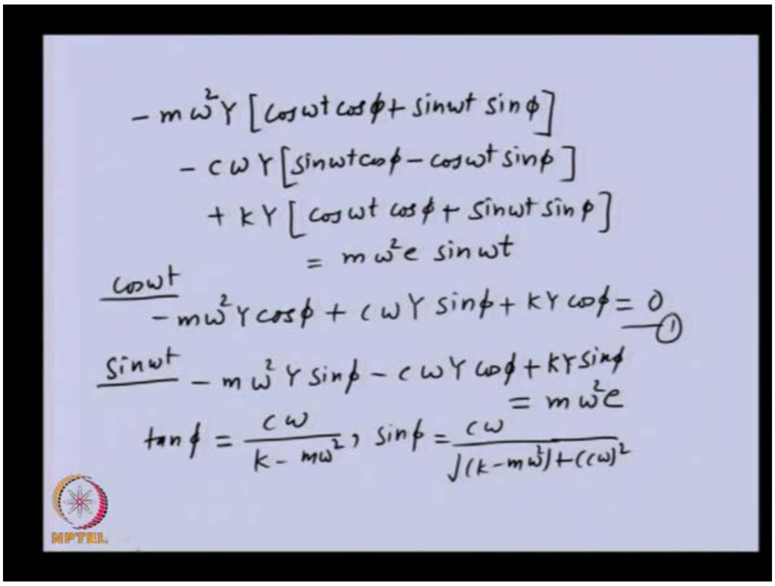

So, m omega square, Y cos omega t, cos phi plus sin omega t, sin phi. This was the first term, the inertia term. Then from damping term we can have sin omega t cos phi minus cos omega t sin phi. Then from the third term, elastic term we can able to expand that as cos omega t cos phi plus sin omega t sin phi is equal to the unbalance force, just wait t and sin omega t terms. So, I am first taking the cos omega t terms separately and I am equating it to 0. First term then from damping, then from stiffness is equal to 0, because right hand side there is no cos omega t terms. Then sin omega t term is from inertia, then from damping, then from inertia and then from stiffness and here in the right side wave of unbalance force so we will be having this term.

Now, we can able to see from first equation we can able to get the phase. We can able to write this as whereas, from second equation if we substitute this phase in the second equation because it is containing sin phi. So, that can be written as sin phi because we know the tan phi from that sin phi we can able to write as c w by k minus m omega square plus c w square sin phi.

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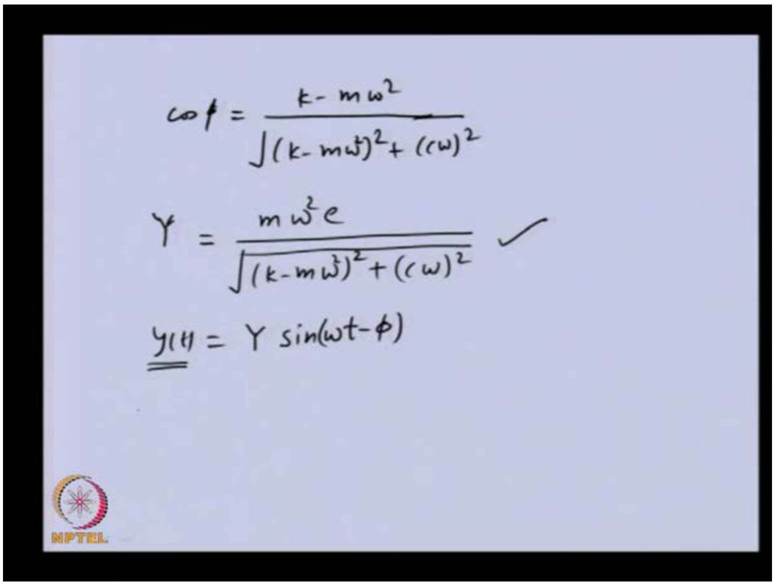

Similarly cos phi we can able to write, I am writing in this next page. Cos phi can be written as, this can be obtained from the tan phi. And this cos phi and the previous sin phi terms can be written in the second equation so that phi terms can be eliminated from the second equation that will give us the y. And y finally you should able to get of this form. Now, we can able to see that this particular, this is the amplitude which is amplitude of the vibration. The response can be written as which is time dependant sin omega t because we took, and the phi. So, phase we have obtained earlier and amplitude is given by this expression. So, the this particular response can be now obtained using this expression. Now, we will plot these responses which we have obtained in the amplitude and the phase and we will see how they vary with the frequency.

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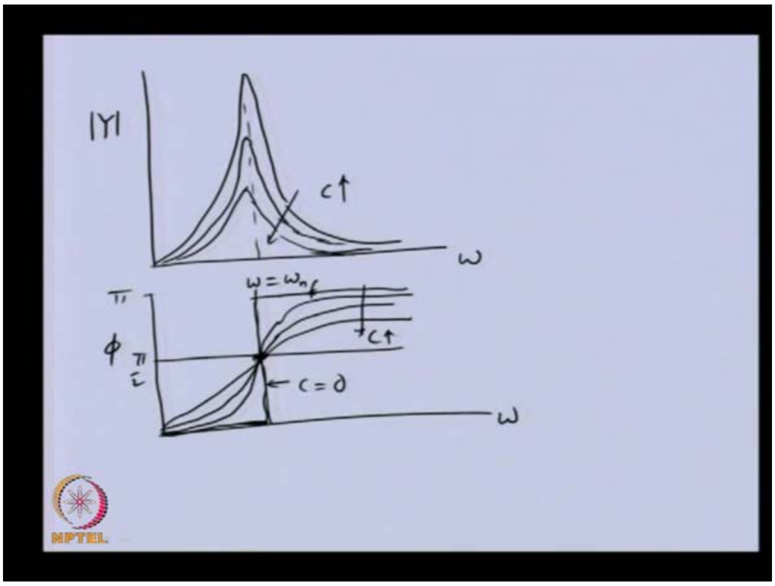

So, if we plot the Y with respect to omega, typically we will be having at resonance. We will be having a large amplitude but it will not be infinity. Now nearly at natural frequency it will be infinite, but as we increase the damping, the maximum with a curve slightly below the natural frequency. So, these are from different damping ratios. So, this is the increasing order of damping. The response would be seen like this. It will not be 0 here at the corners. When we see the phase, let us say this is pi by 2. This is pi. So, the phase would change for different damping at this. Even for high damping it will cross at the same point that is corresponding to the 90 degree phase always this will cross.

So, this is increasing order of damping. We can able to see even at a 0 damping this will come to here, then it will cross then it will go in this direction. This is corresponding to 0 damping. So, you see that at here it is always, the phase is crossing always at the ninety degrees. This is a particular characteristic of the critical speed at that location the phase between the force and response becomes 90 degree.

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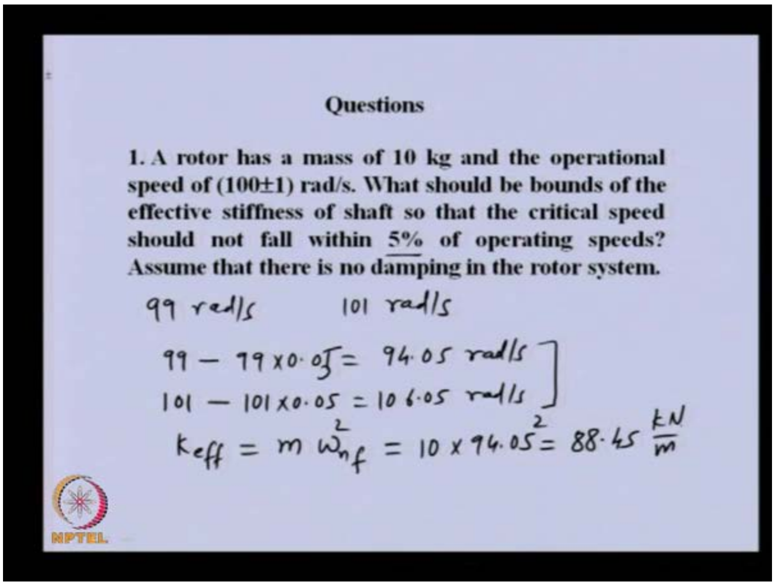

Can I take one example? Three minutes I can take one example, I will speed over the three minutes. So, let us take one question. A rotor has a mass of 10 kg, the operational speed is 10 plus minus 1 radians per second. What should be bound of effective stiffness of the shaft, so that the critical speed should not fall within 5 percent of operating speed? Assume that there is no damping in the system.

So, we know the mass of the system, operational speed range is that is 99 radians per second and 101 radians per second from this we can able to see. And if we see that the operational speed should have 5 percent. This 5 percent, the critical speed should not fall within the operating range 5 percent, that means the lower frequency would be. So, the lower frequency say frequency range will be 99 minus 1 percent of this that means 0.01, so that will give us 0.05 percent of that. That will give us 94.05 radians per second, the upper limit that is 101, 5 percent margin. So, we are giving this as 106.05 radians per second.

So, the critical speed should not fall within this range. We need to design the stiffness such that. That means the effective stiffness of the system should be, is given as, because this is the natural frequency formula. That is the from this we can able to see that the lower limit will give us 10 kg into 94.05 square, will give us the stiffness as 88.45 kilo Newton per meter.

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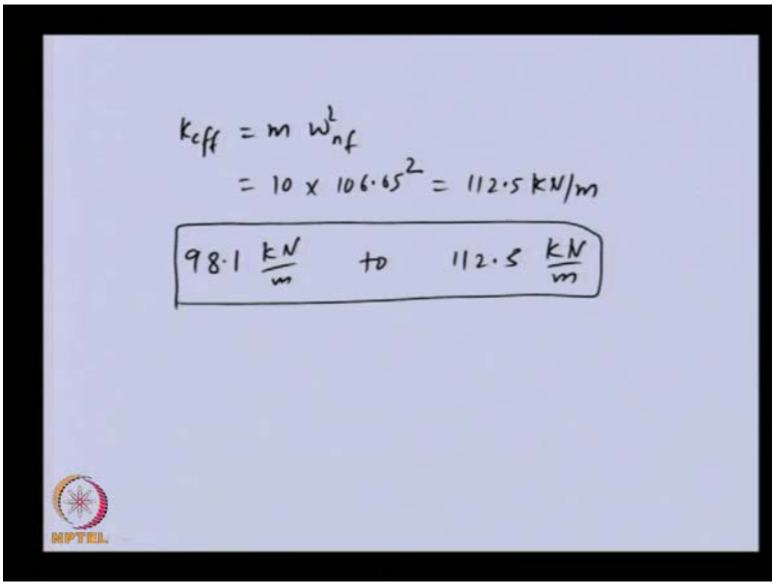

And if we see the upper range, so that is again you can able to substitute this. But in this particular case we will be having the frequency upper limit of the operating speed this much. This will give us 112.5 kilo Newton per meter. So, that means the natural frequency should not fall between range. The previous one is 91, 98.1 radians per second. Again I am repeating this.

So, that means the effective stiffness should not fall within the 98.1 kilo Newton per meter to 112.5 kilo Newton per meter to avoid the resonance. So, if you were choosing the effective stiffness in this range we will be meeting these requirements that, the our operating the critical speed is not falling within the 5 percent of the operating speed to avoid the resonance. So, in this particular lecture we have seen through very simple single degree of freedom, spring mass system and spring mass damper system.

How the critical speed can be obtained. How the force response can be obtained. And specially at the resonance condition how effect the damping in the system. We have seen that the at resonance the damping limits the amplitudes. Also it makes the phase 90 degree. In this particular phenomena, we have seen it. In the subsequent lecture will be taking more complex model to analyze such single mass rotor systems.