Theory and Practice of Rotor Dynamics
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Module - 6
Transverse Vibrations
Lecture - 30
Finite Element Method

Previous lecture we started finite element formulation for transverse vibration. We could be able to develop the elemental equation with the help of the governing equation related with the Euler-Bernoulli beam model. Now, in today's lecture we will try to take up some numerical example. So, that the method of finite element method is more clear, specially the assembly of the equations, application of the boundary conditions, how to solve the Eigen value problem to get the natural frequency and mode shape? Even the force response for a given unbalance into the system. Then, if time permits we will go for calculation of the proportional damping.

Generally, in the Euler Bernoulli beam model there is no damping terms. So, generally your Rayleigh's damping is considered in the finite element method and some reduction techniques in which we can be able to reduce the degree of freedom of the system, if it becomes large to solve the Eigen value problem. So, with this, let us see what are the things we will cover?

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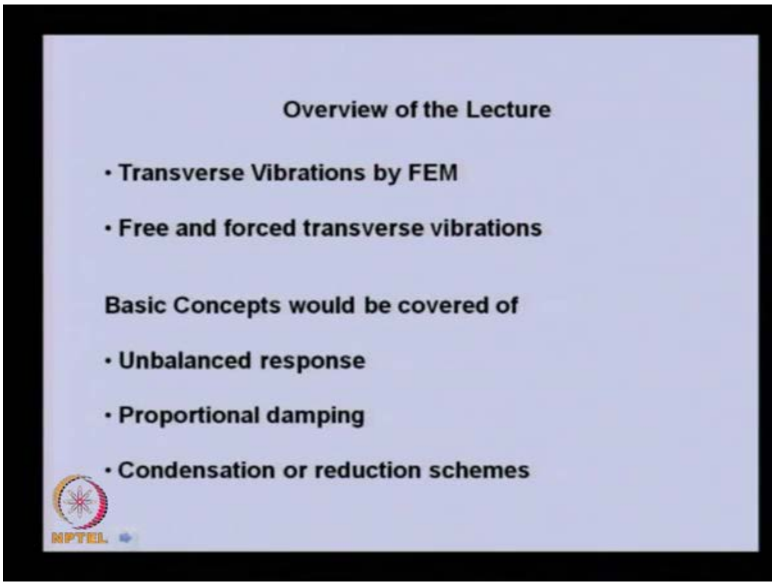

So, basically we will be focusing on the transverse vibration only by a finite element method, both free and forced vibration numerical problems. We will be taking up even the unbalance response. We will be obtaining proportional damping and some kind of reduction schemes. We will try to discuss in this present lecture. So, we will take up 1 example.

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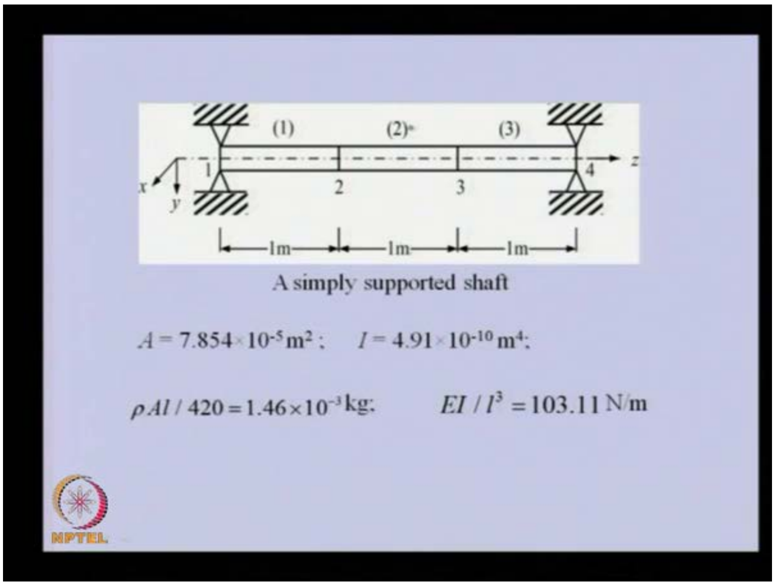

In this particular example, basically we have 1 beam or a rotor or a shaft. In this particular shaft we are considering the shaft property, both mass and stiffness, which is distributed over the length of the shaft. That means the shaft is having enough inertia property. We are considering simply supported end conditions for analyzing and for illustration purpose I have divided this shaft in 3 elements. We could have divided into more number of elements to get more accuracy, but for showing the matrices for elemental and assembled, this 3 element model will be more meaningful.

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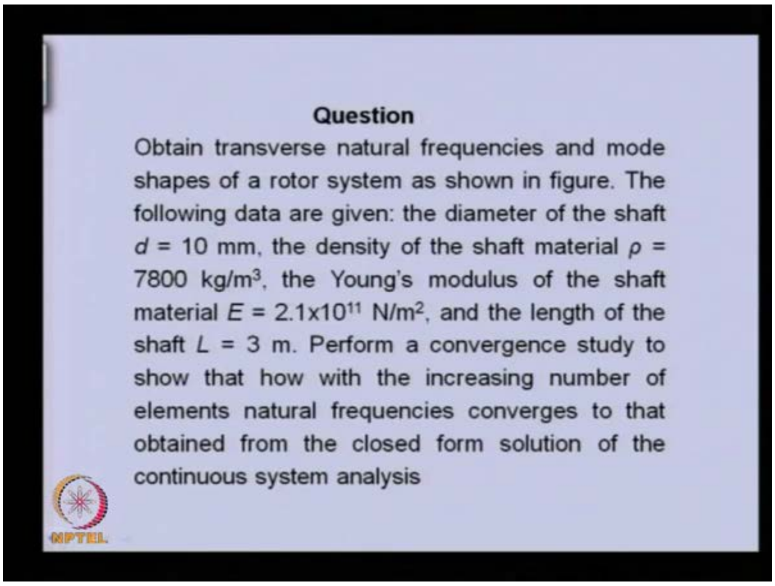

So, for this particular rotor we have all the property including the diameter of the shaft, length of the shaft and property of the material of the shaft. Basically we need to obtain the natural frequency. Once we will develop the assemble equation and apply the boundary condition as I mentioned, we can be able to take more number of elements to get better accuracy in the results. So, we will see that if we increase the number of element, how this natural frequency converges close to the actual values.

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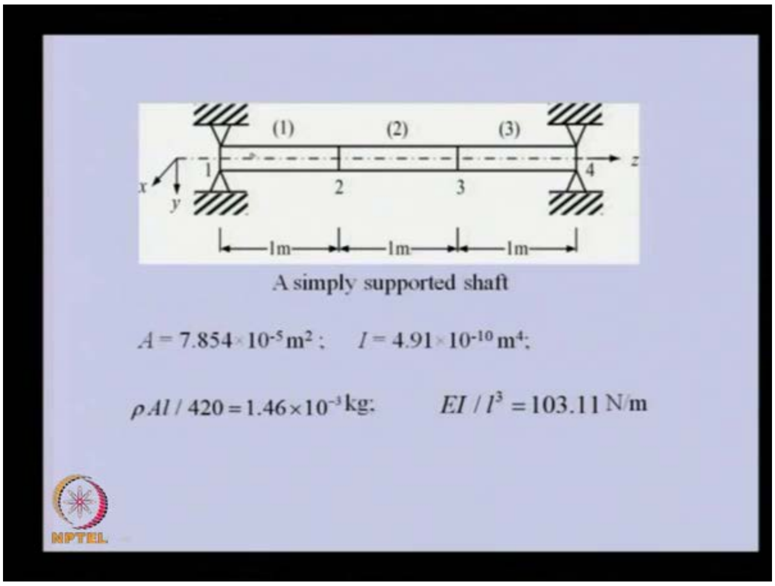

For this particular simply supported case, we have already obtained the continuous system closed form solution. So, we will try to compare the results with the closed form solution of the finite element method. So, with this given property we can be able to obtain further property, like area, second moment of area of the shaft and for the beam value we will be obtaining the mass property. So, this will be requiring for the mass matrix and this will be required for the stiffness matrix.

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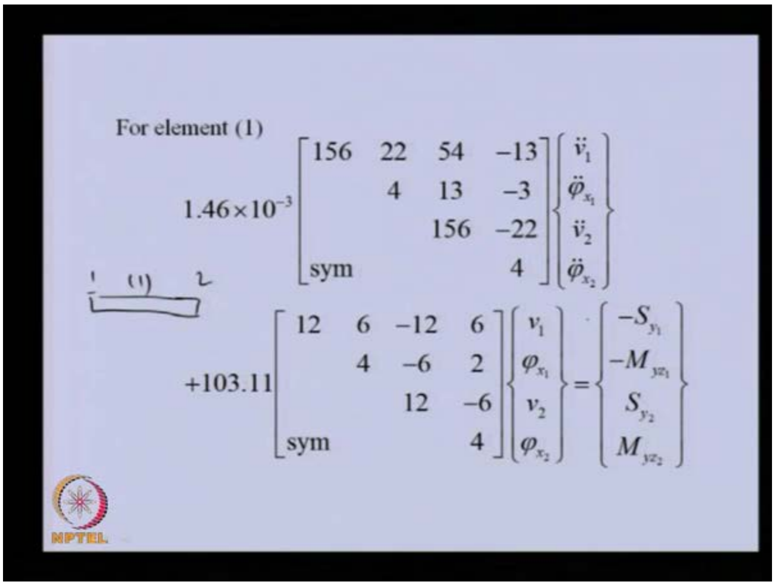

So, for element 1, this is the elemental equation. So, for element 1 we have node 1 and 2. Correspondingly, these are the nodal displacements. This is the mass matrix. This we already calculated earlier for the mass. This is stiffness matrix and this is the reaction force and moment. So, we are not considering any external force. So, that particular vector is 0 here. So, this is the elemental equation for element 1.

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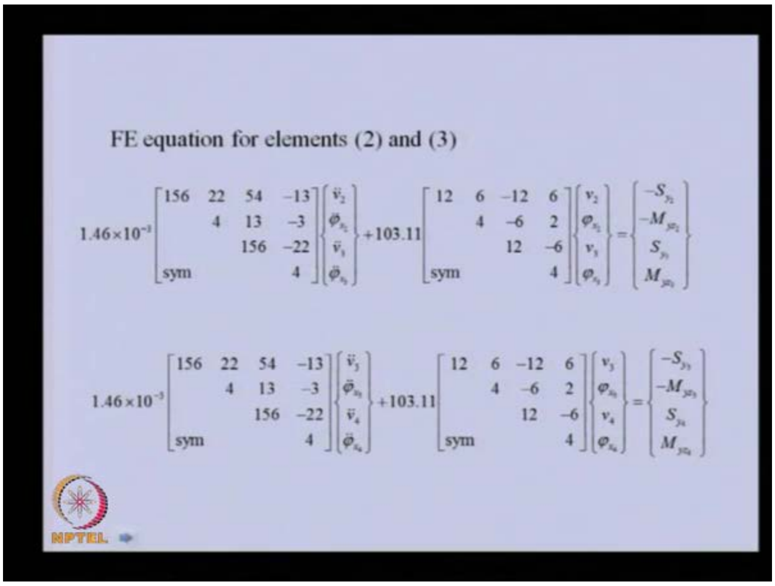

On the same line we can be able to obtain equation for element 2 and 3. In this, because the size of the element is exactly same, so mass and stiffness matrix will be exactly same as the previous one. Only, the nodal displacement vector, here 2 and 3 will be there and here 3 and 4 will be there for element 2 and 3 respectively. Similarly, here these are the reaction forces and moment of the element 2 and 3. So, once we have obtained the elemental equation for all 3 elements we can be able to assemble them.

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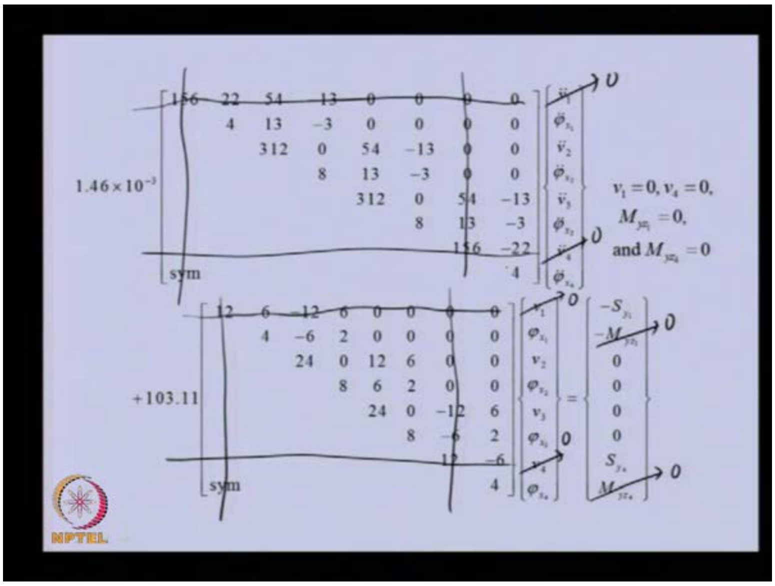

Now, assembly we have already discussed in detail in torsional vibration also and for this case also. So, I am not giving more detail of the assembly procedure, but you can be able to see that this particular vector will now contain all the degree of freedom of the element. So, we have corresponding 4 nodes because each node is having 2 degree of freedom. So, this vector is having size 8 into 1. So, mass matrix is having size 8 into 8. This is symmetric. So, only 1 portion of the diagonal terms have been shown because this is symmetric.

Similarly, this is stiffness matrix. This is assembled stiffness matrix in the reaction force, the sheer force and bending moment at the common mode, that is 2 and 3, where this reaction force and moments will cancel each other. These are at mode 1 and 4 and they will not be 0 and that will be specified by the boundary condition of the problem because node 1 is simply supported and node 4 is simply supported. So, displacement at this locations will be 0.

The bending moment also will be 0 at these nodes. So, we have this displacement 0. It is a boundary condition of the simply supported case. Also, these bending moments are 0. Now, we can be able to see that in this particular right hand side vector, the first term and the seventh term is containing the reaction of the support that is unknown. So, among this we will be eliminating, the first and the seventh row equations. So, basically you can be able to see that we will be having in this mass matrix, we will be eliminating the first column and first row. Similarly, in the stiffness matrix first column and first row, we will eliminate and seventh column and seventh row we will eliminate here. So, remaining terms will be carrying into the reduced form of the assemble equation.

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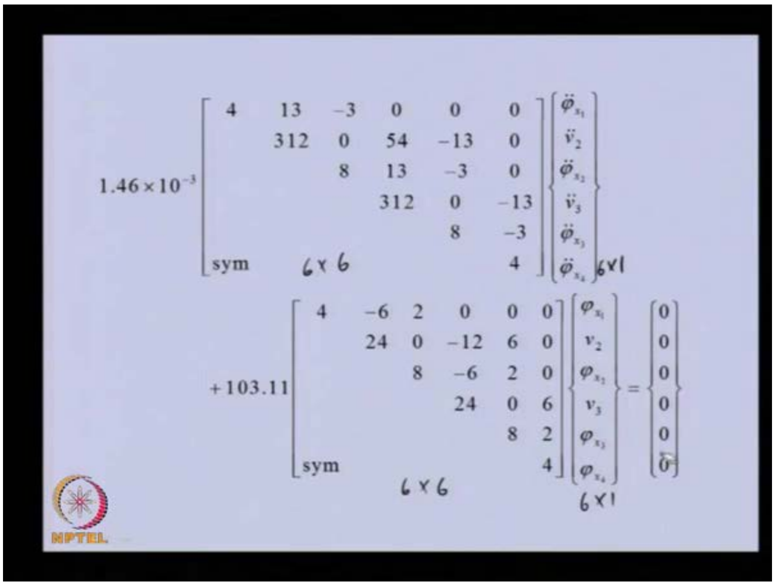

If we see the next equation, this is the reduced form and this is the form because we have eliminated already 2 rows and 2 columns. So, this will be instead of 8 into 1, now 6 into 1 will be there. The mass matrix will be 6 into 6. Similarly, stiffness matrix will be 6 into 6. This vector is also 6 into 1. Because we have eliminated the equation corresponding to the reaction force at the support, that is the sheer force. So, those term and this factor will be 0. Because deliberately we eliminated them in which we had the unknowns in the right hand side, now we can be able to see this particular equation.

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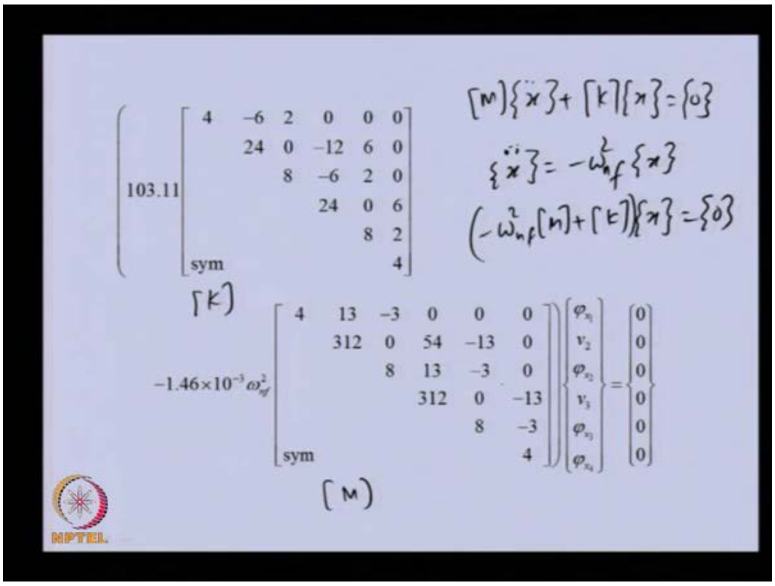

For simple harmonic motion, we can be able to write like this. Basically, we have already seen this kind of conversion. So, we have this kind of equation. For simple harmonic motion, x double dot can be written as this, where omega n f is the natural frequency of the system. So, if we substitute this here, we will get the form of the equation like this and this is the basically that particular form in which this is the mass matrix. This is minus omega squared n f. This is the k stiffness matrix. So, this is stiffness matrix. This is mass matrix, so this equation is of this form basically.

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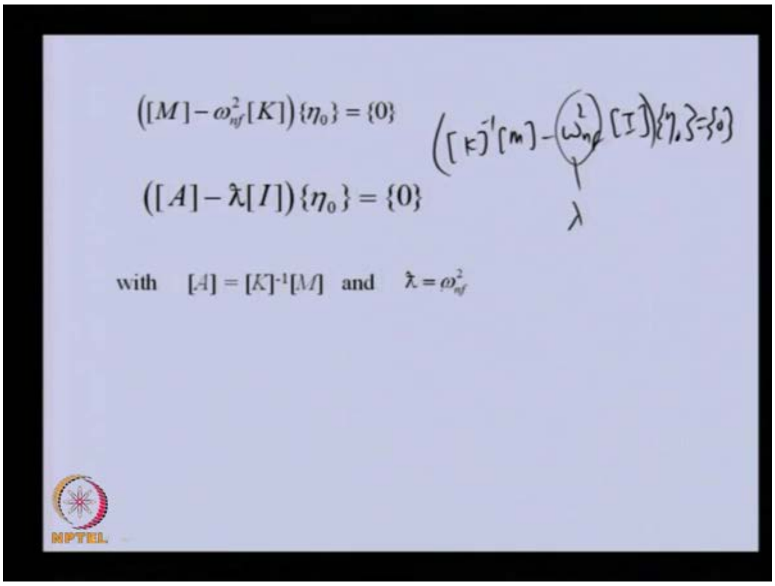

Because this is a homogeneous equation for non trivial solution of this, we need to have determinant of this equal to 0 or what we can do if we can able to convert this into Eigen value problem. So, here we had multiplied by k inverse here throughout. So, basically we got k inverse into m minus this. Also, k inverse into k inverse will give the identity matrix into mu. This is not equal to 0. So, basically this k inverse, where 2 m, I am writing as a matrix. I express this as lambda. So, this is the standard Eigen value problem. So, if we have obtained the Eigen value of a matrix, the square root of that, you give as the natural frequency and Eigen vectors will be the mode shape. So, let us see in more detail the calculation of that.

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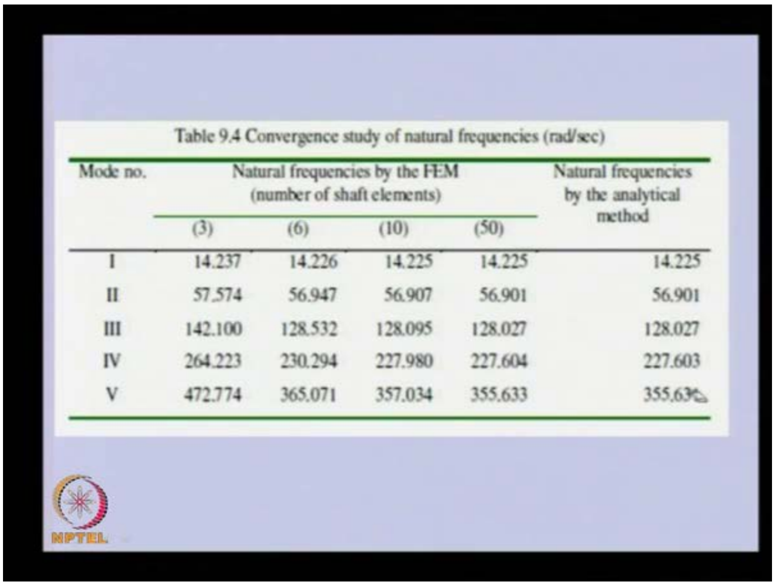

Table 9.4 Convergence study of natural frequencies (rad/sec)

Mode no.	Natural frequencies by the FEM (number of shaft elements)				Natural frequencies by the analytical method
	(3)	(6)	(10)	(50)	
I	14.237	14.226	14.225	14.225	14.225
II	57.574	56.947	56.907	56.901	56.901
III	142.100	128.532	128.095	128.027	128.027
IV	264.223	230.294	227.980	227.604	227.603
V	472.774	365.071	357.034	355.633	355.63

So, if we obtain the Eigen value, this is for the 3 element which I have shown the illustration. So, if we take the Eigen values square root, we will get the natural frequencies. So, these are the natural frequencies for first mode, second, third, fourth, and fifth. If we do the same exercise for 6 element, generally for this, maybe we need to program the assembly of the equations in a computer, so that we can be able to increase the size automatically.

Just we need to change the number of elements and the program should able to take care of the how we should assemble the equations, if the number of elements are more. How to apply the body condition, that should be incorporated in the program. So, if it is 6 element, then these natural frequency will be this. So, we can be able to see the first, second is close, but third there is a difference, fourth also there is difference and there is also lot of difference in the fifth. So, that means with 3 element whatever the natural frequency we are getting up to fifth mode, they are not accurate, but up to second mode they are reasonably, okay?

If we want to find out whether 6 element is enough or we need to have more elements. So, let us take 10 elements. If we take 10 elements, we will see that third is now reasonably okay, not much change. Fourth is also not much changed, but fifth there is a still change. So, that means 10 element is still giving a better natural frequency. The calculation for fifth mode and even for fourth, there is a slight difference. Generally, we will find that if we are taking more number of element, whatever the natural frequency we will get that will be more accurate. So, just to test whether these are accurate, take 50 elements.

So, we can be able to see now up to fourth element. I think with fourth mode up to tenth element itself, we have got that there is slight improvement in the fifth mode as compared to the tenth element. So, with 50 element we could be able to get quite accurate results up to the fifth mode, but let us compare this with the closed form solution because that is what we obtained in the previous lectures using the continuous system approach. Using separation of a variable method also, we obtain closed form of this and with that these are the values. So, you can be able to see that with 50 elements we could be able to get quite accurately the natural frequency calculation up to fifth mode.

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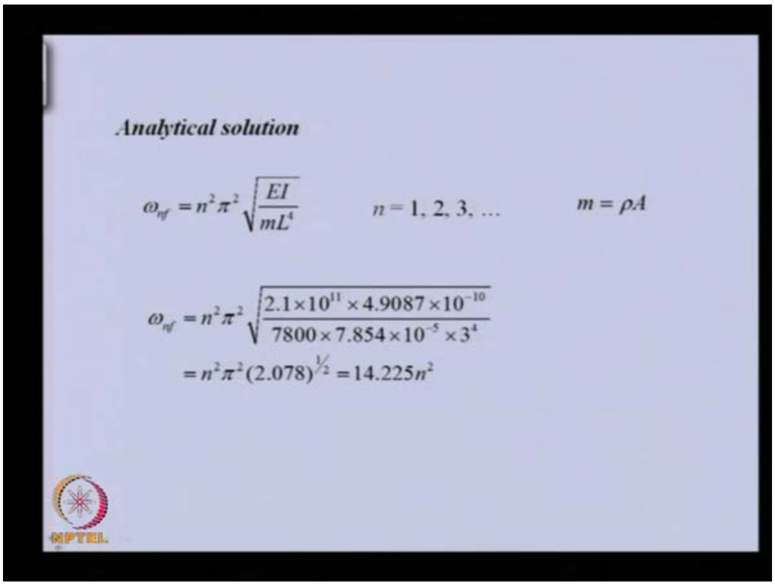

So, this was the analytical expression for the natural frequency where and 1, 2, 3 and 4, we can be able to substitute and get the natural frequencies. So, basically natural frequency is given by this. For simply supported beam case, we need to put 1, 2, 3 and 4 one by one here to get the closed form or analytical solution of the this particular problem. Now, once we obtain the natural frequency, let us see how we can get the mode shape.

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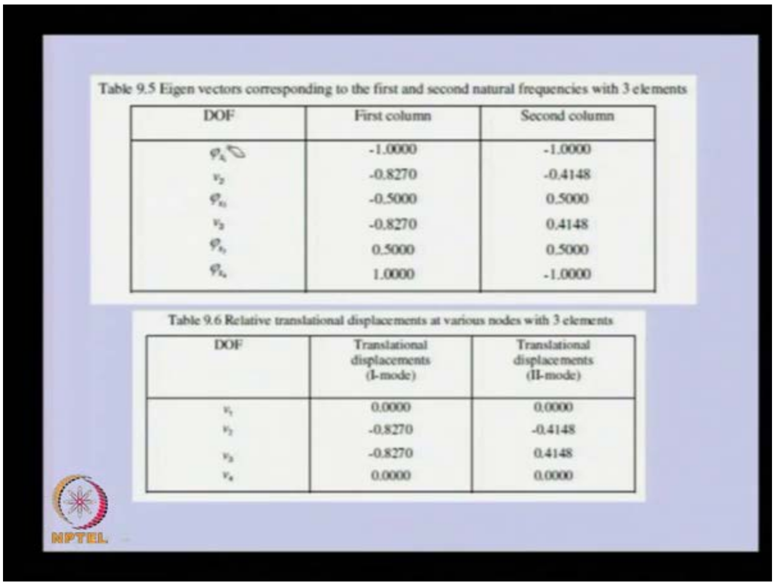

Table 9.5 Eigen vectors corresponding to the first and second natural frequencies with 3 elements

DOF	First column	Second column
φ_{x_1}	-1.0000	-1.0000
v_2	-0.8270	-0.4148
φ_{x_2}	-0.5000	0.5000
v_3	-0.8270	0.4148
φ_{x_3}	0.5000	0.5000
φ_{x_4}	1.0000	-1.0000

Table 9.6 Relative translational displacements at various nodes with 3 elements

DOF	Translational displacements (I-mode)	Translational displacements (II-mode)
v_1	0.0000	0.0000
v_2	-0.8270	-0.4148
v_3	-0.8270	0.4148
v_4	0.0000	0.0000

So, in mode shape basically because we will be getting Eigen value as well as Eigen vector, if we are using any software for this, so Eigen vector stacking will be like this. So, the first column of the Eigen vector whose Eigen vector will be for 3 element, the size of the Eigen vector matrix will be 6 in to 6. The first column of that is this particular values and the corresponding variables, if you see which we got after the application of the boundary condition in the equation of motion.

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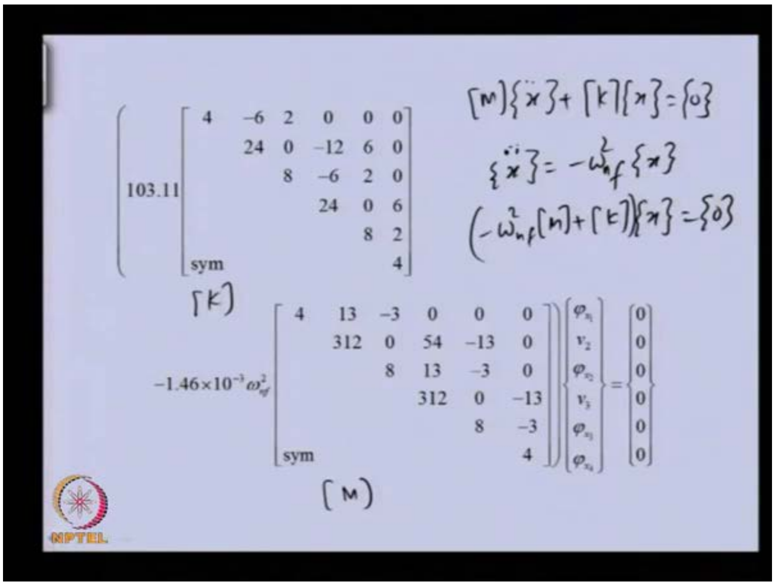

So, we apply the boundary conditions. This is the reduced form of the equation. So, this order, this vectors and this particular field variables are corresponding to the Eigen vectors which we are getting.

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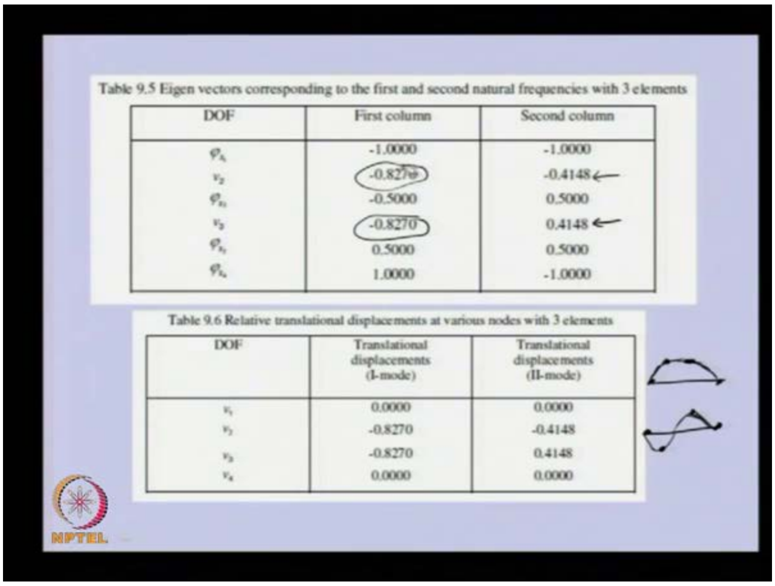

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φ_{x_4}	1.0000	-1.0000

Table 9.6 Relative translational displacements at various nodes with 3 elements

DOF	Translational displacements (I-mode)	Translational displacements (II-mode)
v_1	0.0000	0.0000
v_2	-0.8270	-0.4148
v_3	-0.8270	0.4148
v_4	0.0000	0.0000

So, you can be able to see that the first column of the Eigen vector matrix will belong to this particular field variables. Similarly, for the second column if we want to obtain the first mode shape of our translational displacement, so obviously we need to pick up the v 2 and v 3. So, correspondingly we need to pick up this term and this term for linear displacement. Others are corresponding to the slope, but if we want the mode shape corresponding to the translational displacement. Obviously, we need to pick up these 2. So, this we have picked up v 2 and v 3. Here, we already know at node 1 and 4 that because it is simply supported and condition the displacements are 0, so that we can adopt here.

So, now we can be able to see that for 3 element, only 4 nodal displacement informations are there. With this, if we plot the mode shape we may not get that much accurate, but if we want to plot this we will see that at node 1 and 4, it is 0 and for 2 and 3 is some value. So, in between this we can be able to join or we can be able to predict the displacement using shape function. So, shape function is cubic in nature. So, we will get something like this for first mode, but only these are intermediate points. We know from the finite element method in between that we need to predict using shape function. So, this is for the first mode, for second mode, the second column, we need to pick up. So, corresponding this and this we need to pick up.

So, in this we can be able to see this is negative, this is positive, or this is 0. So, if we want to plot the mode shape, this 1 and 4 are 0. This is, let us say negative at element 2, node 2 and for this is positive in between, we will be using the shape function. So, we expect the mode shape like this, but for plotting the mode shape it will be better if we take more number of elements and get more intermediate points. But, this for illustration purpose here, I have shown with 3 element how to pick up these Eigen vectors variable for translational displacement for mode 1 and mode 2.

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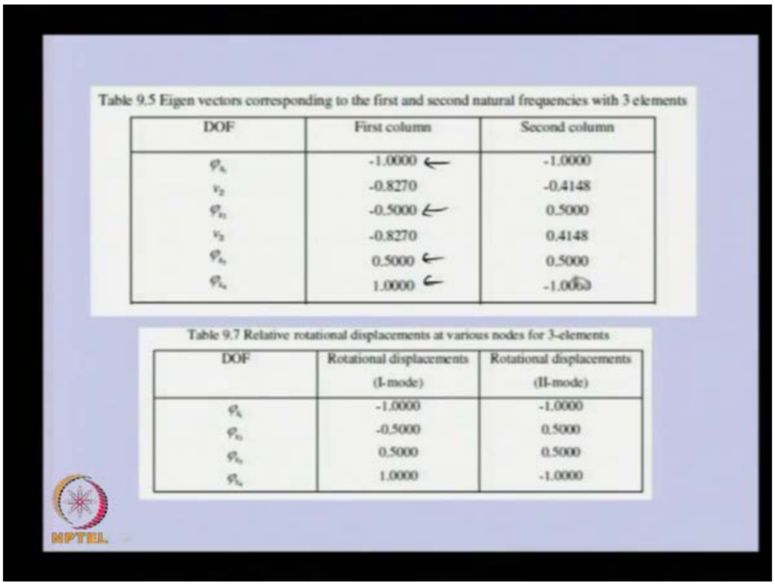

Table 9.5 Eigen vectors corresponding to the first and second natural frequencies with 3 elements

DOF	First column	Second column
φ_{z_1}	-1.0000	-1.0000
v_2	-0.8270	-0.4148
φ_{z_2}	-0.5000	0.5000
v_3	-0.8270	0.4148
φ_{z_3}	0.5000	0.5000
φ_{z_4}	1.0000	-1.0000

Table 9.7 Relative rotational displacements at various nodes for 3-elements

DOF	Rotational displacements (I-mode)	Rotational displacements (II-mode)
φ_{z_1}	-1.0000	-1.0000
φ_{z_2}	-0.5000	0.5000
φ_{z_3}	0.5000	0.5000
φ_{z_4}	1.0000	-1.0000

If we want to plot the slope, corresponding to the slope the mode shape, then we need to pick up that we can be able to see this, this, this and this corresponding to 4 nodes. We will get the slopes, these are corresponding to that they can be drawn separately. This is for the first natural frequency corresponding to the first natural frequency and this is corresponding to the second natural frequency.

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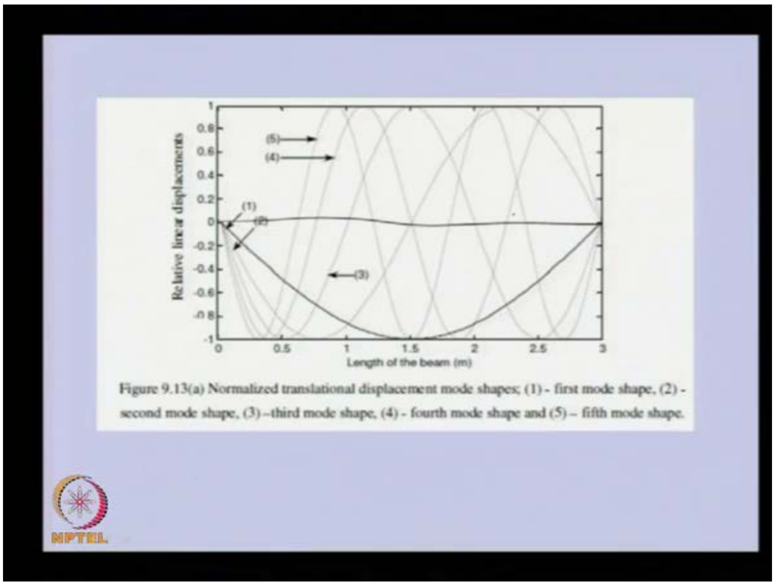

This mode shape you are plotted for more number of elements, so we can be able to see that the first mode is this one. So, this is not for 3 element. We took more number of elements and we have drawn. So, this half sign we can be able to see for first mode, then second mode is full sign. This is full sign second mode. Similarly, third mode will be this one and half sign. Similarly, four five mode shape can be identified. So, here if we take more number of element, if we use shape function we will able to get better mode shapes. Basically, we can be able to see that this is 0 line.

The modes are varying between the 0 and 1. If we draw this line, we will see that this line will intersect the mode shapes at various positions. Those are nothing, but the node positions where the displacements will remain 0. As we have already seen in the torsional vibration case also, this nodal positions are more critical. Even the displacements are 0, but stresses will be more and at these locations this is corresponding the slope variation for mode 1, 2, 3 and 4.

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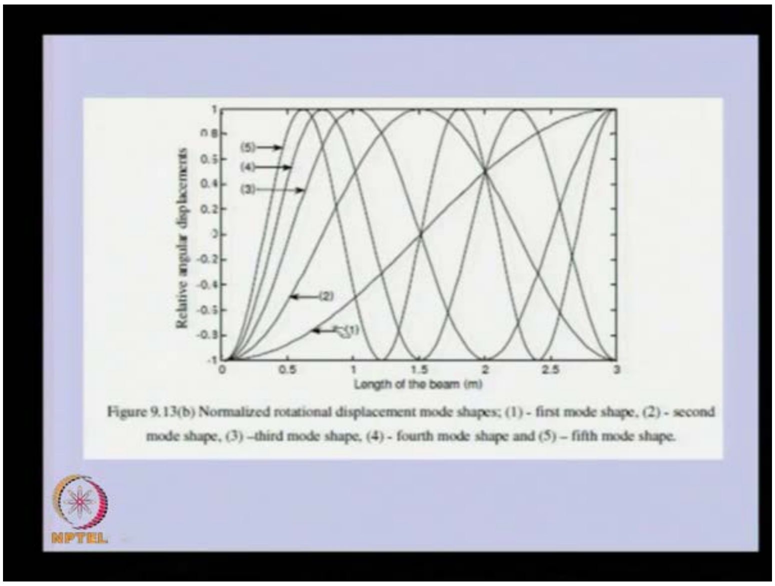

Figure 9.13(b) Normalized rotational displacement mode shapes; (1) - first mode shape, (2) - second mode shape, (3) –third mode shape, (4) - fourth mode shape and (5) – fifth mode shape.

So, we can be able to see this is first mode, second mode. This is the slope variation. As we have seen that the previous Eigen vector contains not only the linear displacement, but also the slope. So, that can be extracted and this slope variation along the length of the beam can be seen. Basically, this horizontal excesses representing the length of the beam and this the amplitude of the displacement because mode shapes are always relative.

So, this we have scaled up scale to a nice number like 1 and minus 1 and this is the length of the beam. Through a very simple example, with 3 element we illustrated that how we can be able to calculate the natural frequency and how we can be able to abstract the mode shape. Now, we will take up some more examples. Especially while dealing with the transfer matrix method, we saw that when there is an intermediate support in the rotor system, then we have difficulty in solving such system. So, let us see in finite element method, if there is intermediate support in the rotor system how we can be able to handle that particular case.

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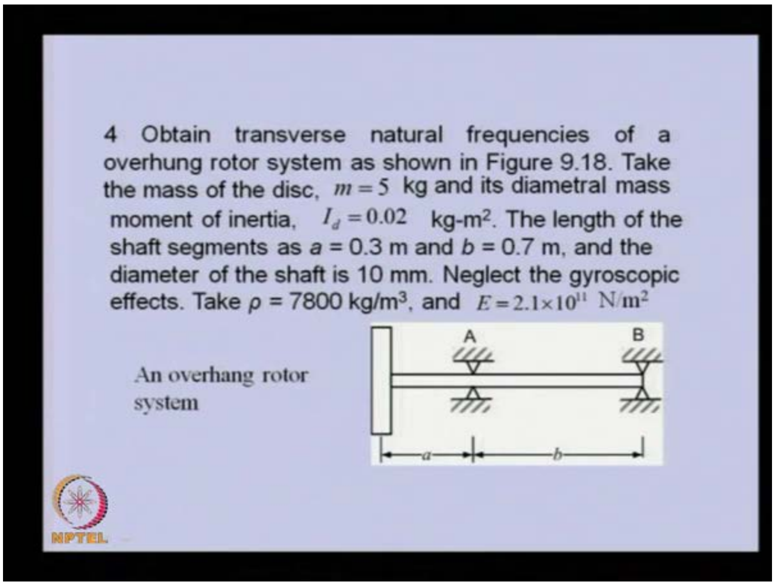

So, this particular case example is exactly same as we did in the transverse vibration case. So, this is an intermediate support and this is basically simply support and this is overhung portion of the rotor. The various properties are given here such as mass, diametric mass moment of inertia of the disk, various lengths of the shaft, diameter of the shaft and the mass and elastic property of the shaft. We are not considering the gyroscopic couple effect.

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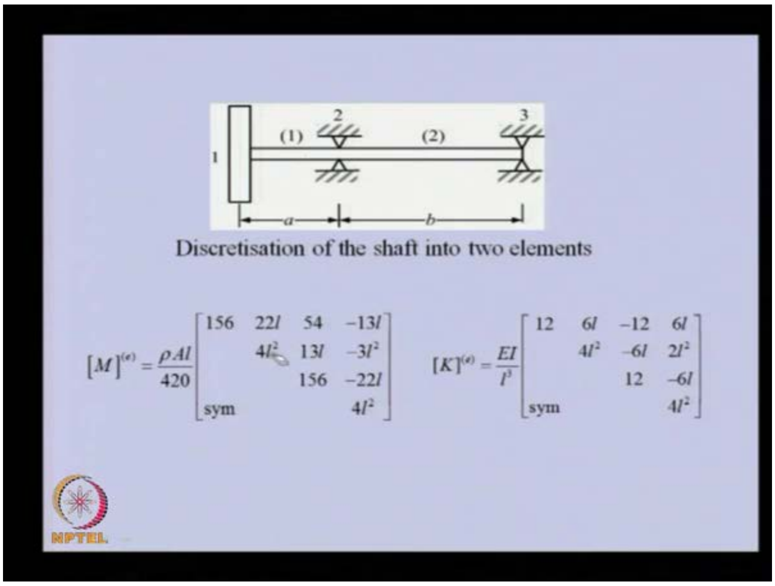

So, in this particular case again for simplicity we are dividing in to 2 elements. So, this is the first element and this is second element. Corresponding, we have 3 nodes and now we are writing elemental equation for element 1 and element 2. So, this is a standard mass matrix and stiffness matrix for Euler Bernoulli beam element or this will be using it for obtaining the elemental equation. So, basically we are considering the mass of the shaft also along with the mass of the disc.

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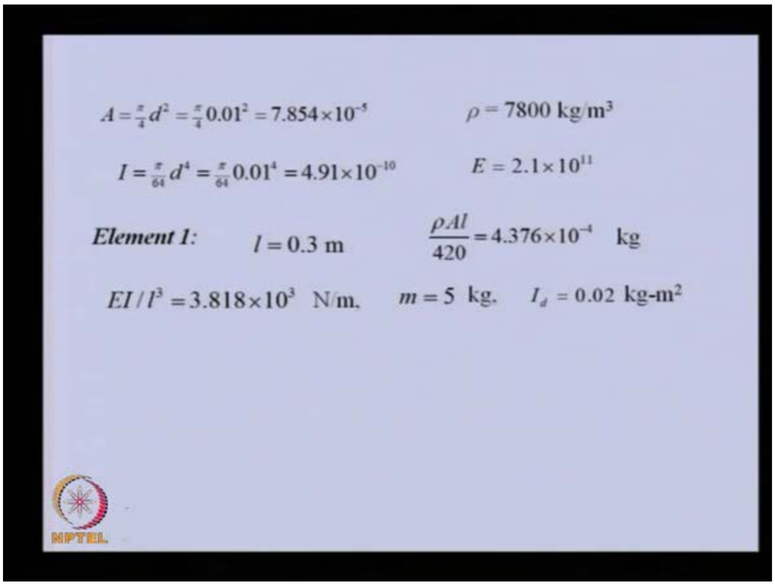

So, various property we can be able to calculate area, certain moment of area from the given data for element 1 because the dimensions are given. So, this is the elemental length and now this is the elemental equation.

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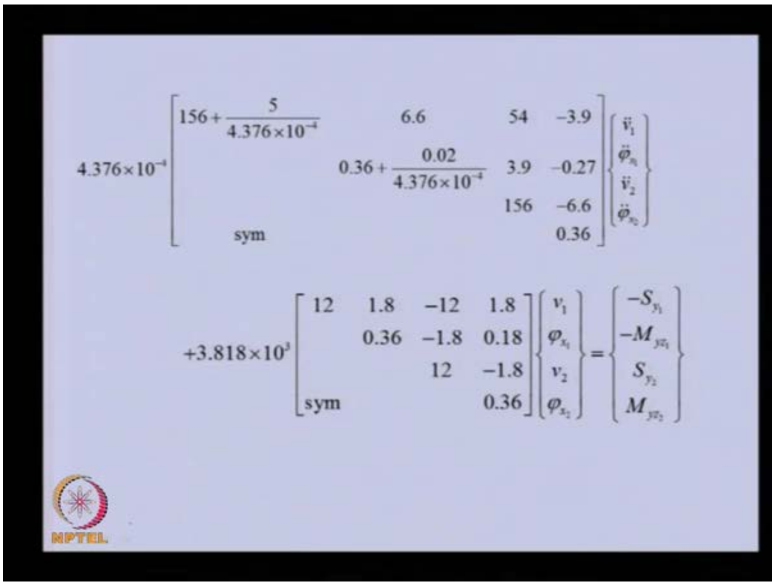

Basically, this is a mass matrix, this is stiffness matrix and this is the reaction torque. In this particular case we are not considering any external force. So, now we can be able to see that apart from the mass of the shaft, the mass of the disc, this is corresponding to the mass of the disc and this is the diametric mass moment of inertia of the disk. Because, that particular disk is at node 1, so at diagonal terms they are appearing rigid diametric mass mode of inertia of the disk. So, at node 1 they are appearing, but not at node 2. So, this is for element 1.

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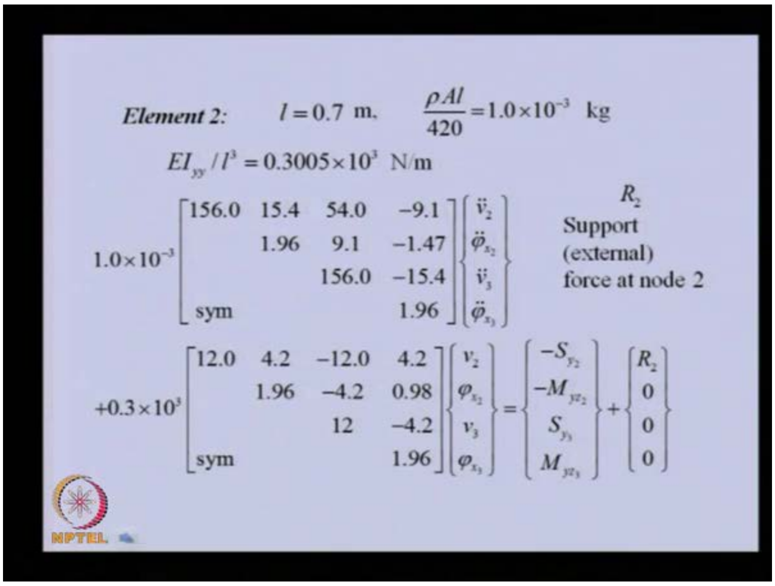

Similarly, for element 2 because element length is different, so it will be having property like this. In element 2 there is no concentrated mass. So, mass matrix, stiffness matrix will be like this. In this particular case reaction, this is the reaction torque. Reaction forces arise from this because once we take out the shaft from the support there will be reaction from the shaft or reaction from the bearing. This reaction from the bearing and this particular shear force, they are different. This is coming because we have cut the shaft in 2 piece; that is why it is coming.

If we join 2 shafts this will cancel because this is at the common node, but this will remain because this is coming from the bearing. So, this is important to see how to handle the intermediate support that we need to add the reaction force from the support which is unknown as external force because this is not giving moment. So, it will not come here, it will come in the node 2. In this position only now we obtain the elemental equation from 1 and 2. Only 2 elements, we have considered. Now, we can be able to assemble them.

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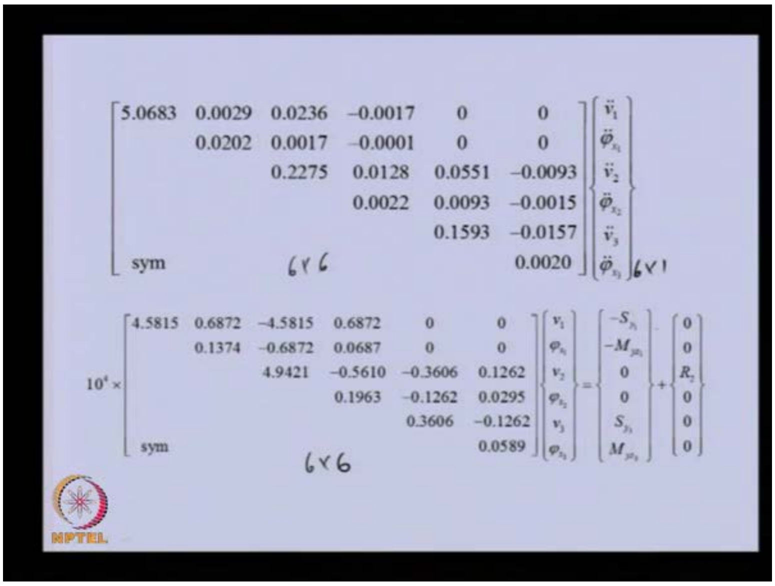

So, once we assemble them this is the mass matrix we can be able to see them. The total degree of freedom of the system is 6. So, field variables are via 6 size of this factor. It is 6 in to 1. The mass matrix size is 6 in to 6 and stiffness matrix 6 in to 6. So, you can be able to see intermediate reaction force. The moment will cancel each other, but this bearing reaction will remain there, now you can be able to see. Now, this is ready for application of the other boundary conditions, that is at node 1 and 3.

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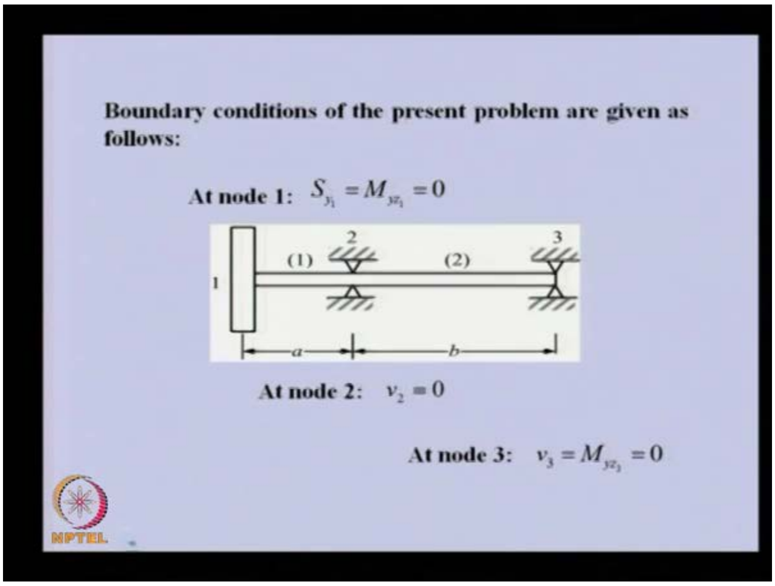

So, let us see what are the boundary conditions at node 1. So, the shafts node 1 is this one. Here, we know that shear force and bending moments are 0. So, this is at node 1. At node 2, we have displacement 0 because, this is a support and at node 3 here not only the displacement, but moment is also 0. In this particular support, the moment is not 0 as said. This is not a simply supported condition. That is why the moment is not 0, but so this is very important to observe the difference between these 2 support conditions.

This is because in this the right hand side there is no further extension of the shaft. So, the moment on this bending moment is 0 and because of that this we are getting it, but here this shaft is continuous. So, as such there will not be 0 bending moment at this location. Now, let us apply this boundary conditions into the previous problem.

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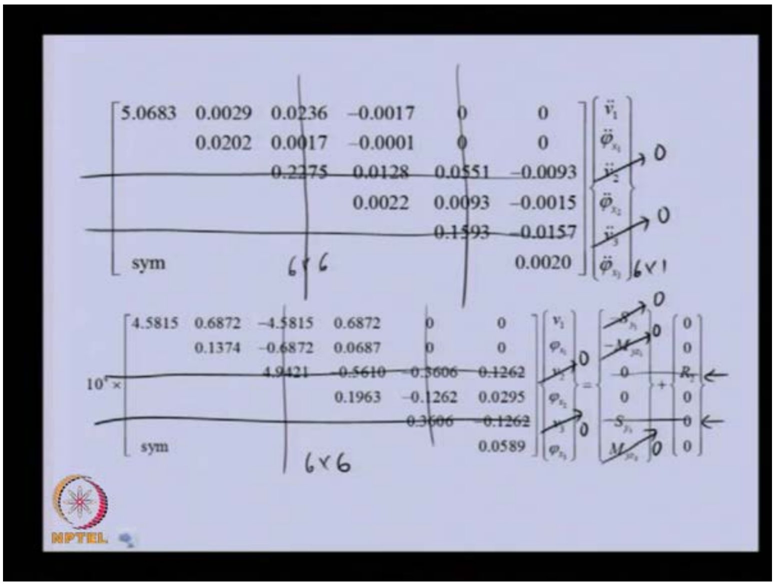

So, we have bending moment and shear force. Bending moment at node 1 is 0. At node 3 bending moment is 0, v 2 is 0 at support and v 3 is zero at second support. So, that means once we apply this boundary conditions. Now, we need to look into the right hand side, so in right hand side those terms in which we have zeroes only we will be picking those equations. For first equation, second equation, third is having one unknown. So, we will eliminate this particular equation. From the reduced form of the equation, this will be picking up there is unknown here.

So, basically we will be removing the third equation and the fifth equation from this and the remaining equations will be keeping with us. So, basically if you see the third row and fifth row will be eliminating from both mass and stiffness matrix, third row and third column and four fourth column and then corresponding rows also. This row and this row we will eliminate. This third row and fifth row we will eliminate. So, remaining terms we will see that will get eliminated. So, this is the reduced form of that.

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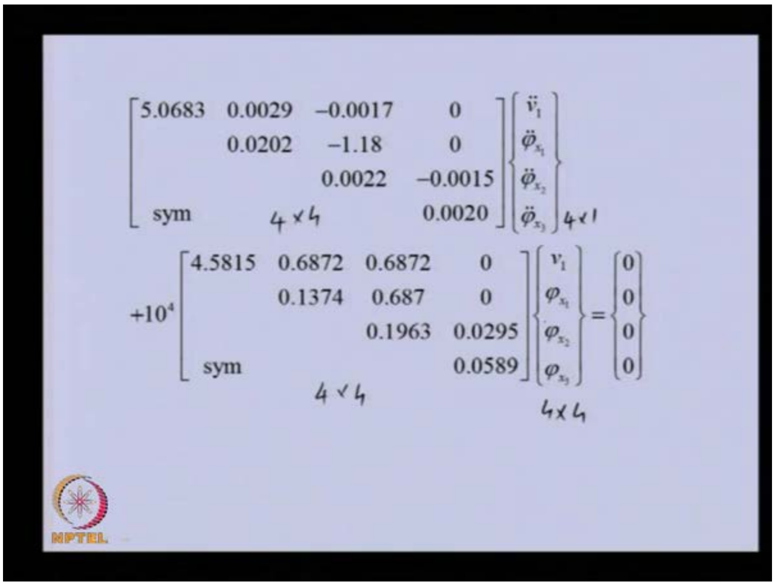

So, reduced form will be this factor, will be 4 into 1 because we have eliminated 2 variables here. Here, 4 into 4 size will be there. This is also 4 by 4. Now, this is ready for Eigen value problem solution.

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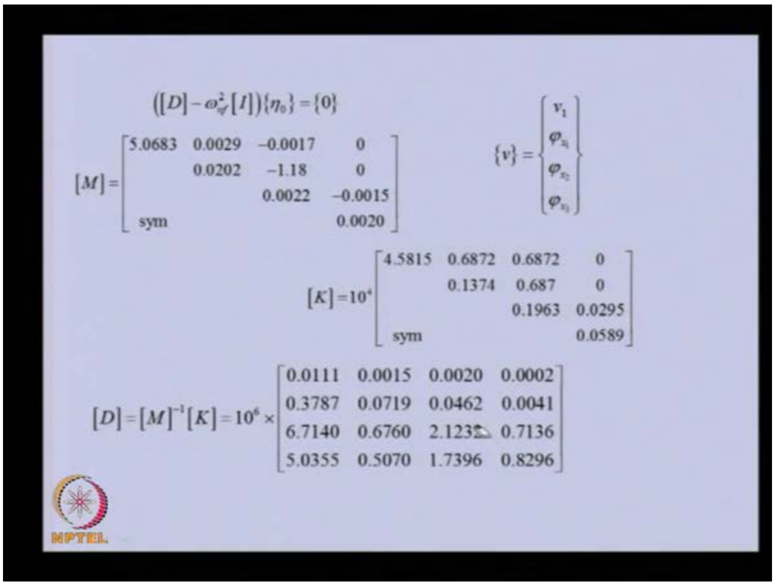

Now, basically again we can be able to take up the inverse of K, so that we will get the D matrix. So, D is defined as inverse of M into K. So, this particular is you can be able to obtain the Eigen value of D matrix, which is M inverse into K. So, this will give us the

Eigen value, Eigen vector square and the root of the Eigen value will give us the natural frequency.

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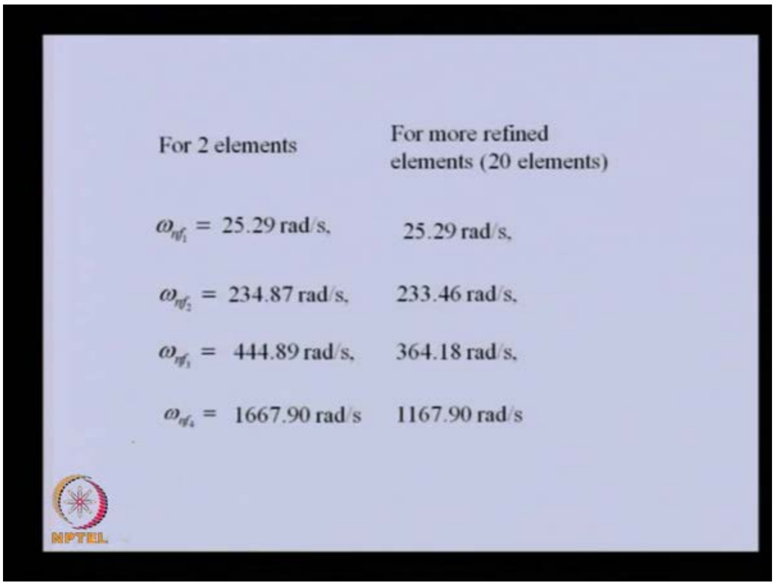

So, in this particular case because we have illustrated with the 2 element, so with that if we calculate the natural frequency we will get this 4 natural frequencies. But, if we take more number of elements, let us say 20 elements, by dividing this into 20 elements ee will get the refined value of this. But, you can be able to see that the first natural frequency and second, with even 2 elements we are getting reasonably well, but third onwards we are not getting that much accurate. So, if you are interested in first few, then even 2 elements are good enough.

From this we can be able to see, but it is always better to check how the natural frequencies are changing by increasing the number of elements. If there is not much change up to the mode we want, based on that we can be able to decide how many number of elements we should choose. So, this is called convergence study. In this we gracefully increase the number of element and see the change in the natural frequencies. If there is not much change in the natural frequency up to the mode we require, we stop the number of element at that position because if we increase the number of elements obviously computational time will be more and more. So, we need to have some kind of trade off between the number of elements and the computational time and the accuracy.

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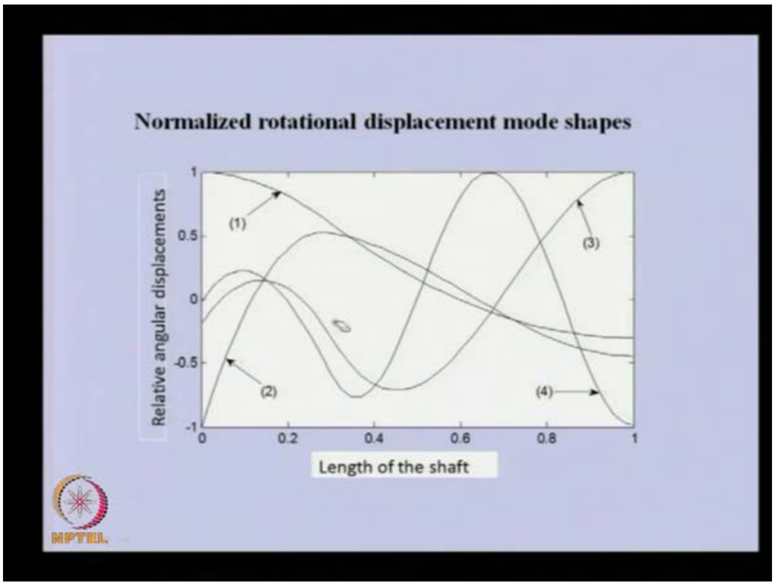

This is the mode shapes, the relative linear displacements as we discussed in the previous case on the same lines. We can be able to extend the mode, this mode shapes. So, you can be able to see the first mode basically, this is the intermediate support. This is the other support because this is the total length of the shaft. So, first mode is like this, disc position is here. Second mode will be like, this third mode is this one. Because this is support, so all the displacements are 0 here. So, you can be able to see that we can be able to get the mode shapes, the fourth mode shape. Then, we have interpolated in between using the shape function. That is why, in these curves the mode shapes are very smooth also.

We normalize from minus 1 to 1, so that we can be able to comply these mode shapes easily. This is corresponding slope variation along the length of the shaft. You can be able to see at the support that the slope will not be 0. That is why, here there is a variation of the slope. So through very simple example of 1 intermediate support, I illustrated how the intermediate support can be handled easily. So, as compared to the transfer matrix method now we can be able to appreciate that the finite element method is quite convenient even for difficult boundary conditions. Now, we will take up another very simple example in which we will be obtaining the unbalance response.

So, how to obtain the unbalance response? We will be taking up very simple example as we basically do. This example will be exactly same as we have done for the simply supported beam case, the only thing is now we will be adding the unbalance in the system.

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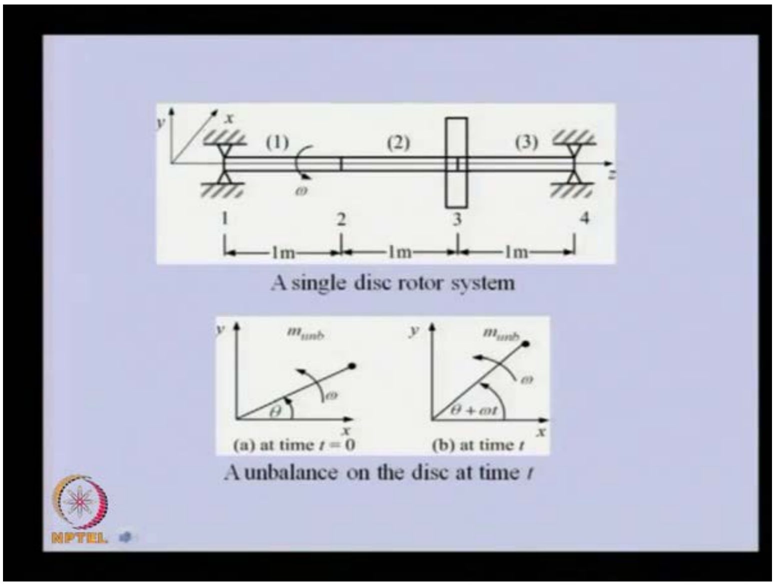

So, in this particular case basically this is the similar simply supported beam. The only thing is that we are not keeping a disc here. For the illustration, again I have divided this into 3 elements and at node 3, I am keeping 1 disc and in this disc we will be keeping some unbalance.

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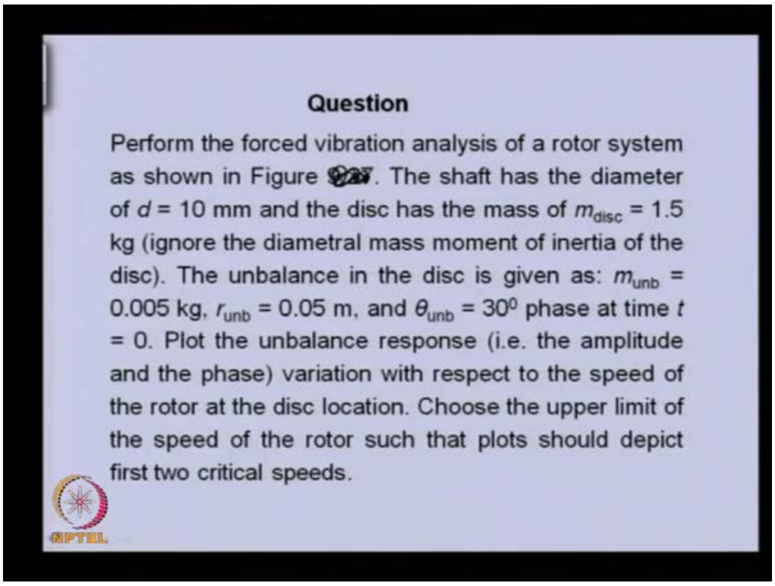

So, let us see various properties, especially of the unbalance. So, unbalance is on 0.005 kg at a radius at of 0.05 meter and the location of that unbalance with some reference position from the shaft, is let us say 30 degree phase. This is with respect to some physical reference mark from the shaft. Now, we need to plot the unbalance. That means the response with a speed. That means how this amplitude of the response at 1 particular location, let us say at the disc location, how it changes with speed? We expect that whenever there will be meeting of the speed with the natural frequency, there will be resonance there.

We expect very high amplitude of vibration. Even we expect some kind of change in the phase information. So, not only we will be plotting the amplitude, but also we will be plotting the phase. Basically, we will be obtaining all the critical speed of the system by changing the speed from low value from 0 to base value. Gradually we will be increasing it, so that we can be able to get 4 or 5 critical real. First two critical speeds we will be obtaining. Because, this is a continuous system we can have infinite number of critical speed, but we will be illustrating here only 2 critical speed.

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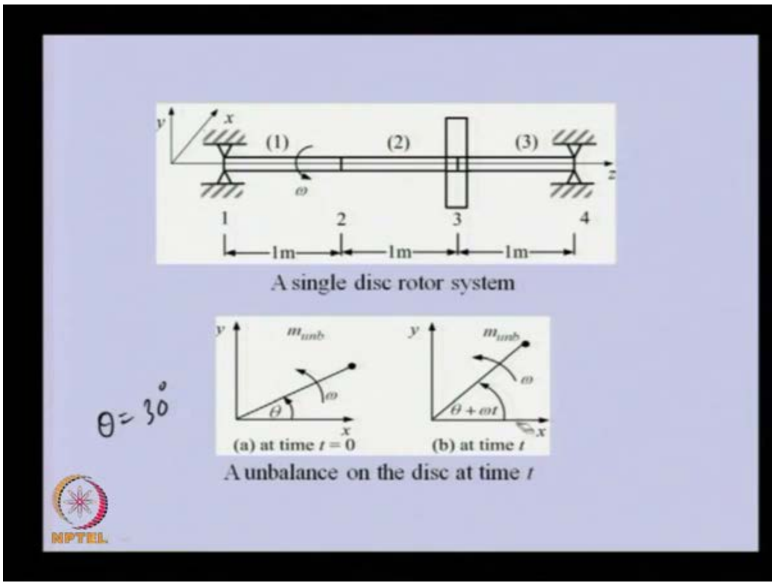

So, in this particular case because we are considering the unbalance which will be acting in basically the two planes, so we are considering the motion in 2 planes x and y. As such there is no coupling in the 2 planes because bearings are rigid, but still for illustration purpose we are combining the 2 plane motion now. For unbalance we are considering, let us say horizontal axis x as a reference and the direction of the rotation is positive is counter clockwise direction.

At time t is equal to 0, we are assuming that this unbalance is in the horizontal position, and at time t it occupies some hard rigid position were this angle is theta for some time t. So, this is phase. So, basically this is the phase which we are talking about. The theta for this particular case is 30 degree. So, that means at t time is equal to 0 the unbalance position will be 30 degree, but with time t then it will increase from that position. So, the initial phase is 30 degree. So, this initial position of the unbalance is here and with time then it increases in the counter clockwise direction as positive rotation.

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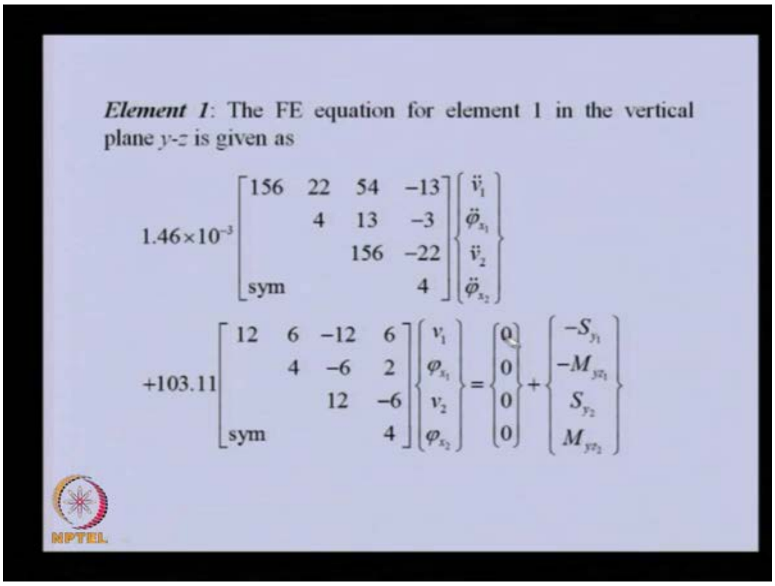

For element 1, because we have divided this into 3 elements, so this is the elemental equation for first element. Because, there is no disc, so there is no change as compared with the free vibration. It is exactly same. There is no external force in this particular element because the unbalance is acting only at node 3.

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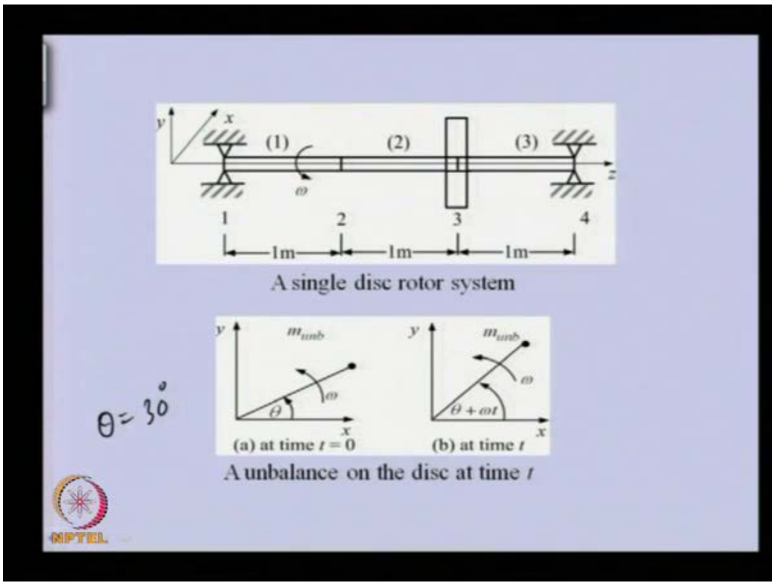

So, either we can consider that unbalance node element 2 or element 3 in one of them. So, let us see this is for element 1.

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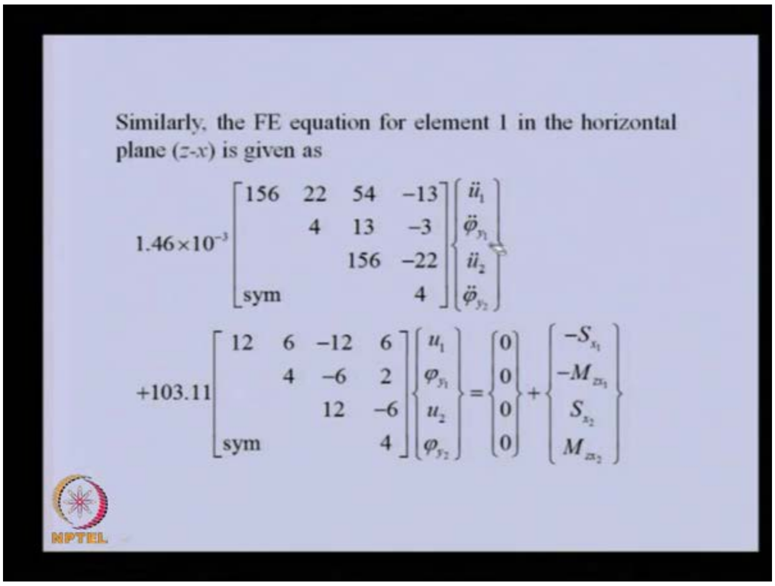

There is no external force here. Element 2 also we have not considered, so this is also again element 1. But, in y z plane this is in vertical plane. Element 1 is in the horizontal plane z x plane. The mass and stiffness matrices are same. Only, now you can be able to see the change in the displacements. Now, these are corresponding to displacements in the horizontal plane and they can be combined.

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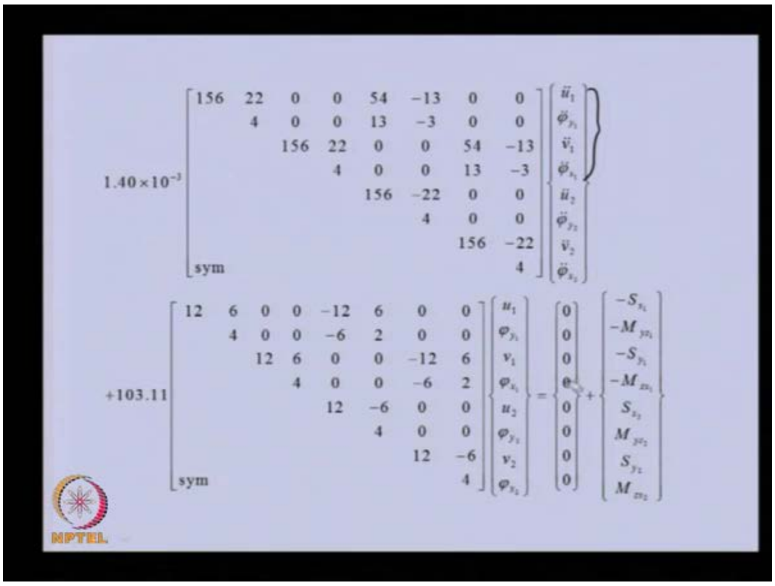

So, this is the elemental whirl equation when we are considering both plane motion. So, you can be able to see here first 4 or corresponding to node 1, but horizontal plane and vertical plane. Similarly, this is for a node 2 horizontal plane, vertical plane, mass matrix and stiffness matrix. There is no external force. These are the reaction forces and moments. Similarly, we can be able to write for element 2, in this element 2 we have considered the disc. You can able to see the disc mass is also appearing.

The diametric mass moment of inertia of disc, we have not considered. The stiffness matrix, this is the reaction force and torque, but now you can be able to see this is the unbalance force. This is the unbalance force corresponding to node 3, because disc is at node 3 of the element 2,we have both plane motion. So, this is the horizontal force component and this is the vertical force component of the unbalance force.

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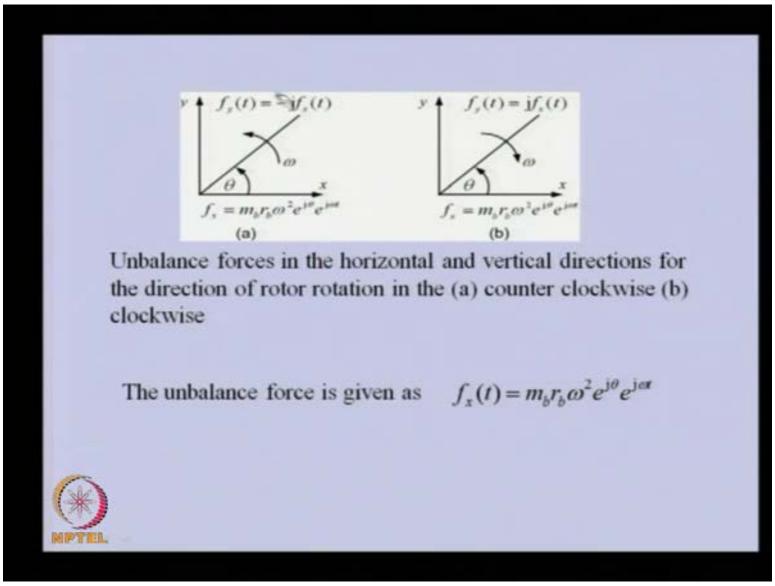

Let us see, how this we have introduced? So, basically unbalance force is m b that is mass of unbalance into radius omega square, the phase of that m is e j and omega t is the frequency. Because, this is rotating with speed of the shaft, now this is f x. This particular is f x. Now f y, this is because f y is lagging behind by f x in the direction of the rotation of the shaft by 90 degree. Because, you can be able to see the direction of rotation is clock wise, so this particular y axis is lagging behind the x axis by 90 degree. So, f x and f y, because they are component of the same force and their phase difference is of 90 degree, so they will be related by minus j, because this represents the minus 90degree phase .

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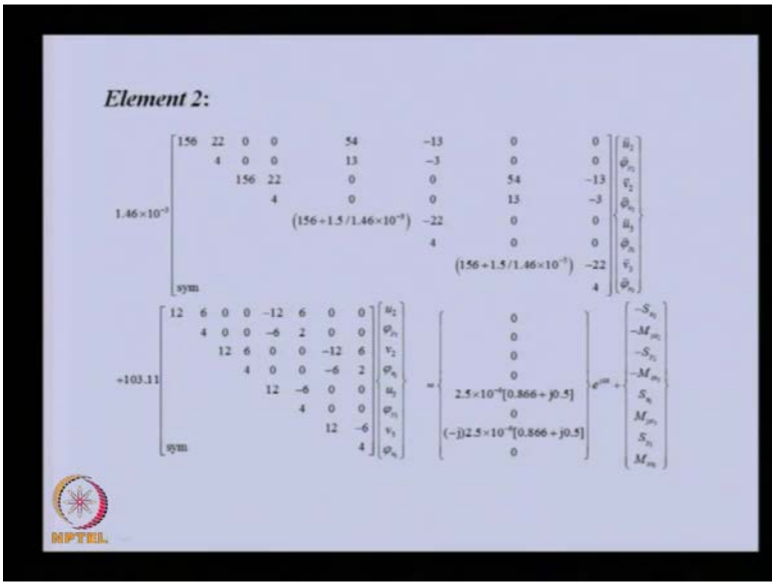

So, now you can be able to see this is the horizontal force and if you multiply by minus j that will be the component of the force in the y direction. This is the phase which we are talking about at the initial, the 30 degree phase which we have considered.

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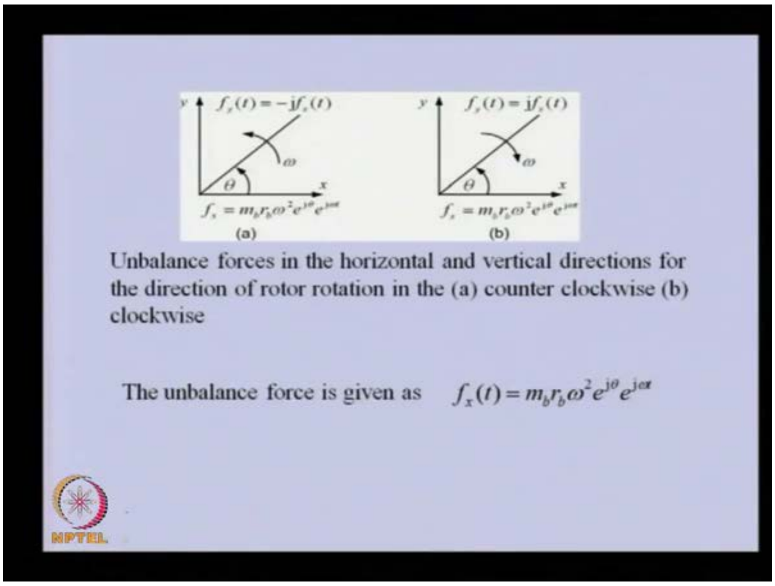

So, this is the case when the shaft rotation is in other direction. So, in this particular case y axis will be leading x axis by 90 degree. So, the force component in the y direction and x direction will be related by this. So, instead of minus j, now only j will be there.

So, we have considered this particular case element 3 is identical to element 1 in which there is no disc.

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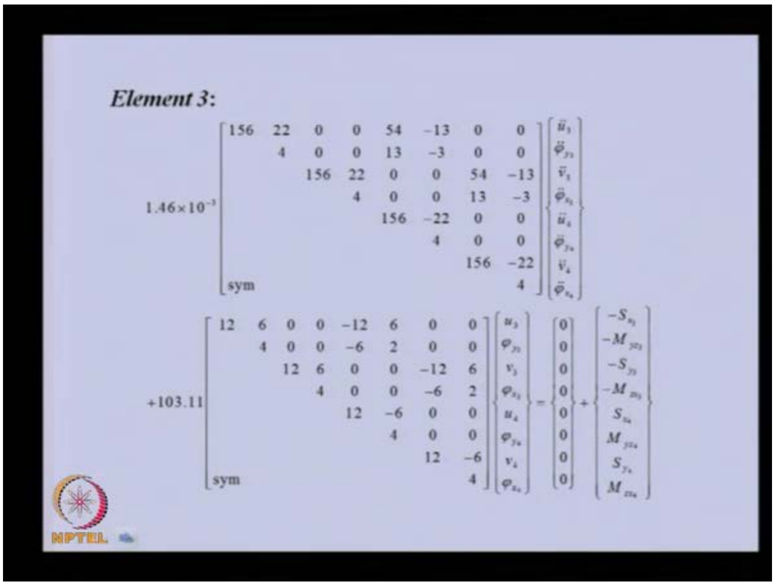

The only change is in this field variable. So, here corresponding to node 3 and 4, the field variables in the horizontal and vertical direction will be there. There is no external force here because. We have considered the disc in the element 2. Now, these 3 elements can also be added.

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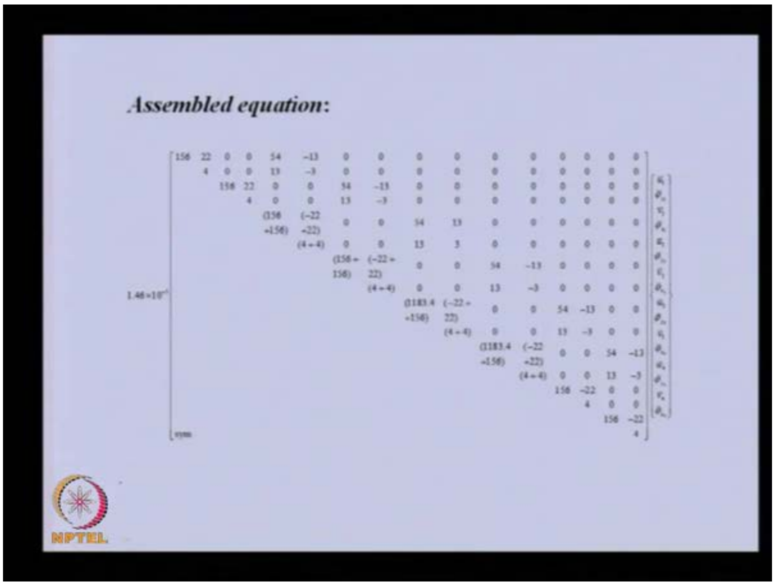

So, if we add this, just as I am illustrating, you may not be able to see the various numbers. But, this is the mass matrix for the whole 3 elements. So, this will give just indication that how the size of the matrix will be if we assemble it. This is the mass matrix.

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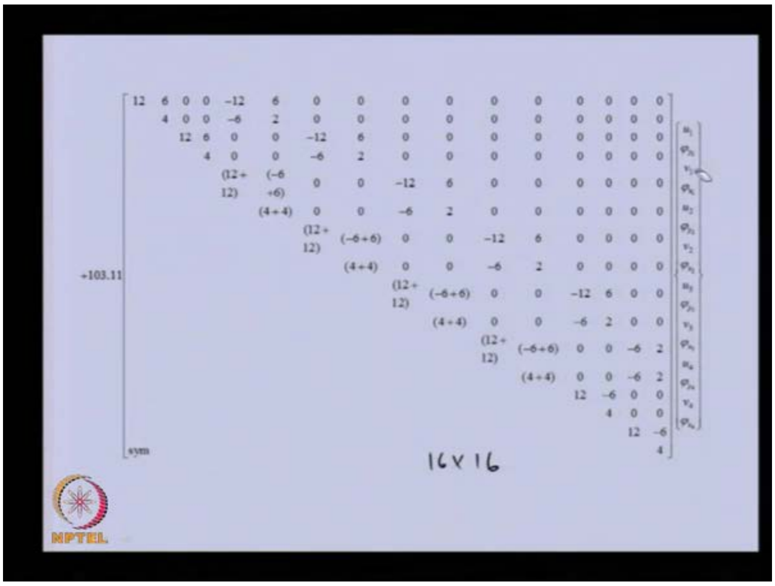

This is the stiffness matrix. So, again the size are very small to look into the values. But, the over all procedure I am trying to explain that now. The size of this will be because we have now 4 elements and each four nodes are there and each node is having 4 degrees of freedom. So, we have now 16 into 16 size of this and this vector is having 16 to 1 size.

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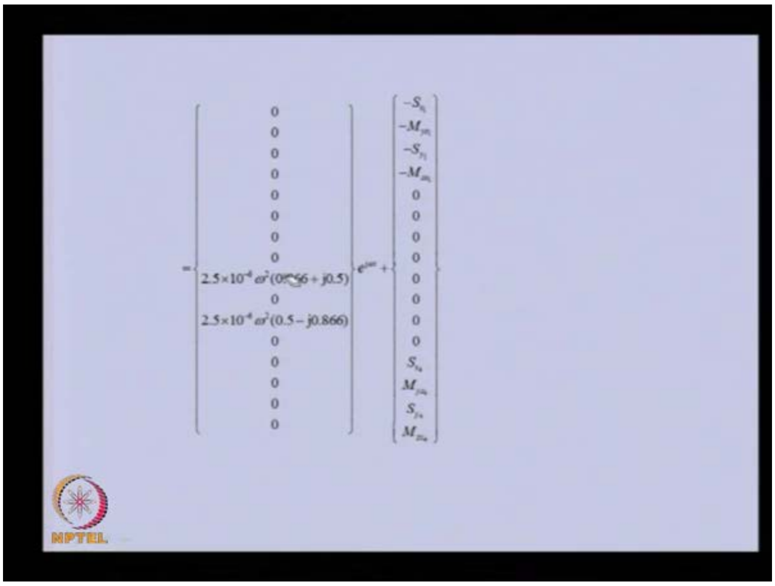

Similarly, reaction torque and reaction force will be having this only term. We will get reaction force and moments will cancel unbalance force which is there at this node x component. Horizontal component and vertical component, there is no external force in any other nodes. Only at two locations corresponding to the horizontal and verticals are there.

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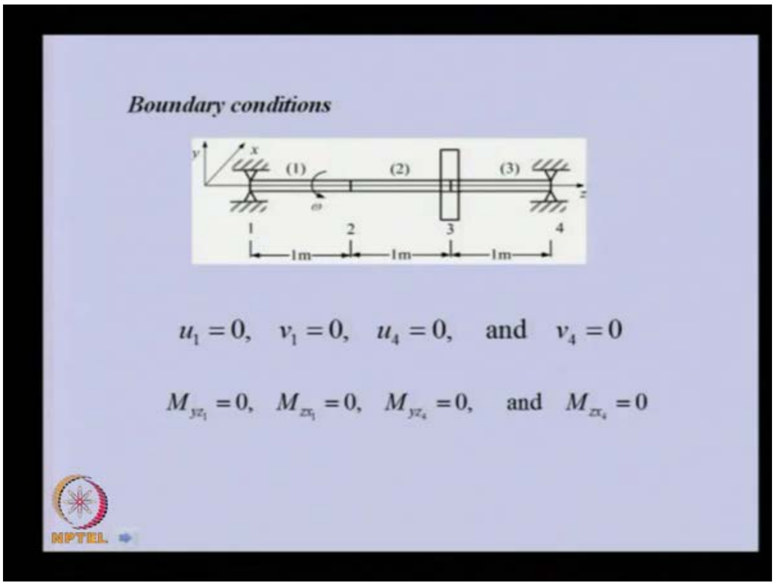

Now, let us see the boundary condition of the problem. So, here simply supported boundary conditions are there. So, displacement in horizontal and vertical direction will be 0. Similarly, here at node four displacement in horizontal and vertical direction will be 0. The corresponding moments are also 0 at these locations.

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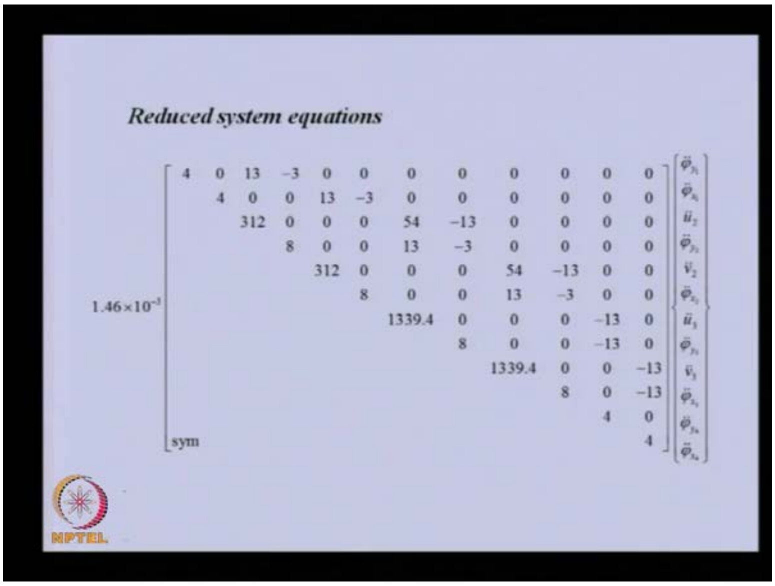

So, that means this boundary conditions we can be able to apply in the previous equation and this is the reduced form of the equation. So, basically in this particular case, now we have equivalent to 1 size and 12 into 2 size. So, this is a reduced form of the stiffness matrix and reaction unknowns.

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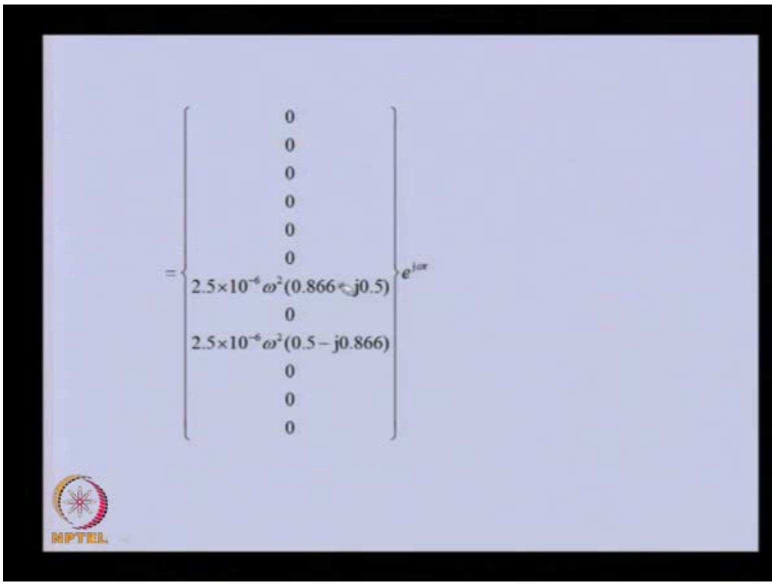

We have already the need to eliminate. Only the terms containing the unbalance force will be there in the right hand side, but other reaction force and moments if they are appearing like at node 1 and 4, then shear force will be 0. So, corresponding rows and columns we have eliminated. So, only known unbalance force are here.

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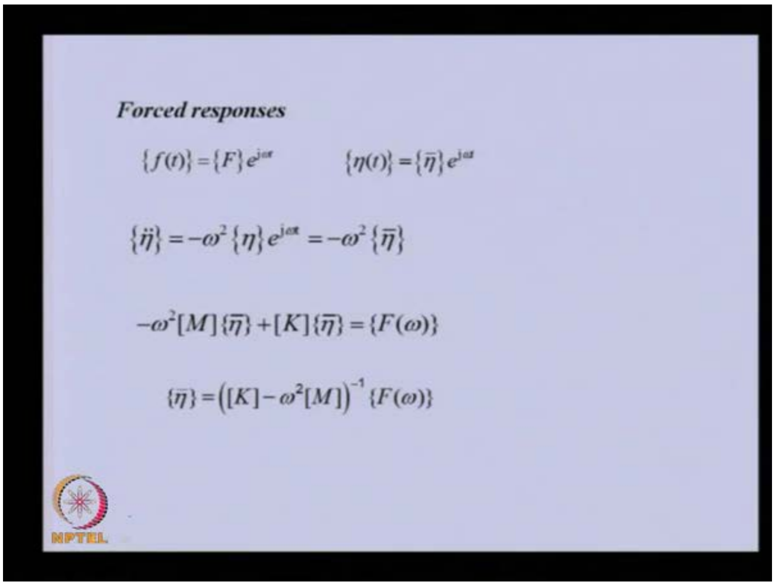

So, basically this particular force we have of this form in which f is a complex quantity. Here, f is the frequency. Similarly, this is the response in which this is a complex response and this is a frequency component. If we take double derivative of this, we can be able to relate the response like this. Now, basically we can be able to see this 1 and this is the displacement. Now, if you substitute this in the equation of motion, the equation of motion will be of this form.

All the e j omega t terms will be common. That will go out. So, basically this equation is now in the frequency domain and because now this is common, so we can able to take this common and whatever the terms within that we can take inverse of that multiplied by the force. This will be the unbalance response.

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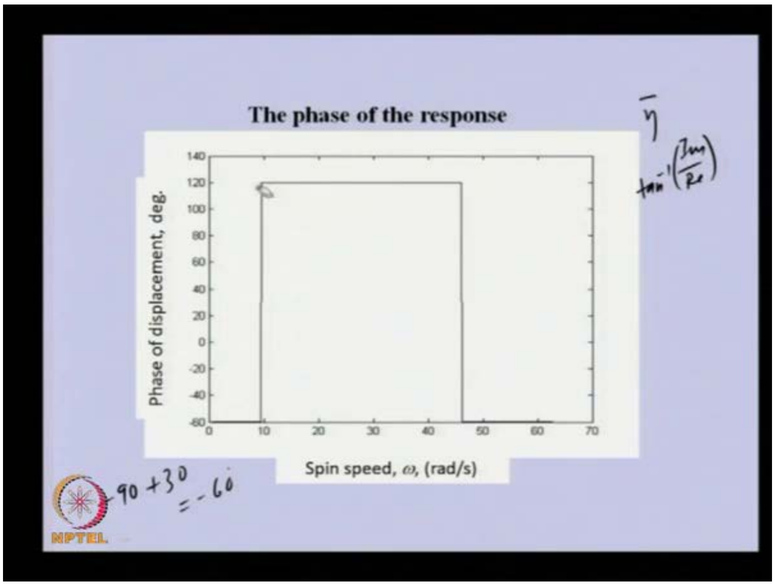

Now, let us see we are taking at the disc location, how the amplitude of the response changes with changing the speed? So, if you change the speed we will see that we will get peak at some location. This is at peaks. These are the critical speeds and if you see the change in the phase, because the response is a complex quantity and is having real part imaginary part. If we take the amplitude of this the previous part, we will get phase plot. It will be the tan inverse imaginary part divided by the real part. So, this will be the phase. So, at the disc location the imaginative part divided by real part tan inverse is the phase.

So, you can be able to see that we have we change in the phase corresponding were the peaks were there. So, here there is a change. You can be able to see there, were peaks are at that location. So, wherever the critical speed occurs there will be change in the phase and this particular phase is this response is in the vertical direction. Basically, the phase was 30 degree from the x, that is the axis horizontal axis. This response is in the vertical direction. So, obviously that is minus 60 degree because y axis is lagging behind by minus 90 plus 30, which is giving us minus 60 degree phase. So, initial phase is minus 60. Then, it is changing by 180 degree at critical speed 1 and then again it is changing at critical speed 2 by 180 degree.

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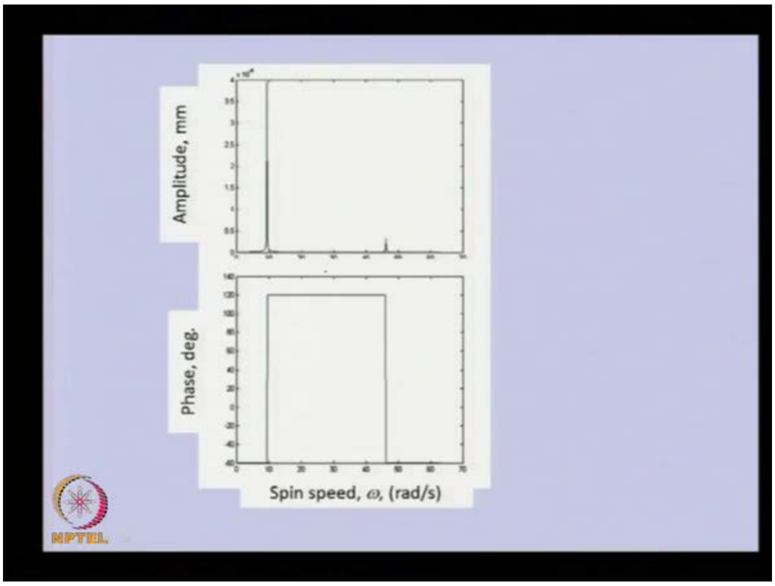

This is the same plot, so you can be able to see that wherever there is a peak there is a change in the phase. So, in fact this peak is not small because we have taken some step size which was not very close to the critical speed, that is why this amplitude is low. Otherwise, if you take more refined value of the step size of the speed you may get this as more, but this phase will indicate nicely the location of this particular critical speed. Today through some simple example, we tried to illustrate how we can be able to solve the free vibration to get the natural frequency and mode shape and how we can be able to handle if there is a intermediate support in a rotor system.

Even we tried to illustrate how we can be able to get for simple rotor unbalance response. Now, with these examples we can be able to appreciate that if we take more number of element, we have the size of the mass matrix and stiffness matrix which are larger in size now. In the subsequent class, we will not only try to incorporate the gyroscopic couple and even some kind of damping into the system like Rayleigh's damping, but also we will try to see some methods by which we can be able to reduce the size of the matrices without compromising on the accuracy.

So, that means some of the reduction schemes we will be taking up in the next class apart from that because till now in the finite element formation we have not considered the flexibility of the bearing. So, that also we will try to see how the bearing equations we can be able to club along with the shaft equations. So, that will give over all idea about how the rotor bearing system can be analyzed using finite element formulation.