Theory and Practice of Rotor Dynamics

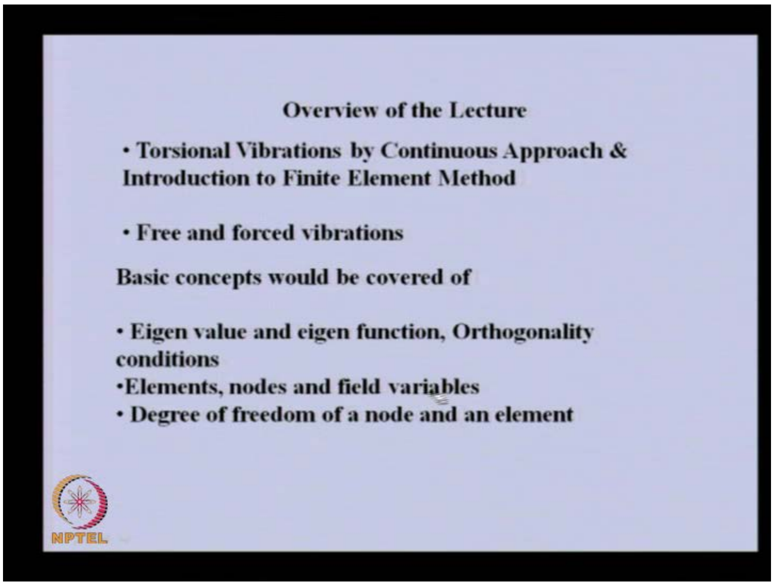
Prof. Rajiv Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 5
Torsional Vibrations
Lecture - 20
Continuous System and Finite Element Method

Till now in the Torsional vibration we have studied simple by Newton's second law, how to obtain the equation of motion? Also we have dealt with transfer matrix method for multi degree freedom system. In these cases all the rotors which we considered generally they the shaft of such rotors had only the stiffness property; we neglected the mass of the shaft. But in practical rotors, we find that not only the stiffness also the mass of the mass or the inertial property of the shaft is distributed throughout the length. So, these shafts are very heavy. And, because of that they have appreciable amount of polar moment of inertial.

And, that is distributed throughout the length; in such rotors generally when we want to model using discrete mass analysis, the analysis is not that much accurate. And, for such cases we have approach that is called continuum; continuum continuous system approach generally, we deal with such systems with continuous system approach. And today we will see very briefly this particular approach how to obtain the governing equation for the torsional case when the both mass and elastic property on the shaft is distributed. And, our main focus would be in the subsequent lectures to have finite element formulation of such system. So that it can be applied to real system.

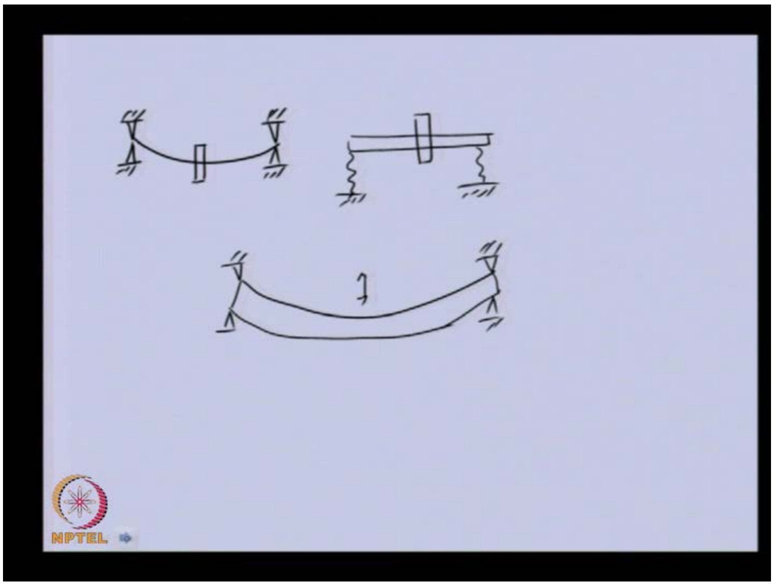
(Refer Slide Time: 02:28)



So, let us see what are the things we will be covering today. So, basically we will be developing the Torsional vibration analysis using continuous approach. And, we will introduce some basic terminology of the finite element method; in this free and forced vibration it is the main analysis which we do. And, some of the concept like Eigen value which is nothing but natural frequency, Eigen functions in discrete system it was nothing but the mode shape. Some other properties like orthogonality property; we will see which is there for the Eigen function especially for the continuous system.

Then, some basic definitions which generally we use in the finite element method like elements, nodes, field, variable degree of freedom of node, an element; these are basic definitions we will be introducing in this particular lecture. And, in subsequent lecture we will be dealing with finite element method in more detail.

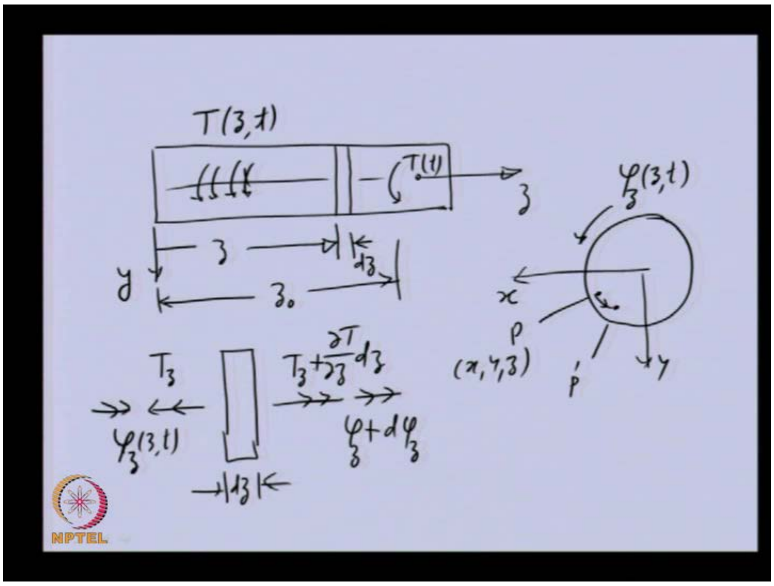
(Refer Slide Time: 03:47)



Let us see how this particular continuous system approach works and for this let us see what are the type of shafts we have considered. So, earlier we had one flexible shaft especially, for transverse vibration it is more clear. So, I am trying to explain this continuous system approach by transverse vibration and then we will be analyzing. So, this is a mass less flexible shaft and a heavy disc replaced. And, such rotors we analyzed by considering only the flexibility of the shaft and the mass of the shaft. In another case we took rigid shaft and flexible bearing; in this particular case the shaft we considered is rigid only the flexibility was there in the bearing. But in some cases the rotors are relatively heavy and flexible.

So, when we found them on bearing they not only have appreciable amount of inertia also they have elasticity; for such rotors we cannot able to distinguish where a particular mass we should able to concentrate at one location or several locations. So, generally in this particular case; we will see the elastic property and the mass property of the shaft we need to consider continuously. This was the example given for the transverse vibration in which we are interested in his kind of vibration. But the present study; we are interested in the Torsional vibration of such rotor system. So, let us see how this can be modeled.

(Refer Slide Time: 05:48)



So, for this particular case if we have a shaft which is; let us say this is the axis of the shaft z and this is the one of the plane of the shaft as shown here and at a distance z. Let us say there is a plane; the motion of that we want to analyze. In this particular shaft; we have various kind of loading like we can have distributed torque external torque acting. So, the torque will be representing as function of z; the axial position of the shaft also it is time dependent. And, if we have some kind of concentrated torque at some location; which is let us say at location z naught that is the location of that is fixed. So, this will call at as a concentrated torque.

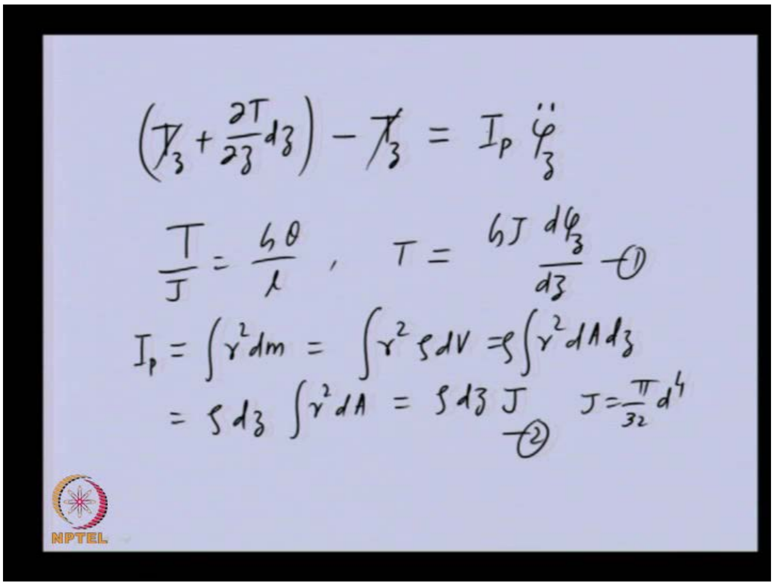
Now, we are interested in a shaft segment which is here. So, if we want to see that particular plane of the shaft with a distortion. So, let us say this is y and this is X axis this is basically center of the shaft. So, this will come like this. And, if we on towards steady motion of the point p; let us say the co-ordinate of that is x y z. I f we have angular twist of this particular plane psi z which is again function of z and time. So, this particular point p will have displacement in the radial direction; which we will call at p prime location due to angular displacement. And, in this particular hypothesis; we are assuming that whatever the loading supplied on this torque. Because of this particular plane which we are considering that will remain a plane; it will not distort during the motion.

And, a particular particle on this particular plane like p moves in that particular plane. So, you can able to see that in this plane itself it is moving; it is not going out of this plane. Now, if we want to analyze the torque balance for a particular shaft segment. Let us say we are taking a very small shaft segment having thickness d z, and if we want to draw the free body diagram of that to obtain the equation of motion of this. Let us take this particular shaft segment and this particular shaft segment as we have seen the thickness in d z. And, once we have taken out from the shaft; the reaction torques will be obtained on both sides of this particular thing.

So, let us say in this particular direction the torque t z is acting which is reactive torque which is coming from the other end of shaft. And, we have because inertia property is continuously changing. So, we expect that the torque will change at further plane. And, this will be given by this expression even the displacements angular displacement I am representing that; let us say positive direction this one psi z which is function of z and T. And, this direction also because from left to right is the positive direction for the z.

So, here we have the displacement as psi z plus d psi z; change in the angular displacement. Because of this inertia property there will be change in this particular angular displacement also. So, now these 2 are the torqueses which are coming out this particular; these are the 2 torques which are coming out of this particular shaft segment. And, now using Newton’s law second law; we can able to obtain the governing equation and for that we will be equating the torques. So, I will take in the next slide.

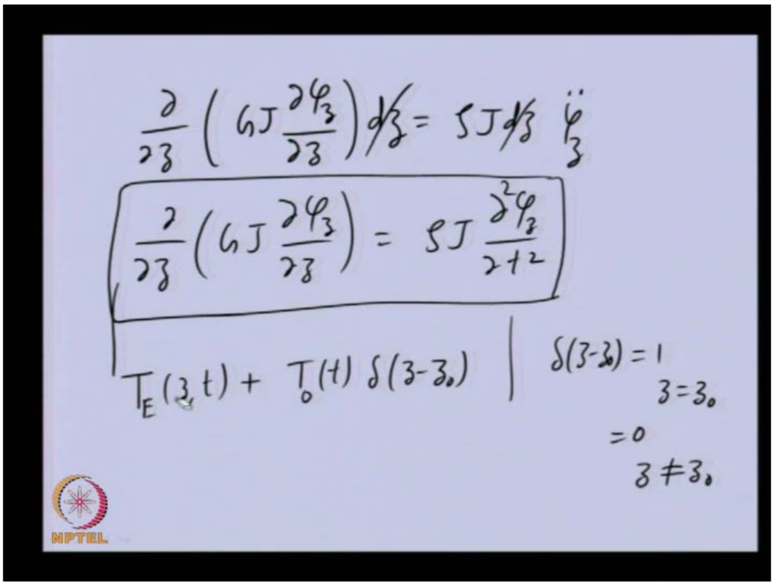
(Refer Slide Time: 11:43)



So, T z plus delta T this is the torque; which is acting on the right side of that shaft segment minus the shaft segment torque in the left plane. And, that should be equal to the inertia, rotor inertia of the shaft segment. And, this now in this because from Poisson theory we know that T by J is equal to G theta by l. So, we can able to get the torque as G J. And, this theta is the twist of the shaft segment which is this for this particular case is this much and the length of the shaft segment is d z. So, you can able d z. So, you can able to see in the previous slide; the related twist of the 2 planes is difference of this angular displacement minus this. So, that is the d psi z and the shaft segment length is d z. So, we got this one.

And, the I p which is mass polar moment of inertia is given as r square d m; d m is the mass of the shaft segment. So, we can able to write this as r square rho d v this is the mass. And, then this we can able to write it as area and d z and rho; we can take outside and this can be simplified as rho, d z. Because d z will come out r square d A and r square d A is nothing but the J second moment of inertia that is polar moment of inertia of the shaft; which is given as J is phi by 32 power d 4 for circular shaft. So, this expression and this expression; we can able to substitute in this governing equation. In this governing equation you can able to see that this2 terms are getting cancelled and here we will substitute torque and here we will substitute I p.

(Refer Slide Time: 14:29)



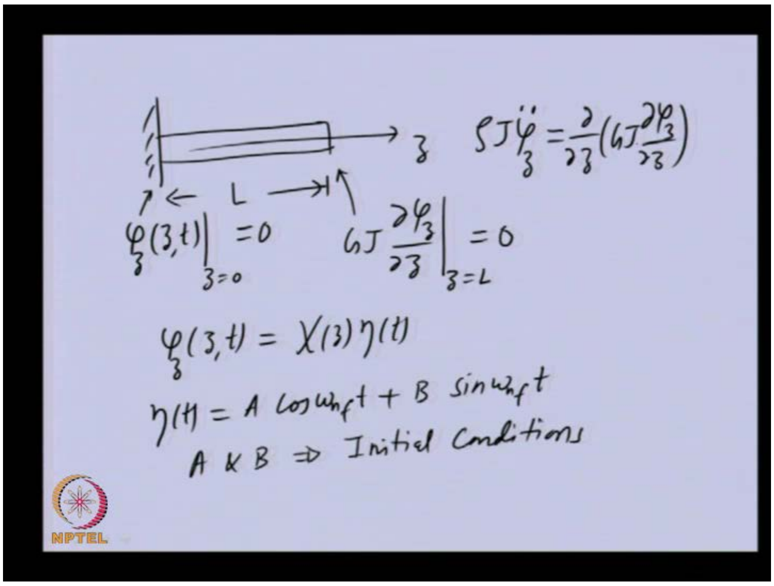
So, d by d z or for torque; we will write G J this for the torque and d. So, again you can see; we have substituted for torque and this on right side we need to substitute for I p. So, rho, J d z and phi double dots. So, this is double dot is representing the time derivative with respect to time derivative and this is a double dot is a double derivative with respect to time. So, you can able to see that this will get cancelled and we were left with equation of motion of the continuous shaft for the Torsional vibration; which will be given as like this. So, this is the equation of motion of the continuous system; in this particular case we did not considered the external torque.

But if we want to consider the external torque; in the previous slide here the we will be having external torques. So, you can able to see that if we want to consider the external torque here; we will be having distributed external torque and concentrated torque which was at the location of the z naught. So, we will be using a direct delta function; delta to specify that particular torque. So, you can able to see this direct delta function; z minus property that when we have this will be equal to 1 and we have z is equal to z naught; and this will be 0 when z is not equal to z naught.

So, you can able to see that when we are mentioning; we are specifying the z is equal to z naught then only this will act otherwise it will not act. Because this is a concentrated load and this is the external load which is distributed over certain length that will be specified; we have obtained the equation of motion of a continuous shaft for which Torsional vibration; we have seen that this particular equation of motion is partial differential equation. It is having derivative with respect to z the special derivative and as well as derivative with respect to time.

And, this particular equation of motion represent because till now we have not considered any boundary condition. So, if we consider boundary condition the solution of this particular differential equation; which we will be calling as boundary value problem will be unique. So, let us take very simple example of a cantilever shaft for that we will solve this particular differential equation. And, this particular equation is having similar form which generally we study in the mathematics that is the; we call it as a wave equation. So, this equation is exactly same in form as wave equation; only thing is the variables are different. So, now let us see we will solve this particular wave; differential equation for the boundary condition of the cantilever beam.

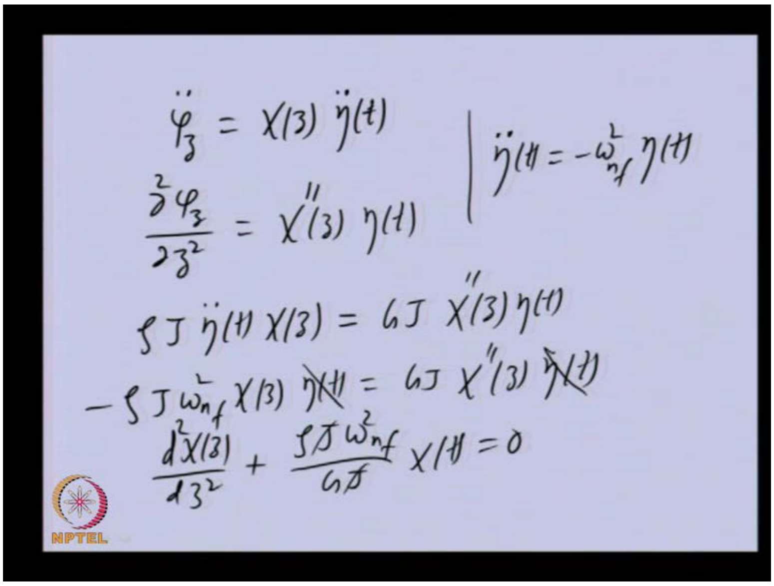
(Refer Slide Time: 18:38)



So, we have 1 shaft is having cantilever condition; you can able to see this is the Z axis length of the shaft is L. And, it is undergoing Torsional vibration; in this particular case you can able to see that at the fixed end the boundary condition is that this particular displacement which is function of time and z; at z is equal to 0 this displacement is 0 at this fixed time. And, other end is free end; there is no torque and we have represented torque earlier like this. So, at z is equal to L we have the condition that torque is 0; because this is a free end. Now, to solve the differential equation which we had obtained earlier that is the I am writing again phi z is equal to G J phi z d z; we will be using the separation of variable method to solve this.

And, in this method we assume the solution in such that we have 2 parts of the solution; one function is purely special coordinate dependent that is z another is time dependent. And, this time dependent function is generally harmonic in nature. And, the form of this particular function harmonic function; we know is cos omega n f t plus B sin omega n f t where, omega is the frequency t is time A and B are constants, and A and B; we will be obtaining based on the initial condition of the problem. And, the initial conditions will be obtaining these 2 constants. Now, if we take that is because in this particular equation; we can able to see that we need to take time derivative of this particular function and also the special derivative here.

(Refer Slide Time: 21:43)

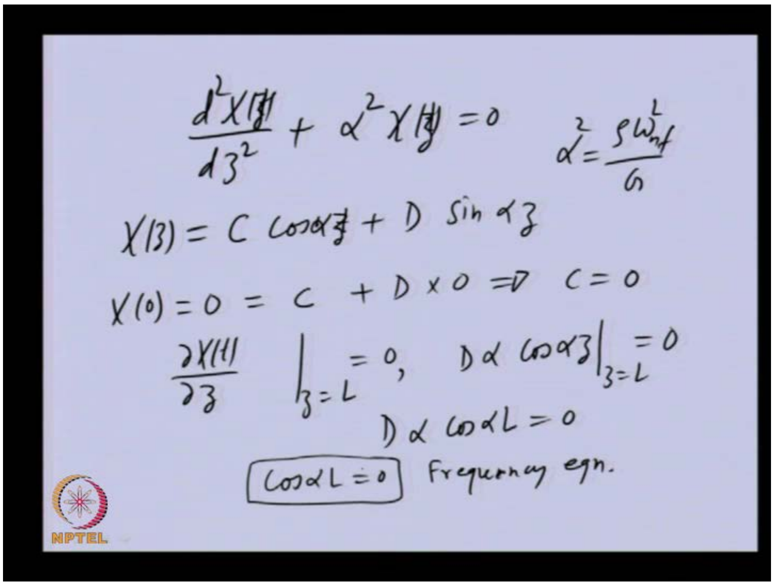


And, so now I want to differentiate this twice with respect to time. So, obviously this will not derivate this chi function only it will derivate the theta function; which is time dependent and if I want to differentiate this with respect to z. So that will go into the chi I am representing that as prime; to represent the derivative, partial derivative with respect to z and this will be as it is. So, these 2 we can able to substituting in the equation of motion. And, if you substitute this in the equation of motion; we will in the previous equation here. So, we will get rho, J, eta double dot (t) this chi (z). So, this in the left side is equal to the right side; we have because this particular shaft we have considered uniform.

So, G J will be constant. So, it will come out. So, we will be having G J and double derivative of that function. So, we will be getting chi, double derivative with respect to (z) and eta (t). Now, if we see the harmonic function is having property this particular; if we take double derivative of this with respect to time. Basically, we will get a relation as plus this is the condition of the harmonic motion. And, if we substitute this here; we will get rho J minus omega square n f and chi (z) eta (t) is equal to G J chi.

So, you can able to see that this chi will be common and it will be get canceled. So, we will left with a differential equation which is now only function of z; that means, this is now ordinary differential equation; we could able to convert the partial differential equation to ordinary differential equation. So, this will be having the form of this and here this J will also get cancelled.

(Refer Slide Time: 24:53)

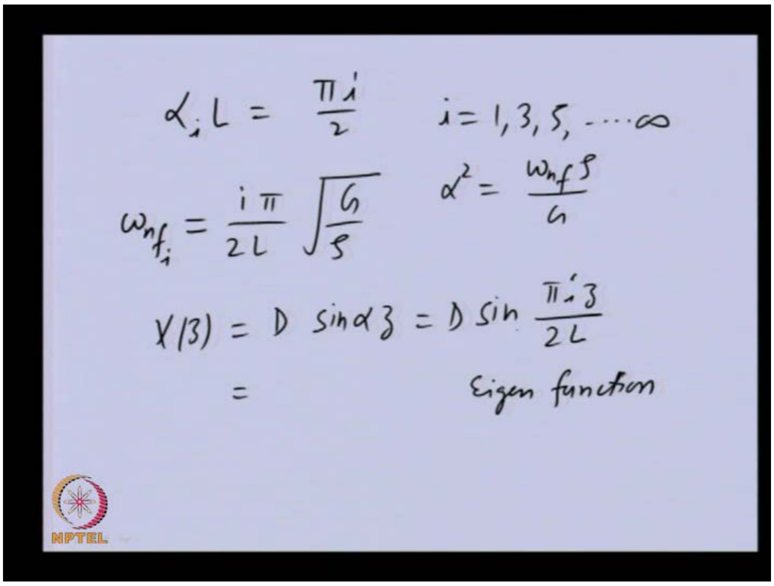


And, this we can able to write it as chi (t) by z square plus I am calling that constant as alpha square chi (t) is equal to 0; where alpha square is as we can able to see here is rho omega square n f by capital G, G is the modulus of rigidity. Now, this particular differential equation which is now ordinary differential equation; we can have the solution of this is only z, function of z, not function of time. So, the function of the solution of this will be of this form; C constant cos alpha t alpha z plus D sin alpha z. So, this is the solution which we are assuming of this differential equation; where C and D are constants and that will be obtaining from the boundary condition of the problem. And, for a cantilever case which we are dealing with; we have seen that these are the 2 boundary conditions. So, from these 2 boundary conditions we will be obtaining the C and D value.

So, let us see the first condition that at z is equal to 0 this displacement is 0. So, we will get this as 1 plus D into 0. So, you can able to see that we are getting C is equal to 0 from first boundary condition is fixed end boundary condition second boundary condition we have is this torque is 0 as eta here in case of phi z we are writing the assumed solution this solution and because there is a differential differentiation with respect to z. So, this will be differentiated this will be outside this one. So, this is 0 so; that means, we need to derivate this with respect to z first and we need to substitute here. So, we can able to remove this quantity also this is this will not be 0.

So, I am differentiating this and we know that C is 0 itself. So, only we need to differentiate this D alpha cos this. So, this is the derivative of this particular term and at z is equal to 0, z is equal to L, this quantity is 0. So, if we substitute D alpha cos instead of z; we will write L is equal to 0. So, you can able to see either D if it is 0 then the whole motion will be 0. Because C is already 0 d is 0 motion will not take place we are not interested in that. So, D cannot be 0. So, for that case this alpha in general cannot be 0. So, we need to have a function this cos alpha is equal to 0. So, this is the frequency equation this is the frequency equation and the root of this will give us the Eigen values. And, from there we can able to obtain the virtual frequencies.

(Refer Slide Time: 29:00)

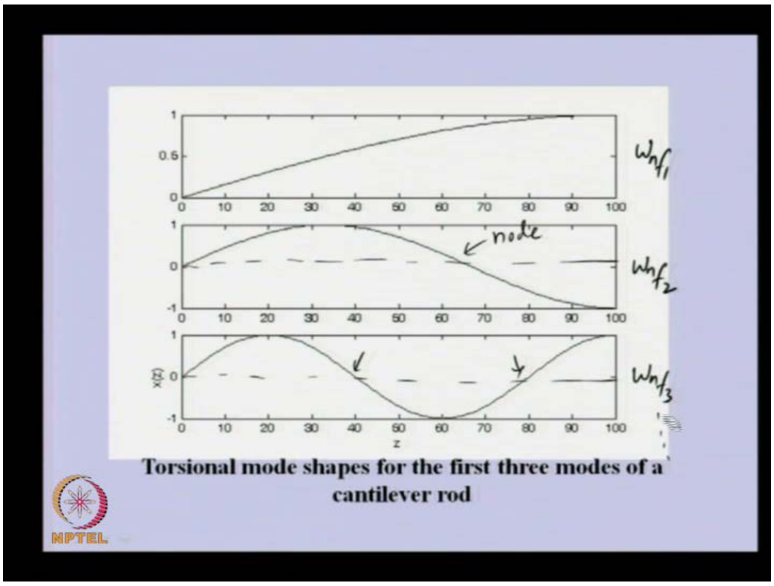


So, you can able to see the solution of this alpha, L is equal to pi i by 2; where i can be odd numbers and the infinite number of solutions. So, infinite numbers of Eigen values are possible. And, earlier we have related alpha with natural frequency of the system by this relation. So, we can able to get the natural frequency from this that will be as i pi by 2 L. So, after simplification; we can able to get the natural frequency from these 2 expressions like this. And, here i is again varying from 1, 2, 3, 5 these are the various natural frequency of the cantilever shaft.

Now, the Eigen function; the assumed solution we had C is equal to 0. So, Eigen function is left nothing but psi (chi) is equal to sine alpha z that again you can able to and here some constant; we can able to attach and alpha is already given there. So, this will be pi i z by 2 L. So, this will be the Eigen function; an Eigen function will give us how the relative parts in the whole shaft will be having displacements. Because in this particular case each and every particle point on the shaft; will be having relative displacement with respect to each other. So, this function will give us the relative displacements of various particles in the shaft system.

In continuous system approach as we obtain the natural frequency and the Eigen function which represent the mode shape, in continuous system because the mass property is continuously distributed. So, basically this system can be considered as an infinite number of degree of freedom. Because each and every particle is are independently moving with respect to each other. And, we have seen that we are getting infinite number of natural frequencies; because we have infinite degree of freedom in the continuous system approach. So, now whatever the Eigen function we have obtained; let us see the plot of that for first 3 modes.

(Refer Slide Time: 32:11)



So, this is the plot of the Eigen function. So, you can able to see this is the position of the shaft this is the fixed end of the shaft and this is the free end of shaft and this is the angular displacement and this is a relative angular displacement. So, we have taken the maximum displacement as unity; which is for this particular case for first mode. So, this is belonging to the first natural frequency; corresponding to i is equal to 1. So, this is the first mode and for second natural frequency; the mode shape will look like this is the fixed end maximum is taking place here and here also, here plus, minus and here minus 1; this we have normalized because this is a relative displacement. So, the maximum we are we are taking as 1. But here you can able to see that if we draw the 0 line; there is a one place where the shaft displacements are 0. So, and either kind of this particular shaft is having opposite angular displacements this particular friction is called load.

As we have seen earlier also. So, there will not be any angular displacement of the shaft at this particular case. But either side of these 2; we will be having opposite motion again I am repeating we are talking about the Torsional angular displacement. So, the shaft at shaft particles which are left side and right side they will be having opposite motion; similarly, this is the third one is the third natural frequency. And here if we draw the 0 line; we will see that there will be 2 nodes where 3 will not be any angular displacements.

But there will be angular displacement in either side and these 2 will be having opposite motions and these 2 will be having again opposite motion. So, and we can able to plot this for higher natural frequencies also. And, we expect for such cases additional loads will be coming as we will increase the natural. For the continuous system approach we have obtained the natural frequency and mode shape; for 1 particular boundary condition this was cantilever case; in continuous system approach we can able to obtain this natural frequencies for simple boundary conditions and few more boundary conditions solutions I am providing here.

(Refer Slide Time: 35:04)

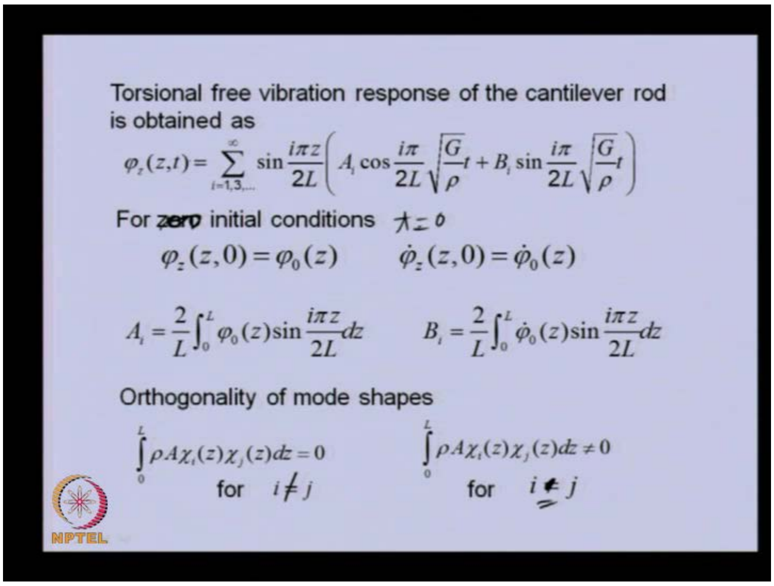
Table 7.1 Natural frequency and mode shapes for torsional vibrations of rods

S.N.	Boundary conditions	Natural frequency (rad/s)	Mode shape
1	Fixed-free	$\frac{i\pi}{2L}\sqrt{\frac{G}{\rho}}$, $i=1,3,5,...$	$\sin\frac{\pi i}{2L}z$
2	Fixed-fixed	$\frac{i\pi}{L}\sqrt{\frac{G}{\rho}}$, $i=1,2,3,...$	$\sin\frac{\pi i}{L}z$
3	Free-free	$\frac{i\pi}{L}\sqrt{\frac{G}{\rho}}$, $i=0,1,2,...$	$\cos\frac{\pi i}{L}z$

So, like we have a Free free v shaft is there is no support at both ends then the natural frequency will be given by this expression; here the Eigen function will be having this function sine function. And, for another case in which both the end of the shaft is fixed; in this particular case natural frequency will be given by this and the mode shape will be like this is also sine function. But there is some difference; for this is the fixed free case this is fixed fixed case and this free free case. So, this is fixed free this is similar to the cantilever; this is a cantilever this is fixed fixed, and this is free free and these are the natural frequency and mode shapes.

So, basically we need to satisfy the boundary conditions to get the constant that C and D and depending upon that we can able to get the this equations; once we obtain the Eigen function for a particular disturbance. If we are talking about free vibration, if we are giving a disturbance to the system how the system will vibrate will be given by the expansion theorem; in which we mention that any free vibration we can able to express in terms of these Eigen function contribution from these basic Eigen functions. And, contribution of various Eigen function will depend upon the initial condition.

(Refer Slide Time: 36:50)

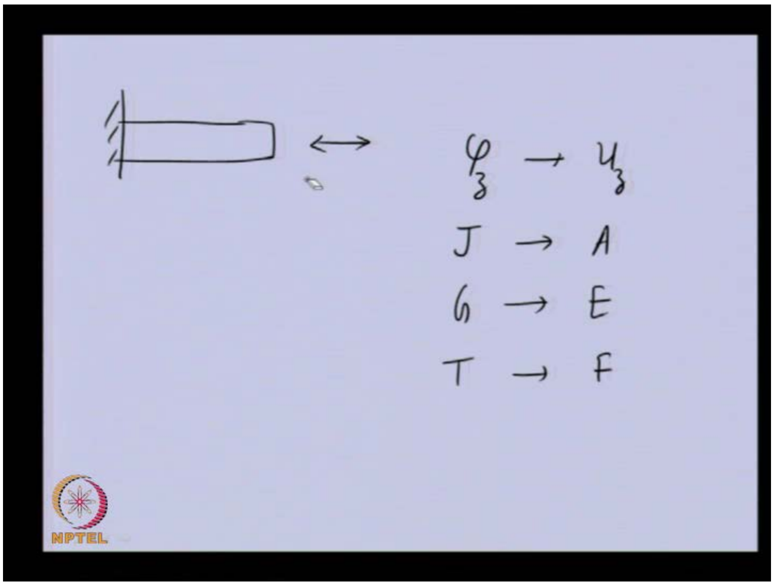


So, let us see this particular torsional free vibration; how we can able to express in terms of the Eigen function. So, this is for the case, for the simply support, for the case of the cantilever beam case. So, this was the Eigen function and this is the harmonic function. So, you can able to see that we are expressing any general free vibration in summation of various Eigen function and the harmonic function; where A and B is the contribution of various modes. And, this will depend up on how we are giving the initial conditions and for 0 initial condition like this if our initial conditions is not 0, initial condition at time t is equal to 0; for initial condition t is equal to 0.

If we have these angular displacement and angular velocity this constants we can able to get by this basically to get this we need to multiply both side of this equation; by the Eigen function for 1 particular mode and we need to integrate over the domain. And, while integrating we will be using the orthogonality of the mode shape that means when these 2 mode shapes are same when they are not same then this is 0 and this should be equal to. So, when they are same then it will not be 0. So, when 2 Eigen functions are same this quantity will not be 0; and when this 2 Eigen functions are different then this will be 0. So, you can able to see that once we are applying multiplying both sides by Eigen function of the ith mode.

The terms here will be 0 corresponding to all modes except i and that is the case here. So, these have been obtained using this orthogonality condition of the motions. So, you can able to see for different initial conditions; we can able to get these constants and the free vibration can be described. Now, we have analyzed the Torsional vibration; the analysis for the axial vibration for continuous system is exactly similar to the Torsional vibration only thing is the some of the variables we need to interchange.

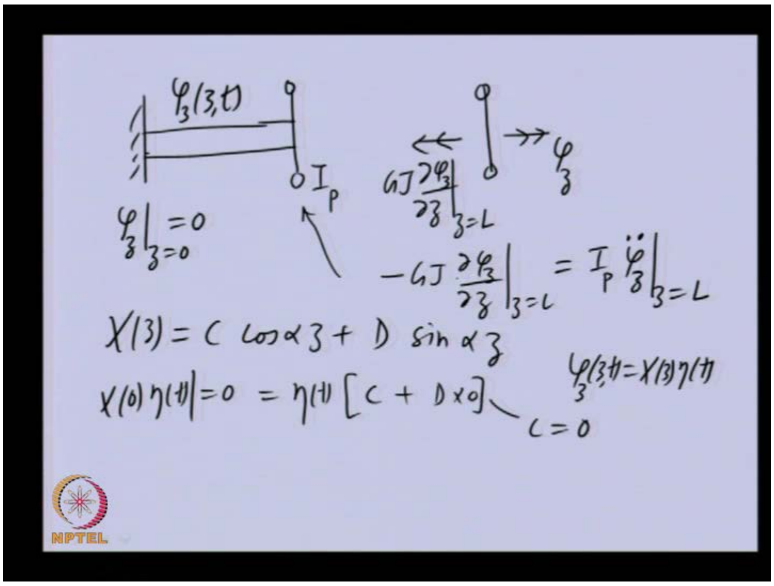
(Refer Slide Time: 39:49)



So, for axial vibration; so if we are talking about a shaft having axial vibration that means logarithmical vibration. So, if you want to analyze this system; we need to replace the angular displacement which we took in the torsional vibration by axial vibration that is a linear vibration. And, the J second moment of area the polar moment of area; we need to replace with A and G we need to replace with E and torque we need to replace by force. So, these will be the changes in the equation of motion and otherwise the analysis is exactly same differential equation Eigen function everything is same; only we need to replace these variables then we can able to analyze the axial vibration. So, we will not be repeating the axial vibration analysis here. But just we are trying to show the analogy here.

In the continuous system approach as I mentioned; for more complex boundary condition or for multiple disc or multiple support the obtaining solutions are not easy. I will just show 1 particular case in which the same cantilever beam; in which we have the shaft is having continuous continuously distributed stiffness and mass property. But along with that there is a concentrated disc at the free end. And, this additional of the disc how the complexity increases because of this we will try to see.

(Refer Slide Time: 41:50)

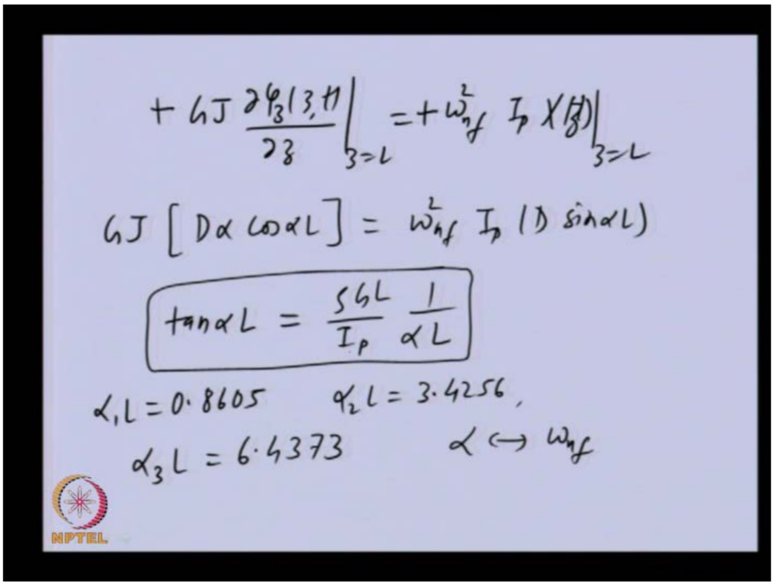


So, we have in this particular case a cantilever shaft having distributed mass and stiffness property and there is a concentrated disc at the end and let us say the polar moment of inertia of that is I p. So, in this particular case we can able to see the boundary conditions; this particular disk either we can take in the boundary condition. So, fixed end boundary condition is same that is this is exact is equal to 0 is 0; here to obtain boundary condition let us take the free body diagram of the disc. So, in this particular case; if this is the positive direction for the angular displacement there will be reactive torque from the shaft on to this that will be obtain, that will be acting and that will be this at z is equal to L.

So, from this we can able to get the boundary condition here; that means if we take the equation of motion of this. Because this particular torque is acting in the negative direction; this is the only torque acting on this particular disc should be equal to this is at z is equal to L should be equal to inertia, this is the polar moment of inertia of the disc and psi z, z at z is equal to L. Because this disc location is at z is equal to L; this angular displacement which is basically in general representing the displacement, angular displacement of the whole shaft at any location of the z. So, we need to specify that this angular displacement which we are talking about for this disc is at z is equal to L. So, this is the equation which we have obtained this is the boundary condition of the problem earlier because there was no disc. So, right hand was 0. But because there is a disc this is having right hand side which is this.

Now, let us see how this boundary condition gives the complexity in the solution; we have the equation like this in which C and D we need to obtain with the boundary condition. So, for this boundary condition; we because this phi z we choose as multiplication of this and time function. So, in this particular case time function is not there. So, basically we can able to write this as. So, I am satisfying here. So, psi at (0) eta (t) at this at z is equal to 0 is 0. So, I can see eta (t) C plus D into 0. So, this we already seen that for this particular case, because eta cannot be 0, C this gives the C is equal to 0; this was the similar condition like the previous one the complexity is there in the second boundary condition now. So, now we need to substitute here the solution. So, I am taking this in the second slide.

(Refer Slide Time: 46:01)



So, the boundary condition I am rewriting like this and because we have the harmonic function eta. So, we can able to write. So, basically here if we substitute this here; we will get negative of that we will get negative of this. So, they will get cancelled. Now, we will satisfy the boundary condition. So, you can able see that if we take the first derivative of this because C is already 0. So, this will be this similar to the previous one we have substituted for psi is equal to. So, here also we are substituting for this. So, we can able to see that now this equation; we can able to simplify as tan alpha L is equal to rho G L by I p, 1 by alpha L.

So, with some rearrangement we can write this equation like this. So, here you can able to see that the solution of this is not easy; because we need to solve for alpha which is here also. So, this is a transcendental equation. So, this we need to solve numerically and few solutions of this I am providing here. But will be having infinite number of solutions. So, first 3 solutions are these and once we have obtained this; because we know alpha and omega n f are related. So, natural frequency can be obtained. But you can able to appreciate that with a addition of the mass itself the frequency equation which was earlier simpler now it has became a transcendental equation and solution of those roots are not easy now.

So, in the present lecture we introduce the continuous system approach for torsional vibration; this particular system is most accurately we can able to model the system, we can able to get the natural frequency and Eigen function. But the difficulty is for very simple boundary condition we can able to get the solutions for more complex conditions; we need to go for approximate solutions And, the subsequent class we will choose finite element method for solution of such system; in which we will be because we know that finite element method is having flexibility in incorporating more complex boundary conditions.

So that particular method will be exploding in 2 lectures just 1or 2 lines more I will tell; in this continuous system approach as we have seen that we solve the system equations as a whole. In the subsequent lecture we will see when we will be using the finite element method; in that particular method we break the system in various small segments that we call it as element and we obtain the elemental equation. And, once we have obtained the elemental equation then we obtain the system equation by assembling such elemental equations. And, then finally we apply the boundary condition to get the governing equation for the whole system. So, that particular method we will see in the subsequent lecture.