Theory & Practice of Rotor Dynamics
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Module - 5
Torsional Vibrations
Lecture - 17
Transfer Matrix Approach

In the previous class, we introduce a new approach of transfer matrix method to analysis the torsional vibration. Today, we will extend the that particular method and we will try illustrate the methods capability through some examples, so that the method application is more clear. Part one that we will see in the part matrix in the field matrix which we known of the earlier, in which we do not consider the dumping, torsional dumping. So, what will be the changed in the part matrix and field matrix if we include the dumping also, in the shaft system ordinary, in the disc system we would see.

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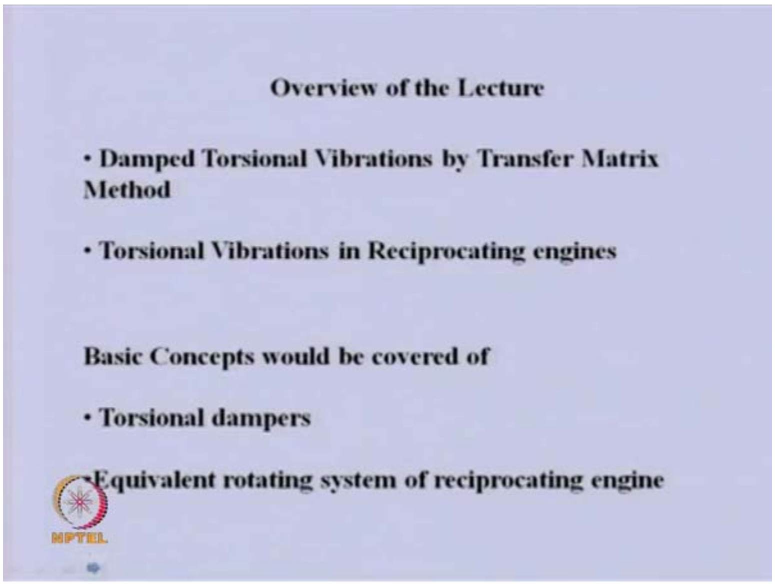

Now, let us see the over view of what will be covering, so dumped torsional vibrations will develop the transfer matrix for that. Even we will see torsional vibration in reciprocating engines, some of the basic concept, which will cover is torsional damper. Equivalent rotating system of reciprocating engines, so these are the things we will try to cover in the present lecture. At the beginning, let us start with a simple example, so the m m method is more clear. So, for that propose I am taking very simple example of a mass less shaft.

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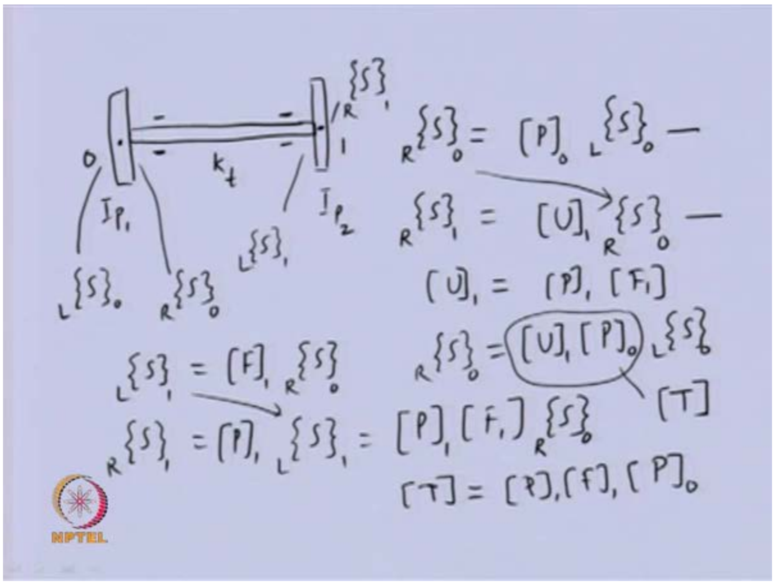

Both ends there are two rigid disc and there over all moment of inertia are as I p 1 and I p 2. This is bounded on factualness bearing. It is free to assault on this bearing, so such bearings are not resisting the motion of this particular rotor system. In this particular case, if you want to earlier solve this using the Newton's law, now will be using the transfer matrix method to obtain the natural frequency for such system. Let us say this stiffness of the shaft, which is torsional stiffness is k t. Now, if we want to attach stations, so let us say this is 0 stations and this is 1 station, because the rotor system is simple, so we can have only two stations.

Now, we can able to relate this state factor or let us say this state vector at 0 station; that means left of 0 disc and we want to obtain state vector right of 0 station. So, we know that the state vector here is left of 0 disc and here will be right of 0 disc and in between this there is a disc. So, we need to multiply this particular state vector with point matrix. We know the form of the point matrix you already obtain earlier. Now, if you want to transfer this state vector right of 1, if and we want this state vector from right of 0 stations.

So for this propose we already seen that we need to multiply by the transfer matrix for U 1. U 1 we already seen is F 1 into P 1. So, you can able to see that this ordering is very important. How we obtain, let us see again for clarity. So, basically we first need to obtain state vector left of 1 from state vector right of 0. So, state vector left of 1 is the small, so if we want to transfer the information of state vector from here to here, there is a shaft. So, corresponding we need to multiply by the field matrix. Now, we can able to get the state vector right of station 1 is left of 1, so right of 1 is here. So, you can able to see that in between the right of 1 and left of 1, there is a disc.

So, we need to multiply by point matrix and if we substitute this here you will get P 1 F 1 S early 1; that is R naught, this is subscript of the S. So, you can see that the you matrix is having not this particular form the ordering using important P 1, P 2, F 1, so this techniques clarified through systemic approach that U, which we obtain earlier was having in form of this, in which P into F 1 was obtained. Now, we can get the state vector or you can able to see that here you can able to substitute this.

So, you can able to get the state vector right of 0 is U 1 into P naught S left of 0. So, basically the first equation, this equation substitutes in the second equation here. So, we got the this particular thing, so you can able to see that this matrix, which is obligation of three matrix is we let us called this as over all transform matrix. The form of the t matrix is U is P 1 F 1 and P naught. This the oral transfer matrix.

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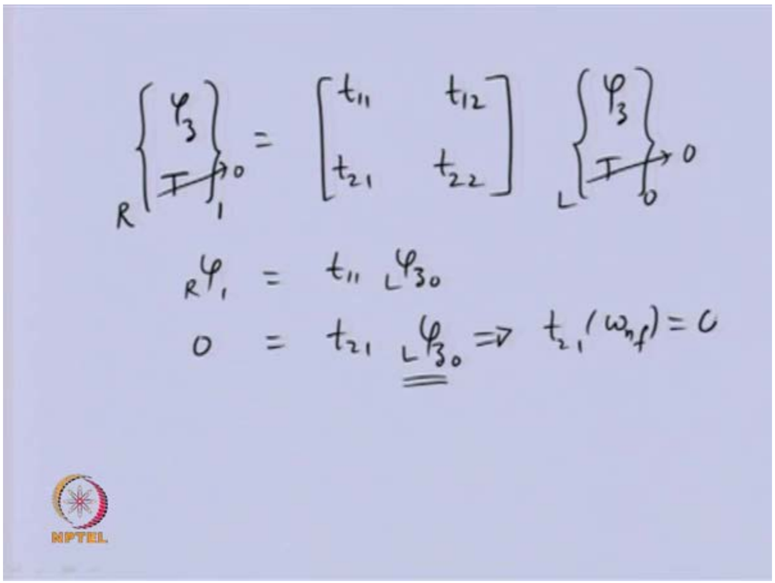

Now, let us see this state vector in expanded form which we obtain earlier. So, we derived right of 1 as t matrix and this is P z t left of 0. So, let us say this are after multiplication of 3 matrix is we get this t 1 1, t 1 2, t 2 1 and t 2 2. In the previous case because both the ends of the disc are free.

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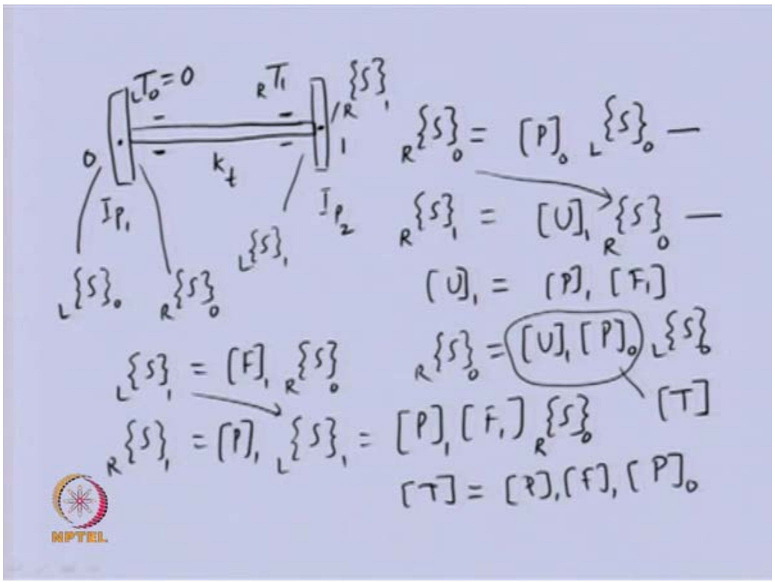

So, torque at 0 station is 0, also target 1 stations right of 1 is 0, left of 1 is 0. So, right of 1 is 0 and left of 0 is 0. So, you can able to see that if you expand this, you will get from first equation. This is from the second equation and because this displacement cannot be 0 at free end, so we need have from here t 2 1, which is function of natural frequency is equal to 0. So, we have seen that how we can able to apply in the boundary conditions once we obtain the oral transfer matrix. How the frequency equation can be obtain? Now, let us see in more detail after multiplication of various part matrix and field matrix. How the this oral transfer matrix take the form, so that the things are more clear in the form of equations frequency equation. How we can able to get?

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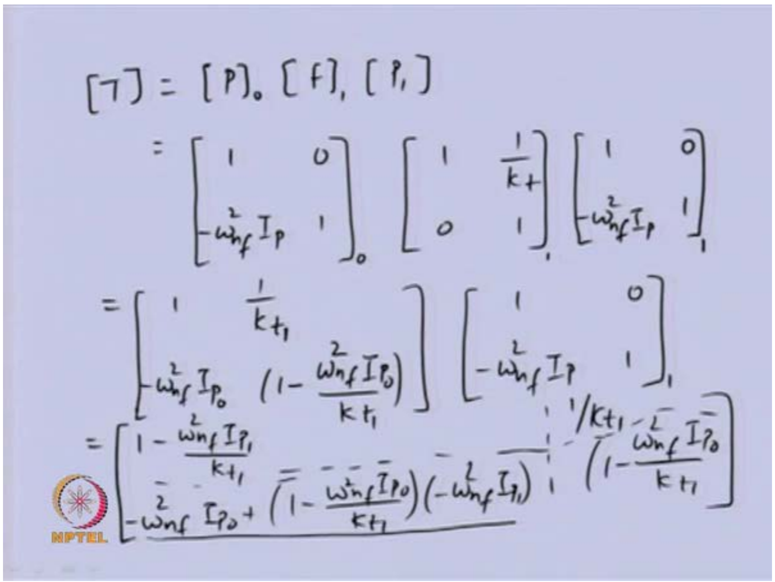

Now, let us see the T matrix? T matrix we had is P naught, F 1 and P 1, so if we the same doing for illustration, but larger system we need to program. Then so that we can able to easily obtain the roots of the frequency equations. So, this is the point matrix for 0 stations field matrix is having this form for shaft one. Then the field matrix of part matrix for disc two or disc one is station 1 is this 1. Now, you can able to multiply this, so if we multiply this two part matrix at 0 station and field matrix at station 1, you will get.

Now, you can able to attach the knots and the one here, because this is belonging to the 1 and here will be having natural frequency is for the system. So, we will volatize in a subscript for that and the part matrix for this stations 1. Now, if you multiply this you will get I p 1 by k t 1, then 0 1 by k t 1, then minus omega n F I p naught and plus 1 minus omega square n f I p naught by k t 1 and minus omega square n f I p 1 and the last term is 1 minus omega square n f I p naught k t 1.

Basically this be can able to shift here, so that is more clear 1 by k t 1. So, here 2 by 2 matrix is the division of the matrix supplication of the matrix. Now, you can able to see that, if we go back to the this k 2 1 is the frequency equation that need to be equated to 0. So, you can see that this particular equation is the frequency equation we need to equate this equal to 0.

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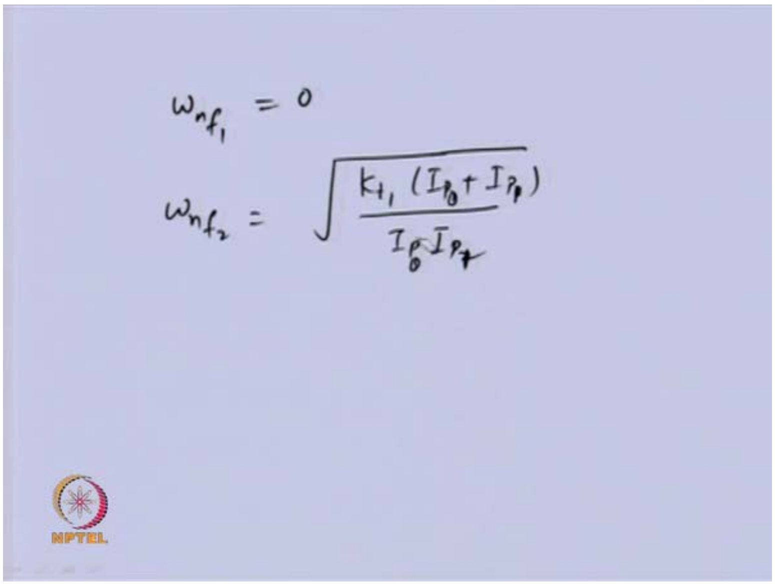

If we equate this equal to 0, we will get two roots; one will be 0 and another will be exactly same which we obtain earlier. This one here basically I have replaced or I should have written 0 1 0 1 earlier. It was 1 and 2, the analytical expression when we derived there were 1 and 2, but now because we have attaching the 0 and 1 stations. But the form of the equation is exactly same as we obtain using the close form solution. Then natural when the system is very large, then we will be having difficulty in multiplying. This matrix is a analytical form or symbolic form, so in that particular case generally we use numerical methods to solve the frequency equations.

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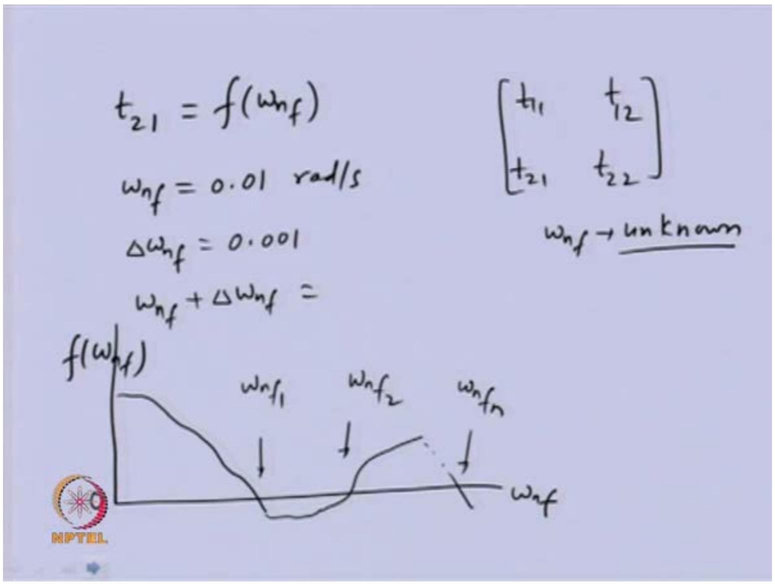

So, let us see how it can be solve? So, let us say the frequency equation which is, which will be function of omega n f is this one. Now, we cannot have explicit form of this particular component, so like for previous example we had t 2 1 of the oral transfer matrix, which is function of omega n f is equal to 0 be aspect form to get the natural frequency. So, let us say we are equating this particular component as a function of omega n f and because the T matrix, which we obtain there multiplication of various part matrixes and field matrixes.

In all of them basically natural frequency is are unknown, so unless we know this numerically, we cannot able to multiply the part matrix in fields matrix. So, for that propose when we solve the natural frequency using numerical technique, we assume some initial gas of the natural frequency. Let us say small value on 0 1 radians per second. Once we assumes this, then we can able to see with a this particular function the frequency equation with the… It is satisfying this particular function, because this initial case. So, we do not aspect that, let the first initial case this will be satisfied.

They will be some residual and a obviously, then we need to take increment in the natural frequency assumes natural frequency; that inclement can be further less depending upon accuracy that we need. So, this will be the new natural frequency predict there is a for the second step. We will again substitute this in the function and will see whether reduces and increases. In fact we can able to have a plot of the natural frequency chosen and this function because now you can see that when we assuming some natural frequency this oral transfer matrix.

We can able to get in numerical form because all the part matrix and field matrix will be known to us. We need to see just the this particular t 2 1 component for the example which we are shown for different boundary condition different component we need to observe. So, t 2 1 for free condition, we found as a frequency equation. So, once we plot this function, we may see that may intersect the 0 line of the natural frequency. So, that mean this are the, this is the one root of the natural frequency of the frequency equation. Depending upon the degree of freedom it maintain separate different places, so in general if we have n degree freedom, we aspect n number of such intersection to get the n natural frequency of the system.

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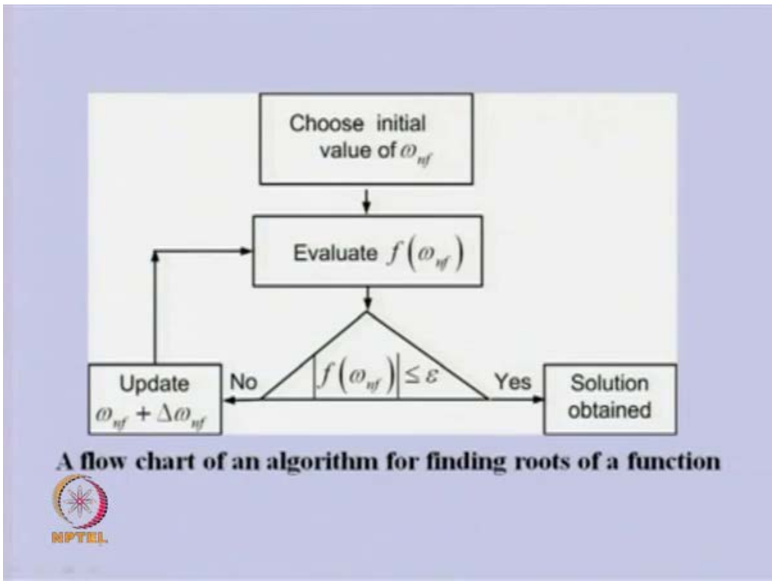

A flow chart of an algorithm for finding roots of a function

This can be illustrated through this particular flow chat that we assume some initial value of the natural frequency. Estimate the frequency equation value and we will we see that this particular frequency value, because numerically it cannot be perfectly 0. So, very small quantity if it reaches, if it is very small then there is solution, if know then we need to again take another step of the natural frequency. Again we need to evaluate the this particular frequency equation, so this cycle will repeat till will get desire accuracy of the natural frequency.

That will be the solution and once we get one solution, we can search for more depending upon the degree freedom of the system. So, we will be seen that for larger system what would be the processer to solve the frequency equation in this particular case, obviously numerical competition is will require to obtain the all the roots of natural frequency. If there is some torque external torque in the system then, the form of the par matrix and fields matrix. We have seen that changes, even the oral transfer matrix forms will be changing, so let us see how that the particular form will take place when there is external torque in the system?

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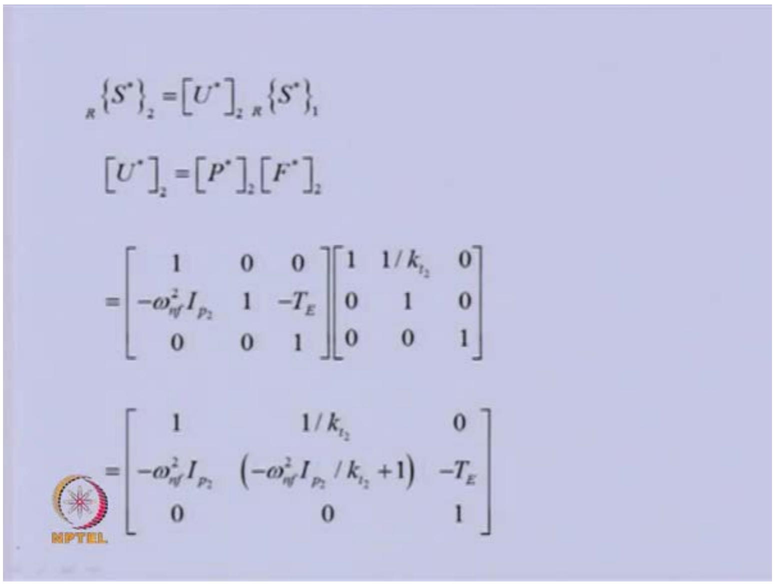

So, you can able to see that earlier we developed this oral transfer matrix, sorry the transfer matrix between one shaft and the disc. The form of the p matrix modified part matrix and modified field matrix where… So, this is the modified field matrix, this is the modified hard matrix and external torque was here. If we multiply them, it will be having this particular form there is 3 by 3 matrix and external torque is here. As such this 2 by 2 first show and columns are same as the previous once. Only this is coming.

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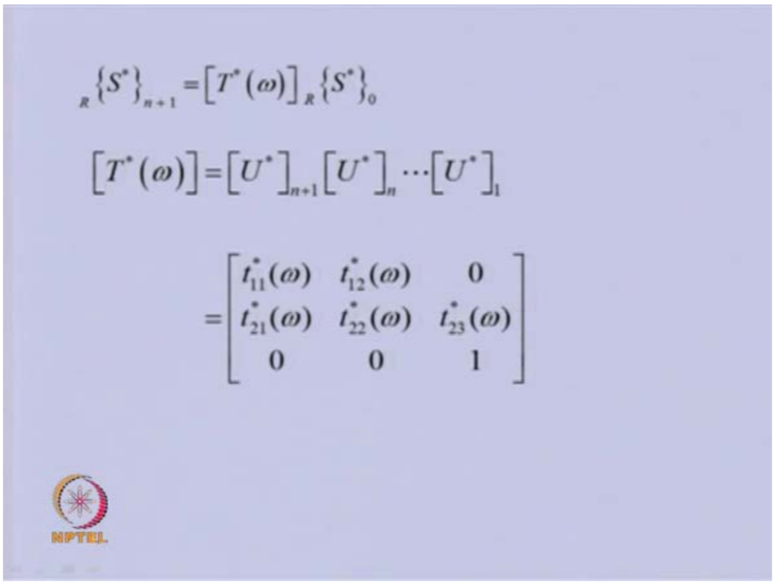

Now, for particular large system if we obtain the oral transfer matrix, so such transfer matrix is we will get multiplied. If we multiply them finally, the form of the oral transfer matrix is will be of this form in which this will contain the external torque component. Now, because we have the external torque will be having some frequency, so those frequency will be appearing in each of this terms. So, you can able to see that if we want the response of the system, because in this particular case, we will be having the omega as the spin speed of the shaft or the torsional external torque frequency, external torque frequency.

This is excertation frequency, so if we want the response for various external torque frequency, we can able to vary this. Once we substitute in the various transfer matrixes they will be in the form of numbers they can be multiplied to get the oral transfer matrix; that will also being the form of number. So, we can able to get the response from these particular equations for various value of external torque. So, you can able to see that when this external torque will conside with the one of the natural frequency at the system, you will be having reganess condition.

So, that also calculate is in this particular equation, because this state vector contain not only the angular displacement also the torque, depending upon the boundary condition, we can able to get various displacement angular displacement at various station. Now, we take more some example to illustrate the capability of the method for more complex boundary conditions.

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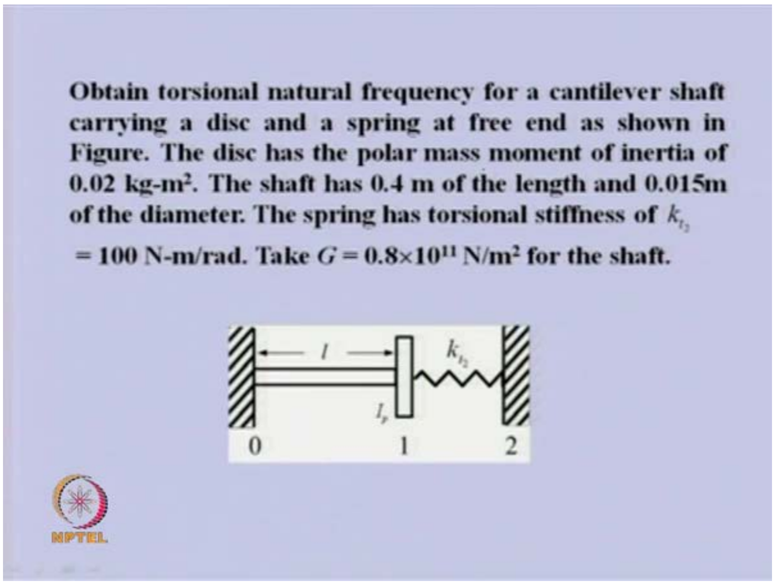

So, in that connection let us say take another problem in which there is a mass less shaft and disc is there at end this end. Up the shaft is fixed and free end of the shaft there, disc is there. We have one torsional spring of this stiffness k t 2 and various dimensions of the shaft and the diameter of the shaft are given here, given here, given the models of rigid it is given to us of the shaft when neglecting the mass of the shaft. So, for this particular case first when we want to analysis using transfer matrix method, we need to assign stations.

So, let us say this is 0 station, this location is station 1 and this fixed end of the showing is station 2. Once we have this, now we want to relate the state vector at various stations in the left and right side of the disc and try to relate the oral the state vector relation. That means let us say the state vector which is there at the stations 2, and the state vector which is there at the station 2, and state vector which is there at the station 0, so and in between we will be having this field and point matrixes.

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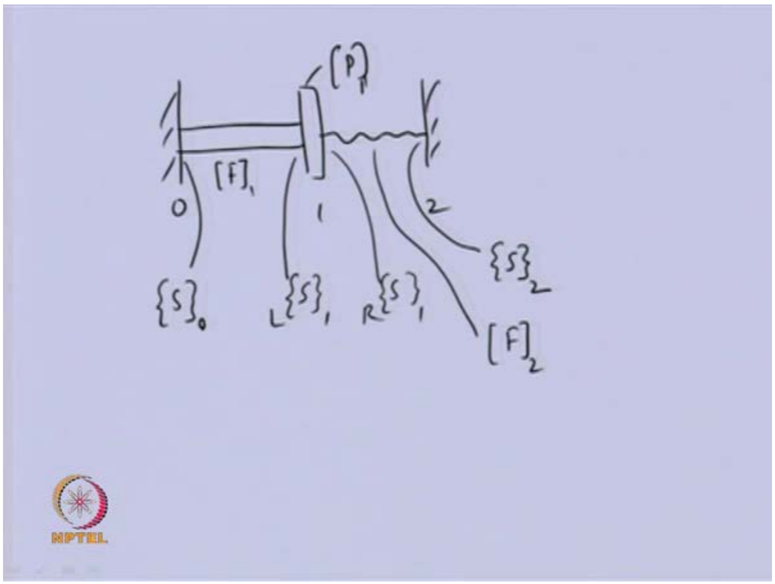

So, let us see this particular thing how will be having various state vectors? So, 0 stations 1 and 2. So, here I am again repeating this is 0 stations, because there is no disc so we not attaching any left and right to the state vector. Here will be having state vector, left of 1 station. Here will be having state vector right of 1 station. This is state vector at station two because there is no disc here, so it will be no left and right will be attached here. Now, you can able to see that between this two state vector, there is a field matrix. Here between this two, we have point matrix. This spring is identical to field matrix because this is acting just like a shaft with the same with the stiffness k t is now k t 2. So, this can be consider as shaft directly. Now, you can able to see that we need to relate all these state vectors.

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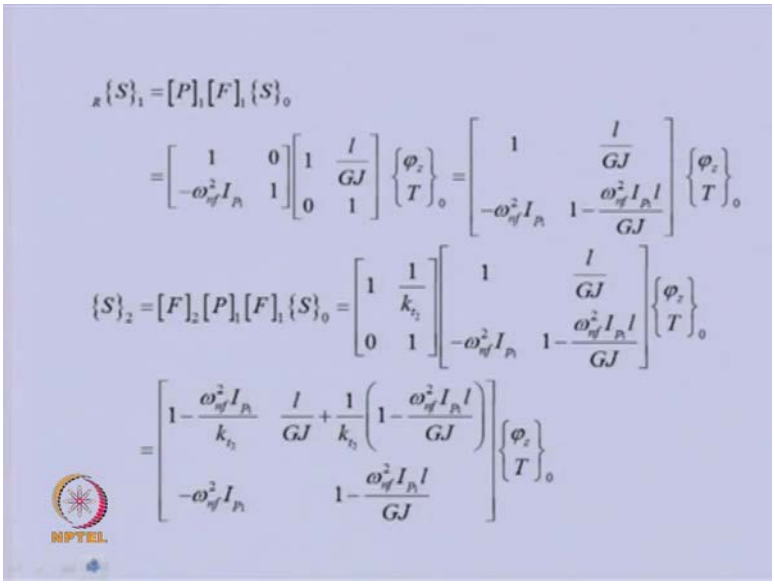

Let us see the station 0 and right of disc one can be related whirl the multiplication of the field matrix and point matrix. Multiply this will be the expanded form of the transfer matrix and this you can see the natural frequency expression are there, which is unknown. All other quantities are unknown, this is the state vector 0 stations and this state vector at right of disc wall now even the state vector at station 2. So, for that we need to have multiplication of field matrix part field matrix part this corresponding to the spring.

So, if we multiply this, we will get this 2 by 2 matrix, which is nothing but the oral transfer matrix for the resign problem. But in this particular case the boundary conditions are different, now is fix fix case, so as we equated t to 1 equal to 0 for free free case. This will not be the case now let us see for fix fix case, what will be the… Now, we obtain the oral transfer matrix for this particular problem, now we will see because now boundary condition is different, what will be the frequency equation?

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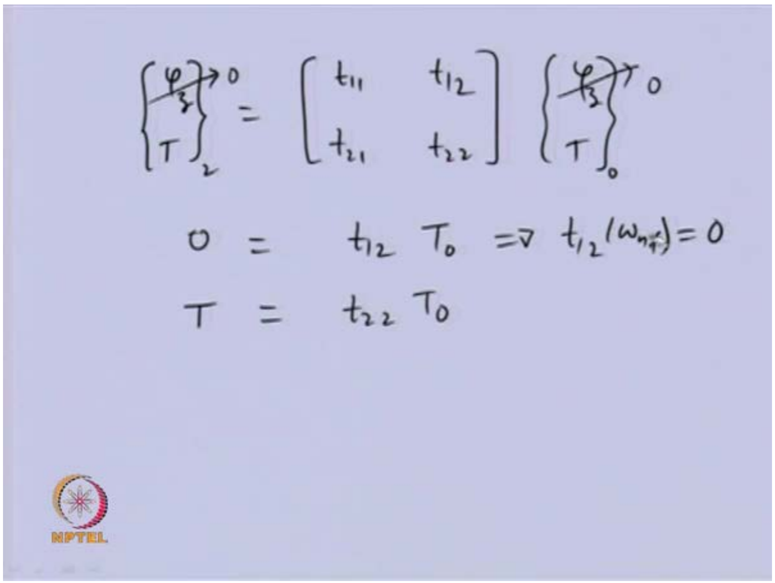

So, for the that particular case we have state vector at station 2 is equal to t 1 1, t 1 2, t 2 1, t 2 2 and state vector at 0 station. Now, as we can able to see both the ends are fixed, so that means here angular displacement 0 at station 0. Here also there 0, so this boundary conditions we need to apply here. So, this is 0 and this is also 0, so if you expand this two equation, you will get this form. First equations, second equations is you can able to see that the first equation because this torque cannot be 0 at fix support. So, we need to have this t 2 1 equal to 0, which is function of natural frequency at this. So, this is the frequency equation, so you will go back to the previous case.

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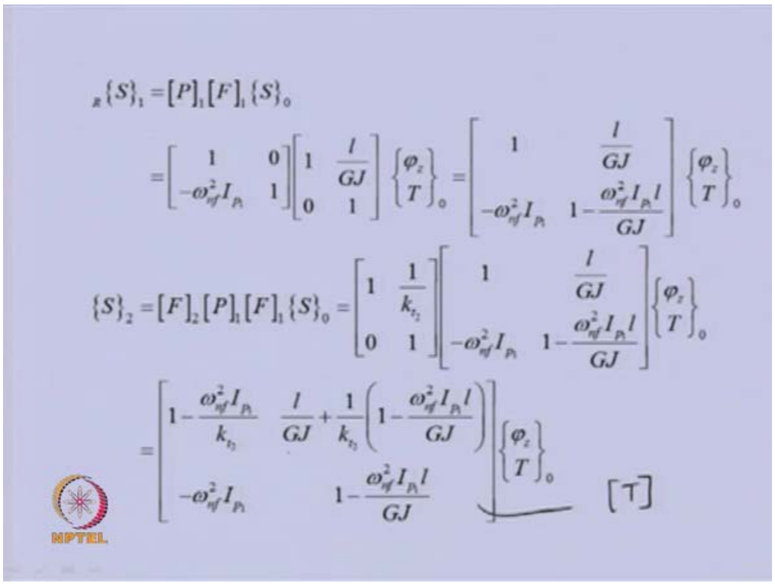

So, t 1 2 is this one, so we need to equate this quantity equal to 0 that is the frequency equation.

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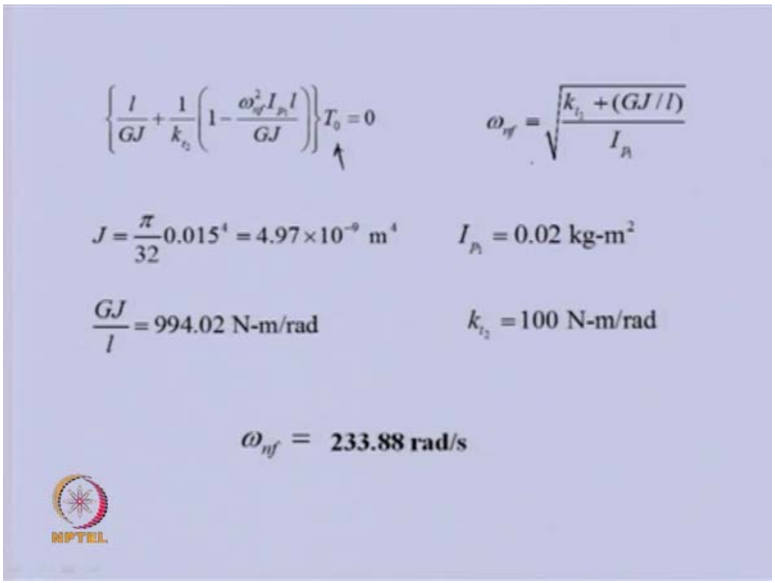

So, you can able to see this is frequency equation in which we are equating this because this cannot be 0. So, if we equate this term 0, you will get the natural frequency is as this and you can able to see this. This the stiffness of the spring this stiffness of the shaft, so as such they this two springs in the form of actual spring and in the form of shaft basically connected in parallel. So, that is why there stiffness are getting at the directly and this the cooler moment inertia of the disc. Now, for given values we can able to obtain the second moment of area.

This is already given so all these quantities are known, so they can be obtained on the torsional. This is the spring of torque is also given is to us. So, if we substitute this in the natural frequency expression, we will get this as the natural frequency radius per second. So, we seen very simple examples and this particular case aim was that how to apply boundary conditions? How to get the frequency equations? Now, I will see slightly more complex example in which we have two discs.

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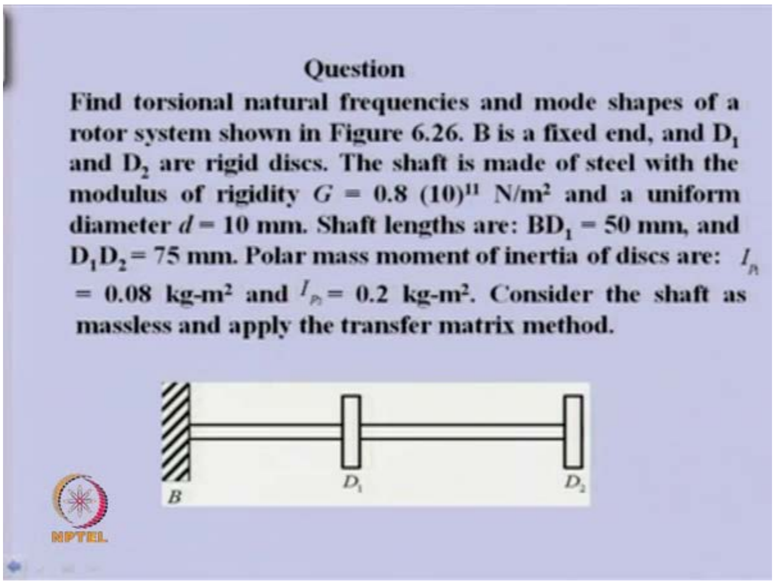

In this particular case the example is like this, so example is this one, we have a canteen liver beam in which two disc are there. Various property of the disc lens and everything is given here. We are considering the shaft as mass less only inertia are there at the disc. This end is fixed and other end of this is free.

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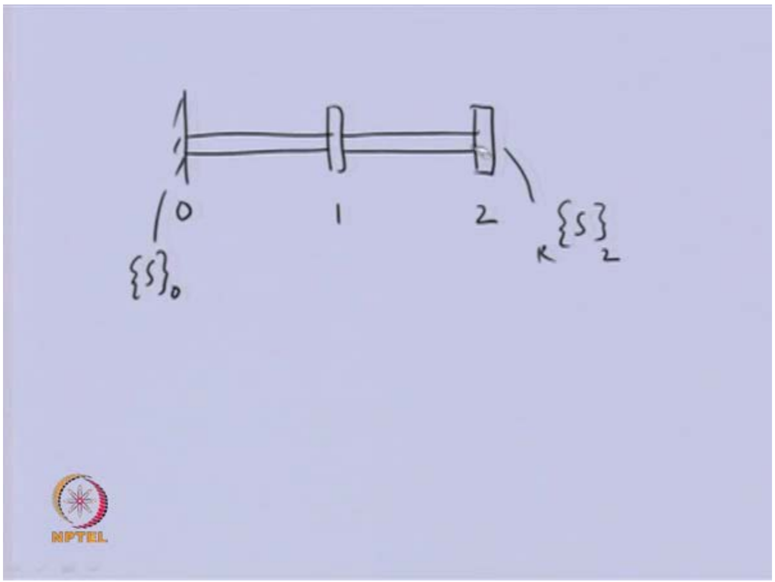

So, in this particular case we let us see, if we have two discs? So, this can be 0 station, this 1 station, 2 station. So, you can able to see that we need to relate the state vector, which is here right of 2 2 state vector right of or there is no disc. So, at 0 station once we related this two state vector, you can apply the boundary conditions. Only thing will be here the matrix will be more, we will be multiplying more number of matrixes. So, let us see how this take place?

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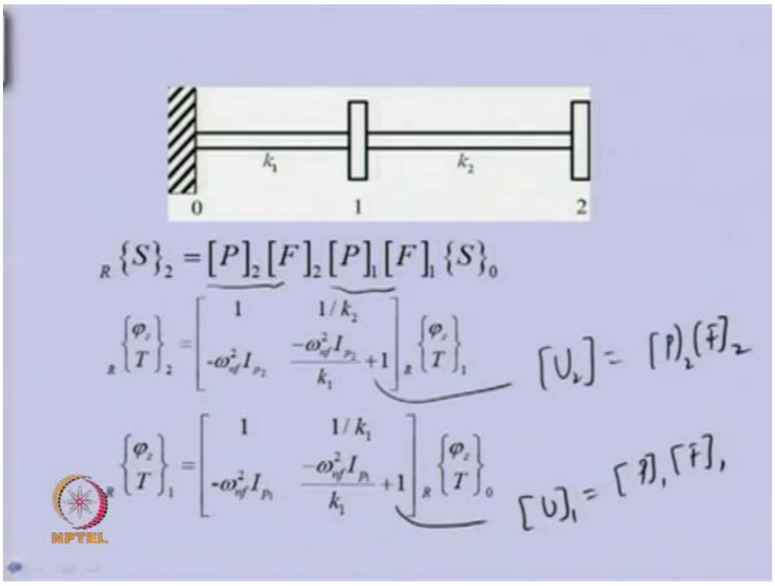

So, this the oral transfer matrix, so is multiplication of P 1 and F 1 for the first shaft segment and P 2 and F 2 for the shaft second shaft segment. In this particular case you can able to see that, if you want to get the state vector, let us say for right of 2, that means here and right of 1, so for that we know this the U matrix, which is multiplication of P and F. Similarly, from station the station 0, if you want the right of 1 state vector here, we need a another transfer matrix corresponding to the wall. This is U 2, this is I think, but P 2 into F 2 and this nothing but P 1 into F 1.

So, you can able to see that we could to able to multiply this two and this here. Now, you can able to multiply, so basically this is nothing but if we go back here you can see, this is U 2 multiply by U 1. This is 2 by two matrix, but there having more terms. Now, in the present case the boundary condition is that the stations 2 is free. So, there will torque and stations 0 is fixed. So, displacement will be 0 there, so this boundary condition you can able to apply here. So, this is different boundary condition as compare to the previous once. So, if we expand this in the form of let us say t, so the first equation will give us t 1 2. That is torque naught.

So, t 1 2 is on this quantity and second equation will give us t 2 2, t naught. So, t 2 2 is this one. So, you can able to see that the second equation because this cannot be 0 at free end at the fix end. So,, we need to have t 2 2, which is function of natural frequency should be equal to 0. So, we need to equate this quantity equal to 0, which is function of natural frequency. That will give us the natural frequency at this system. So, the form of the expanded form of this t 2 2, because that contains t 2 1 also is this one.

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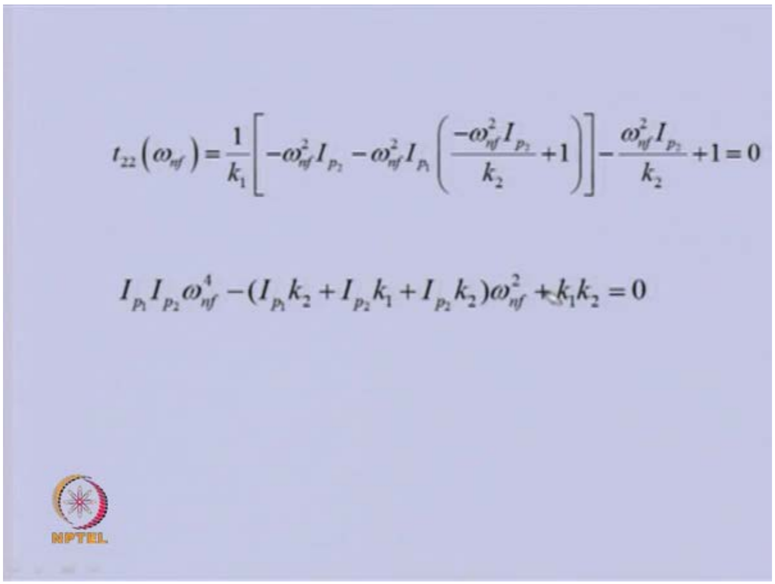

If we equate this equal to 0, we can able to get a polynomial is quantity polynomial. Basically in terms of omega n F square this can be solved to the two natural frequency of the system.

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$$J_1 = \frac{\pi}{32} d^4 = \frac{\pi}{32}(0.01)^4 = 9.82\times 10^{-4}\ \text{m}^4 = J_2$$

$$k_1 = \frac{GJ_1}{l_1} = 1570.79\ \text{Nm/rad} \quad \text{and} \quad k_2 = \frac{GJ_2}{l_2} = 1047.19\ \text{Nm/rad}$$

$$I_{P_1} = 0.08\ \text{kgm}^2 \quad \text{and} \quad I_{P_2} = 0.2\ \text{kgm}^2$$

$$\omega_{nf}^2 = \frac{I_{P_1}k_2 + I_{P_2}k_1 + I_{P_2}k_2 \pm \sqrt{\left(I_{P_1}k_2 + I_{P_2}k_1 + I_{P_2}k_2\right)^2 - 4k_1k_2I_{P_1}I_{P_2}}}{2I_{P_1}I_{P_2}}$$

$$\omega_{nf_1} = 54.17\ \text{rad/s} \quad \text{and} \quad \omega_{nf_2} = 187.15\ \text{rad/s}$$

So, basically you can able to see these are the, these are the close form expression of that particular polynomial. If you want to use various numerical value is here, which is given in the problem geometrically property is stiffness polynomial definition. We will get two natural frequency, this the first 154.17 radius per second, another is 187.15 radius per second. So, we have seen that how to obtain the natural frequency using the transfer matrix method?

Now, once we obtain the natural frequency, the mode shape can be obtain using the various transfer matrix equation, which we related in between the stations. For that the particular case we need to obtain mode shape, is nothing but relative and the angular displacement at various dislocation. So, we need to obtain this relate too this displacement for each of the natural frequency, which we have obtained. So, let us see how the transfer matrix between various stations like 0 and 1 and 1 and 2, the intermediate transfer matrix is how they can be use to obtain the relate to displacement between various stations?

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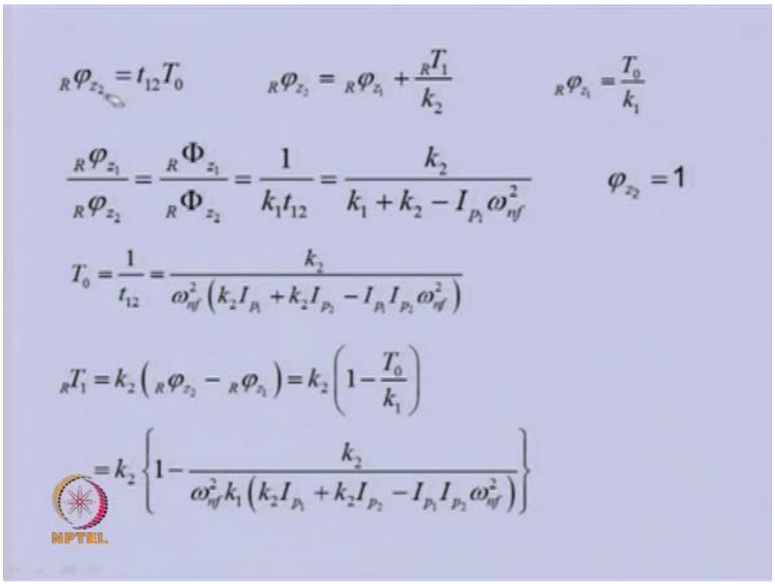

So, for that particular case if we see, this is the equations from the oral transfer matrix but which we did not use.

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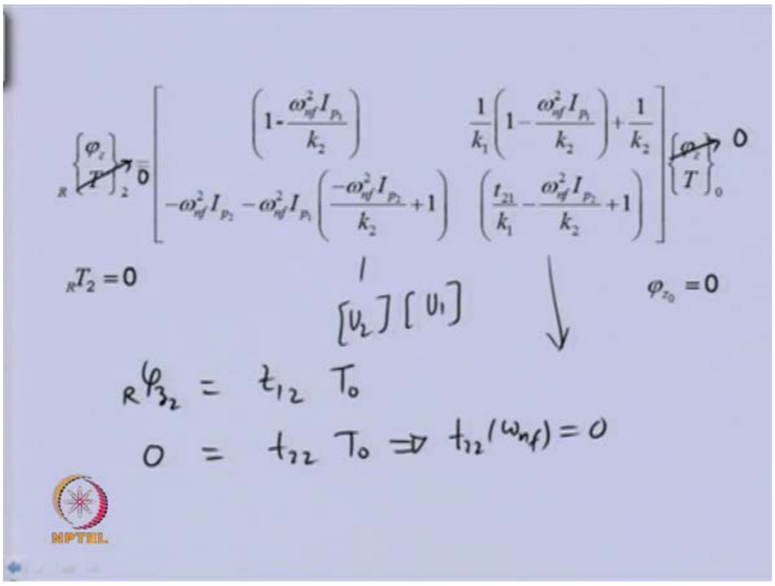

So, if we go back and see this particular equation we did not use. This was the frequency equation and so this can be use to obtain this particular displacement. So, if we for unit torque at a 0 station, you can able to get a this particular angular displacement. So, once we know this that means we have state vector at station 2 naught, this is 0 and this is t 1 2 and t 1 2 is function of natural frequency.

So, if we want o obtain the displacement for particular frequency, you can substitute that frequency here to get the particular displacement. That means we know the all the field variable here, so once we know field variable at once stations using various transfer matrix. We can able to get the field variable at any other intermediate location; that is the thing we are doing here.

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$$_R\varphi_{z_2} = t_{12}T_0 \qquad {_R\varphi_{z_2}} = {_R\varphi_{z_1}} + \frac{_RT_1}{k_2} \qquad {_R\varphi_{z_1}} = \frac{T_0}{k_1}$$

$$\frac{_R\varphi_{z_1}}{_R\varphi_{z_2}} = \frac{_R\Phi_{z_1}}{_R\Phi_{z_2}} = \frac{1}{k_1 t_{12}} = \frac{k_2}{k_1 + k_2 - I_{P_1}\omega_{nf}^2} \qquad \varphi_{z_2} = 1$$

$$T_0 = \frac{1}{t_{12}} = \frac{k_2}{\omega_{nf}^2\left(k_2 I_{P_1} + k_2 I_{P_2} - I_{P_1} I_{P_2}\omega_{nf}^2\right)}$$

$$_RT_1 = k_2\left({_R\varphi_{z_2}} - {_R\varphi_{z_1}}\right) = k_2\left(1 - \frac{T_0}{k_1}\right)$$

$$= k_2\left\{1 - \frac{k_2}{\omega_{nf}^2 k_1\left(k_2 I_{P_1} + k_2 I_{P_2} - I_{P_1} I_{P_2}\omega_{nf}^2\right)}\right\}$$

So, you can able to see that we are obtaining we are relating the field variable at station 1. So, once we know this we can able to get the field variable here. Even we can able to take the ratio of this 2, so this is the ratio. So, ratio is in the form of this in which natural frequency is revolve of the term. So, you can able to see that if we substitute the values of the natural frequency, you can able to get the relate to displacement between station 1 and 2. Station 0 is any way fixed, so let us see this for two natural frequency what is the values it takes?

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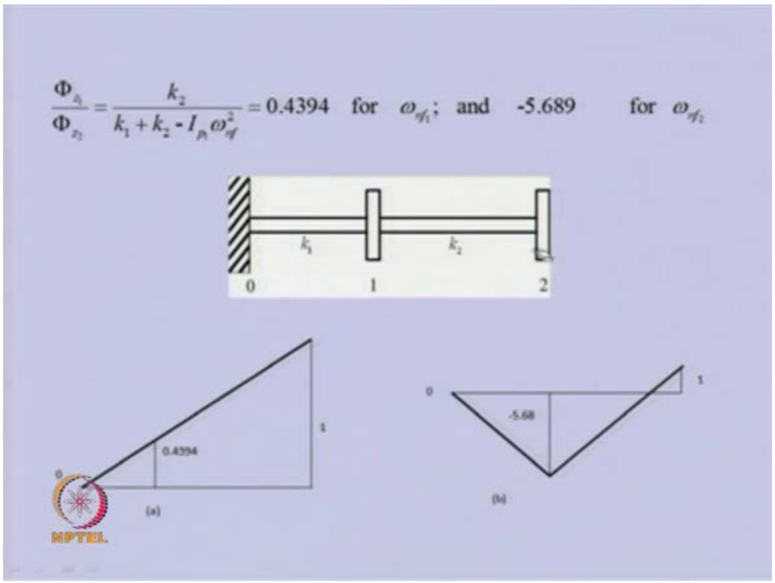

So, for first natural frequency, this is the value and for second is negative. So, you can able to see that we have more check corresponding to the first natural frequency. This is 0 station fixed, disc one location disc displacement, and here if you are taking unit displacement at station 2, the station 1 will be having 0.4394 unit on displacement. Similarly, here you can able to see that these two discs will be having anti phase motion for the second module.

So, in the first mode they both the disc will be having same phase motion, so and in this particular case there will be having anti phase motion. That means if this disc is rotating clockwise, in torsional vibration, this will assaulting the contra clockwise direction, when it second wise rotating in clock wise, just will be anti clock wise for the second one case, so in the present lecture, we try to use transfer matrix method with various examples and various kind of boundary condition.

How we can able to get the natural frequency? How we can able to get the more change? We will be extending this particular method for the next lecture. Also for other cases like damped case and reciprocating engine case, which we could not able to cover in this particular lecture. Then once we have over all idea about this transfer matrix method. Then we will go for the finite element method, which is more versatile and the application of the boundary conditions are easier in that particular case.