Theory and practice of rotor dynamics

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Module – 5
Torsional Vibrations
Lecture - 16
Torsional vibrations: Transfer Matrix Approach

Till now, we have considered simple rotor system analysis, therefore transfer vibration or for torsional vibrations was main focus. Till now, for introducing various terminology in the rotor dynamics and simple concepts which phenomena occurs in transfer vibration or torsional vibrations, and till now our rotor model was simple. Till last class we saw that when we considered three degree of freedom rotor model, we had difficulty in handling the big equations. So, we can expect if the degree of freedom is 4 or 5 or more like 50 or 100, in actual practice we may face difficulty with such methods.

Now, onwards we will be having some focus on the tool to be used for analyzing such rotor system. We know that there are two very popular methods, which are which can handle large degree of system in rotors one is the transfer matrix method. Another is the finite element method apart from this there are some other methods also, but these two methods are developed mainly. So, that we can able to develop them in a more programmable form so that depending upon the complexity we can able to use them much without much difficulty.

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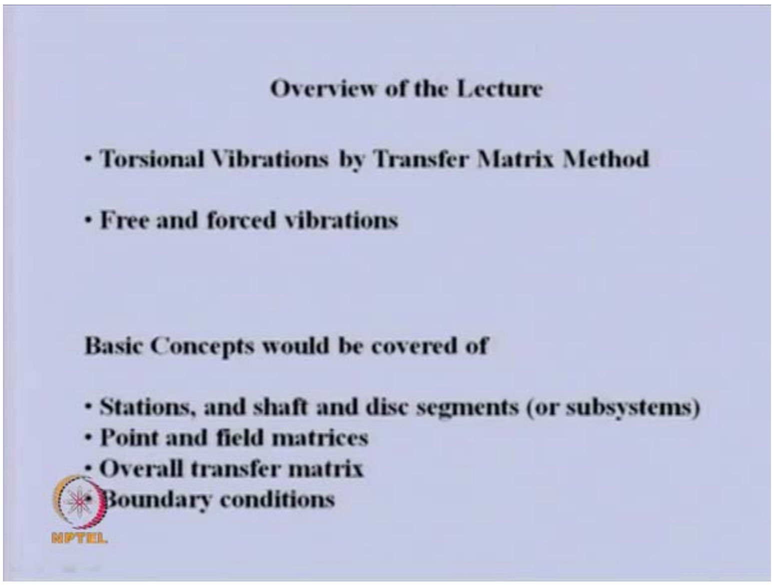

So, today's lecture is in of direction of the transfer matrix method and let us sees what the different concepts we will be covering are in this particular lecture. So, this is the Trans torsional vibration by transfer matrix method and we will be analyzing both free and force vibration. With this method various concepts regarding these methods are like stations shafts disc segments or subsystems then point and field matrices overall transfer matrix boundary conditions.

So, these will be introduced in the lecture coming to the transfer matrix method or TMM method unlike in the previous case like Newton's method or the principle method or any methods like Lagrange's or Hamilton equation or Hamilton principle. In this particular case we tried to derive the equations of some segments of the rotor system like shaft segment disc segment or coupling segment. Then, we have these equations for these segments we can able to use them for any complex system.

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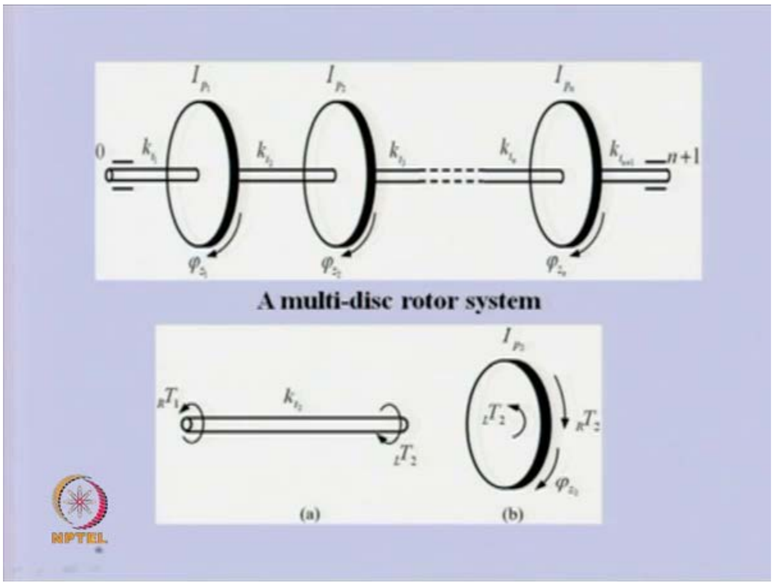

So, let us see that how we can able to use this method and how we can able to develop this particular method. So, here we have a rotor which is having large number of discs, so you can able to see here discontinuity is there, so that means we can able to take any number of disc. So, let us say n number of discs are there, so this is whole shaft supported on friction is support, but as such we will not be having any problem in boundary conditions because boundary conditions will come at the end of the problem.

So, basically in this particular case we have one shaft, so this is the shaft segment free body diagram of shaft segment and this disc this is the free body diagram of the disc. So, using this free body diagram we will try to develop some form of equations governing equations the governing equations will involve. What are the field variable at various points on this segments these points in transfer matrix method are called station so this could be one station this could be another station.

Similarly, we can able to see that we can have n number of stations or in a particular system and we at one particular system the either the angular displacement of the disc and associated torque. So, at one particular station will be having two field variable one is the angular displacement and the torque. So, we will try to relate these two field variable from one end of the shaft to another end using the free body diagram. Similarly, for the disc will be trying to derive the field variable relation which is there in the left of the disc and the right side of the disc because of the inertia, we expect that the torque at the two ends of the disc from the shaft reactions.

So, we will be having some change in the torque and these relations, we will be developing will help us in transferring the information from one end of the rotor system to another end. Now, let us take the free body diagram of the disc and try to relate the field variable at either side of the disc that means left of the disc, and the right of the disc what are the field variables that are there how they are related.

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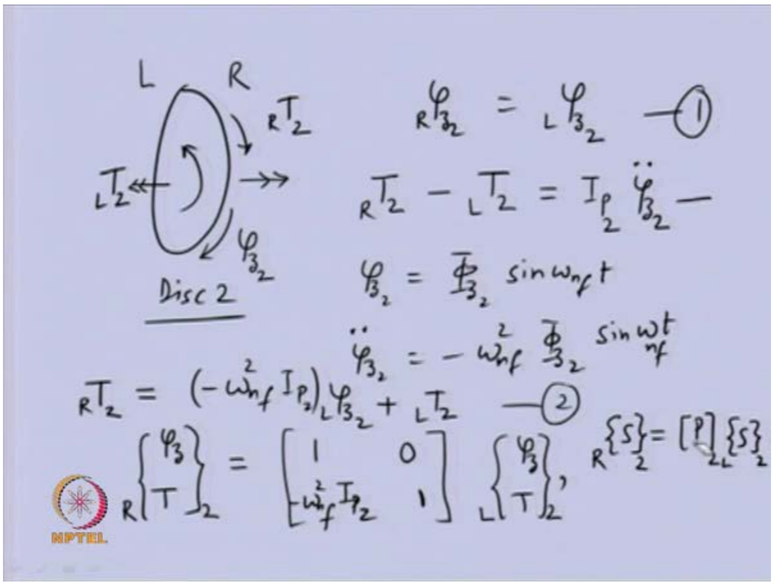

So, we have a disc and from the torque or from the shaft we have, let us say in the left side of the disc we have this particular torque which I am designating as left of let us say this particular disc is any represented to disc. Let us say this is disc two torque left of the disc two will be represented like this and direction of this is when we are looking from right side is actually is having clockwise direction. So, we can able to represent this in this direction the negative direction and the displacement of this particular disc.

Let us say two which is having direction in the positive direction; this is the positive convention we have used for this particular angular displacement. Another torque which is coming from the right side of the disc 2, because the both end of the disc is connected with the shaft. We have removed from the shaft to obtain this particular free body diagram. So, we have let us say torque which is right of this disc two will represent as T bell by subscript R which is representing right side of the disc 2.

So, this terminology we should able to understand carefully because we will be using this now very frequently, so back subscript represent the either left side or the right side of the disc always will be referring with respect to the disc not with respect to the shaft. So, we will see the field variables will be representing with respect to disc either left side or right side of the disc. Now, we are able to see this particular torque is also having similar positive direction as of the displacement, now we will able to relate the field variable which is there in the left side of the disc and in the right side of the disc because this disc is very thin disc.

So, the displacement angular displacement on the either side of the disc we expect will be same so we can able to write that the angular displacement of the right of the disc 2 is equal to angular displacement left of the disc 2, because disc is very thin. So, we expect these two discs angular displacements will be same at either side of the disc. Now, to relate the torque we can see the free body diagram that we have external torques of right of disc two and minus that is left of disc two these are the external torque reactive torque which is coming from the shaft and that should be equal to the inertia of the disc.

So, that is let us say polar moment of inertia of the disc 2, so angular acceleration of disc two so this is relating the torque and this is relating the displacement angular displacement. Now, because as such we are considering the free vibration or force vibration, so if we are considering let us say free vibration we expect that whatever the motion is taking place will be having simple harmonic motion. So, that means we will be having let us say with frequency is natural frequency because if we describing the system we expect that it will be oscillating with the natural frequency of the system.

So, if we this is the amplitude of the displacement and this is the frequency part and if you take the double derivative of this we will get this quantity omega n F is the natural frequency of the system, now this we can able to substitute here. So, we can able to write this equation here as let us say torque at the right of disc 2 is equal I am rearranging the terms omega square n F I P 2 and phi left of disc 2 this substituted in place of this one. We have substituted this one and plus torque in the left of disc 2, so this particular torque I have brought in the right side.

Now, this particular expression I have rewritten here and we have, so this is one equation and this is second equation. So, you can able to see carefully these two equations are relating basically field variable in the right side and the left side of the disc this is the displacement and this is the torque here also. You can see right side torque of the disc 2 and here field variables which are displacement and torque either at the right or left side of the disc. In this particular thing, we have attached the left one because right and left angular displacements are same for our convenience we have added this left here.

Now, we can able to combine these two equations one and two in a matrix form so this is this displacement and torque I am writing in the subscript and by subscript and the subscript in the outside of the this vector. We have one matrix and another field variable vector which is belonging to the left of disc 2, so if we want to enter the first equation you can able to see that we have 1, 0. So, if we expand this first equation we get this one, now the second equation can be inserted here and will see that this will be omega n F square I P 2 and here 1.

Now, we can able to see that we have inserted these two are in matrix form and this particular matrix we can able to write it as let us say s is the state vector which is representing the angular displacement. The torque right of disc 2 and this particular matrix I am calling this as point matrix P corresponding to disc 2 because you can able to see the disc 2 mass moment of inertia. So, polar mass moment of inertia is appearing here, so that subscript is that this point matrix is belonging to disc 2.

Now, another one is the field variable that is left of 2, so you can bale to see that this particular relation is relating the field variable which is at the left of the disc two to the right of disc 2 if we know the field variable at the disc 2. If we multiply pre multiply the point matrix we can able to get the field variable in right of the disc 2 we developed a point matrix which was relating field variable in the left of the disc to the right of the disc. Now, we will take the shaft segment and will try to develop similar relation of the field variable in the right side of the disc shaft and the left side of the shaft and the right side of the shaft.

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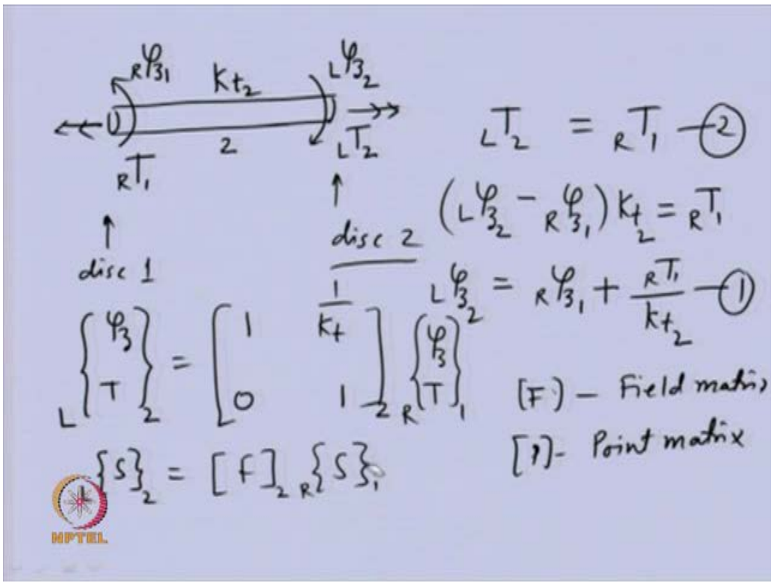

So, again we will refer to the free body diagram of the shaft, so shaft is this and we have let us say torque this is a second shaft and disc 2 is basically we expect it should have been here, so and here we expect disc 1. So, this particular torque will be representing with respect to disc 1, so I will be writing this as torque right of disc 1 because this particular shaft is attached to the right of disc 1. Here, we can expect we will be having torque representation as left of disc 2 because this particular end of the shaft is attached to the disc at the left, so will be representing this particular torque as left of disc 2.

We have torsional stiffness of the shaft let us say K t 2 and now vectorially, we can able to represent basically this torque as this one and this torque. Obviously there should be equivalent opposite and we have apart from this torque field variables, that means that is the displacement here will be having displacement of disc 1 that is right side and here will be having displacement of disc two in left side. So, our aim here would be to relate field variable which will be are here on the left side and to the right side so that we can able to have the similar expression we had for the disc case.

So, let us use these relations, so you can able to see that we have in this particular case the torques from the free body diagrams of this we can able to relate the torques which is left of disc 2 is equal to torque of right one because there is no other torque acting on to the shaft. So, these two shafts should be equivalent equal in the magnitude and regarding the twists, now whatever the relative twists between the two ends of the shaft that is taking place. So, I am representing that as let us say displacement of left of 2 and angular displacement right of 1, this is the relative twist if we multiply this by torsional stiffness of the shaft we expect this should be equal to the torque.

So, let us say torque one of the torque right side of the wall, so it is the relative twist into torque we will be getting to equal to the torque. Now, we can able to rearrange these equations like this, so first I am writing this as left of 2 the angular displacement is equal to right of 1 plus basically divide it both sides by 2, so that I have this expression. So, this is the rearrangement of this second equation, so we have one equation this and another equation this. Now, we can able to see that in this side the field variable related with the left of disc two and this side we have displacements which are right of disc 1.

So, we can able to combine this in a matrix form like this as we did for the disc case, so here I am saying left of 2 is equal to some matrix and we have field variable angular displacements and torque for right of 1. So, always as I told earlier we refer left and right with respect to disc only, so you can able to see that if you want to insert this equation first this is the second equation. So, that will give us 0, 1, so if we expand the second equation you will get this, so the first equation can be inserted and this will give us 1 and 1 by K t and I am putting the 2 outside of the matrix.

So, this representing this particular matrix is related to the second shaft and in more compact form we can able to write both the state vector as s left of 2 is equal to and this matrix, I am calling this as field matrix for disc 2 for shaft 2 for state vector right of 1. So, F is the field matrix and earlier we had P matrix which is point matrix so here we can see that we are relating the state vector which is right of disc 1 that means this side. If we pre multiply the field variable of this particular shaft we will get the field variable in the right side of the of this particular shaft, but left of disc 2 because disc 2 is here, so that will be belonging to the left of disc 2. So, now we can able to use this develop transfer matrices field variable and the point matrices to obtain the equation of the any rotor system, now we will see that how we can put this basic equation or development of the whole governing equation of the shaft.

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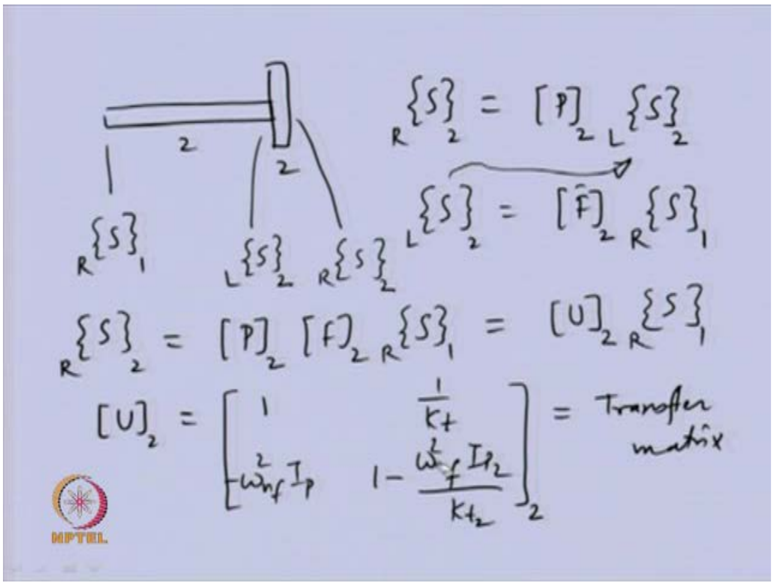

So, now consider the shaft two and the disc two and now let us see the field variable here we had the field variable or state vector which is right of disc one and here we state vector left of this 2 and field variable. Here, we will be having that is right of disc 2, earlier we have related we had these relations; I am writing these again rewriting these expressions. So, right of this two we related with point matrix of the disc 2 and state vector left of disc 1 and in the second case we developed state vector left of 2 with field variable shaft 2 and the field variable right of disc 1 these relations we earlier obtained.

Now, we can able to see that we this I will repeat again, so these state vector we related earlier, so first 1 was state vector right of 2. We related with point matrix of disc 2 with state vector left of disc 2 and in second case we related the state vector left of 2 with field matrix of shaft 2 and state vector right of 1.

These two equations can be combined we can able to say see that this particular state vector can be substituted here it can be substituted here. So, what we will get state vector right of 2 is equal to P 2 point matrix for disc 2 into F 2 and state vector right of 1. So, you can able to see that once we have substituted this here we could able to relate the state vector which is right of disc 1 to the right of disc 2 directly. So, that means the state vector which is here and state vector which is here we could able to relate directly. In this particular matrix, we can call this as let us say U matrix which is transferring from right side of the disc 1 to the right side of the disc another disc.

So, let us say this U 2 which is product of P and F this ordering is important, so U 2 is the product of P 2 and F 2 if we do the multiplication F 2 P 2 that will be incorrect. So, we should be careful about the ordering of this that U 2 is here P 2 into F 2 and we know P matrix and F matrix. If we multiply this we can able to obtain a form of the U over like this so because it will remain by two matrix, so if we multiply P 2 into F 2, we will get K t I am putting the subscript 2 outside and last term I P 2 and K t 2.

So, in this disc we call this as transfer matrix, now we can see that this particular transfer matrix contains the stiffness of the shaft mass moment inertia of the disc polar mass moment inertia of the disc and the natural frequency. Now, we have developed the transfer matrix for free vibration, so if there is some external torque which generally we assume that the external torque will be acting at the disc. If external torque is present and we want to analyze the force vibration also, let us see how the point matrix and the field matrix will gets changed because of this external torque.

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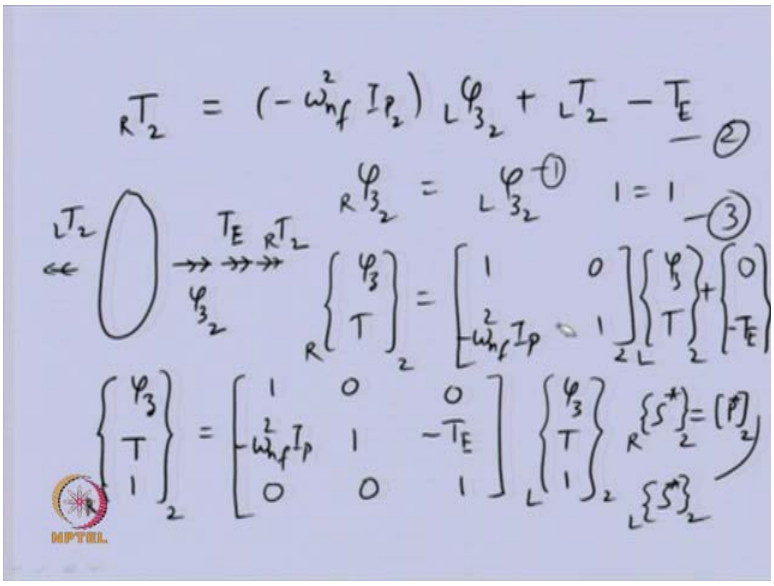

So, if we see the equation of motion which we derived earlier we had this particular expression for the disc from the disc free body diagram. Generally, when we are considering the disc we took positive direction of the angular displacement in this direction let us say the external torque T E is acting on to the disc which is in the positive direction of this particular displacement. Now, because we had earlier this also in the positive direction and the left side of the torque is in the negative direction.

So, you can able to see that this particular torque will find a place or along with the torque at disc 2 in the right side here and if we transfer that other side. So, will be this external torque coming here and we had another relation that is angular displacement in the right side of the disc 2 and the left side of the disc 2 are same. So, now these two equations as we combined earlier can be combined, so will be having field variable or right of disc 2 is equal to field variable left of disc 2 this particular matrix is for disc 2. In addition to this, because of the external torque we will be having another vector this because this is the last the last equation is corresponding to this torque equation.

So, we expect another torque here which is coming from the external torque and we will be having here 1, 0, 1 n minus omega n f I P, so this was the previous point matrix. Now, because of the external torque additional vector is coming the difficulty is because of this is coming that we were earlier able to transfer the state vector from one end to another by just multiplying with the point matrix and with 2. I will write also, to make more systematic we will expand this particular matrix by adding a identity equations that means one is equal to 1 as a third equation, let us see how we can able to do this.

So, we have this and this as field variable and we want to add a identity equation at the third one because this will not effect as such our analysis, but this will be convenient in handling the transfer matrix, so here also we will be phi 3 t and 1. So, basically last equation is nothing but 1 is equal to 1, so that means this, so you can able to see if you multiply this the last equation is 1 is equal to 1. The first equation will have this from this one and this equation of the torque equation will modify as I P 1, this is and then the external torque which was outside, now it is coming inside itself.

So, you can able to see that if second equation if we expand we this one will multiply with this external torque and will be getting that here like this. So, this equation can be get if we expand the second equation so you can see that these equations are belonging to the first equation is this one second is this one and third one is this one. By adding this, identity we our analysis results are not affecting, but here the advantage is here we can able to write state vector by different state vector. So, I am calling this as s star modified state vector are so this is this particular state vector is equal to the new point matrix to this point matrix. I am calling this as star again and here we have stat vector left of 2 on this one this is also modified.

So, I am writing this as star s star P star and they will represent the modified field vector and modified point matrix. So, you can able to see now as for the free vibration case we could able to write this particular expression very conveniently instead of writing the external torque outside the point matrix. We have seen that we modified the point matrix to include the external torque that we can able to do the force analysis, but we are assuming that there is no external torque acting on to the shaft. In the field matrix, components will not change because we have putting this again to the external torque, but we need to change its form because point matrix is 3 by 3 matrix. So, field matrix also changes to the 3 by 3, let us see how we can able to modify this.

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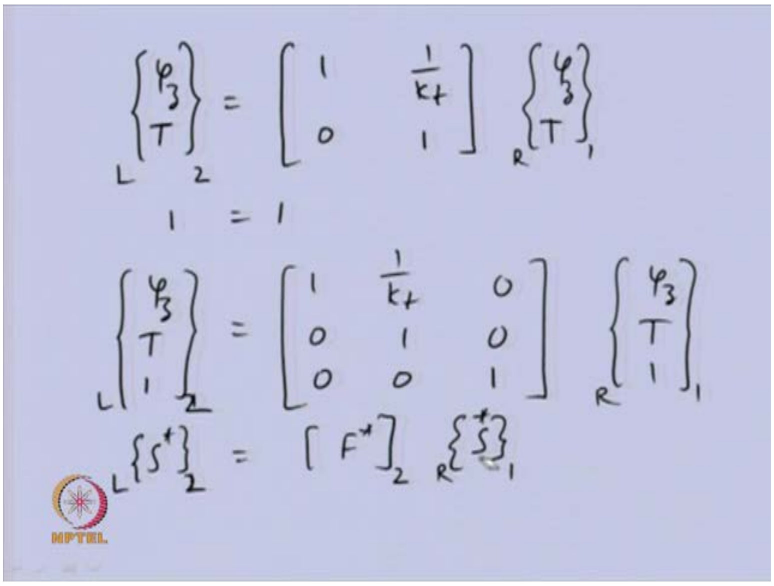

In the previous case we had the field variable which was relating the two state vectors at other kind of shaft, so right of disc 1 to the left of disc 2 and we had this as 1, 1 by K t and 0, 1 and this was the field matrix. Now, we want to add one is equal to one identity equation in this now let us see how this change the form. So, this is the modified state vector left of disc one and this is the modified molar phi z right of disc 1. Now, from the first two equations this will be giving equation like this identity equation we can able to insert as 1.

So, we can able to see that in the last equation if we multiply it we will get one is equal to one this is now modified state vector left of 1 left of 2, sorry this is left of 2 because this is left of 2 this is the field matrix for shaft 2. This is also modified field matrix and state vector right of one this is also modified I am writing as star. So, we can able to see that this particular similar system in the previous one only thing is the size of the matrix is 3 by 3 to accommodate torque inside the point matrix. You can able to see that over all from is similar to the free vibration case. Now, we have developed the basic point matrix and the field matrix corresponding to the disc and the shaft respectively. Now, you can able to use this for obtaining the natural frequency of force vibration for a any rotor system, now through an example will see how these can be used to obtaining the natural frequency.

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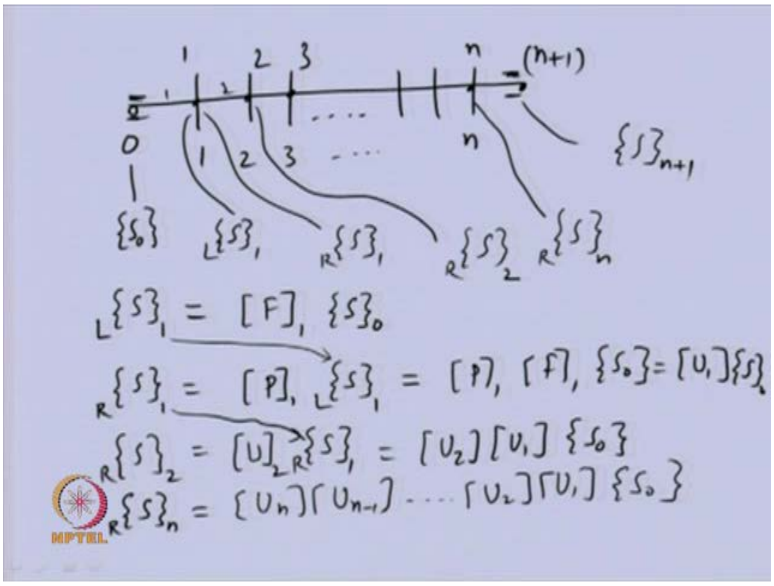

Now, I am taking let us say the rotor system which is mounted freely on friction as bearings this is having several discs and let us say it is having total n number of disc various stations we can assign. So, let us say this is zero station this is station one station two station three and so on this is station n and this is station n plus 1, so these are the various station where basically we will try to relate free variable corresponding to these stations. Now, let us start with the left side the field variable corresponding to the station 0 is let us say s naught because here there is no disc.

So, we are not attaching any left and right disc, but we are just saying it is the field variable corresponding to the station 0, here we can have state vector state vector that is left of disc 1 as well as right of disc 1. Earlier, we have related let us say state vector left of 1 we can able to get using the field matrix of shaft 1, let us say this is the shaft 1 shaft

2 like this. So, if you multiply to the naught the field matrix, we can able to get the state vector left of disc 1.

The state vector right of disc one is related with the point matrix of disc one and the state vector left of disc one and this two state vector is related by point matrix. So, if you want to get the state vector in the right side of the disc we need to multiply with the state vector in the left side. Now, you can able to see that this equation you can able to write as P one and this can be substituted here so will get into F 1 s naught and we represent this as a transfer matrix U 1 of s naught. If we multiply transfer matrix U one which is multiplication of P 1 and F 2 ordering is important again I am stressing with the state vector at zero station we can get the state vector to the right of the disc 1.

Now, if we want the state vector right of disc 2 earlier we related with the transfer matrix state vector rights of the disc 1 this we related earlier while deriving the field matrix and point matrix. Now, you can able to see that we can able to substitute this here, so we have U 2 U 1 s naught. So, if we multiply transfer matrix 1 and 2 again ordering is important with s naught, we can able to get the state vector by this two, so directly we can able to get the state vector right of disc 2 means here.

Now, we are related this state vector by the right of disc 2 with the this one on the same line you can able to say that we can able to relate this state vector right of disc n as U n minus 1 like this U 2 U 1 s naught. So, that means right of disc n is this one this is the state vector. So, this state vector we related with the state vector far end of the shaft using multiplication of these transfer matrices and now another shaft segment is left out, so we want the state vector let us say at n plus 1, so this we can able to do it in the next slide.

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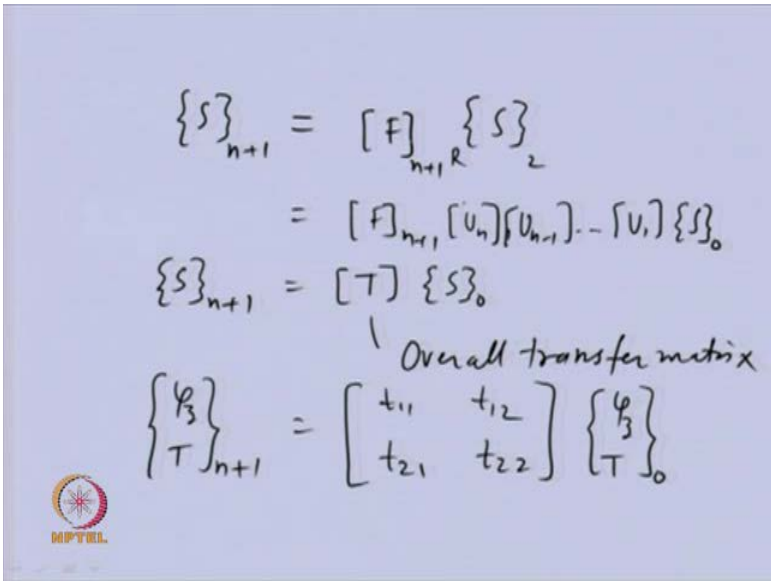

Here, state vector n plus one because there is no disc here so we are not attaching any left and right is related with the state vector or right of this two with field matrix that is n plus 1. So, you can able to see go back so this particular state vector and this are related by field matrix this particular shaft which is n plus 1 th shaft. So, we need to multiply this pre multiply this with field matrix of this shaft to get the field variable of this. Now, we can able to see that we have earlier related this one with this state vector at zero station, so that means if we multiply all these matrices let us say this is a T matrix.

So, we can able to relate the state vector at n plus 1 th station with state vector station to the state vector to the zero station with t matrix this t matrix. We call this as overall transfer matrix the beauty of this matrix is the size remains 2 by 2 matrix for free vibration and for force vibration because will be using the modified transfer matrices the size will be 3 by 3. So, you the size of the matrices does not change with number of degree of freedom system this is the advantage of this particular method.

So, in the expanded form if you want to write in the above equation is having this form let us say t 1 1, t 1 2, t 2 1, t 2 2 because size is 2 by 2 all multiplication. Let us say these are the terms this may be the big equation. So, we can able to see that we are transfer the information of the state vector at zero station by multiplying by overall transfer matrix to extreme right side. So, from zero station to n plus 1 station we can able to get the information and from other relations we can able to get the intermediate station information. Now, let us see how we can able to obtain the frequency equation from which we can able to get the natural frequency for let us say the free boundary condition.

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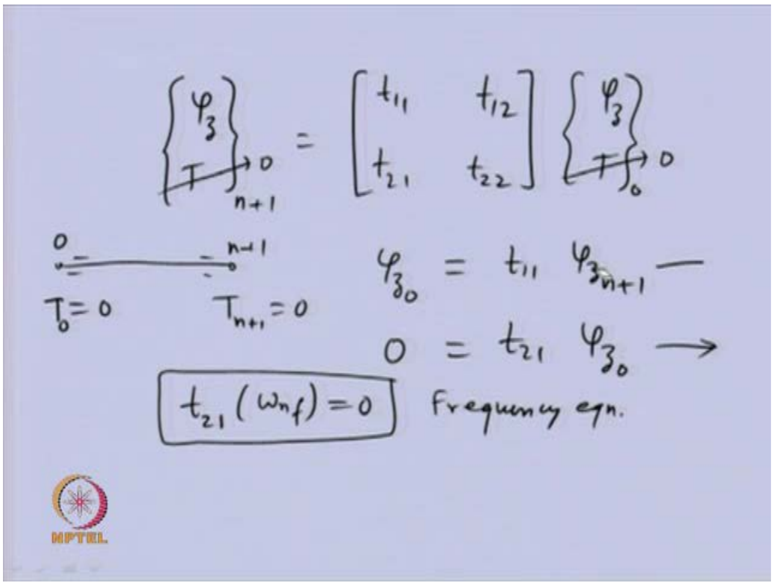

So, we derived earlier this particular state vector and we had multiplication of various transfer matrices like this. Now, for free boundary conditions let us say this is 0 this is n plus 1 station because free the torque at boundary condition is 0 0 and at n plus 1 station also it will be 0. So, this is the boundary condition of the free rotor system, so you can able to see that this term will be 0 and this term will be 0. So, if you expand this equation we will get field variable at 0 station as t 1 1 phi z and n 1 at n plus 1 other term will be 0 and 0 will be the second equation t 2 1 phi z naught and other term will be 0. So, you can able to see that the second equation is the homogenous equation this is the displacement of the free end cannot be 0. So, we need to have this quantity equal to 0 and in fact this will be the function of natural frequency.

So, this is the frequency equation or characteristics equation of rotor system from this will be in the form of polynomial or we can able to solve the simple system using the polynomial solution or by some route searching technique. We can able to find the various natural frequency of this particular equation the second equation this particular equation once we obtain the equation of motion can be used for obtaining the field variable related to each other. So, that we can able to draw the mocked shape today we developed the transfer matrix method and first basic thing was the field matrix and the point matrix development. We have seen that they are very useful once we have the information regarding these two matrices.

We can able to develop the system equations and once we have the system equations, we can able to apply the boundary condition and from there we can able to frequency equation which can be solved to get the natural frequency of the system. This particular method will be illustrating in the subsequent class through some examples. Then, it will be clearer how this particular the technique is very popular in industry for solving those problems in fact during the seventies and eighties this method was more popular than the finite element method.

After eighties the finite element method is more popular because of its inherent diversity and easy to implement the boundary conditions, but still this particular method is having the advantage of regarding size remains the small. In the this next lecture we will not only consider some example we will try to see how damping in the system in the torsional system can be incorporated and how the this point matrix and field matrix will modify because of that.