Theory and Practice of Rotor Dynamics

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Module – 5
Torsional Vibrations
Lecture - 22
Torsional Vibrations: Finite Element Analysis

In the previous lecture, we started Finite Element Analysis of Torsional Vibration. In finite element method we started with very basic concepts like element, node, and safe function and gradually we developed finite element formulation for an element. And we developed the mass matrix and the stiffness matrix of an element in which, what we started with the equation which we developed for the continuous system of torsional vibration of the particular shaft.

And today we will be extending that particular concept of for development of the system equation for using those elemental equation. And apart from that we will see several example, through several example how this finite element method can we can able to apply for various kind of boundary conditions. Especially the boundary conditions are complex, the finite element method can handle such complexity; and when we will develop some kind of equation or elemental equation for geared, because in torsional oscillation previous we have studied geared system and also the branch system. So, some initial development of the elemental equation we will try to develop in the present lecture.

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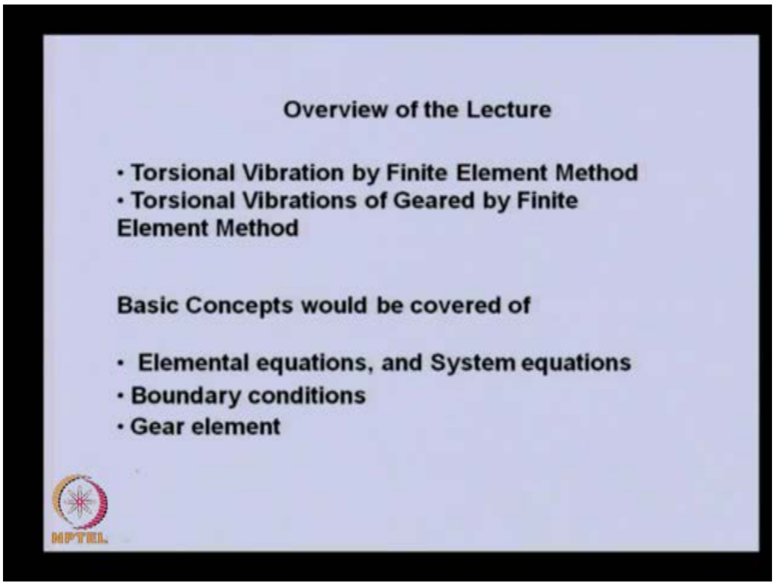

So, let us see what are the things we will be covering so basically we will be extending the finite element method which we developed in the previous class lecture and through some examples will be illustrating this particular method; even will try to develop some kind of elemental equation for geared system. So, in this is basic concepts would be like elemental equation, system equation, boundary condition how we can able to apply and gear elements, these are the things will try to cover in the present lecture, so will start with the equation which we left in the previous lecture.

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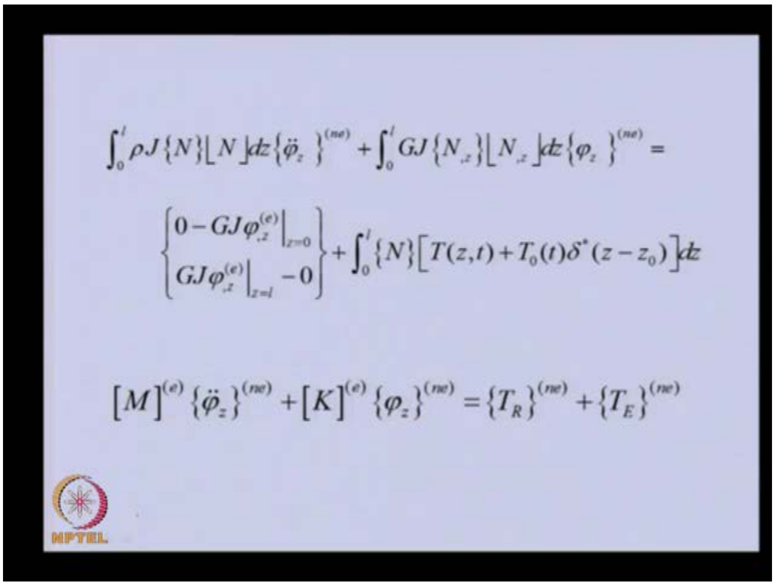

So, this was the elemental equation which we developed, so the form of the equation is mass matrix inertia term, the acceleration term, angular acceleration stiffness and displacement. This is the reactive torque and this is the external torque and these mass matrices, we the previous lecture we explained how we can able to obtain using the safe functions.

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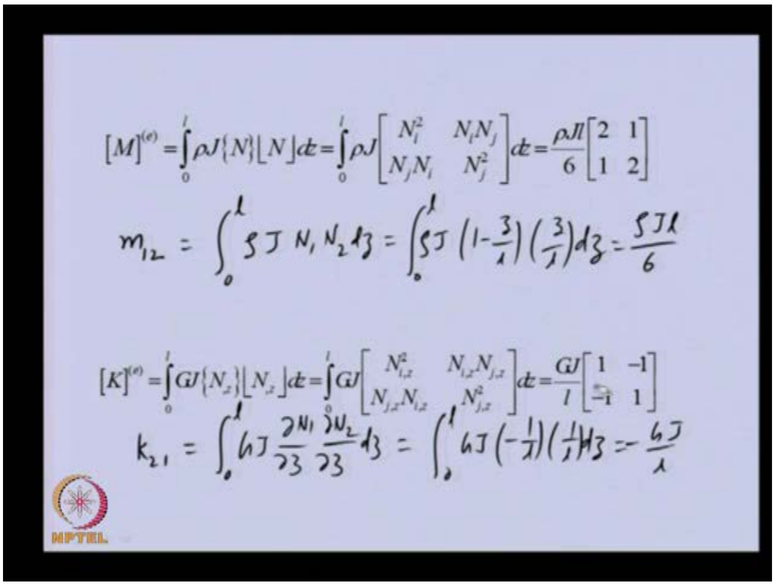

So, because this is a row this is column vector and this is a row vector. So, if we multiply it we will get the multiplication of these 2 vector as a matrix and for a representative case let us say if we want for the elemental matrix if you want the M 1 2 element; that means, here we need to integrate from 0 to l over the length of the element rho J N 1 N 2 then d z and N 1 N 2 we already obtained earlier. Now, these are the safe function, so if N 1 was 1 minus z by l and N 2 was z by l integration. So, if you integrate this we will get this particular term; that means, we will get rho J l by 6.

Similarly, we can able to calculate other terms by substituting the relevant safe functions and we can able to get the mass matrix. On the same length we can able to obtain the stiffness elements like if you want K 2 1 this particular term. So, we need to integrate from 0 to l G J and in this we have derivative of this safe function. So, N 1 with respect to z and N 2 with respect to z and if we differentiate this quantity; obviously, we will get minus 1 by l and from second term if we differentiate this with respect to z will get 1 by l d z and this will give us or this particular G J by l minus term. So, that is K 2 1 similarly other terms we can able to obtain in the similar lengths.

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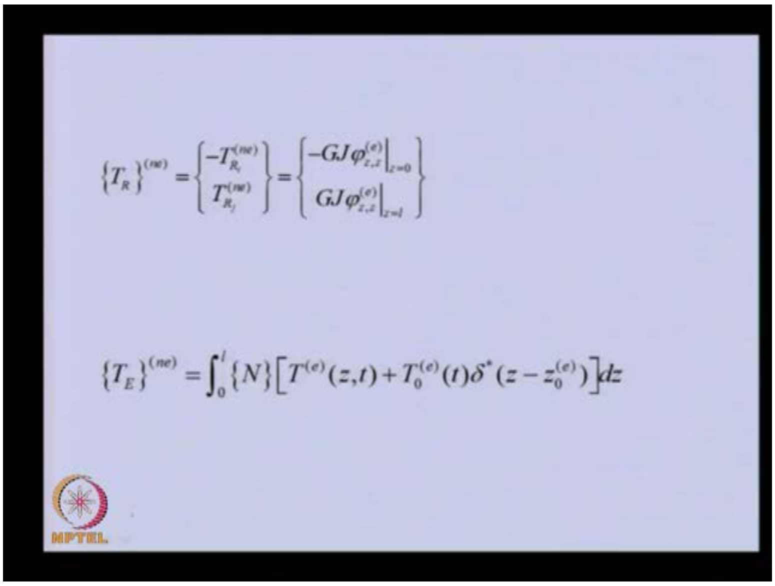

Till now, in these where the other term the reaction terms which after substituting this we obtain them in this particular form. So, in this the reaction at or let us say i th node re reaction torque at the j th node. And these are the external torque if we know how this distributed torques are vary then we can able to put that particular variation and we can able to integrate with respect z to get the in the form of explicit form of the this kind of external torque case.

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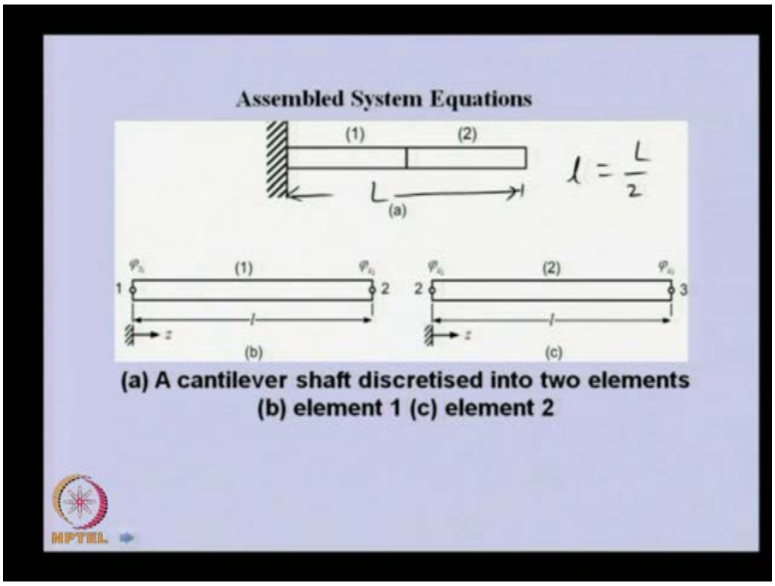

Now, once obtain this elemental equation now, through one example of a cantilever shaft will develop a system equation because in this particular for railway station purpose we have divided the shaft into simple 2 elements. So, this is the first element then this is the second element both the element size is same. So, if total length of this is l, so the element size l is for each of these element would be l by 2 not necessary we should always take the same length, but in this particular case we have taken 2 elements same length.

In this particular case this property of the stiffness and the mass is distributed through out the length and during discretization is in the finite element method. We are considering them this distribution property of the mass and stiffness both as we have developed the elemental matrices for that particular case. And so this is a one of the element this element and in this element we assigning the node one in the left side node two in the second other side.

And this is a local coordinate system and at each node we are assigning the fuel gearable that is angular displacement p z 1 and p z 2 corresponding to the nodes. So, in finite element method basically, we will be obtaining the displacements at these 2 locations and intermediate locations will be obtaining using the safe function. When this is the second element, in this is the common node to this, so this node is 2 and this particular node is 3.

And we have for this also we have local coordinate system z and angular displacements here is p z 2 and here p z 3 because this is a common node. So, we expect or during assembly we should have the same displacement at this location and compatibility of this we have already seen in the while choosing the safe function. Now, once we have divided the whole system into two elements the boundary conditions will take care of subsequently once will get the system equation. So, now, let us concentrate on the element, so for element one if you want to obtain the equation of motion for element one or the form of the equation will be exactly same as we obtained earlier.

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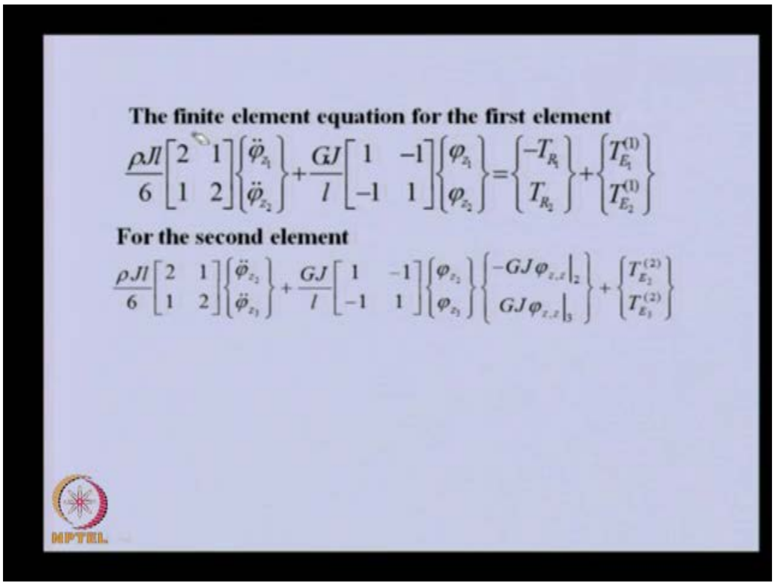

So, this is a mass matrix, here for element 1 we had the node 1 and 2. So, corresponding the acceleration, angular acceleration terms are there p z 1 and 2 this is the stiffness matrix term here p z 1 and p z 2 displacements are there reaction torques are T r 1 and T r 2 this is minus and this plus as we have seen it earlier according to the convention. It comes out like this or external torque if it is there they will take they will be coming here at node 1 and node 2.

Now, the second element second element the mass matrix because the size of the element is same, will remain same or here there will be change we can able to see, now here angular displacement 2 is there and angular displacement 3 is there angular acceleration. So, this is the only difference otherwise this stiffness matrix remain same as the previous one, so here also p z 2, p z 3 corresponding to the r 2 and 2 angular displacement of that particular element.

Now, these are the reaction terms, so this we can able to write it as T r 2 and this is minus T r 2 and this is T r 1 or T r 3, because this is corresponding to the third node this is the external torque at node 2 and node 3 of element 2. So, superscript is representing the element and subscript is representing the node number. Now, once we obtain elemental equation 1 and 2, we can able to combine them in a in a system equation.

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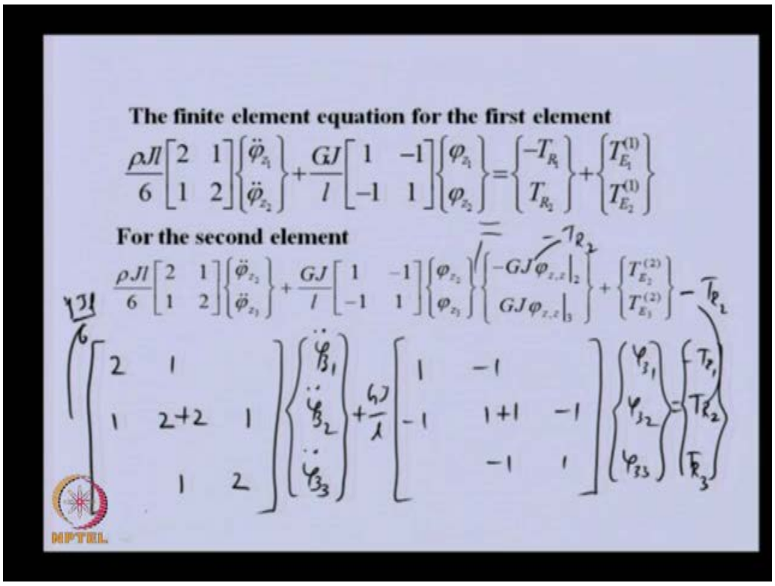

So, how we can able to assemble them, so let us see this is our mass matrix now, stacking up the Eigen vector will be like this because there 3 nodes in the system. So, in the system equation we expect these 3 field variable we will be there plus will be having stiffness matrix and here p z 1 p z 2 p z 3. So, now, I am putting I am in the extreme case we will be having the torques. So, now I am inserting the first equation in this, so we will be having the common turn that is rho j l by 6 will be common which will be outside.

And here 2 1 1 2 this is corresponding to this we can able to see p z 1 p z 2 p z 1 p z 2. So, they will be in the first row first column and the first two row and column they will come similarly, the stiffness matrix this is common. So, this will be outside, so this will be placed in the 2 by 2 size. So, where the p z 1 and p z 2 will be multiplied they will occupy that position.

Here in the torque we can able to see here we will be having T r 1 and T r 2 and this minus T r 2 will be added sorry this will be T r 2 because at present we have inserted only the first element. So, now we can able to see we have inserted the first element remaining elements are 0 at present. So, if we expand this basically, we can able to get the first equation. Now, I want to insert the second equation, so second equation this particular displacements are 2 and 3 nodes.

So, 2 and 3 nodes are here, so this particular two term of the mass matrix will come here because this will get multiplied with the z 0. So, plus 2 and the 1 is multiplied by p z 3, so p z 3 in the same row it will come here and these 2 will come then at the lowest position at the here. So, you can able to see there is 2 1 2 1 is corresponding to this particular mass matrix element corresponding to the second element. And similarly, for stiffness we will be having plus 1 this is plus 1 minus 1 minus 1 1.

So, you can able to see this particular 4 elements are corresponding to this 1. Now, in the here right hand side basically, we should have equality here, so this 2 will come here. So, we will be having here plus T r 2 and sorry this is minus T r 2 and t r 3. So, you can able to see that here T r 2 was from element 1 and this T r 2 is from element 2 they are same. So, they will cancel each other at this particular node. So, this is the assembled form of the equation of element 1 and 2 and the same things have written here.

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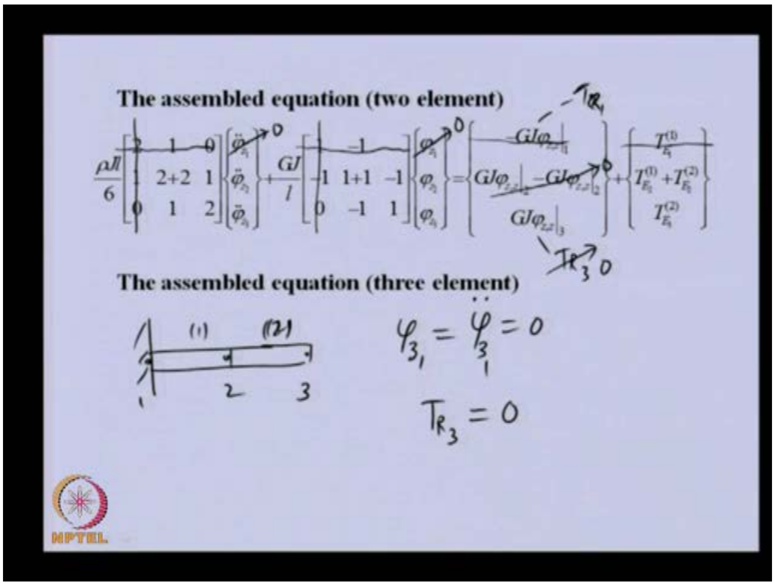

So, this is the assembled mass matrix, this is assembled stiffness matrix and these are the internal torques. So, basically these are nothing but T r 1 and these are T r 2 and this one is t r 3 and these are the external torques. So, this is corresponding to the element 1 and this is corresponding to element 2. Now, once we have assembled equation now, we need to see how we can able to apply the boundary condition of the problem, in the boundary condition we had 1 is the element 1 element 2 and this is node 1 node 2 node 3.

So, this at node 1 we have fixed end. So, there displacement and acceleration we expect to be 0 at node 1 and this at node 3 we have free end. So, it cannot take any torque, so that is T r 3 reactive torque will be 0. So, in the system equation if we want to substitute this we will see that this particular term will be 0 this will be 0 because this is a fixed end and this one is this term will be 0 and this two are equal and opposite. So, they are anyway are canceling to each other.

So, you can able to see we could able to apply the displacement and a displacement condition also the torque at the node 3 equal to 0. Now, this equation we can able to this we already applied the boundary condition. So, this equation will reduce in size because some of the terms are 0, so you can able to see this particular term is 0, so this will get multiplied with the first row. So, whenever we try to expand it the first row will not contribute any anything. And so this particular row we can able to delete similarly here this will multiply to this particular row this will get deleted.

Now, in the right hand side in the reaction of this torque if you see and the second one is 0 third one is 0 only thing is the first one is containing the reaction torque at the support, which is unknown to us external torques will be known to us. So, because in right hand side we have one unknown, so this particular equation which is containing additional unknown we will eliminate from the equation. So, first row we will be eliminating in this; that means, the first row we can able to delete, this equation we can able to use subsequently for obtaining the reactions at this location.

Once we have solved the angular displacements of the system, so we have removed the first equation or first row first column and first row in each of this matrices. And what were the remaining is left out that is that will be 2 by 2 matrix that we will call it as a reduced form of the equation.

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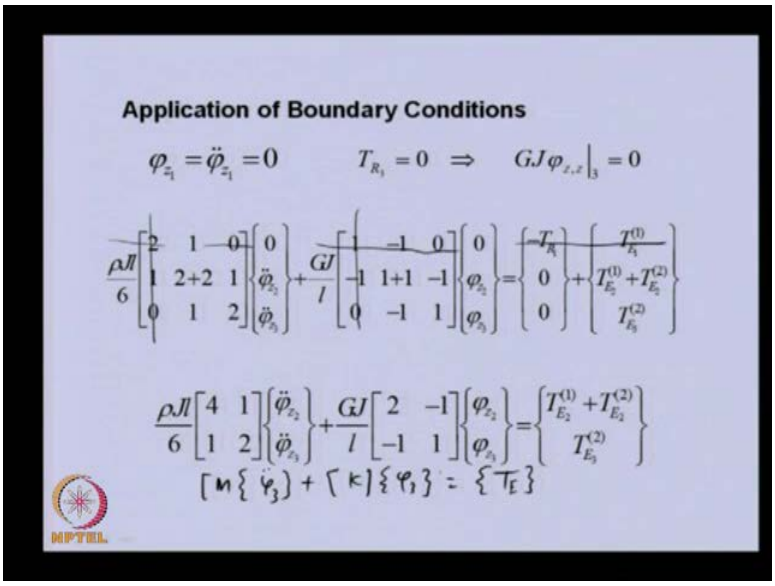

So, that will be basically of this form, this is the reduced form, so these were the boundary conditions which we applied in this and in this particular case we removed the first row and first column of this. So, basically we can able to see reduced form is this 4 is coming here 1 and 2 is coming here similarly, this 1 plus 1 2 is coming here. So, this is the reduced form of the equation and these are the external torque, which are known; so now, this equation is ready for solution.

Because, now you can able to see this is having standard form as m pi double dot z plus stiffness pi z is equal to some external torque. So, this is a standard form of the equation for multi degree of freedom system. So, that can be solved either for forced vibration or free vibration, in this particular case we divided the shaft in two element, but if we want to have a more accuracy we need to have more number of elements in the system. So, I am just explaining if we have three elements how the system equation will change.

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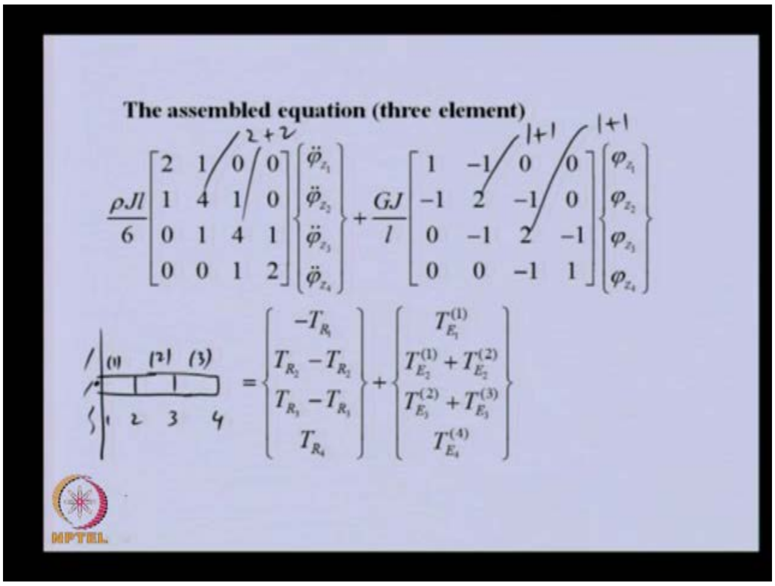

So, for that particular case let us say we have this and we have 3 elements. So, we will be having node 1 2 3 4 nodes will be there. So, will be having 4 angular displacements and the assembled equation, because we need to first try the elemental equation for first element, second element and third element and then we need to combine them. So, we already done up to the second one, but if third is also there then we will be having can able to see earlier it was 3 by 3 size, but now four nodes are there.

So, will be having 4 angular displacement and mass matrix will be again can able to see form will be similar to the previous one and this contribution from the second element and the first element both are there. Similarly, here contribution of this second and third element is there and this is the from the third element. So, basically, this is coming from 2 plus 2 and similarly this is coming from 2 plus 2 and in this stiffness matrix also this is from one element and the second element and here also from second element and one element.

So, this is just extension of the previous assembled equation from three nodes, three elements and here in torque will be having 2 common nodes the node 2 and 3. So, will be having contribution from element 1 and 2 they will cancel each other during the assembly because they are reaction torque and similarly, at the node 3 they should cancel each other because this is a reaction torque. So, when we are assembling these 2 parts, 2 elements; obviously, the reaction torque should cancel each other and these are the external torque.

So, this is just a illustration that if we have more number of elements how the equation would change in the assembled format, will continue with the same 2 element for the free vibration analysis of the this particular cantilever shaft. We developed the reduced form of the elemental system equation with the help of 2 element will continue with the 2 element and will try to solve the free vibration; that means, we will try to obtain the natural frequency of the system using the finite element method.

So, for that our reduced form was 2 by 2 matrix earlier. So, we are assuming the solution of that in this form in which this is the amplitude part which could be complex and this is the harmonic part in which the frequency is there and because that particular equation the previous equation was containing derivative with respect to the time. So, if we derivate this we will get this one.

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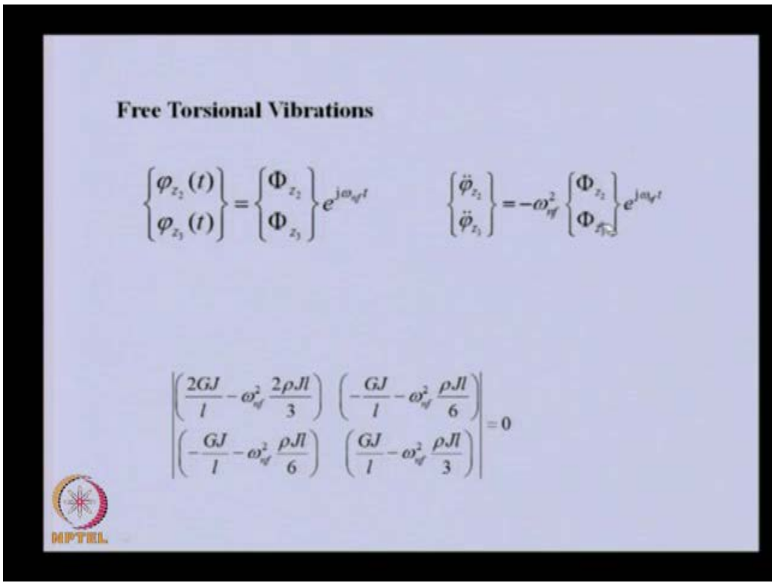

So, basically you are able to see that.

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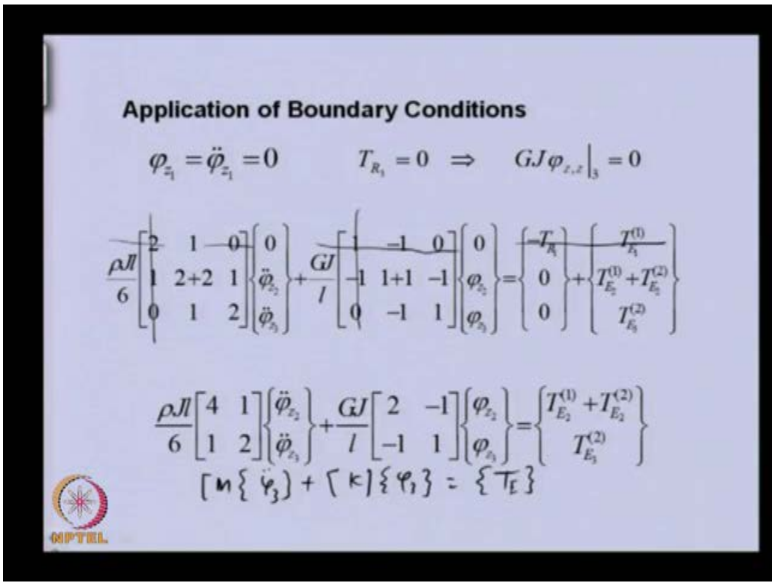

In previous equation or in the reduced form, if we substitute these solutions.

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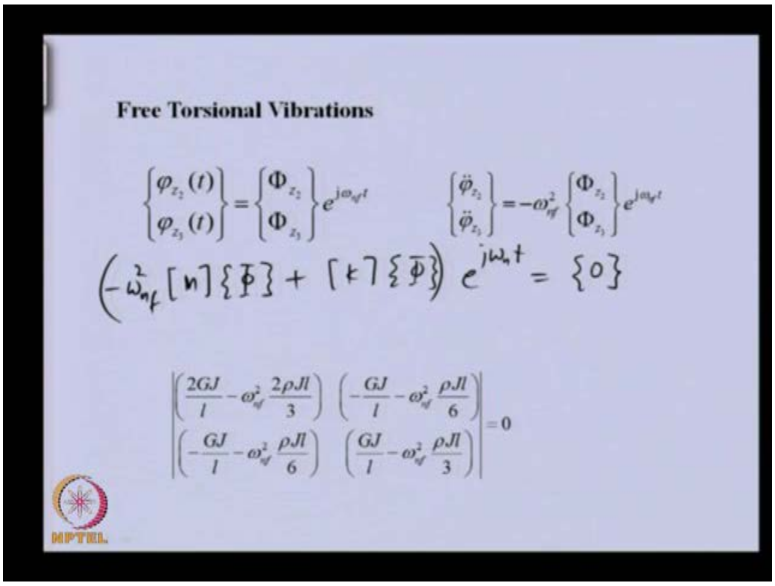

So, we are substituting this solution in the previous mass matrix. In this we had and then the stiffness matrix and time dependent term there is e j omega n of t is outside and this is for free vibration. So, on the external torque we are keeping at 0. Now, you can able to see back this equation.

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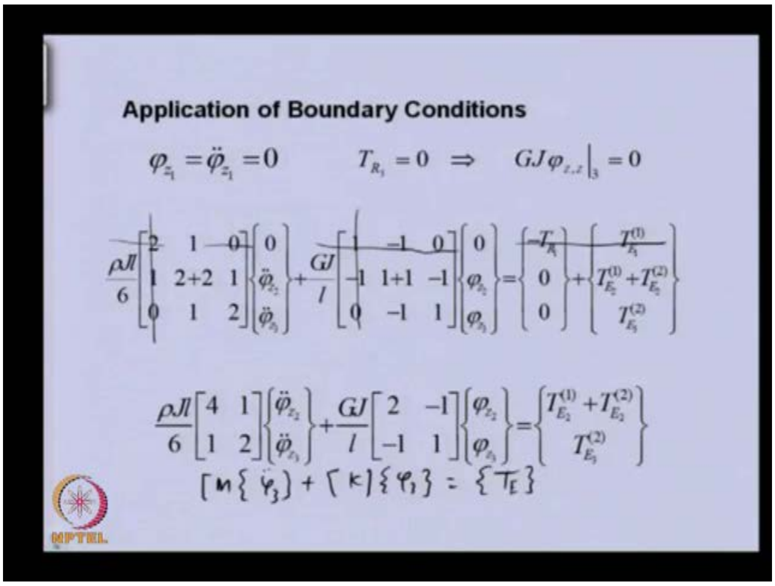

This is the mass matrix this is the stiffness matrix this is 0 because free vibration.

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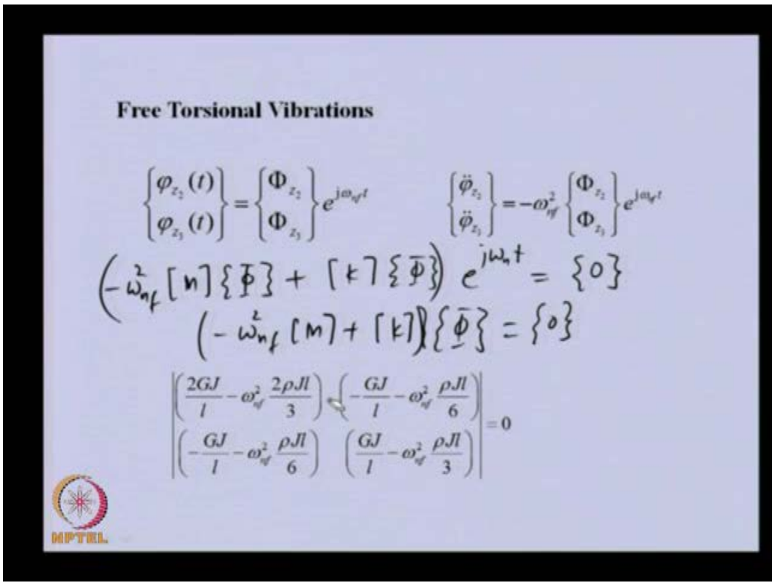

Now, if you substitute those equations those matrices here, and here and because these two vector are common. So, we will be having the form of the equation is minus this and plus k and this will be outside and because this time dependent term will not be there. So, now, you can able to see this is a homogeneous equation and if we equate if we put this equal to 0 there is no motion of the system. So, this cannot be 0, so the determinant of this to be 0 and that is that particular term.

In which we are a combined the mass matrix and the stiffness matrix multiplied by the this particular natural frequency square term. So, if we equate this we will get a polynomial of this form.

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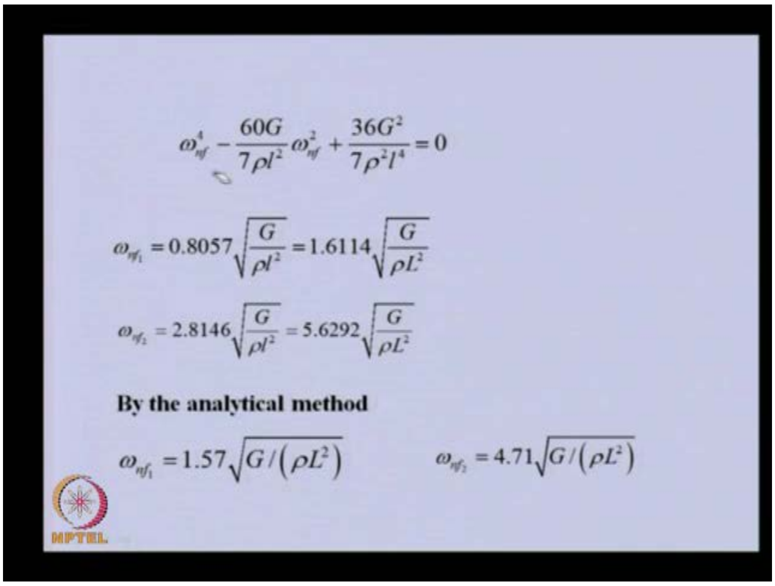

In the quadratic in omega n f square and if we solve this quadratic equation we will get the natural frequency 1 and 2. So, these are the natural frequency of the system or using finite element method, earlier we obtained the close form solution for this particular case and we expect that close form solutions are exact values and from that these are the natural frequency and if you compare the first natural frequency was 1.57 and this factor and from finite element method we are getting 1.6114.

So, they are close second natural frequency was 4.71 exact 1, but with finite element method we are getting slightly higher. And so second natural frequency is not that much accurate here, but if you take more number of element we expect the finite element results will be giving better results closer to the closed form solutions.

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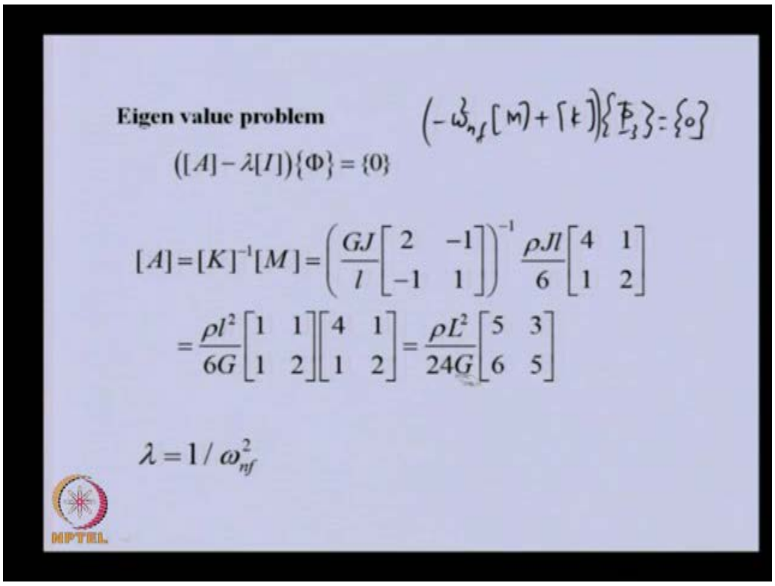

The same problem I am we are solving here in the form Eigen value problem. So, the previous equation which we had as omega n f square m plus k and this is equal to 0. So, I am multiplying both sides by k inverse of the matrix and dividing by the natural frequency square. So, if I do it we can able to see that k inverse into M is A A is k inverse into m and lambda is 1 by omega n f square because I have divided throughout by omega n f square, so this lambda is 1 by omega n f square, and because there is no other term. So, this is a identity matrix and the A matrix is the inverse of k into M we can able to do the multiplication of this, we will get this. Now, if we solve the Eigen value problem of this matrix.

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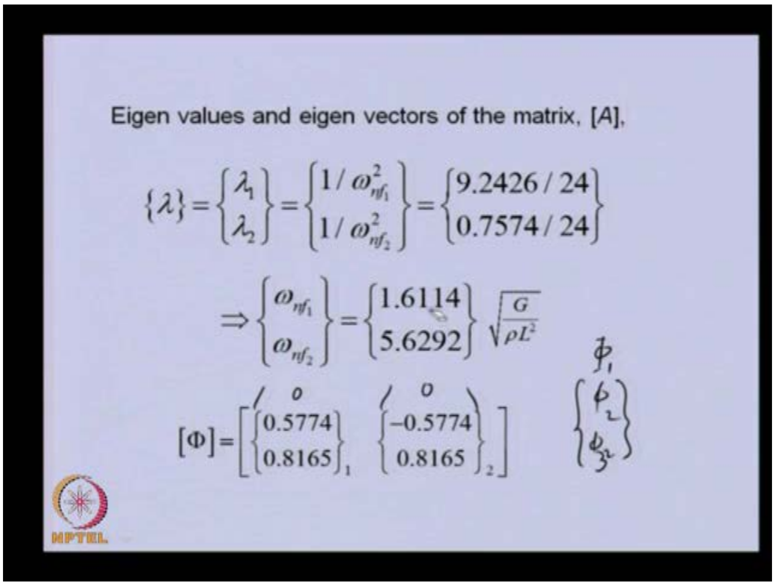

We will get 2 roots are like this and these 2 roots are nothing but inverse of the natural frequency square because here Eigen values are defined like this. So, you can able to see that if you inverse this and take a square root we will get the same natural frequency values which obtained earlier. So, directly from Eigen value problem we can able to obtain the natural frequency then corresponding to that particular A matrix this is the Eigen vector with software like mat lab directly we can able to get the this Eigen vector.

And this Eigen vector is representing how this, those two particular node the second and third are getting displacement because in the previous equation we had 2 field variable 2 and 3. So, they are belonging to that and we already know that at node 1 we have 0 displacement. So, this vector can be extended like this to represent the how at node 1 2 and 3 on the displacements are taking place and we can able to draw the mode shape this will be more clear if we take more number of element.

So, number of the size of the Eigen vector will be more and we can able to see the shape of the mode shape, more clearly. Similarly, if we when we are taking more number elements we will be getting more number of natural frequency and we expect the natural frequency which will be the lowest 1 or the lower ones will be better as compared to the higher ones. We will take up another example, now we will see a slightly more complex boundary condition.

And we will see how the finite element method can tackle more complex boundary condition such, boundary conditions if we want to apply using the continuous system approach it will be very difficult if not impossible, but it will be very difficult to apply and get the closed form solution. So, now let us see another example in which we have let us see the first problem.

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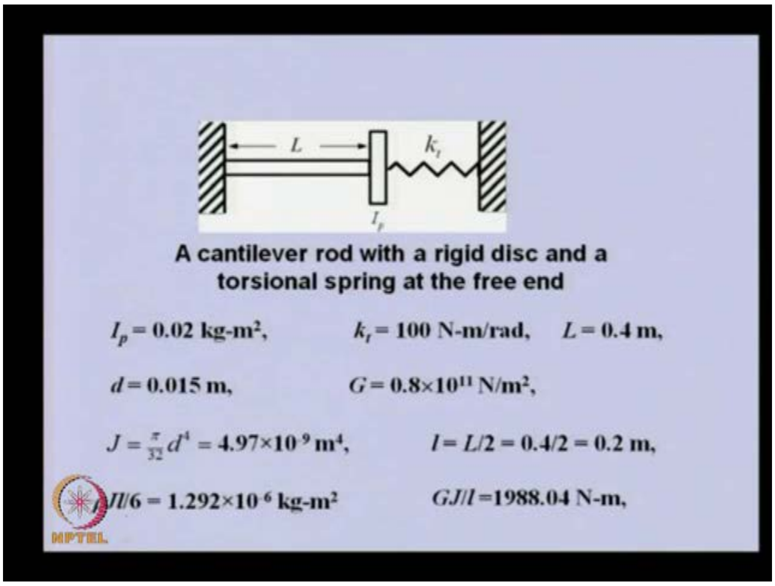

So, fixed end this is shaft having distributed property this is a disc at the free end and also there is a spring attached at the free end of the disc. So, in the previous case we had only cantilever shaft, but now disc is there apart from the, we have one spring a torsional spring attached to the free system.

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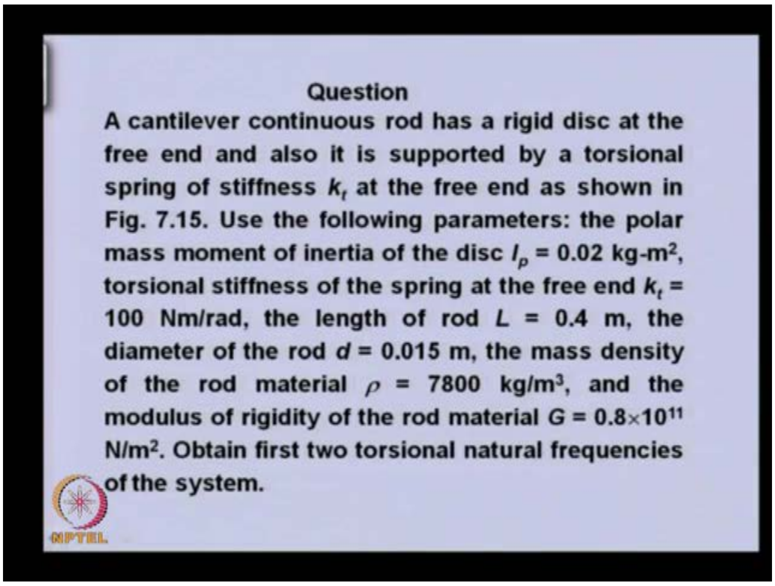

And for this we have various property like the shaft property is given, length of the shaft is given, torsional stiffness of the spring is given, diameter of the shaft the property the density of the shaft because we are considering the distributed property of the shaft then modulus of rigidity is also given. So, we need to obtain a torsional natural frequency at the system using the finite element method.

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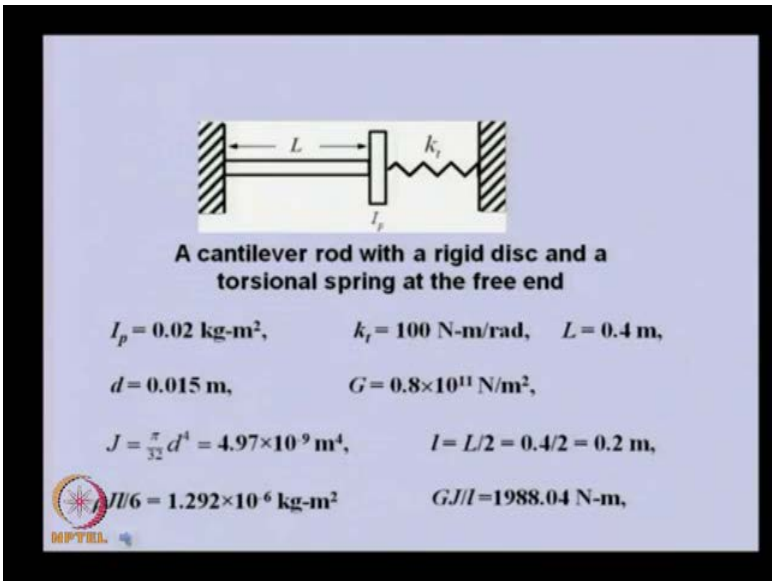

So, these are the various property which of the shaft and spring which are given and other property which we can able to calculate using this are second moment of area that is polar second moment of area element, in this particular case we will be dividing this particular shaft in 2 element only. So, the element size of that will be half and these are the other mass property and the stiffness property.

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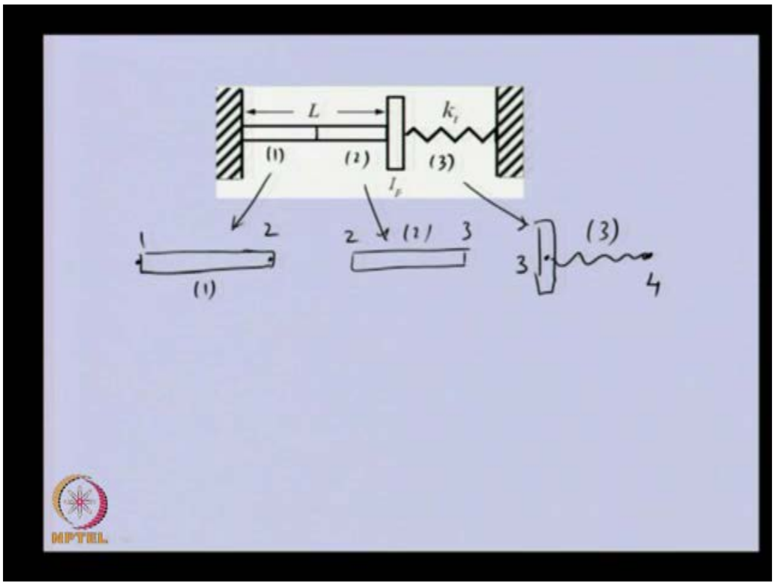

Now, let us see how we will be descretising this particular system. So, as I mentioned I will be descretising this particular shaft in 2 elements and this will be the third element this particular disc either we can able to take up along with the element 2 or along with the element 3. So, for this case we will take this particular disc along with the spring the element 3. So, now, I am drawing the 3 element separately. So, this is the first element having node 1 and 2 this is the second element having node 2 and 3 this particular disc I am carrying along with the third element.

So, let us say this is the third element this is node 3 and this is node 4. So, you can able to see this particular system I have divided into 3 element. So, first element is this second element is this and third element is this. So, total 4 nodes are there, and now we can able to obtain elemental equation for each of them. So, first step would be that.

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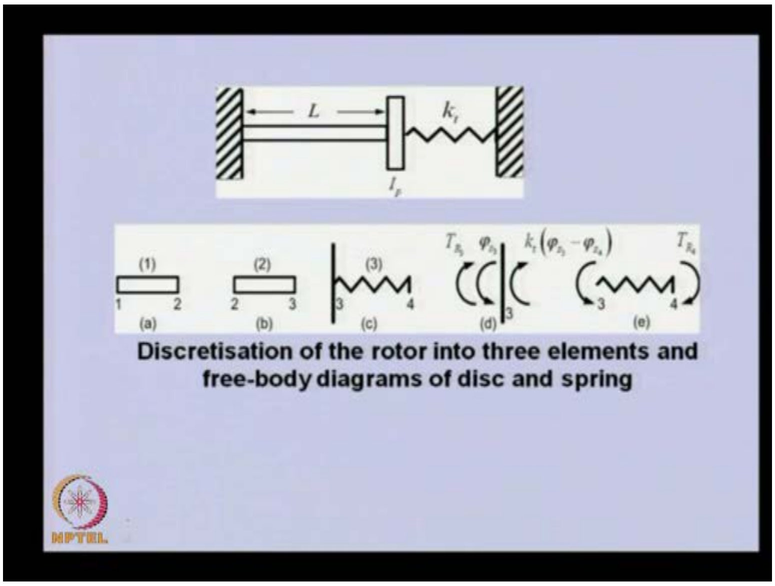

So, this is the same the division of the element and in this itself we will see that, how we can able to develop the elemental equation for this for that this is a free body diagram we will see later this how this free body diagram can be useful for development of the elemental equation for this.

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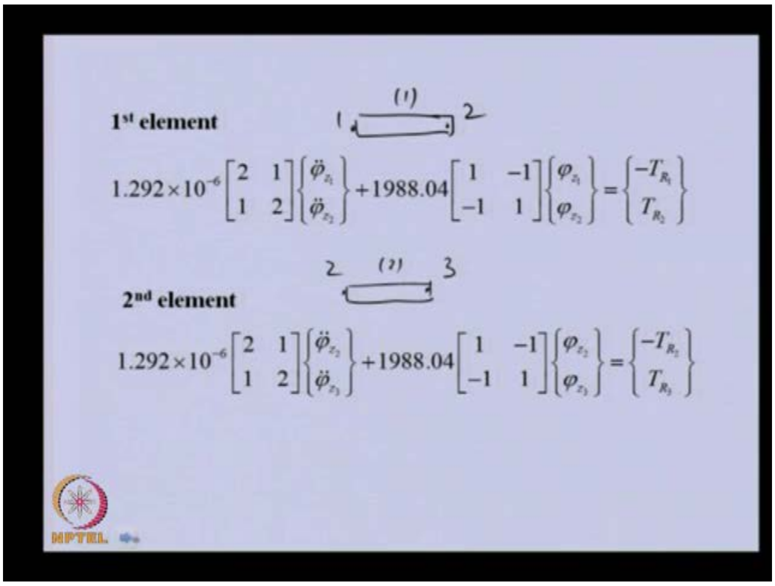

So, let us write the elemental equation for first element which is simple this is the first element having node 1 and 2. So, correspondingly this will be the mass matrix these terms we already calculated earlier here the this field variable is corresponding to node 1 and 2 this is the stiffness matrix here again the corresponding to node 1 and 2 these are the reaction torque at node 1 and 2. So, they have this particular form similarly, for second element which is identical to the first 1 shape and size, but only the node number are different. So, you can able to see here remaining things are same only thing is this particular field variable is having subscript 2 and 3 and here 2 and 3. Corresponding to the 2 and 3 nodes of the element 2 and similarly here we have the reaction torque T r 2 and T r 3. So, we have developed the elemental equation for element 1 and 2, now will see for third element how we can able to develop the element.

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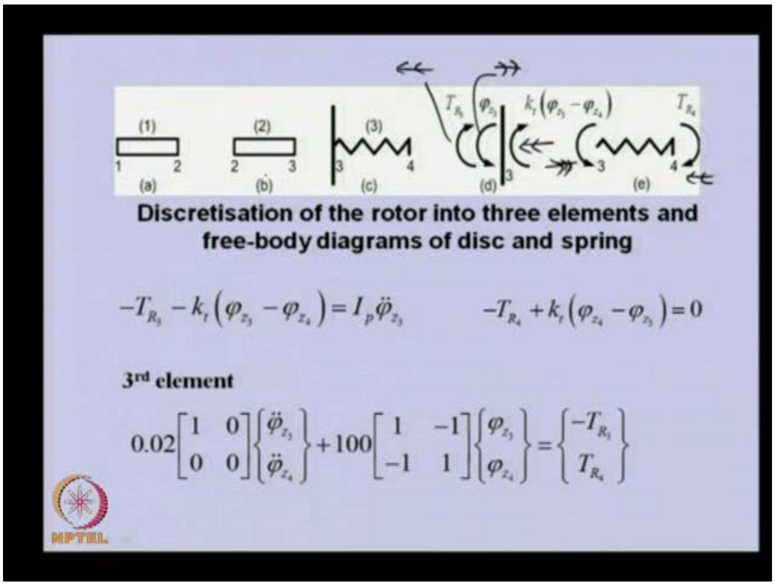

So, now we are concentrating on this to develop the elemental equation for this. So, for this I have drawn the free body diagram of these 2 parts disc separately and the spring separately. So, disc is having let us say this particular angular displacement is positive in this direction. So, the reactive torque will act opposite to that in this direction and this will also act in the opposite direction. So, this reactive torque is from the shaft you can see this particular disc is not only attached to the spring, but it is attached to the shaft also.

So, this reactive torque is coming from the shaft and this reactive torque is coming from the spring and this spring is having relatitive stuff theta at 3 minus theta 4 because 2 field variable are there. So, pi z 3 minus pi z 4 is the relatitives into the torsional stiffness will gives the reaction torque from the spring. Now, this is the free body diagram of the disc and the same reactive torque will act here, but direction will be opposite and this will be opposite, here also the same reactive torque will act because there is no other is here.

So, the torque here and torque here they should be same. Now, let us take the free body diagram of this particular disc. So, we have reactive torque from the shaft from the spring they should be equal to the inertia of the disc similarly, there is from this reactive torque of the spring is nothing but the is coming from the relatitives of the shaft. So, there is no inertia, so this will be 0. So, now, you can able to see that these 2 equation we can able to combine in this form.

So, the first equation is here if we expand the first equation we will get this we substitute the value of the I p and t also k t also. So, the first equation is this equation if we expand the second equation there is this equation. So, basically, we assemble these 2 equation and this is the elemental equation for the third element. And now, we have obtained elemental equation for 1 2 and 3 element.

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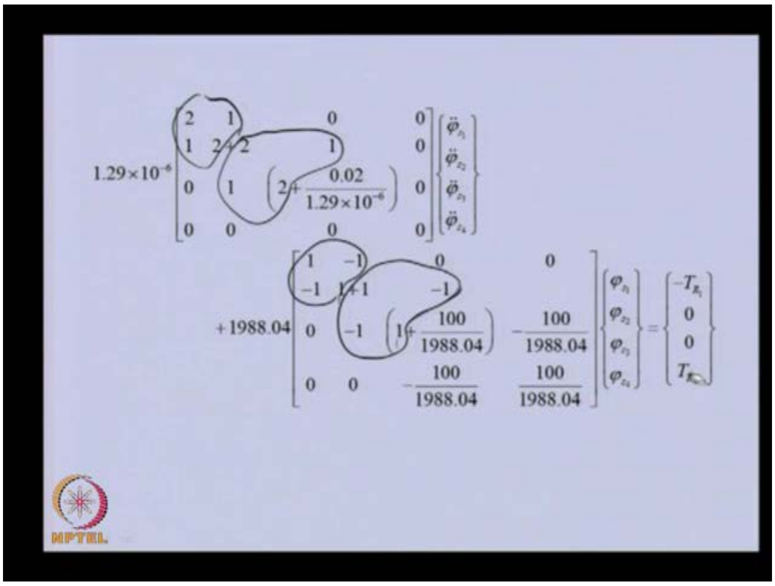

Now, we can able to assemble them and assembly; obviously, because we have 4 nodes. So, this vector will be having subscript 1 2 3 4. So, this is the assembled mass matrix, so you can able to see this is coming from element 1 and this is coming from element 2 and this term is coming from the element 3 and there is no term here because in the element 3 only 1 mass term was there. Similarly, in the stiffness this first 4 are corresponding to element 1 and these are corresponding to element 2 and remaining 4 are from element 3.

And you can able to see at intermediate node 2 and 3 reaction torque will cancel each other this is the reaction torque at fixed end and this is at the other extreme end of the spring.

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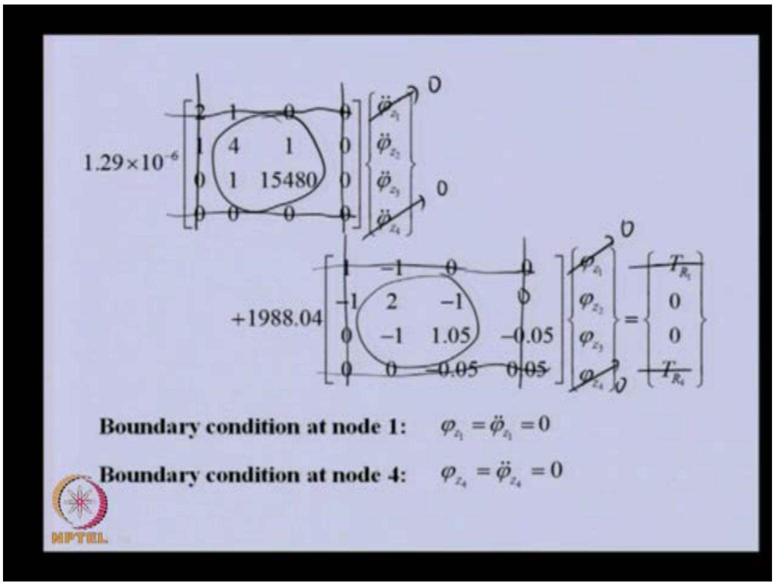

And this is the simplified form of the previous equation we added those coefficients. And so this is the simplified form of the previous equation now, we can able to apply the boundary condition at node 1 is that displacements are 0 and at node 4 also the other end of the spring the displacements are 0. So, that means; we can able to put this equal to 0 is equal to 0 is equal to 0 this equal to 0, so these are 0. So, you can able to see that once we have these 0. So, they will multiply with the first row. So, they will not contribute anything and the first and the forth row are containing a node.

So, they can be eliminated and they can be used subsequently to obtain the reaction torque but at present first and the fourth row can also be eliminated. Because, in the right hand side they are containing a known similarly, this forth these 4 will get multiplied with the forth row you can see this will multiplied with the forth row. So, the we need to eliminate them here we have 0 I think. So, we can able to see that we have only 2 by 2 matrix that is this 1 for mass matrix and 2 by 2 matrix for stiffness matrix and right hand side is 0.

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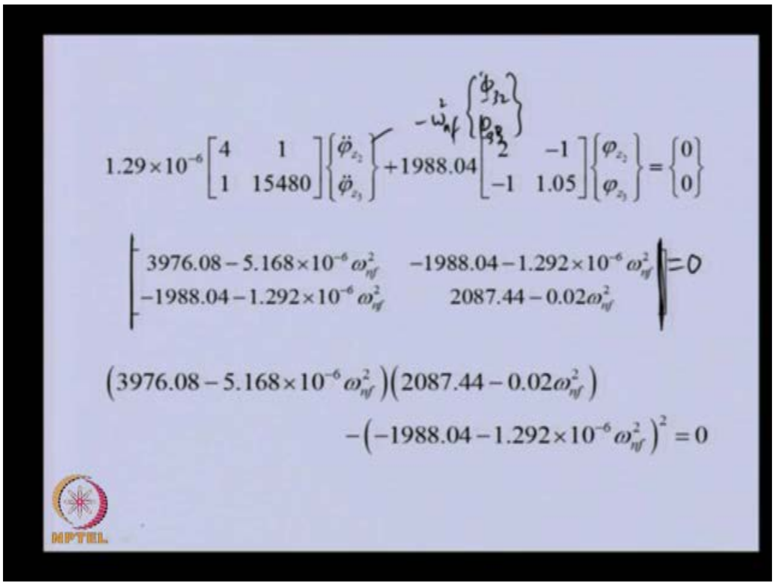

So, if we write them separately this is the reduced form of the equation. We can able to see this was between, the after deleting rows and column this is the mass matrix this is the stiffness matrix they are this and right hand side is 0. Now, for free vibration we can able to convert this particular term as minus 1 omega square and angular displacements 3. So, and this will be common. So, as we did earlier we need to have the determinant of this term to be 0 for non trivial solution and this will give us a polynomial in terms of omega n f square.

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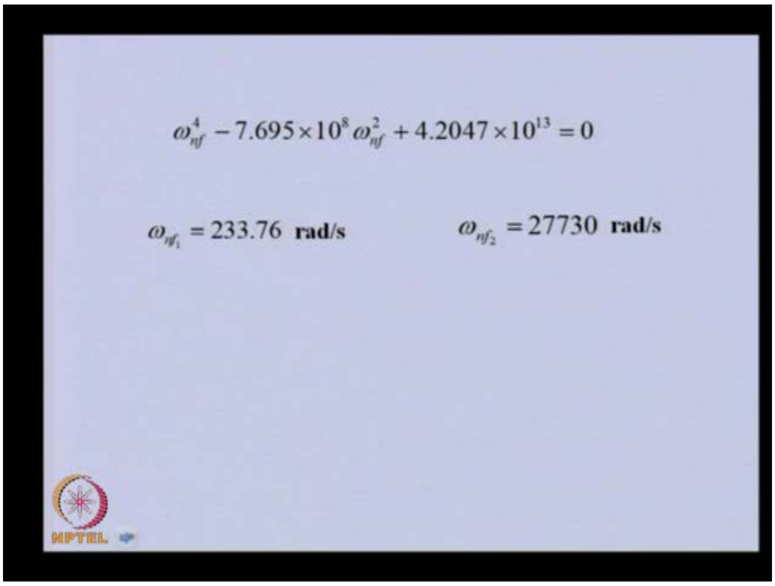

And that will be having this from and if we solve this polynomial we will get 2 natural frequency of the system. So, you can able to see that using finite element method we could able to get 2 natural frequency close form solution of this is not easy. But with finite element method the same elemental equation was helpful in obtaining the natural frequency of the system quickly. Now, we will take up another example, in which we will be taking more number of element and we will try to study the conversions of the solution.

So, we will see that how many element numbers we should choose for particular application with this exercise it will be more clear; that means, we expect generally when we are taking more and more element the solution should converge towards the exact value. And this particular study is also called convergence study.

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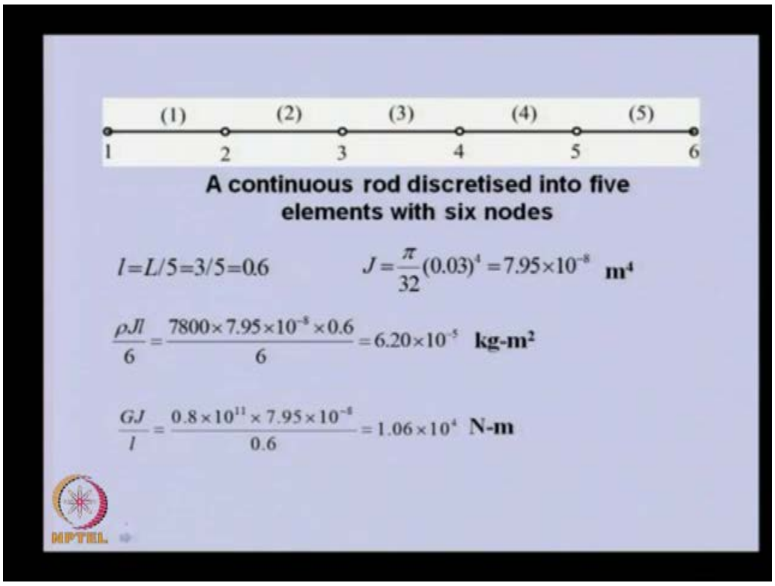

So, let us take the case of a simple a continuous shaft and this particular shaft is having.

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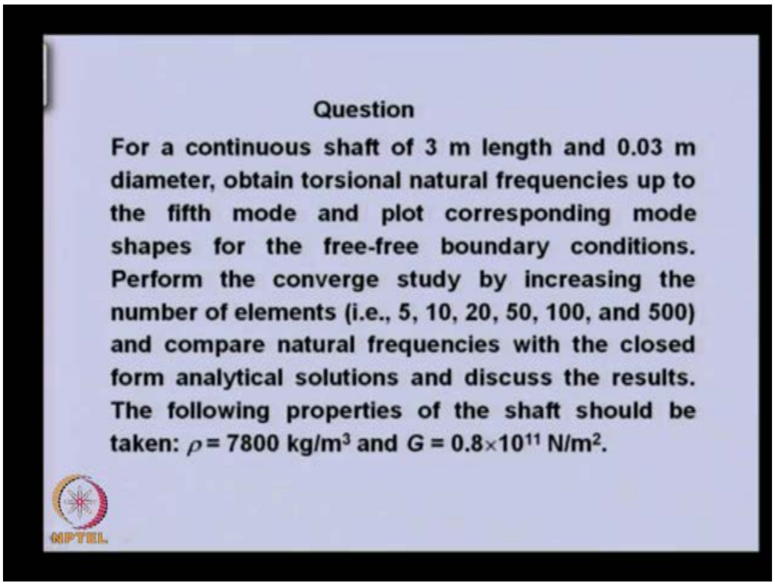

So, we have a continuous shaft of 3 meter length diameter is given. And this is having distributed property this particular shaft is supported freely and we need to obtain up to the fifth natural frequency or the natural frequency as well as the mode shape. So, you can able to see that, we will be taking 5 element, 10 element, 20 element, 50 element, 100 element, even 500 element just to see how the result changes with the number of element and will be comparing the natural frequency with the close form analytical solution. Now, basically, in this particular case the boundary condition is free-free boundary condition and for that we have the close form solution available. So, these are the various property of the shaft.

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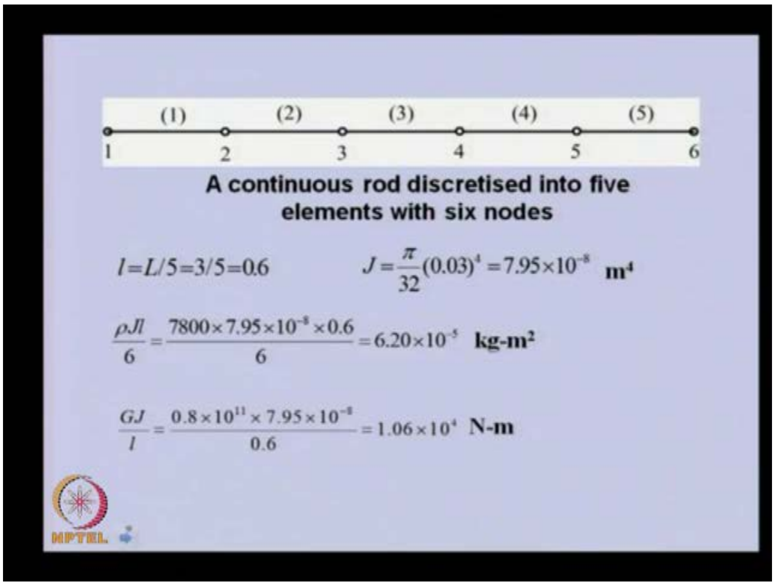

So, for illustration purpose I have divided the shaft in 5 element and because of that we have 6 nodes 1 2 3 4 5 6; these are the various property because the elements are same. So, will be getting the element length various geometrical property, mass property and stiffness property from the given data we can able to obtain.

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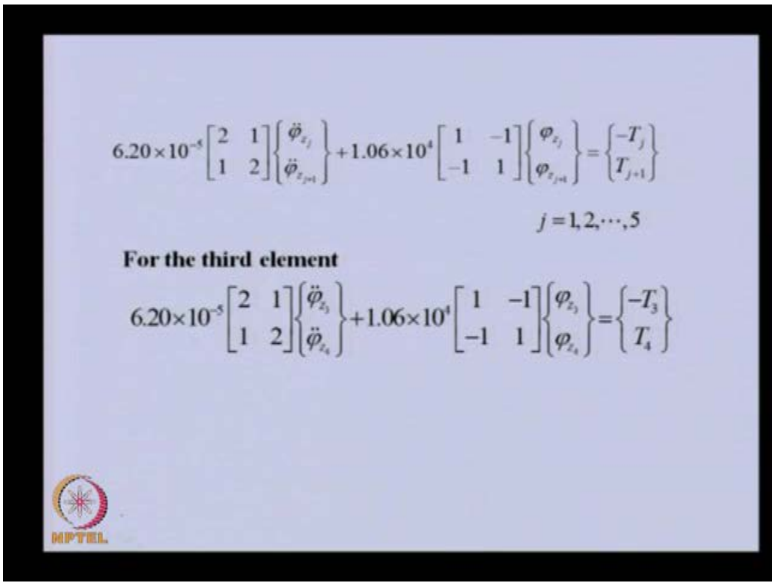

Now, we can able to write the elemental equation for any representative element because all elements is same. So, we can able to see that j can take from 1 to 5 value. So, that means; we can able we need to write this equation 5 times with different node displace node in displacement vector because we have divided into 5 element. And let us say for the third element this particular element we will take pi z 3 pi z 4 and here we will be having torque 3 and 4. These are the reaction torque, we are considering free vibrations. So, there is no external torque. So, these 5 equations such elemental equation we can able to combine.

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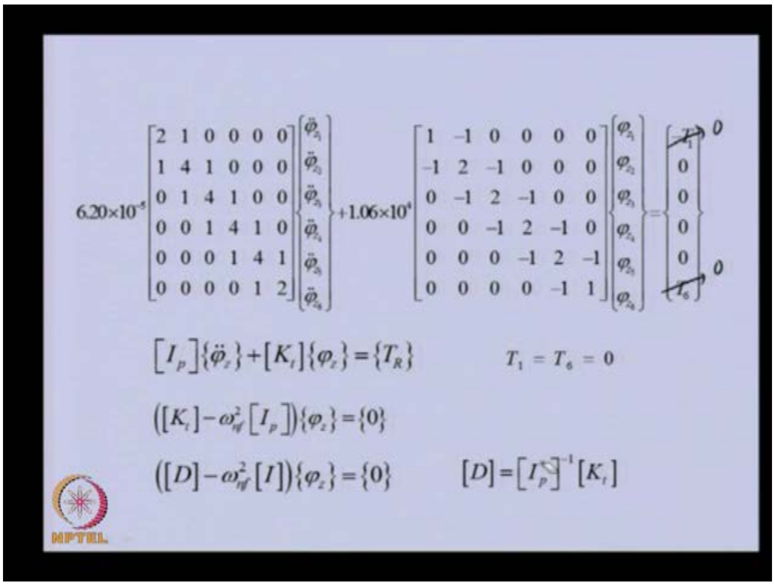

And this is the assembled equation, in this we can able to see all the nodes 6 nodes were there. So, we will be having field variables corresponding to all 6 nodes and here the assembly as we discussed here will be having the first 2 by 2 as of from first element and remaining will be having second element third element forth element and fifth element. Then the form of this equation is of this which is the mass matrix inertia this is the stiffness.

And this is the reaction torque for this because this free-free condition both the ends are free. So, it will be having boundary condition as torque at 1 and 6 are 0. Here intermediate torques at common nodes will be 0, but because of free-free end this torque will also be 0. So, this is the application of the boundary condition, so because all vector here are 0. So, none of the rows and columns we need to eliminate them and this particular for free vibration this acceleration term we can able to write like this hmm this we already seen it.

And we can able to formulate the Eigen value problem as we did earlier. So, here we have inverted the mass matrix instead of the K matrix. So, Eigen value will be this 1. So, whatever the Eigen value will be getting square root of that will be the natural frequency. So, for D matrix which is inverse of mass matrix into K will be solving the Eigen value problem.

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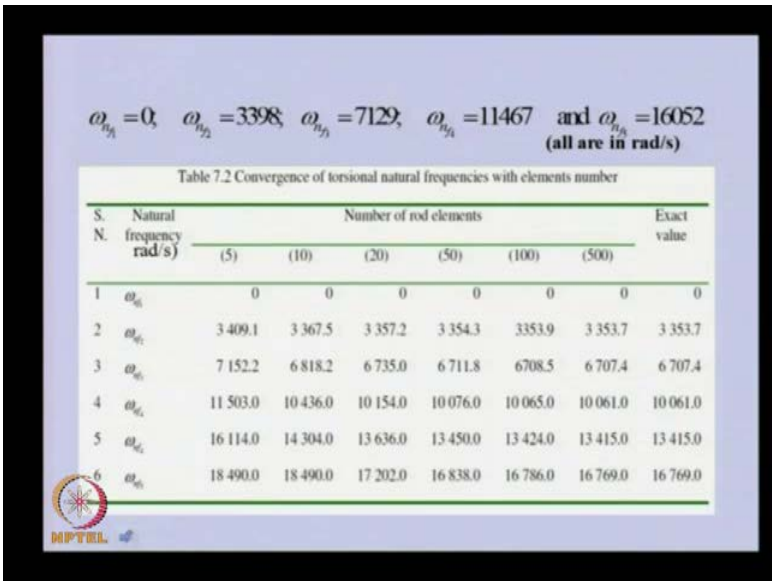

$\omega_{n_{f_1}} = 0;\ \omega_{n_{f_2}} = 3398;\ \omega_{n_{f_3}} = 7129;\ \omega_{n_{f_4}} = 11467$ and $\omega_{n_{f_5}} = 16052$
(all are in rad/s)

Table 7.2 Convergence of torsional natural frequencies with elements number

S. N.	Natural frequency rad/s)	Number of rod elements						Exact value
		(5)	(10)	(20)	(50)	(100)	(500)	
1	ω_{nf_1}	0	0	0	0	0	0	0
2	ω_{nf_2}	3 409.1	3 367.5	3 357.2	3 354.3	3353.9	3 353.7	3 353.7
3	ω_{nf_3}	7 152.2	6 818.2	6 735.0	6 711.8	6708.5	6 707.4	6 707.4
4	ω_{nf_4}	11 503.0	10 436.0	10 154.0	10 076.0	10 065.0	10 061.0	10 061.0
5	ω_{nf_5}	16 114.0	14 304.0	13 636.0	13 450.0	13 424.0	13 415.0	13 415.0
6	ω_{nf_6}	18 490.0	18 490.0	17 202.0	16 838.0	16 786.0	16 769.0	16 769.0

And square root of that Eigen vector will give these natural frequencies and that is for the 5 element. So, these are entered here, so you can able to see this for 5 elements these are the natural frequencies. Then if we want to take 10 element we will get these natural frequency will get more natural frequency, but I am summarizing only the first 6 1 of the natural frequency is 0. So, you can able to see that for 10 element there is hardly any very small change in the first natural frequency, but higher frequencies are get getting change more this is not changing.

So, this is for 10 element, but as we increase the number of element to 20 50 100 we will see that most of the value are changing very little; that means, we can able to see up to 20 element hardly 1 to 1 change is there 1 here there is some change here. So, if 100 is good enough I think for the convergence, but just for checking up to 500 if we take the element you will see that we the values are quite stable and these are the values corresponding to the close form solution.

So, we can able to see for 500 number of elements we are getting the exactly as the close form solution. So, this is the convergence study as we increase the number of element we. There is not much change in the natural frequency; that means, we can able to assume that we have achieved the convergence.

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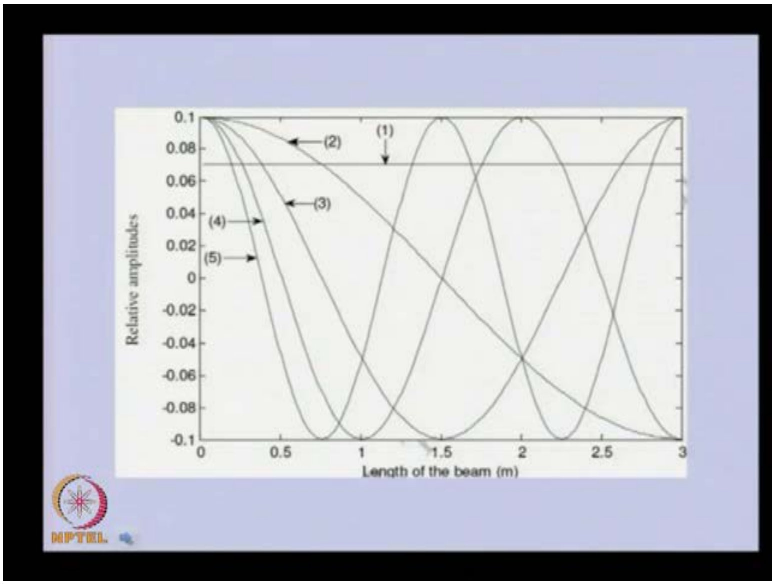

And these are the mode shapes which we can able to plot it. So, for first mode this is the rigid body mode, second mode is this 1, third mode is this 1, forth mode is this 1. So, you can able to see these are coming directly from the Eigen vector and the Eigen vector will give us the displacement at the node position in between, we can able to interpret using the safe function. So, these are the safe functions which directly we are plotting from the Eigen vector.

Today's lecture we have seen the how the elemental equation which we developed in the previous lecture can be used for obtaining the assembled equation or the equation for the system equation and once, we know the system equation we can able to apply the boundary condition of the problem at that stage. And then we can able to obtain the natural frequency of the system and also, the mode shape.

And the through various example we have seen how powerful is this finite element method and even though not only the complex boundary condition also based on the number elements if we increase the number of element even we can able to get the natural frequency quite close to the close form solution. And the subsequent class we will explore the finite element method for the gear element, and the branch system which we could not able to cover this particular lecture.