Non-Linear Vibration
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Module - 6
Applications
Lecture - 10
Nonlinear Vibration of Parametrically Excited System with Internal Resonances

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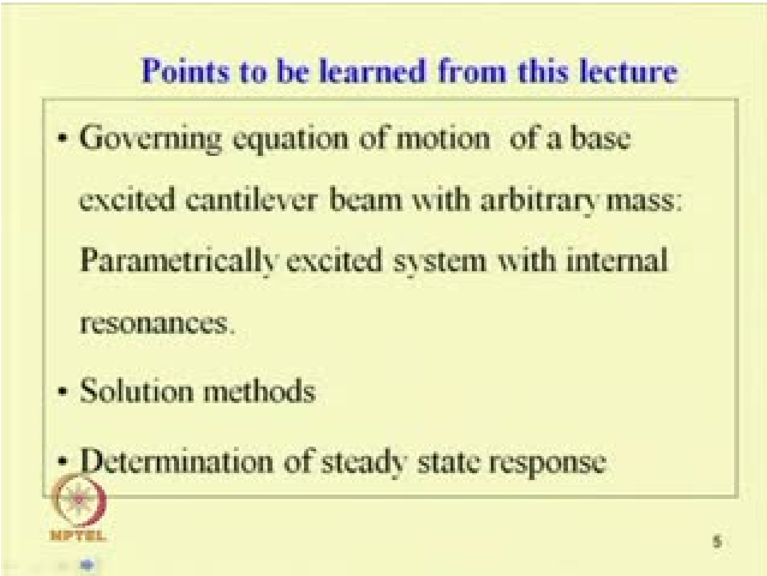

Welcome to the today class of non-linear vibration. So, today is the last class in this NPTEL course and in this class we are going to discuss about non-linear vibration of parametrically excited system with internal resonance. So, will take one example to study how this parametrically excited system with internal resonance behave, and already we have or will study in this class how to derive this governing equation motion of a base excited cantilever beam with arbitrary mass, which will behave as a system with internal resonance. Then how to solve this equation and how to determine the steady state response and also will discuss about different phenomena observed in this type of system.

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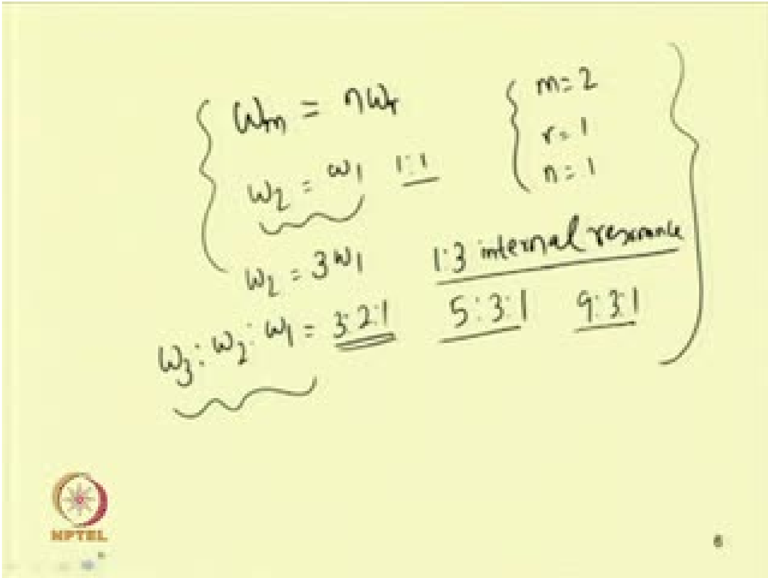

So already we know a system is said to be internally resonance when frequencies, that is the model frequencies let the model frequencies omega m, if omega m can be written as n omega r. So where m and r are the model frequencies for example, m equal to 2, r equal to 1 and let me take n equal to 1. So in this case omega 2 equal to omega 1, so this is known as 1 is to 1 internal resonance. Similarly if omega 2 equal to 3 times omega 1, so this is known as 1 is to 3 internal resonance.

So, if we can take let us take this way so omega 3 is to omega 2 is to omega 1, so let they are in the ratio of 3 is to, so they can be of the ratio of 3 is to 2 is to 1 or they may be of the ratio, we can take 5 is to 3 is to 1. So or it can be of 9 is to 3 is to 1. So we can have different integer relationship between these modes. So when will have this integer relationship between these modes, so due to this non-linear effect due to this non-linear terms present in the equation motion. So one can show that the higher modes are resonated when the system is excited at a frequency near to the lower mode frequency. So in that case as the higher modes are excited by this lower modes or the lower modes are excited by the higher modes without any external force, so that is why the system is said to be resonance internals or in other words one can tell this internal resonance occur in a system.

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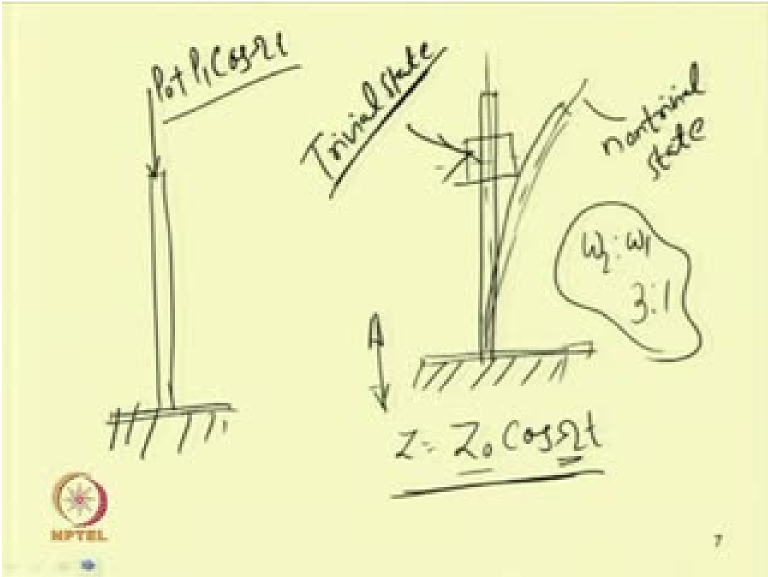

When there is when the model frequencies are commensurate. So in this case one can have this 1 is to 1 internal resonance, 1 is to 2, 1 is to 3 or 3 one can tell this as 1 2 1 is to 2 is to 3 or 3 is to 2 is to 1 depending on the frequency relation. So today class we will study a system which already we know a parametrically excited system we can take a cantilever beam. So let this is the cantilever beam and the base of this cantilever beam is excited is moving up and down. So with periodic force Z equal to Z 0 cos omega t. So already we have discussed the system or we know that for some value of this Z 0 and omega the system will start buckling or the system will try to vibrate in a transverse reaction. So the system will try to vibrate in a transverse direction or it will come out of its trivial state, so this is the trivial state and this state is the non trivial state so this is the non trivial state.

So non trivial state and this is the trivial state, so either one can apply this base excitation Z equal to Z cos omega t. Or one may apply periodic axial load, so if one apply periodic axial load also, so the system may be that of a parametrically excited system, so if a periodic axial load P 0 plus P 1 cos omega t is applied to the system. So for some value of P 1 and omega and P 0 the system start buckling. So we know that in case of static loading, so if it is greater than the critical Euler buckling load then the system start buckling but, in case of dynamic loading, so if this value this amplitude is very, very less than this critical Euler buckling load also, so the system starts buckling for some value of P 1 and omega. So today class will study or will see a system where it is base excited and if we put a mass at arbitrary position by changing this value but, or by changing this position of this mass we can show that for some system parameter or for some position of the system the natural frequencies of the systems are commensurate; that means we can show will take a frequency relation omega 2 is to omega 1, so which is nearly equal to 3 is to 1 and taking cubic nonlinearity in the system will show how the system corresponds becomes. So let us take the system.

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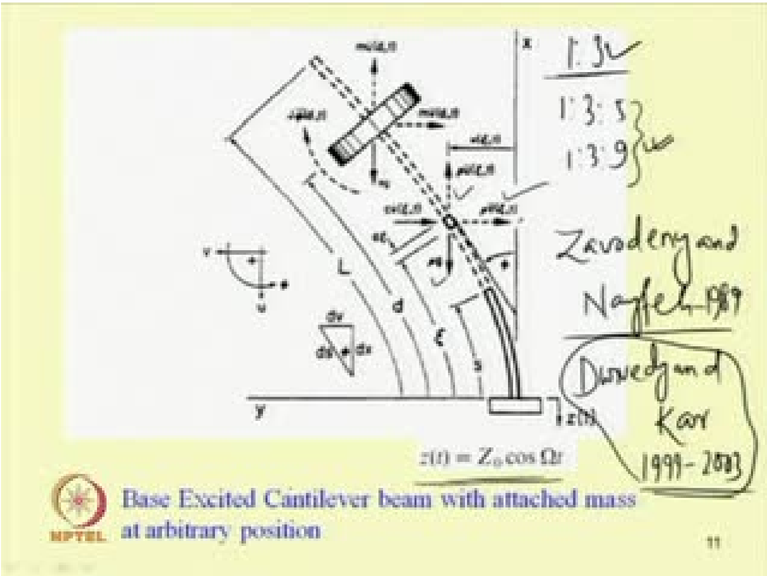

Let us consider this is the system, the system had been considered by many authors for example, the system is considered by Zavodeny and Nayfeh Zavodeny and Nayfeh and many papers on these have been studied by Dwivedy and Kar. So from 1999 to 2003 nine publications are from the system and Zavodeny and Nayfeh they have considered the system in 1989. So they so in there system they have considered a base excited system Z t equal to Z 0 cos omega t and they have taken only single mode approximation, but as we are interested to study the internal resonance case by varying the mass position.

One can show that the system has many different type of resonance condition for example, one can have this internal resonance condition of 1 is to 3, 1 is to 3 is to 5 and 1 is to 3 is to 9. So this is 2 mode interaction and these two are 3 mode interaction. So one can study, so let us study in detail about this 2 mode interaction of the system and for 3 mode interaction. So, some reference refers will be shown and one can refer this paper to study or know more about this 3 mode interaction. So let us first derive this equation motion, so now let us consider let the due to this vibration.

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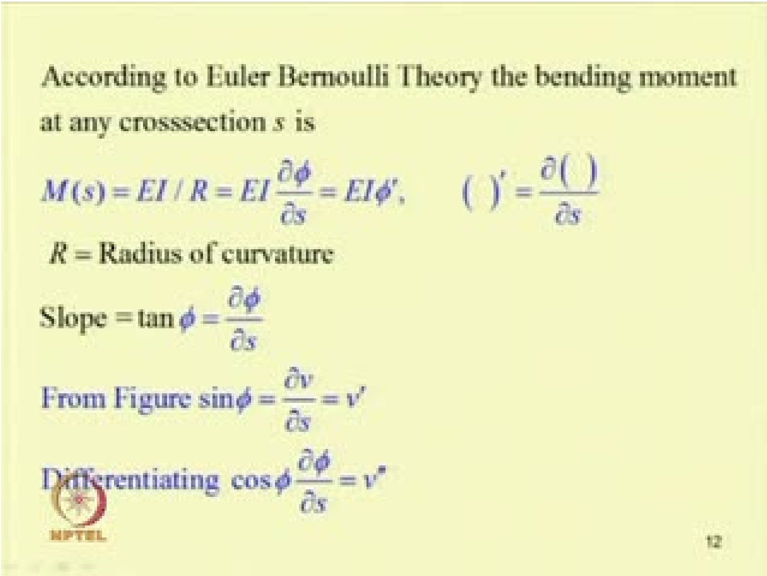

So let it has take a small element, so in the small element one can write what are the forces acting. The force this is the inertia force rho v double dot zeta t then this is the inertia force in x direction. So this is the weight rho g and this is the inertia force in y direction rho u double dot beta t. So we have taken this transverse reflection to be v and this axial direction reflection to be u. So in x direction it is rho u double dot and in y direction it is rho v double dot. So these are the inertia force and dumping force including the dumping force this is c v dot.

So if the mass is taken into account, so this is the n mass so n mass also will have this m u double dot, m v double dot. So this is the position at a distance d from this base so this is the deformed length d, this is un-deformed length l, l is the total length of the beam considering inextensibility condition, so now we can find the equation motion. So we have taken a small element at a distance zeta and we are finding the equation motion of the system with respect to the space coordinate s and time t. So here d is the y direction deflection and u is the deflection in the longitudinal direction. .

So one can use the basic equation of the strength of material to derive the equation of motion, so we know this bending moment of the system can be written in this form, this is equal to E I by R, so which is equal to also E I del phi by del s so where this phi dash can be written as d of that function by del of that function by del s. R is the radius of curvature, so the slope tan phi, so slope this angle is phi, so the slope tan phi can be written in this form, d phi by del phi by del s. And from this figure we can know the sine phi, so we can write the sine phi in terms of this. So sine phi equal to del v by del s, so we have taken so del v so this is v so for a small displacement del v so del v by del s will be sine phi.

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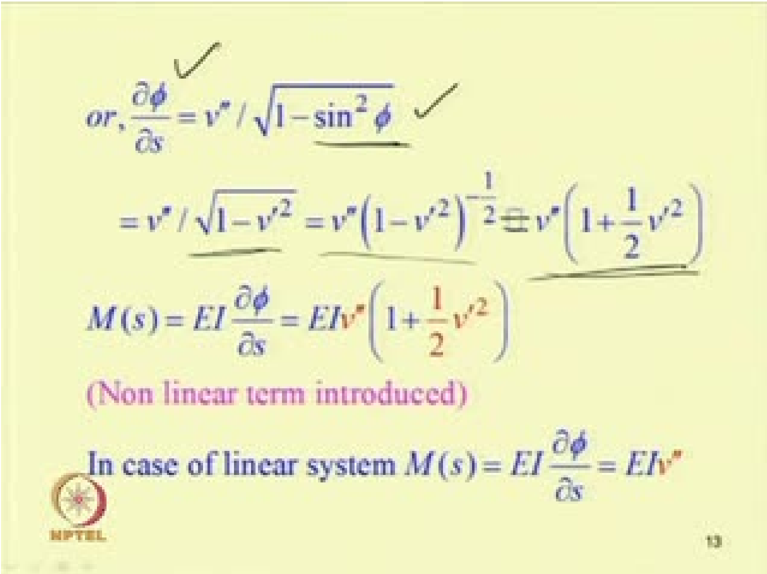

So this is equal to v dash, so if you differentiate the sine phi so we can write this cos phi equal to, so cos phi del phi by del s equal to v double dash. So now from tan phi equal to del phi by del s, so we can write or we can get this del phi by del s equal to v double dash by root over and 1 minus sine square phi. Or we can write this del phi by del s equal to v double dash by root over one by substituting the sine phi equal to v dash that is del v by del s, so we can write this d double dash by root over 1 minus v dash square. .

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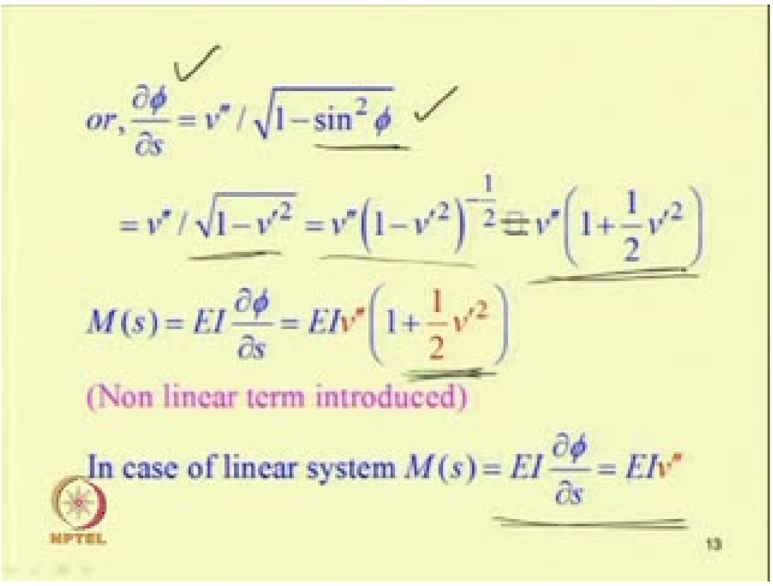

So this can be written as v double dash by taking this or writing this taking this part to the numerator, so we can write this is equal to v double dash into 1 minus v dash square to the power minus half or this is equal to v double dash into expanding this thing binomially and taking the first term only. So we can write this equal to v dash into 1 plus half v dash square. So now we can substitute this del phi by del s in this equation m s, so this del phi by del s we can write in terms of so m s we can write equal to e i del phi by del s equal to e i v double dash plus half v dash square.

So in this way the non-linearity is introduced in the system. Otherwise if you take one take the simple Euler Bernoulli beam equation; the equation will becomes this m s that is the bending. Bending moment can be written as E I del square d by del x square. But, in this case as we have taking this non-linear term so we have one additional term here in this equation, so m s equal to so m s so we can write this m s equal to E I v double dash into 1 plus half v dash square. So here we have introduced this non-linear term. .

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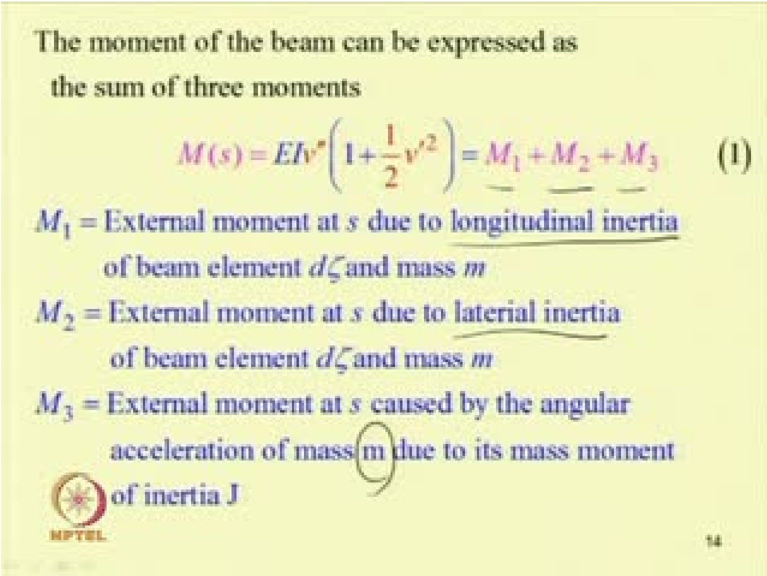

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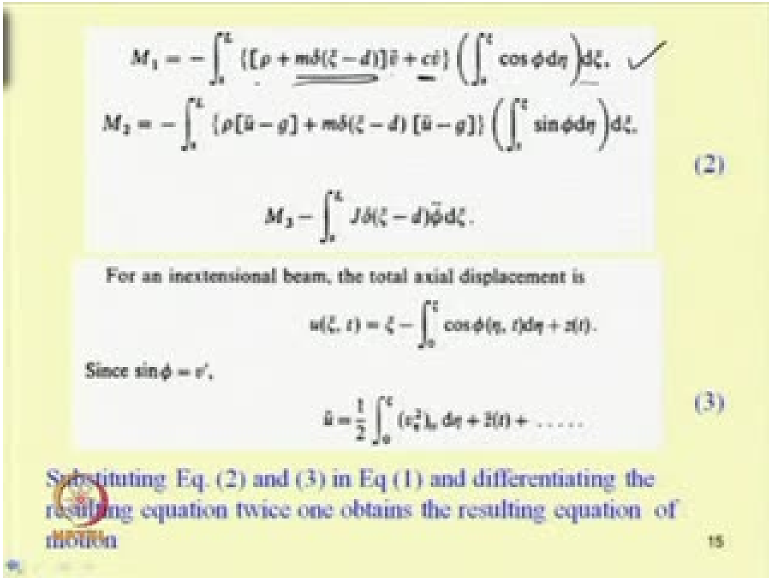

So in case of linear term already we know this is equal to E I v double dash. So now so we can write this the moment of the beam can be expressed as the sum of three moments, one is m 1 plus m 2 plus m 3. So where m 1 is the external moment at s due to longitudinal inertia of the beam element d zeta and mass m; m 2 is the external moment at s due to lateral inertia, so longitudinal inertia we can take that is m v double m u double dot lateral inertia m v double dot. Similarly, external moment at s caused by angular acceleration of the mass m that is the attach mass we are putting, due to this mass moment of and due to mass moment of inertia j.

So taking all this thing into account so our m 1 equal to minus s 2 integral s 2 l, so rho v double dot so we have added the dumping also and we have use this direct delta function to write the position of the tip mass at a at a distance d from the base. So rho is the mass for unit length of the system. So we can write the equation motion or we can write this m one due to lateral moment. So these into integral s 2 zeta cos phi d zeta. So as we are finding so we have to find the component of so where this force into so this is the force so as we are taking the moment about point s, so we have this force so m u double dot. .

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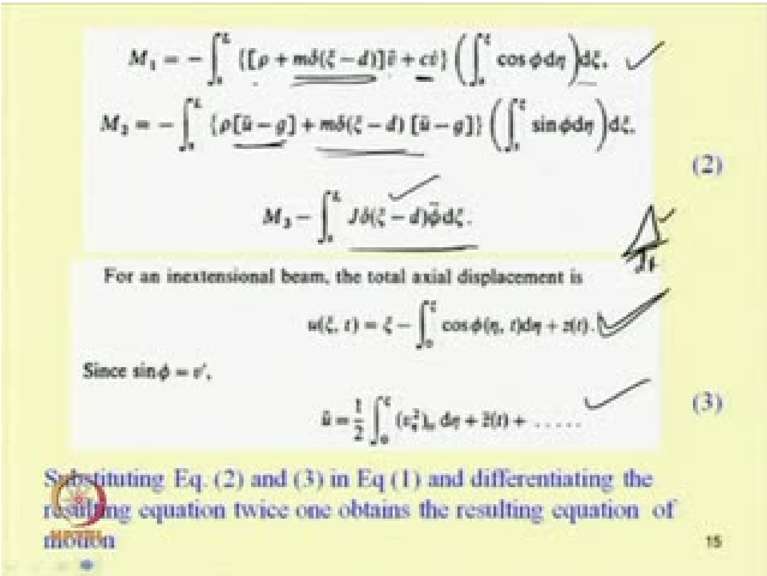

So this is the force and into this distance so, this distance for the small d zeta. So you can find that component along this direction and in that way, so we can find this m 1 similarly, we can find m 2 so while finding m 2 so we have this u double dot minus g along with that. So u double dot minus g and also we should take this mass into account using this dirac delta function and we have this m 3 due to that rotary inertia j delta. So as we are attaching this mass so we are using this direct delta function here. And one can use this in-extensibility condition u zeta t equal to zeta minus 00 to zeta cos phi. So as the length is not changing, so this is d s, this is t x and we can taking this in-extensibility condition, so we can write this equation so you can relate this u with respect to this d of the system.

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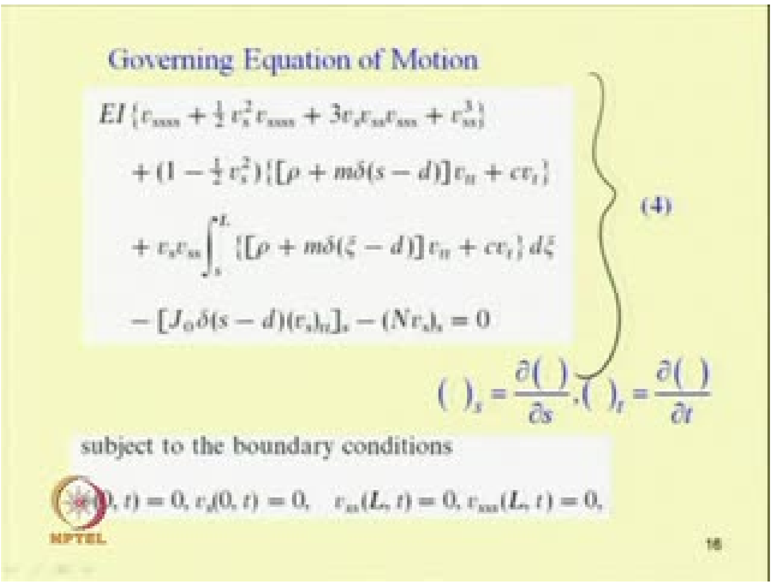

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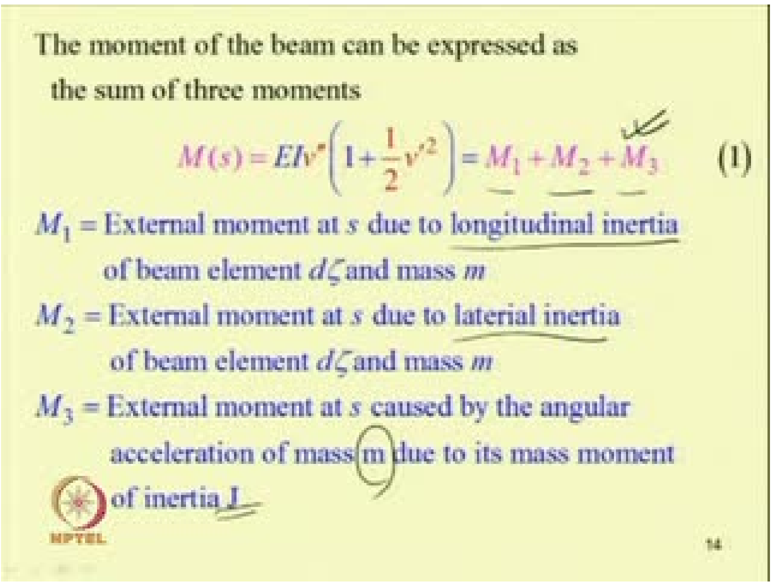

So differentiating it twice, so, you can write this u double dot in this way. So now introducing this we can write the governing equation motion using Hamilton or using this differentiating this equation twice. Now substituting this expression for M1 M 2 M 3 in this equation and differentiating this equation twice as we are differentiating.

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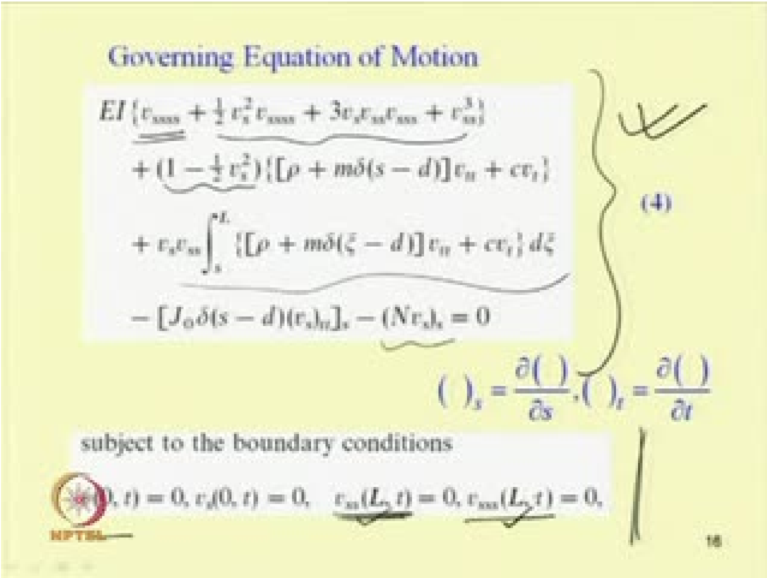

Bending well twice we will get the loading equation or the force equation. So we have this equation motion we obtain the equation motion, which is written in the space and time coordinates. So b is a function of space and time so in this equation contain this del 4th v by del s 4th. So this is the linear term present in the Euler Bernoulli beam equation and one can note these are the additional non-linear terms present due to the term what we have considered in our equation in the bending equation. So differentiating that bending equation twice we obtain this equation.

So these are the additional non-linear terms. So all these terms are non-linear terms present so this is also another non-linear terms so present in this equation. So this is subjected to boundary condition so at base at s equal to 0 the displacement and slope are 0 and also at the end, so we have this sheer force, this is the sheer force proportional to sheer force and this proportional to bending moment. So sheer force and bending moment are 0 as we are not putting the mass at the tip.

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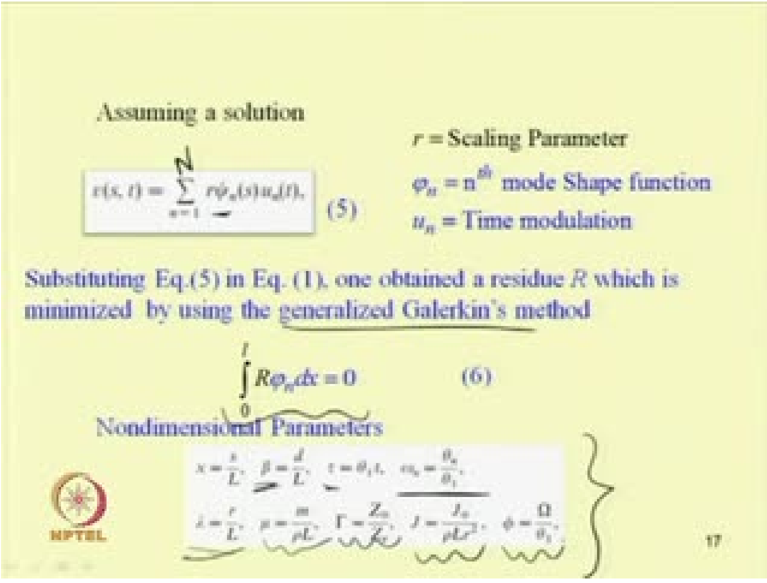

So the mass is put at some arbitrary position that is why at the tip so at s equal to l. So we have this bending moment which is proportional to the s s and sheer force which is proportional to del cube v by del s cube equal to 0. So now we can assume a solution in this form v s t equal to n equal to 1 to infinity, so we can take number of modes, we can limit our number of modes instead of taking infinity we can take up to two mode or three modes depending on the model interaction. So if we are considering only 2 mode interaction, so you can take this term instead of taking infinity. So let me write this n equal to 1 to n so this n can be infinity. If we are taking infinite number of terms or it can be finite if we are taking a finite number of terms.

So this depends on the model participation we are taking in this case. So here r is a scaling parameter so psi n so the psi n is the shape function n s shape function and u n is the time modulation. So we can applying this extended Galerkin method we can reduce or we can minimize the residue as the function what you are taking is not the exact function of the system. So we can so there will be some residue in the system, so that residue can be minimized by using this generalized Galerkin method by using this equation.

That is r is the residue, so r into psi n into d x you can integrate it from 0 to l and equate it equal to 0 and let us take this as the non dimensional parameter. x equal to s by l, beta equal to d by l so this is the position parameter and this tau non dimensional time. So you can take it equal to theta 1 into t where theta 1 is the first mode frequency and omega n equal to omega n equal to so non-dimensional natural frequency can be written as theta n by theta 1, lambda equal to this is the scaling parameter non dimensional scaling parameter, mu is the mass ratio mu equal to m by rho l.

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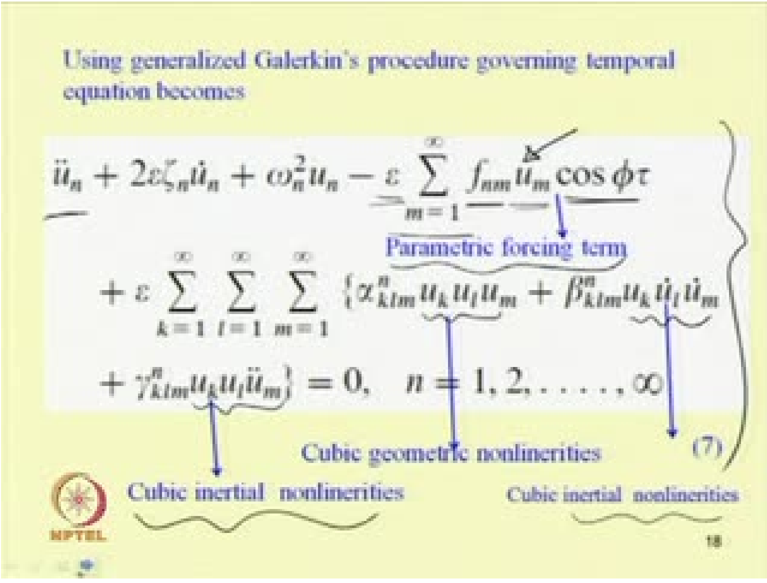

So this non dimensional forcing amplitude equal to Z 0 by Z r. Similarly, j equal to j 0 by rho l r square and external frequency phi equal to non dimensional external frequency phi equal to omega by theta 1, where this theta 1 and other terms. So after writing this thing so we can after applying Galerkin method, so this equation reduce to this form, so this equation can be written as u n double dot, so this is written in terms of the time mode relation. So it is equal to u n double dot plus 2 epsilon zeta n u n dot plus omega n square u n minus epsilon, epsilon is the book keeping parameter m equal to 1 to infinity f n m u m cos psi tau. So this is the so one can note here that these time varying.

So this time varying term that is f n m cos phi, so this time varying term f n m cos phi tau is the coefficient of u m, as this is coefficient or as this parameter is coefficient of the u m that is the response of the system that is why this is known as parametric forcing term and in addition to that we can see the systems contain all these non-linear terms. So this non-linear term where this is the product of three displacement term u k u l and u m and this contains u k u l dot and u m dot and here also it contains u k u l and u m double dot. So the first one that is product of u k u l u m, so if k equal to l equal to m so this becomes u cube, so this is a geometric non-linear term cubic geometric non-linear term and these two, so where this u k u l dot and u m dot. So product of two velocity term gives raise to inertia term that is why this term is known as inertial non-linearity term. So this is also cubic inertial term and this also where this inertia u m double dot is the acceleration term.

It is multiplied with this displacement term and as due to presence of acceleration term, so this is also known as cubic inertia non-linearity. So in the system in addition to this parametric forcing term so we have cubic geometric and inertia non-linear terms. So now so this equation is written in terms of the nth mode and one can note that this forcing function n equal to 1 to infinity contains the interaction of the other model frequencies, other model other modes also, because one can take instead of taking n so one can have this interaction of the other modes.

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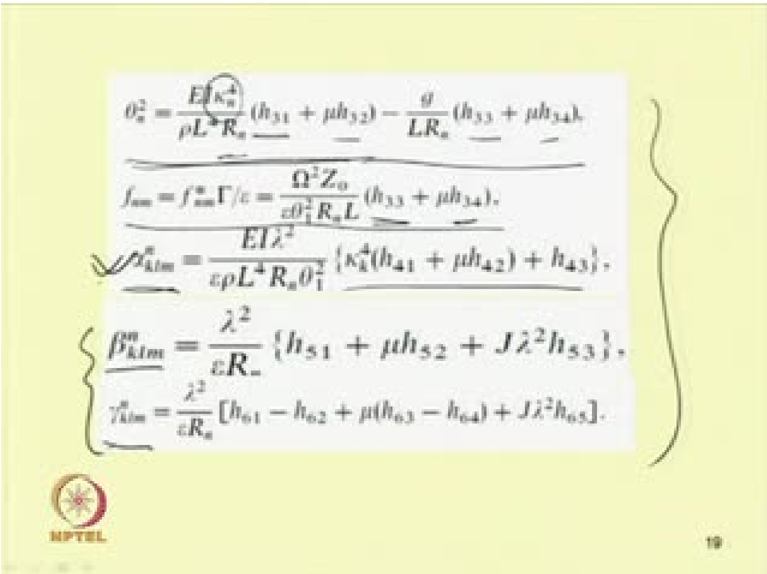

So this term can be taken from this u m term. So u m so it can be u 1, u 2, u 3; that means the nth mode is influenced by the presence of the other mode in the system. Similarly, in this non-linear terms also we have the summation t equal to 1 to infinity, l equal to 1 to infinity and m equal to 1 to infinity, so that means the nth mode is nth mode so nth mode is influenced by the other k l and m mode present in the system. So we can take so here the coefficient of those terms coefficient of this equation can be seen from these, so where theta n square theta n square is the actual non-linear frequency of the system. So this is written in this by this expression.

Similarly, this forcing amplitude of the forcing function can be written by this expression. So this frequency term contain the term with young’s modulus i and then this k n term k n is the k n 4th, so this is the characteristics so this has come from the characteristic equation and rho mass per unit length and this is the scaling parameter. So also this contain this these terms that is s 3 1 these are the integral terms, mu is the mass ratio and these are the integral terms present in the system. So this is the forcing term, so forcing term contains these frequency of the forcing amplitude of the forcing then this first mode frequency then these are the integral term s 3 3 and s 3 4 are the integral term, so which are defined after this slide.

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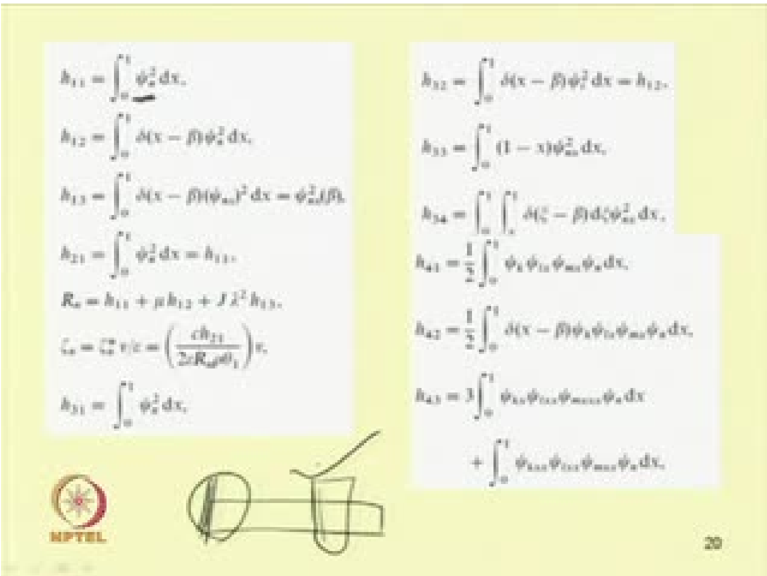

So this is the non-linear term alpha n k l m, and this is beta k l m and gamma k l m. Beta and gamma are non-linear inertia term and this is the coefficient of linear this is the coefficient of cubic geometric non-linearity. So these are the expressions for expression for the integral what we have used in this equation. So one can see so there are many integral terms one has to evaluate for finding the coefficient of the system, where the psi n is the shape function of the system and in this case the step function is taken that of a cantilever beam with thick mass.

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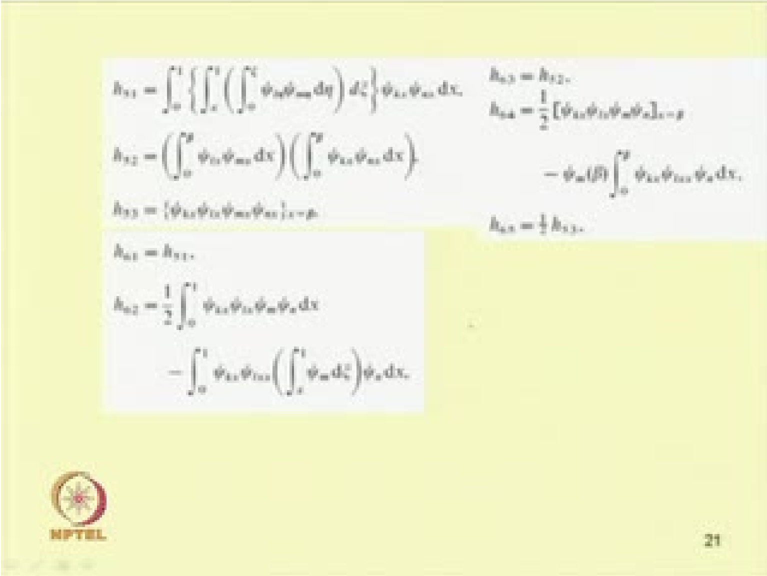

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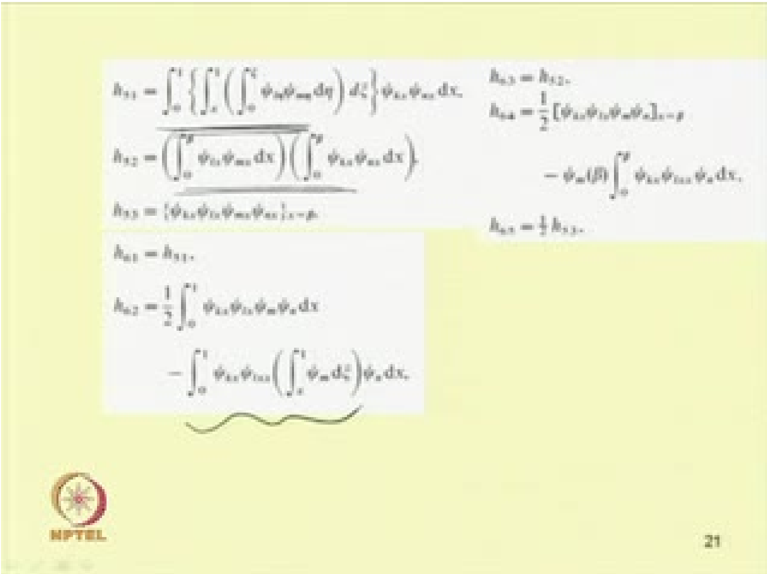

So one can take or cantilever beam one can take this shape function of a cantilever beam also which satisfy the geometric boundary condition; that means it satisfy geometric boundary condition here at the left plane that is displacement and slope equal to 0 or one can take the step function of a cantilever beam with arbitrary mass position. So by taking this so one can find all this integrals, so one can note that is in some places, one can find only single integral and other places one can find double and triple integrals also in the system. So one should be very careful so here one can have triple integral terms so here double integral terms are present so two twice the system so integration has to be performed. So one has to take care while finding this coefficient of the system.

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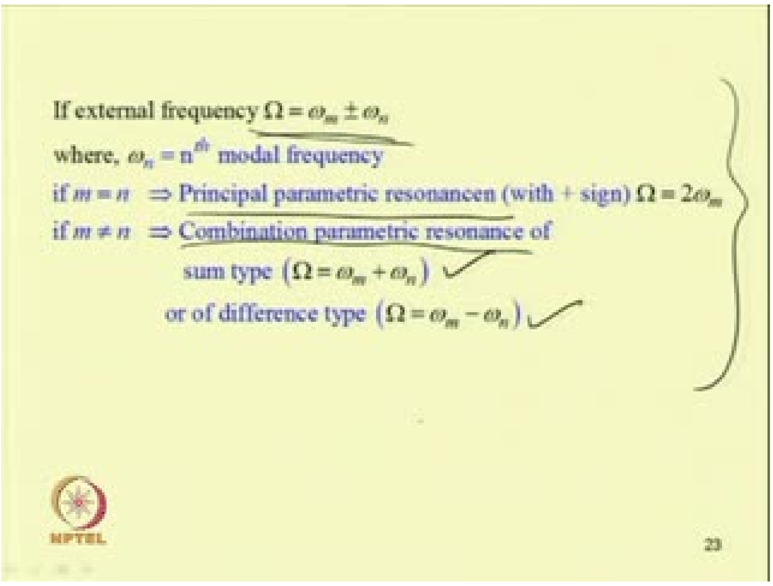

So after finding this coefficient of system, so one can see or we have already discussed that the system can have external resonance conditions. So in case of the parametrically excited system. When this external frequency equal to summation of other modal frequency for example, so if omega equal to omega m plus minus omega n, so we can have resonance condition. So in this case this m and n are modal frequency, if m equal to n then we can tell this as principal parametric resonance condition and f naught equal to n so this is known as combination resonance condition. So in case of combination resonance condition we can have sum type or we may have difference type. So one can have all these type of resonance condition so this principal parametric resonance condition can be of nth mode or n can be of first second third or higher modes.

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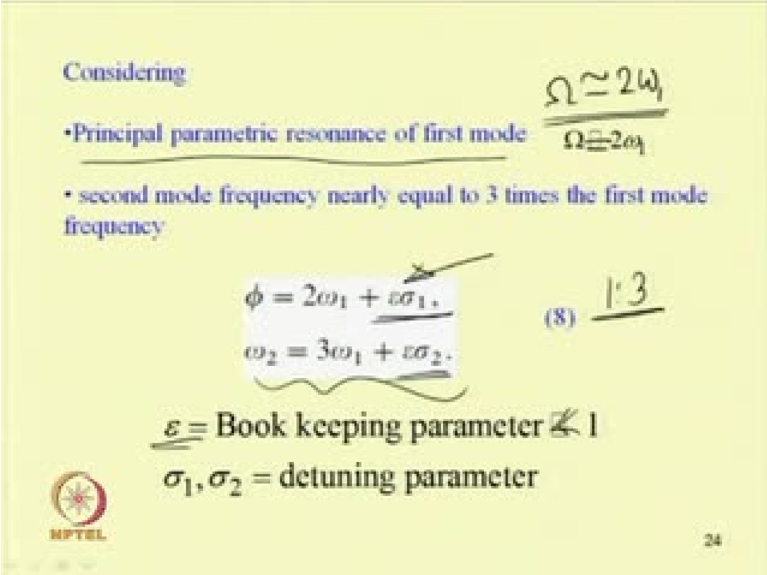

So in that case it will be principal parametric resonance of first mode, second mode, third mode or combination resonance of first and second mode, if m equal to 1, n equal to 2 then it is sum type combination resonance of first and second mode of sum type. Then one may have difference type also, so let us take the condition when we are considering the principal parametric resonance of first mode; that means omega is nearly equal to omega is nearly equal to nearly equal to 2 omega 1. So will consider the case in the neighborhood of twice omega 1.

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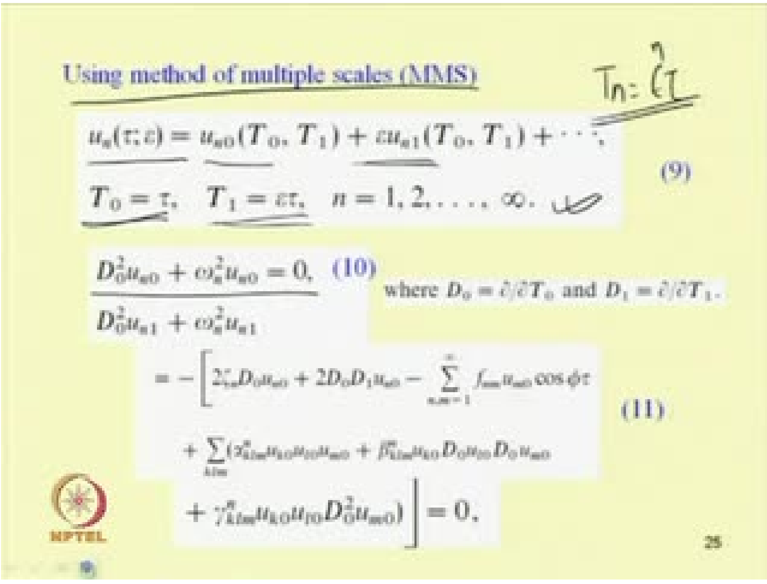

So to take into account the nearness of omega that this is the external frequency to thrice the natural frequency, we can take detuning parameter epsilon sigma 1. Sigma one is the detuning parameter, so and as we are discussing about this 1 is to 3 internal resonance case, so as we are considering internal resonance case of 1 is to 3, so you can take this omega 2 l equal to 3 times omega 1 and this is the detuning parameter which will take care the nearness of this omega 2, that is the second mode to that of three times omega 1. So here we are taking this epsilon as the book keeping parameter and we already know this book keeping parameter should be very very less than 1. So now let us use this method of multiple scale to derive the resulting equation or reduced equation which we can further use for finding the response and stability of the system. So using this method of multiple scale so we can write or we can reduce the temporal equation and for that purpose let us take this u n equal to that is the time modulation u n equal to u n 0 plus epsilon u n 1.

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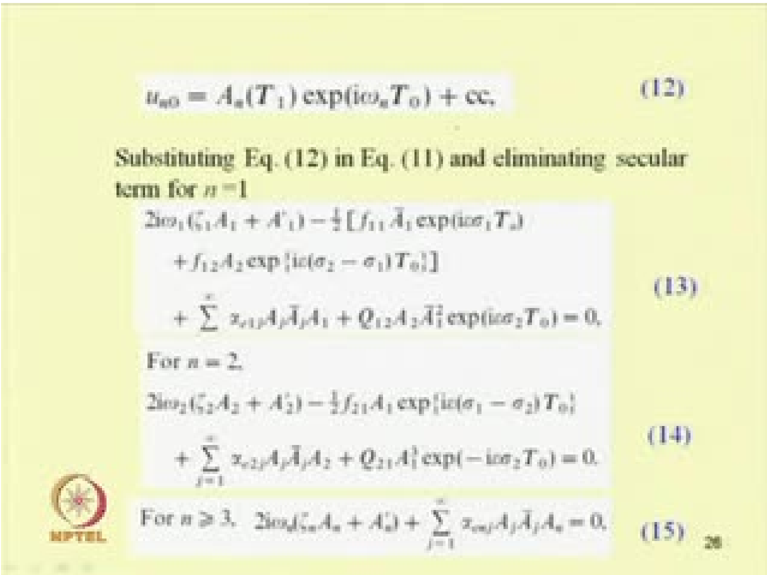

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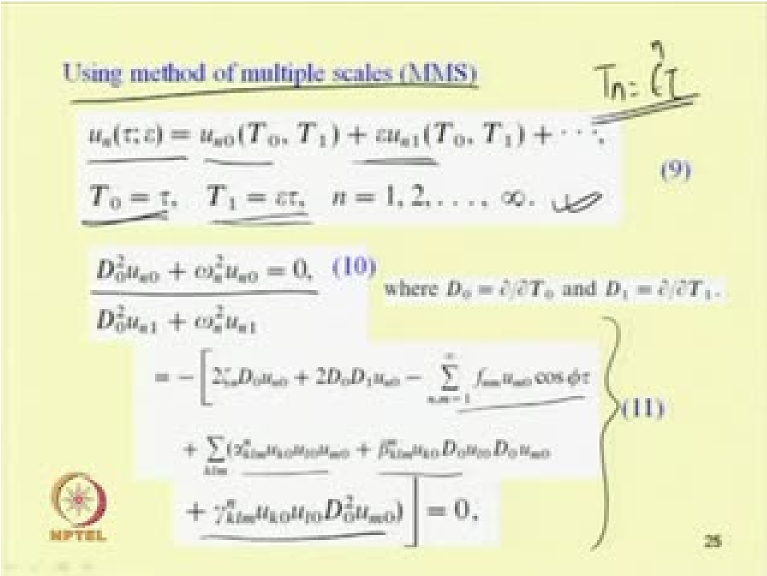

So u n 0 and u n 1 are function of time different time scales that is t 0 t 1, so if you take up to t 1 only of the order of epsilon, so here we are taking this t n equal to already we have discussed this, so t n equal to epsilon n t, so where as we are taking this non dimensional time tau, so we can put this equal to epsilon to the power n into tau. So our t 0 n equal to 0, so this is equal to tau and epsilon equal to 1, t 1 equal to epsilon tau so here we are taking this n equal 1 to infinity.

So by substituting this equation in the temporal equation, so we can and separating the terms in the order of the epsilon 0 and epsilon to the power 1 so we have this two equations. So the first equation is this which can be written as d 0 square u n 0 plus omega n square u n 0 equal to 0. So in that case we know the solution of this equation or we can write the solution of this equation in the form A n t 1 A to the power i omega n t 0 plus its complex conjugate.

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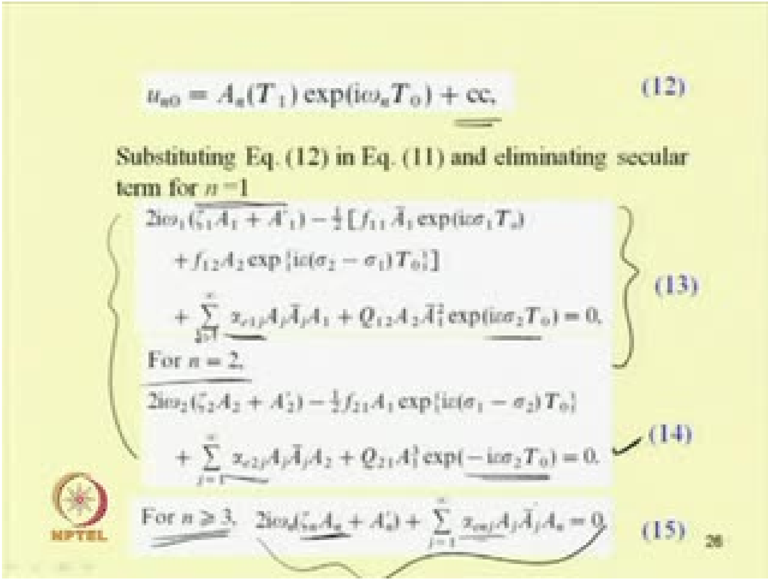

And substituting this equation in the second equation, that is d 0 square u m 1 plus omega n square u m 1 equal to minus 2 zeta n d 0 u n 0 plus 2 d 1 d 0 d 1 e 1 0 minus n m equal to summation n m equal to 1 to infinity, f n m u m 0 cos phi tau plus, so these are the non-linear terms. So substituting that u term here in this equation, so we can have we can have some terms which are known as secular or mixed secular or near secular terms. So as we know that this terms will leads to infinite response and the system has a definite response, so we have to eliminate the secular terms. Now eliminating the secular terms, so for n equal to 1 so we can write this equation, so in this equation for n equal to 1 we can write this equal to 2 i omega 1 into zeta 1 A 1 plus A 1 dash minus half f 1 1 A 1 bar A to the power i epsilon sigma one t 0 plus f 1 2 A 2 e to the power i epsilon sigma 2 minus sigma 1 t 0 plus n equal to or equal to 1 to infinity j equal to 1 to infinity, so this is A equivalent term alpha i 1 j A j into A j bar into A 1 plus q 1 2 A 2 into A 1 bar square A to the power i epsilon sigma 2 t 0 equal to 0. Similarly, as we are taking only 2 mode interaction, we can expand up to n equal to 2, so similarly, for n equal to 2 we can write this term by eliminating the secular term equal to this that is 2 i omega 2 into zeta 2 A 2 plus A 2 dash A .

Where A 2 dash equal to D A 2 by d d 1 minus half f 2 1 A 1 e to the power of i epsilon sigma 1 minus sigma 2 into t 0 plus j equal to summation j equal to 1 to infinity alpha e 2 j A j into A j bar into A 2 plus q 2 1 A 1 q e to the power minus i epsilon sigma 2 t 0 equal to 0 and for n greater than 3. So this equation is written in this form 2 i omega n zeta n A n plus n dash plus j equal to 1 to infinity alpha i n j A j A j bar into n equal to 0 and 1 can note from this third equation, that is n greater than equal to 3 as neither forcing or internal resonance cases are there. So with due to the presence of damping term, so this will the response amplitude will decay and the system will come back to the original trivial state for n greater than equal to 3. So we have to consider only the first two cases that is n equal to 1 and n equal to 2. Because for n greater than 3 due to presence of damping and absence of internal and external forcing the system indicate to the trivial state.

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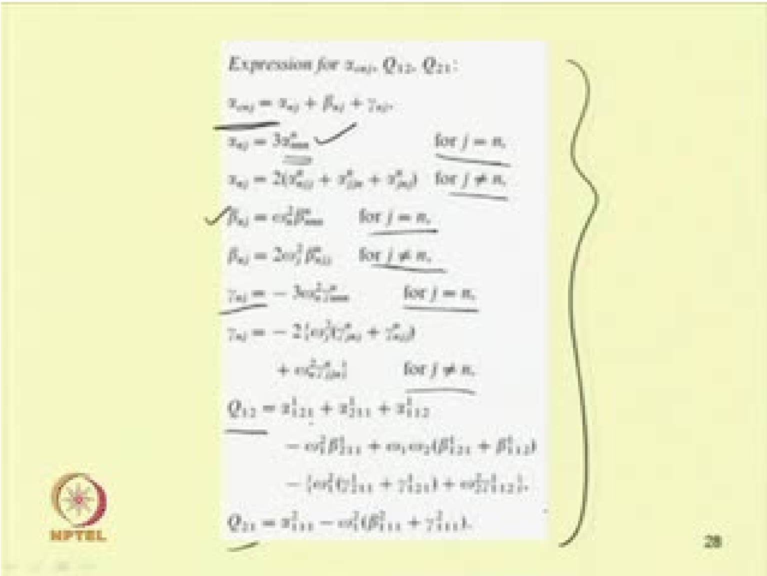

So let us take so in previous expression, so these are the terms alpha i equivalent terms so alpha E n j equal to alpha n j plus beta n j plus gamma n j, where alpha n j can be written as 3 into alpha n n n n. .

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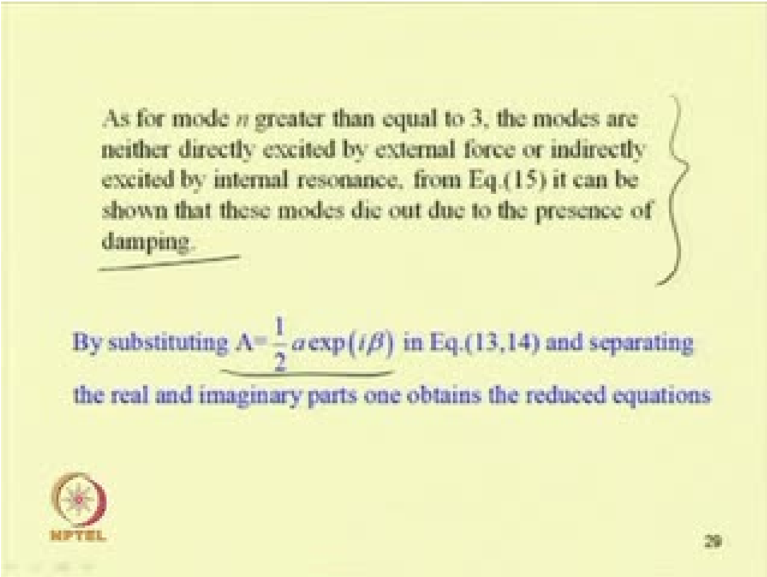

So j equal to n j k l m are written in terms of n. Similarly, alpha n j can be written as 2, so for j equal to n so it is written in this form and j not equal to n. So alpha n j is written equal to 2 into alpha n j j plus alpha n j j n plus alpha n j m j. So in that way one can write similarly, one can write this beta n j for j equal to m j naught equal to n, similarly, gamma m j. So this is the expression for gamma m j for j equal to n and j naught equal to m and q 1 2, which accounts for the internal resonance terms.

So one can write this expressions, so these are coming due to the non-linear terms alpha beta and gamma so writing this terms, so already it is been told that as for mode n greater than equal to 3 the modes are neither directly excited by external force or indirectly excited by internal resonance. So from that equation it that can be shown that this modes die out due to the presence of damping. So now by taking A equal to half A E to the power i beta, so we can write this equation by separating the real and imaginary parts.

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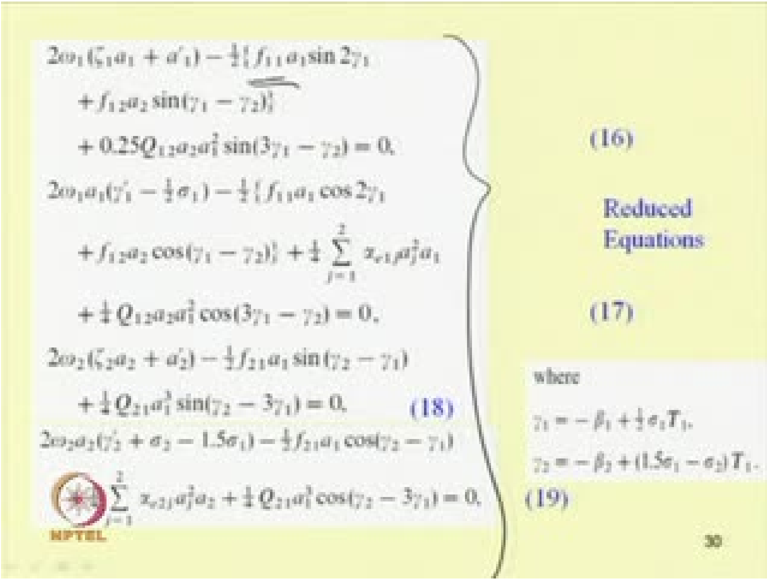

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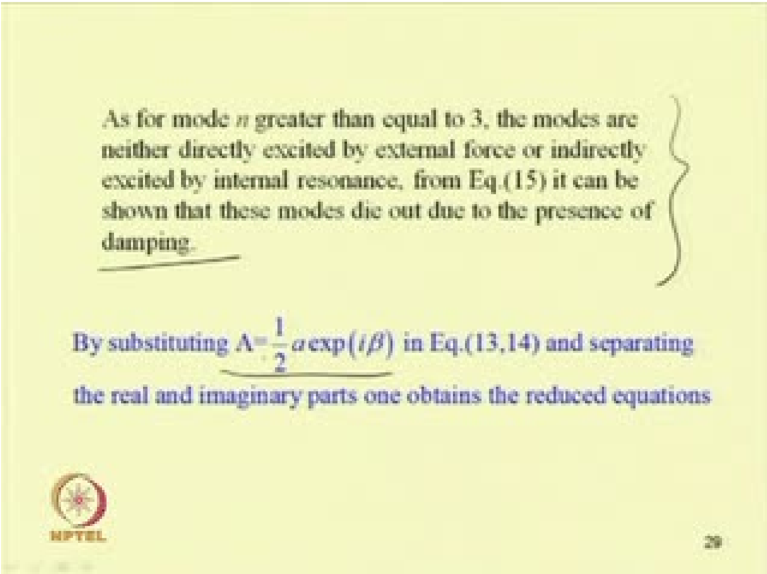

So we can write this equation in this way, so these are known as the reduced equation. So in this case we are getting four equations, so these four equations are known as the reduced equation. So this equation so these equations can be written so this is the first equation, 2 omega 1 zeta 1 A 1 plus A 1 dash minus half f 1 1, so this is the forcing term f 1 1 A 1 sin 2 gamma 1 so here we have initially we have substitute A equal to half A E to the power i beta.

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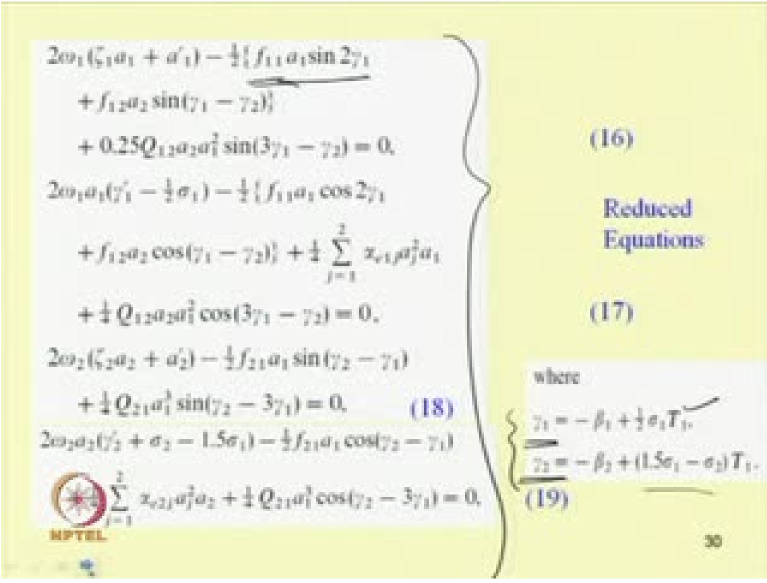

So after substituting E to the power i beta, so you can so as the resulting equation will contain this time explicitly. So we can write those equation in its autonomous form by writing or by transforming this beta in terms of gamma, so we can write this gamma equal to or gamma 1 equal to minus beta 1 plus half sigma 1 t 1 and gamma 2 equal to minus beta 2 plus 1 point 5 sigma 1 minus sigma 2 t 1 so in this way by writing this equation in terms of gamma 1 and gamma 2. So we can write these terms this way. Now we have reduced this equation to a set of first order equation, so this reduced equation so these are the reduced equation we have already written.

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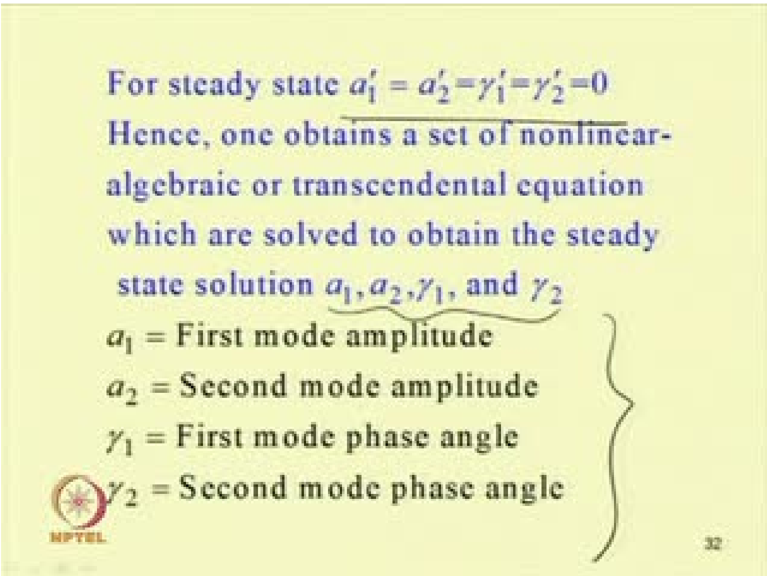

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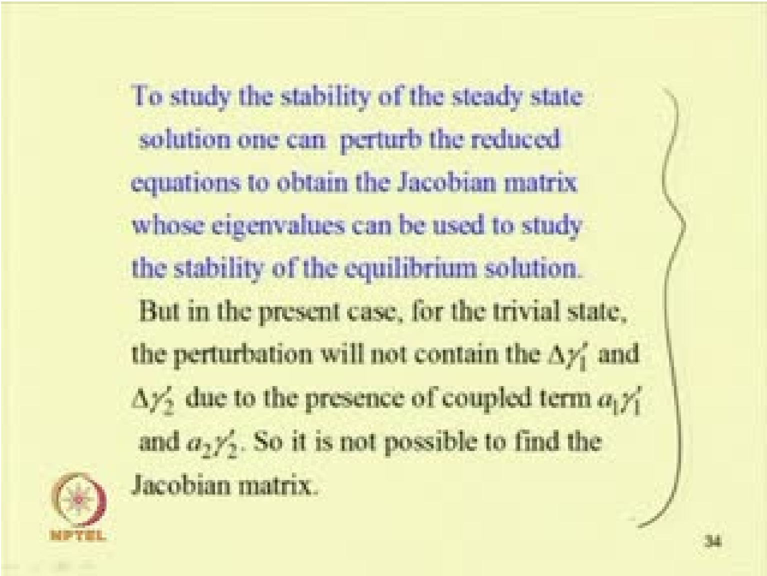

So, for steady state as this a and gamma that is the amplitude and gamma are that function of time, so we can substitute this terms equal to 0, hence one can obtain a set of non-linear algebraic or transcendental equation, which are solved to obtain the steady state solution. So one can obtain the steady state solution A 1 A 2 gamma 1 gamma 2 by solving those transcendental equation. So A 1 is the first mode amplitude, A 2 is the second mode amplitude, gamma 1 is the first mode phase angle and gamma 2 is the second mode phase angle. So and to study the stability of the system to study the stability of the trivial state so the steady state trivial should what we obtain as steady state fixed point response to study the steady fixed point response which may be trivial or non trivial.

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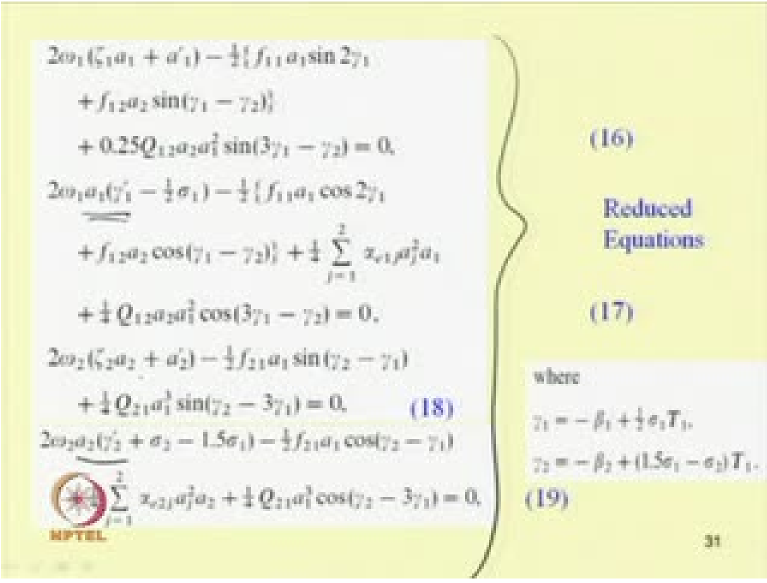

So one can perturb this equation but, while perturbing this equation due to the presence of the terms like A 2, gamma 2 dash, A 1, gamma 1 dash so when will perturb this terms, so it will contain the term A 1 and A 2, so the perturbation will not contain the term delta gamma 1 dash and delta gamma 2 dash.

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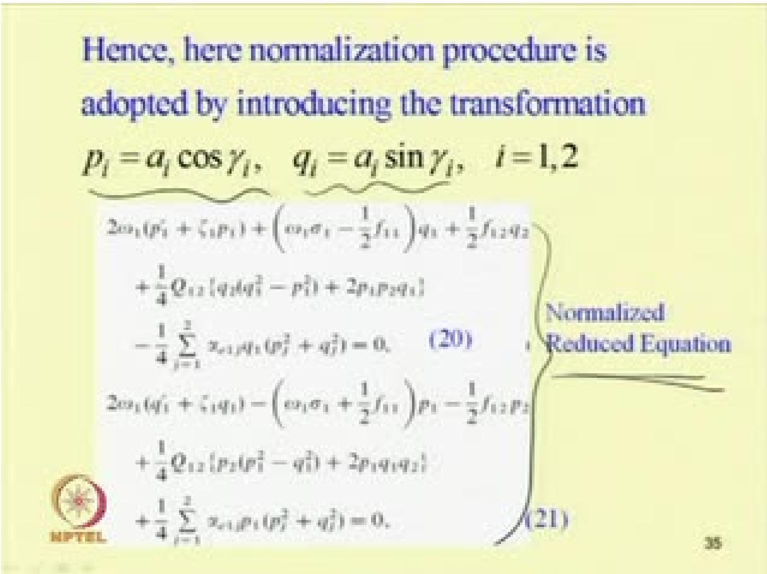

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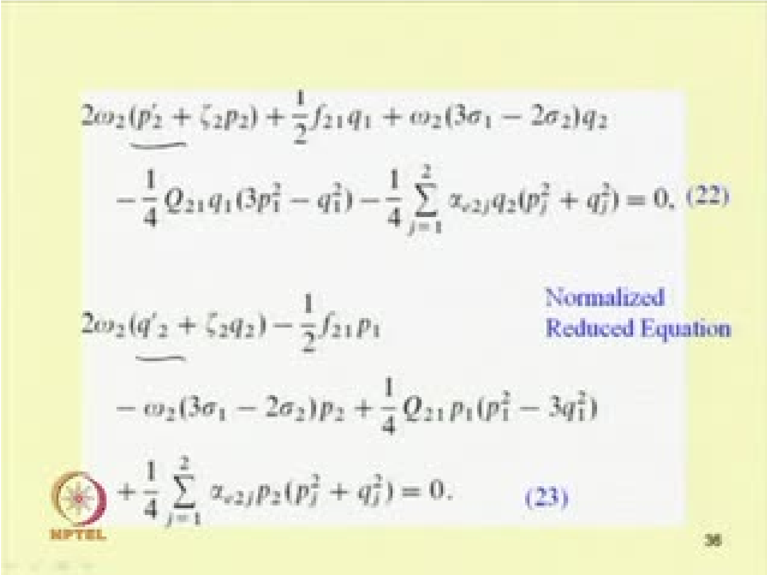

For the trivial state as for the trivial state this A 1 and so A 1 dash and A 2 dash as 0, so for the for the trivial state as A 1 and A 2 are 0, so it will not contain due to the presence of this product term. So it will not contain this term, so one has to transform this into its normalized form. So by substituting this p i equal to A i cos gamma i and q i equal to A i sine gamma I, where i equal to 1 to 2. So by substituting this p i equal to A i cos gamma i and q i equal to A i sine gamma i. So these four equations are now reduced or written in its normalized reduced equation in terms of p 1 and p 2. So we can have a set of equation in terms of p 1 dash, q 1 dash, p 2 dash and q 2 dash p 2 dash and q 2 dash. So these are uncoupled so uncoupled in terms of A and gamma.

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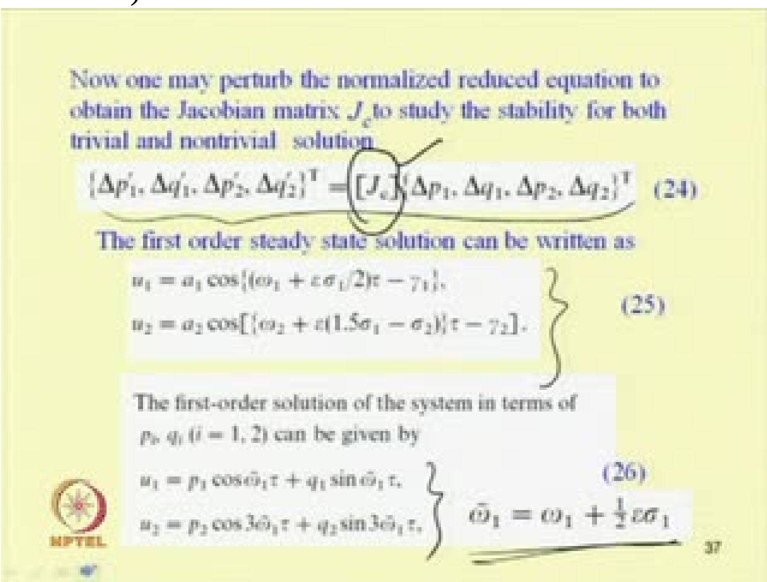

Now one can easily perturb this equation to find the response or stability of the system. So now by perturbing this equation one can write or one can find the Jacobian matrix, so Jacobian matrix j c, so which is the which is obtained by the first derivative of the terms of that equation. So one can write or one can find the Jacobian matrix, so after finding this Jacobian matrix so one can find the solution.

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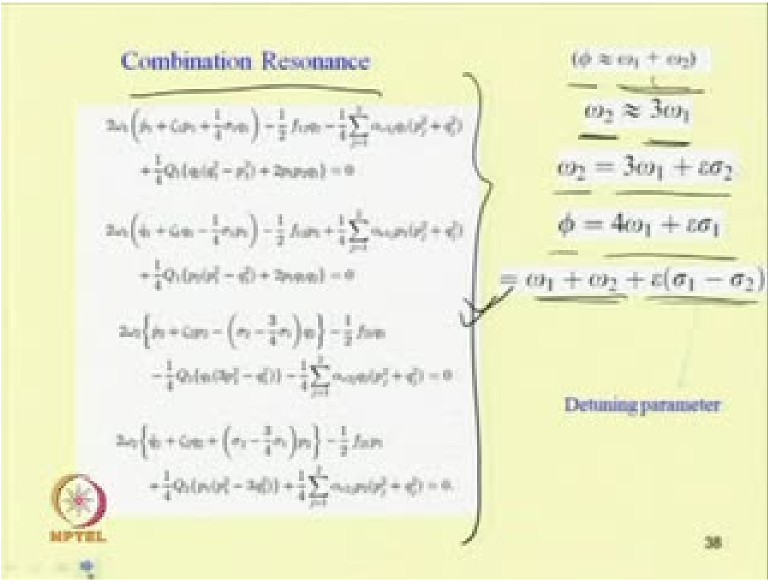

One can find the response by for the steady state response, one can find the response by solving the equations so after solving these equation or after getting the response then one can study it is stability. And first order equation can be written in this form that is u 1 equal to A 1 cos omega 1 plus epsilon sigma 1 by 2 into tau minus gamma 1 and u 2 equal to A 2 cos omega 2 plus epsilon 1.5 sigma 1 minus sigma 2 tau minus gamma 2 or this equation can be written in terms of p 1 and p 2. In this form so it can be written in terms of p 1 p 2 by p 1 cos omega 1 tau plus q 1 sine omega 1 bar tau and u 2 equal to t 2 cos 3 omega 1 bar tau plus q 2 sine 3 omega 1 bar tau where this omega 1 bar equal to omega 1 plus half epsilon sigma 1.

So in this way one can find the first order solution and one can plot the solution to find the response of the principal parametric resonance space. Similarly, one can study the combination resonance. So in case of combination resonance if somebody take this external frequency equal to omega 1 plus omega 2 and internal.

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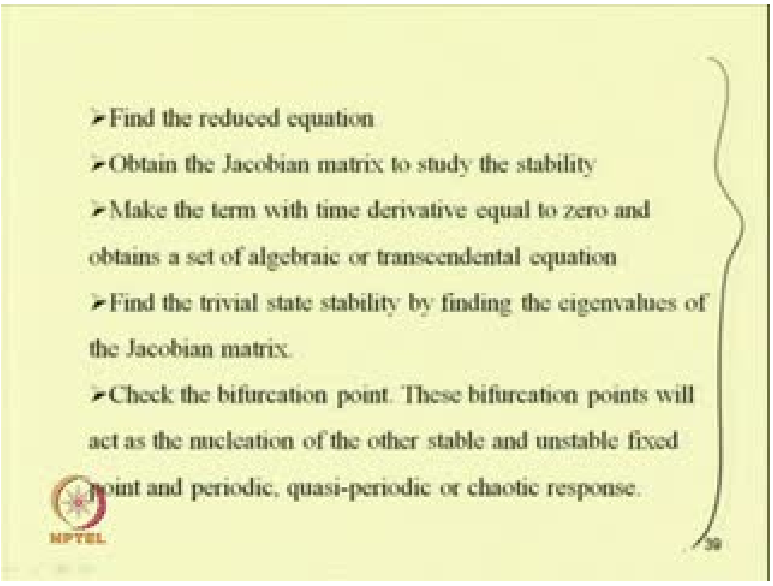

Considering internal resonance omega 2 equal to three times omega 1. So one can write this omega 2 equal to 3 omega 1 plus epsilon sigma 2 and this phi that is external frequency equal to 4 omega 1 omega 1 is 3 times omega 1 omega 2 is 3 times omega 1 plus omega 1. So this becomes four times omega 1, so phi nearly equal to 4 omega 1 plus epsilon sigma 1. So this is equal to omega 1 plus 1 omega 2 plus epsilon into sigma 1 minus sigma 2.

So now in that temporal equation solving the temporal equation in the similar way so one can get a set of this four equation for the combination resonance in terms of, one can initially write in terms of a and gamma and following the similar logic. In case of the principal parametric resonance as the trivial state will not contain the terms in with perturbation terms. So one can write the normalized equation, so one can write this normalized equation in terms of t 1 dash, q 1 dash, t 2 dash and q 2 dash. So these are the normalized equation.

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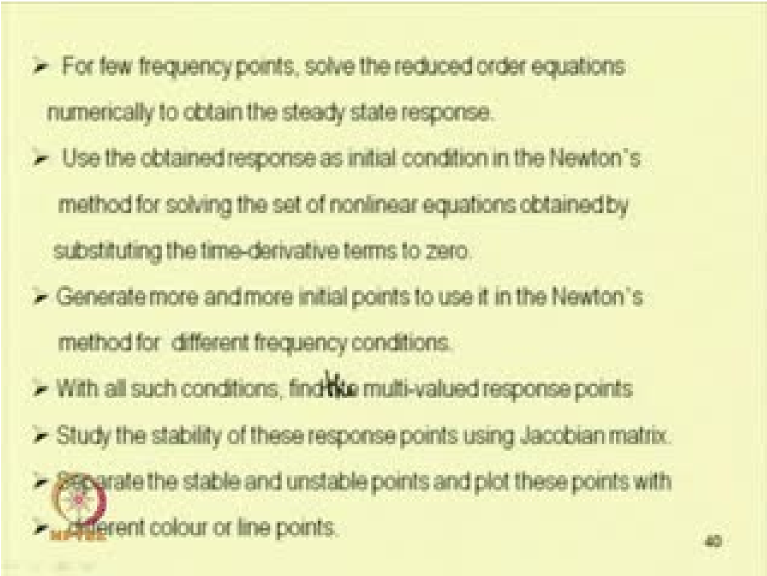

One can so one can follow this procedure that this first find the reduced equation obtain the Jacobian matrix to study the stability make the term with time derivative equal to 0 and obtain a set of algebraic or transcendental equation, find the trivial state stability by finding the Eigen value of the Jacobian matrix, check the bifurcation point. So this bifurcation point will act as the nucleation of other stable and unstable fixed point periodic and quasi periodic or chaotic response. So one has to check this bifurcation points and for few frequency points.

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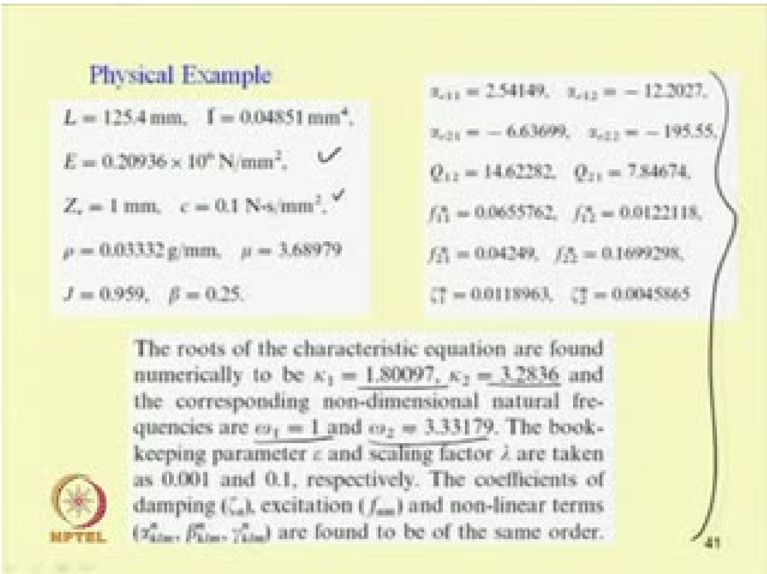

So solve the reduced equation reduced order equation numerically to obtain the steady state response, so use the obtained response as initial condition in the Newton's method for solving the set of non-linear equations obtained by substituting the time derivative terms equal to 0. So generate more and more initial points to use it in the Newton's method for different frequency conditions.

So with all such conditions find the multi valued response point, find the multi valued response points so study the stability of this response points using Jacobian matrix separate the stable and unstable points and plot this points with different color or line points, to show the system response. So let us take one physical example and see how we obtain the response of the system or how this simple system behave in a non-linear way.

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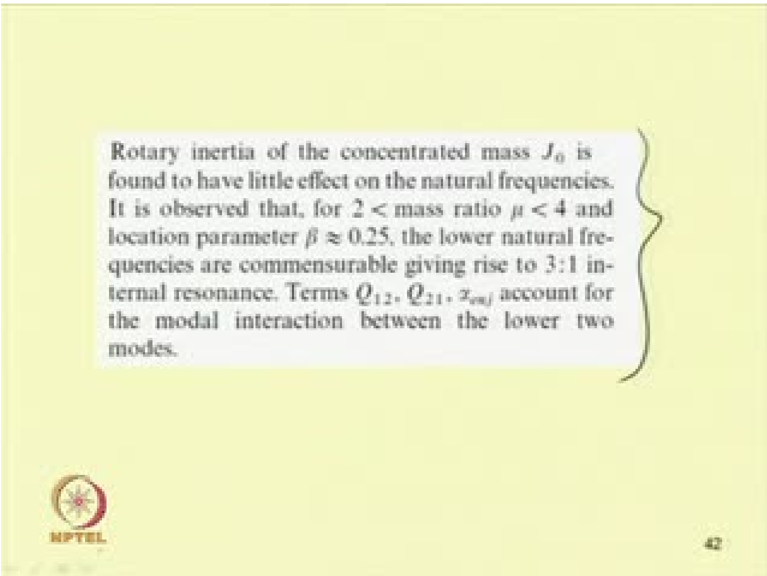

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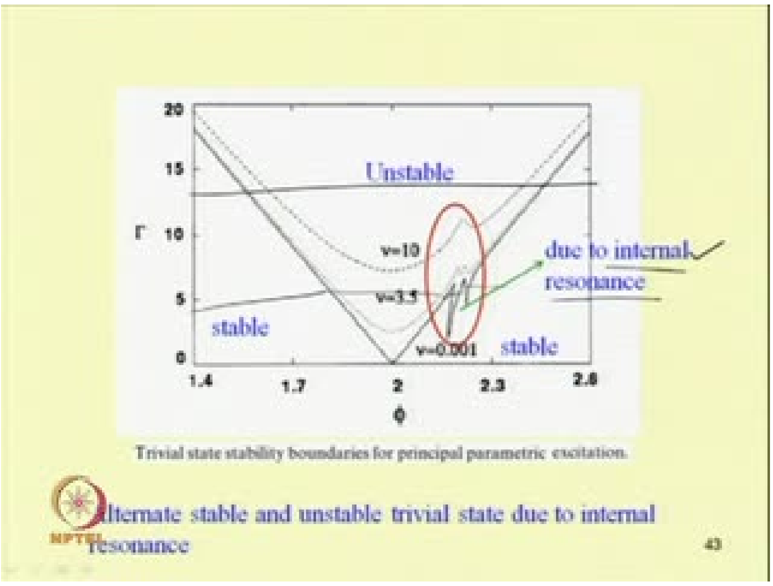

So in this physical example we have taken this length equal to 125 point 4 m m f equal to, so i equal to point 0 point 0 4 8 3 5 1 with m m cube e. So all this parameters are given here, so the roots of the characteristic equations are found numerically to be 1.80097. So these are the first two roots and 3.2836 and the corresponding non dimensional lack of natural frequencies are omega 1 equal to 1 and omega 2 equal to 3.33179 as it is nearly equal to omega 2, nearly equal to 3 times omega 1.

So we have taken these as 3 is to 1 internal resonance conditions here the book keeping parameter and scaling factor are taken as 0.001 and 0.1 respectively. The coefficients of this terms are found out and so these are the coefficients of the terms and so one can plot this in stability regions. So first when one plot this in stability region one can observe that due to the presence of internal resonance.

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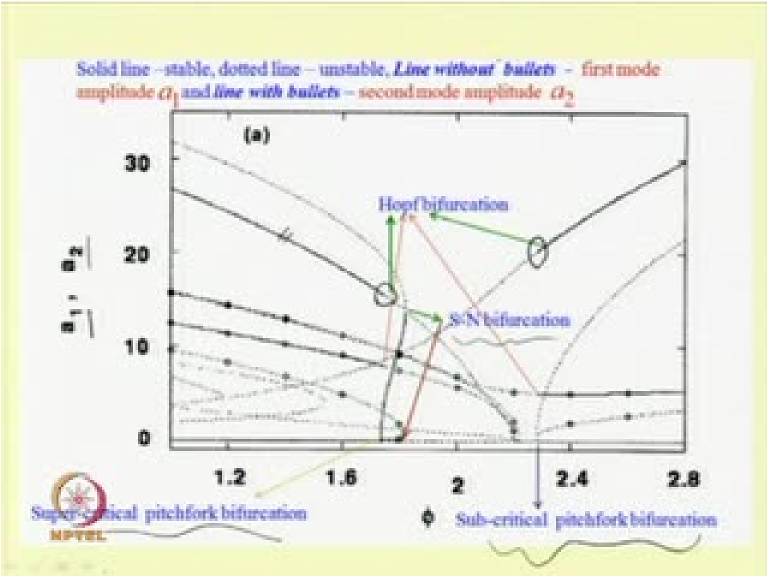

So here one can observe this keen type of phenomena here, so if one take a line here so this show the this part is stable this is unstable. And this is stable but, where do we consider this internal resonance case. So one can see multiple stable unstable branches so up to this system is stable then here to here the system is unstable, then again here it is stable then again here it is unstable, then stable, then unstable then again it is stable. So one can have multiple stable and unstable region in the trivial state of the system, where are consider internal resonance in the system.

So from this one can obtain the response curve, here A 1 and A 2 are response curve is plotted. So A 1 is shown in this line with that circle mark and A 1 with the line without the circle mark. The solid lines are stable branch and the dotted line are unstable branch, so one can observe many bifurcation points in the system, for example, here one can observe the super critical pitch fork bifurcation point and here one can have the sub critical pitch fork bifurcation point. Similarly, here these points corresponds to the saddle node bifurcation point and this point correspond to this hop f bifurcation point. This is also hop f bifurcation point.

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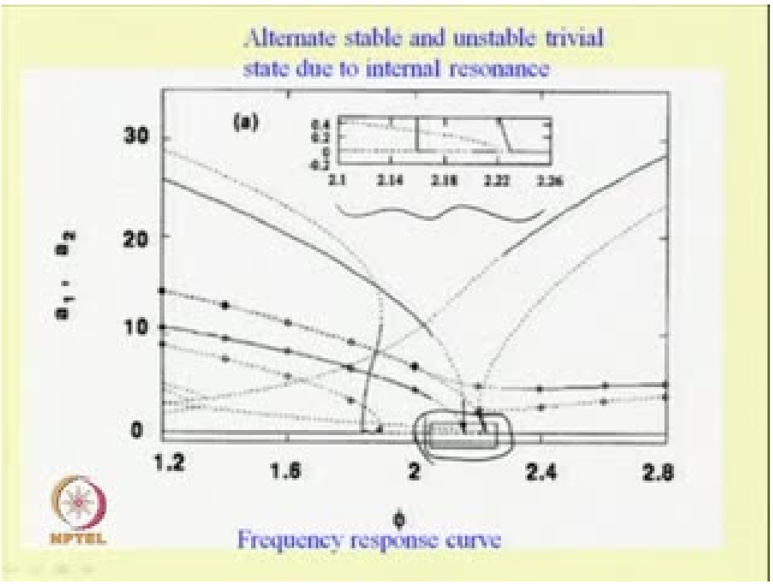

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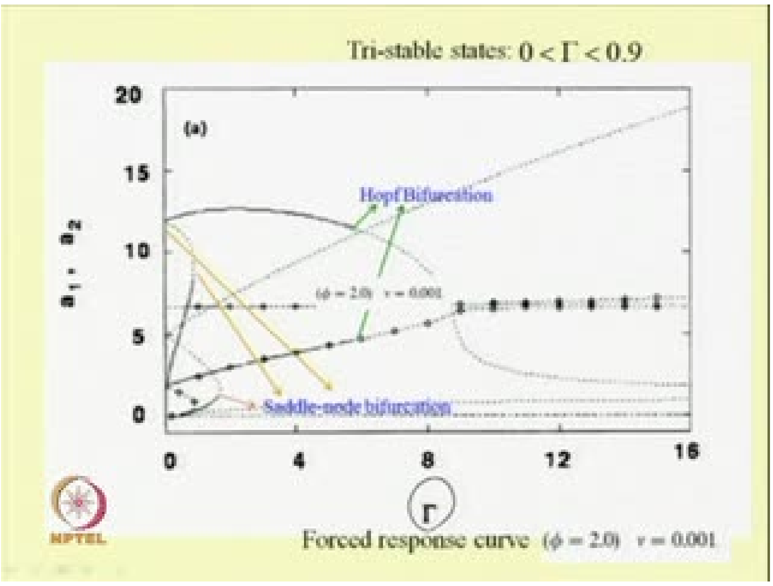

So as we have several sets of bifurcation points, so these bifurcation points will act as nucleation of other type of response of the system. So this is the fixed point response is plotted so one can have other periodic, quasi periodic and chaotic response of the system. So to show the alternate so this part is zoom, so by zooming this part one can see the system has alternate stable, unstable, stable, unstable branches in this trivial state. So this is the force response. In previous curve we have shown the frequency response so frequency response is plotted with respect to this A 1, A 2 versus this frequency for different system parameter, here it is plotted with respect to this forcing parameter.

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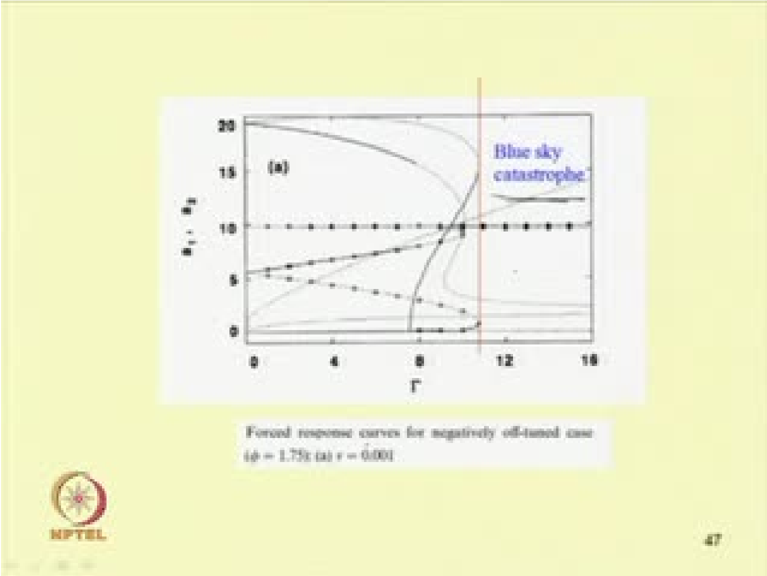

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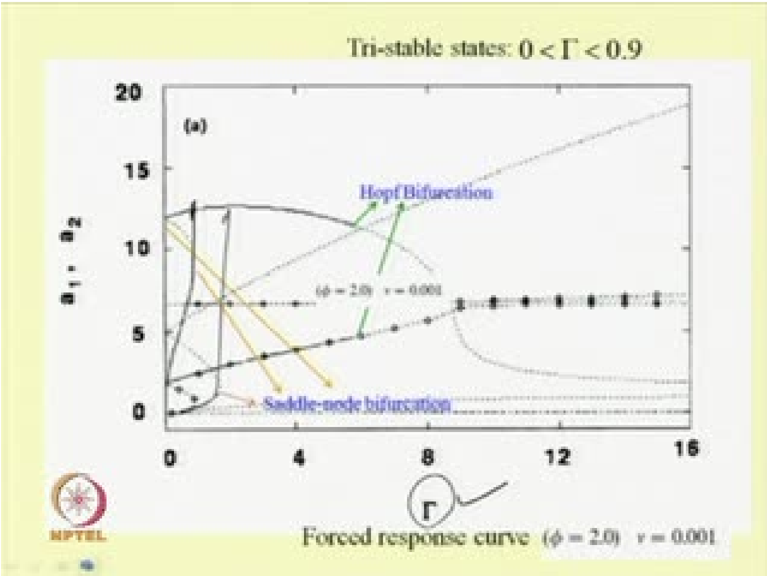

So this is the so here also one can see several bifurcation points and here one may note that after this point as there is no stable state present in the system, the system will fail due to this catastrophic blue sky catastrophe phenomena and here one can explain the jump up and jump down phenomena in the system, for example, by increasing this gamma from this position one can see the system will rest jump from point to this point.

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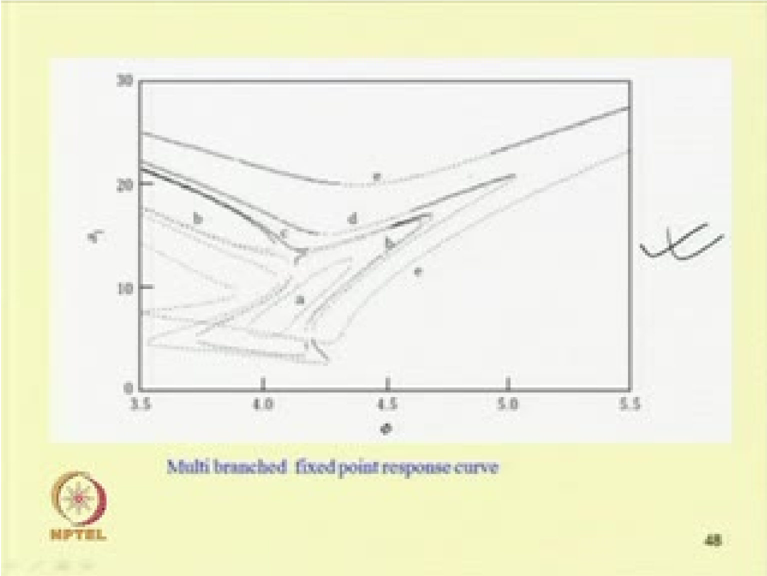

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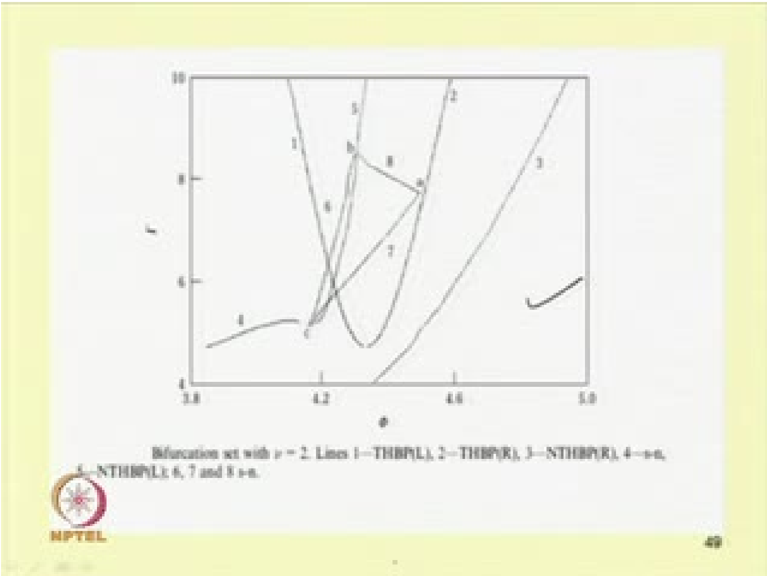

So this is a jump up phenomena similarly, after this so one can observe that it can jump from this to this branch also, there will be a lot of jump up and jump down phenomena present in the system. And this response curve is obtained for the combination type of resonance. So this is one can plot the bifurcation set by plotting different resonant or different bifurcation points.

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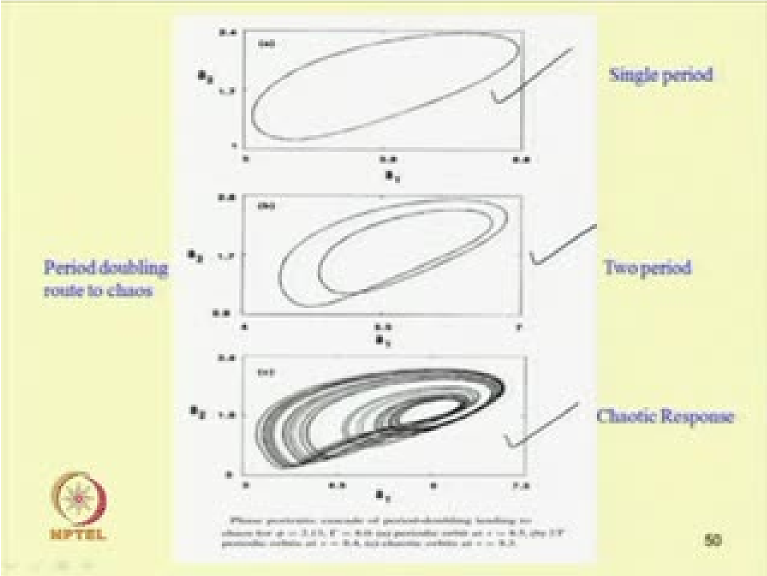

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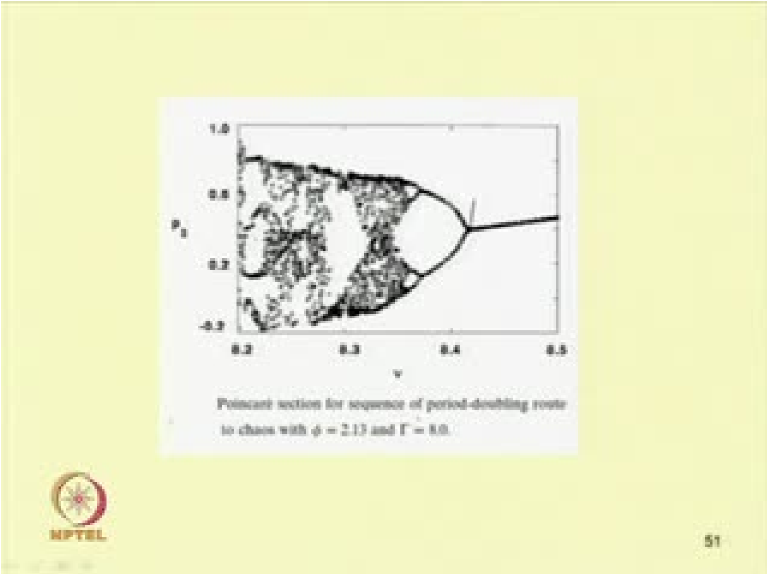

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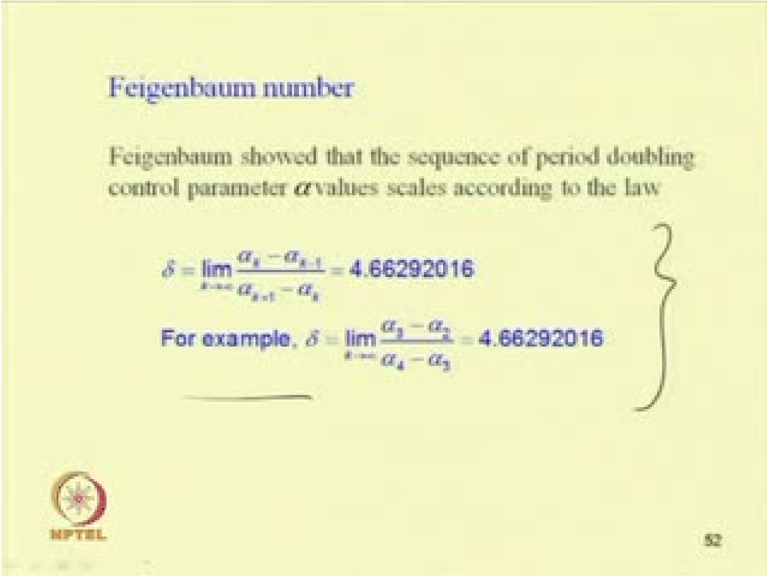

So this is the trivial state bifurcation point, in addition to that these are the non trivial state bifurcation point in the system. So in this way one can plot different fixed point response of the system also by using this reduced equation or by solving this reduced equation one can see near this hop f bifurcation point, one can obtain a set of periodic solution. So by changing the system parameter one can show that this periodic solution becomes two period and further changing this parameter one can observe a chaotic response of the system. So one can plot the Poincare session.

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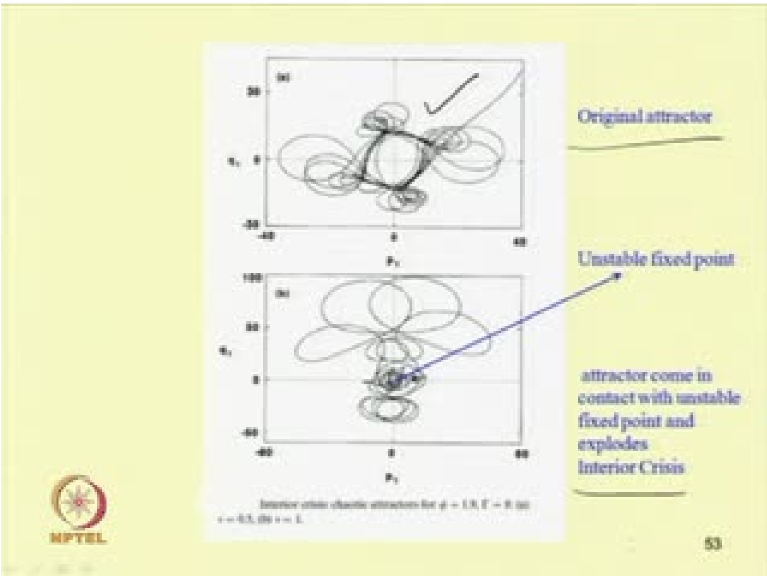

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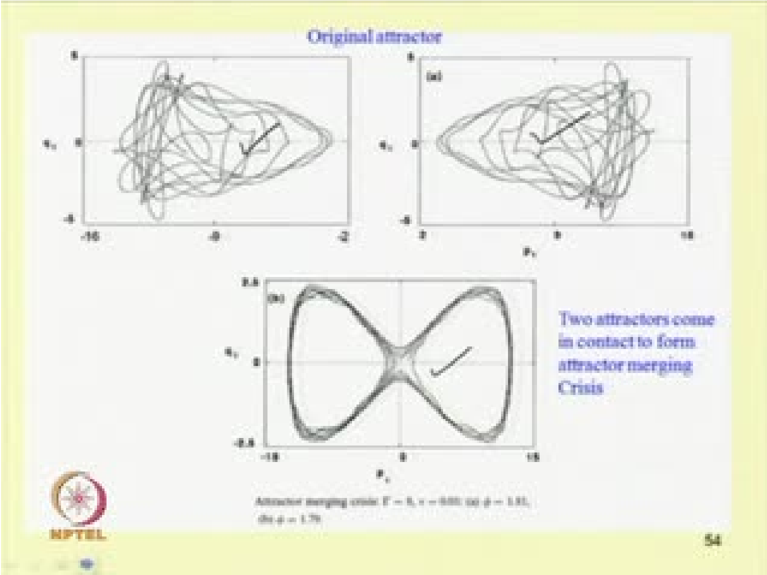

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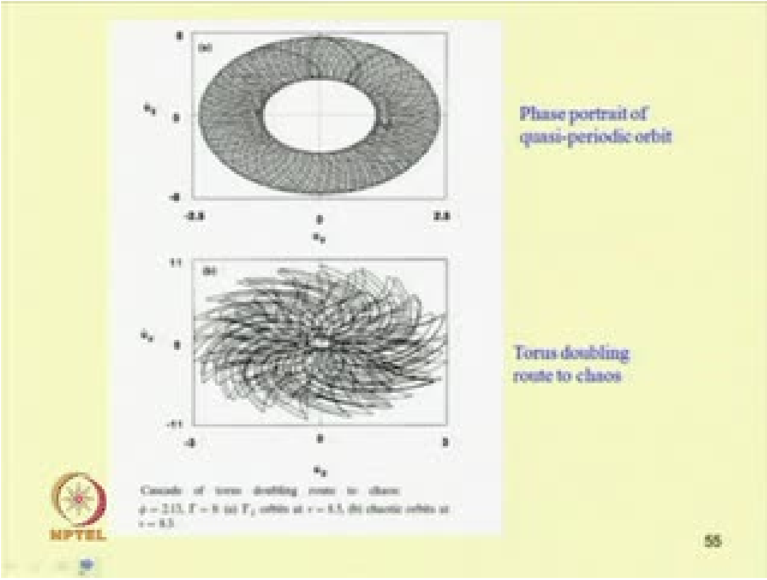

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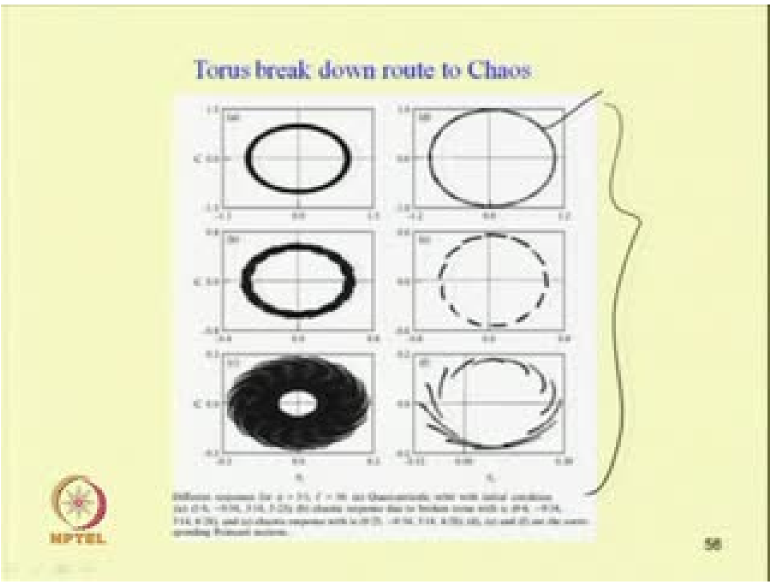

So up to this it is periodic then, 2 periodic, then 4 periodic and then chaotic response of the system and one can use this Feigenbaum number to check or to find whether the satisfy this Feigenbaum number or not in case of the period doubling route to chaos. So one can see different type of crisis also in the system, for example, in this case one has this is one of the attractor and as it is come in contact with a unstable fixed point response then it explodes and one can have this interior crisis of the system because this original attractor is now a part of the bigger attractor or inside this bigger attractor. That is why this is interior crisis in the system similarly; one can observe that these 2 attractor mores to form a single attractor and, that is why this is known as attractor margin crisis.

Similarly, one can find so initially a quasi periodic response and this torus by changing the system parameter, one can show this torus breakdown and finally, it forms a chaotic response. So this is torus break down route to chaos. Similarly, one so this torus break down route to chaos is shown by using this Poincare section, so this is periodic this is quasi periodic response initial quasi periodic response. This is shown by Poincare session in this close orbit and then it breakdown so when it is breaking down. So this discontinuity comes into picture and finally, it becomes chaotic.

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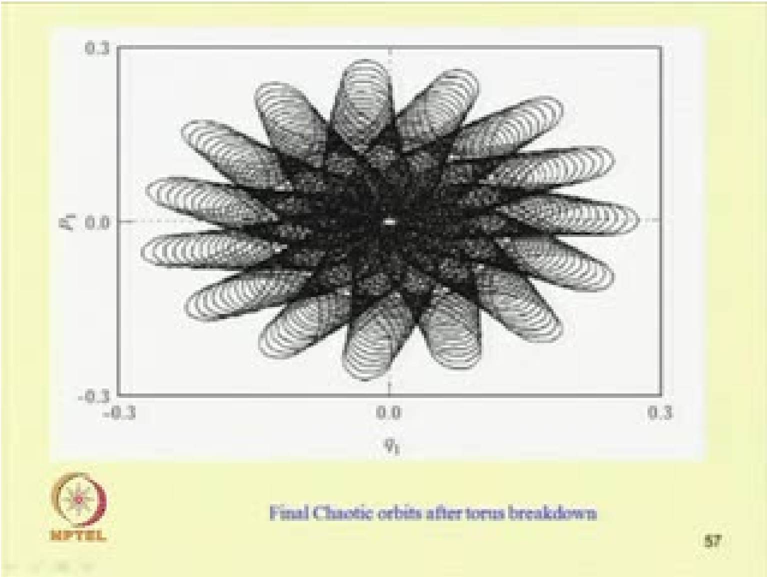

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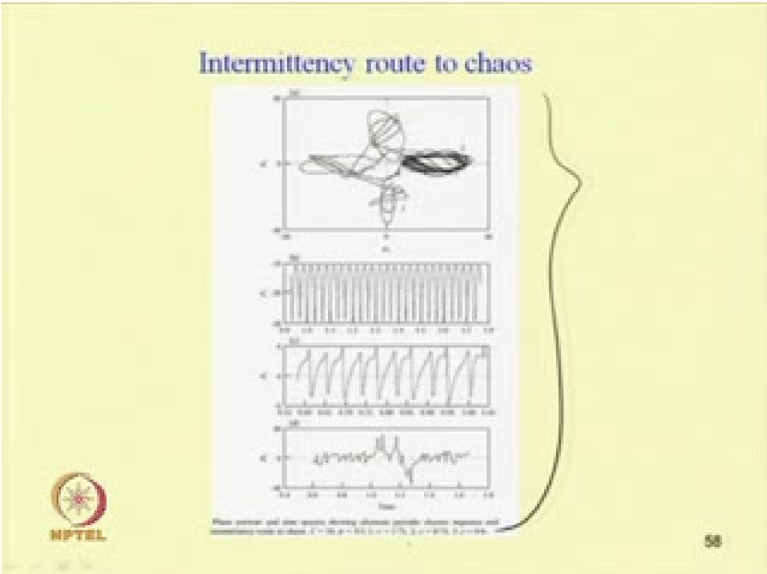

So this is the chaotic response of the system. So, some other intermittency route to chaos also are observed in this type of system. So in this so in this case we have seen that due to the presence of internal resonance and external resonance, in this parametrically excited system though the system looks very simple, but, it behaves or it shows many complicated phenomena like chaotic response, fixed point fixed point response, periodic response and chaotic response. Different type of crisis are also observed in this system. So in this way one can study different type of systems and study the non-linear phenomena associated with this.

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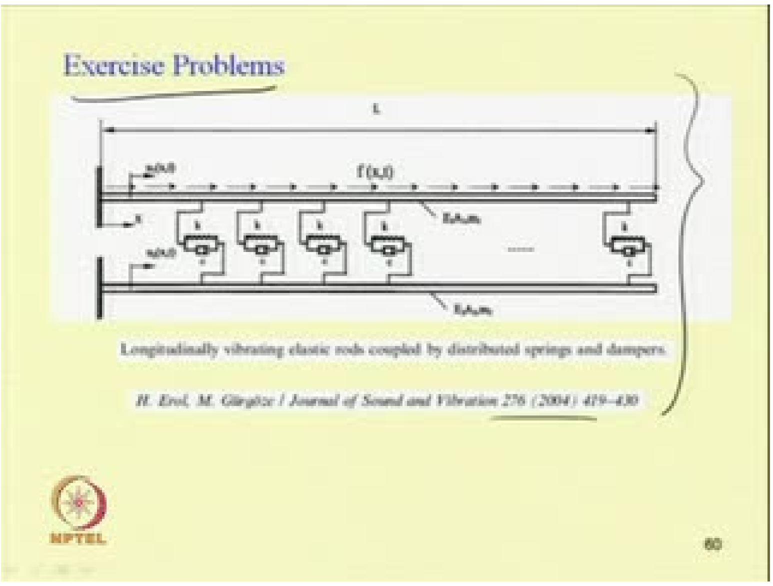

So one can take some exercise problems also, this in this exercise problem one can see this is a longitudinally vibrating elastic rod coupled by, so different springs so one can take. So this is the longitudinal vibration so one can derive this equation motion of the system and one can derive the non-linear equation of motion of the system by taking these springs to be non-linear, and finally, one can find or one can check some conditions for which both the beams will show some internal resonance.

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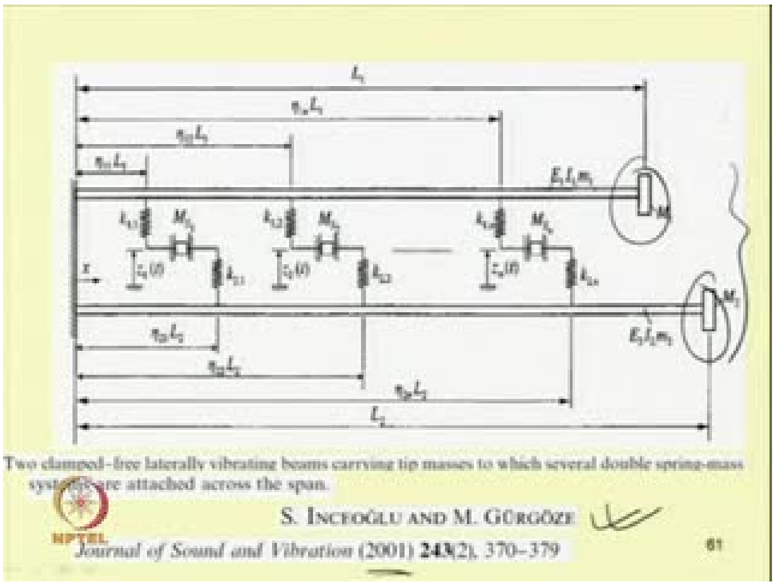

And one can solve this problem to find different non-linear phenomena associated with the system. So the basic equations for this system is developed by Erol and Gurgoze, so it is published in the paper journal of sound and vibration in 2004. So one can solve this or take this as a exercise problem and solve this equation. Similarly, another case similar system with some tip mass by this tip mass can be either at the tip or one can put this tip mass at some intermediate position. So in this case either one can take the system as longitudinally vibrating system or transversely vibrating system as where one takes a double pendulum.

So without complicating the system by taking only single mode approximation one can have two equations and by taking these springs to be non-linear one can derive this equation motion or by taking this beam to vibrate or taking this large displacement as in the present case. So one can study the equation with and without internal resonance. So this paper is also published in journal of sound and vibration in 2001. So one can take this as a exercise problem and one can find many non-linear phenomena in this case though the original papers are on nonlinear systems where the where only natural frequency of the systems and linear response of the systems have been found.

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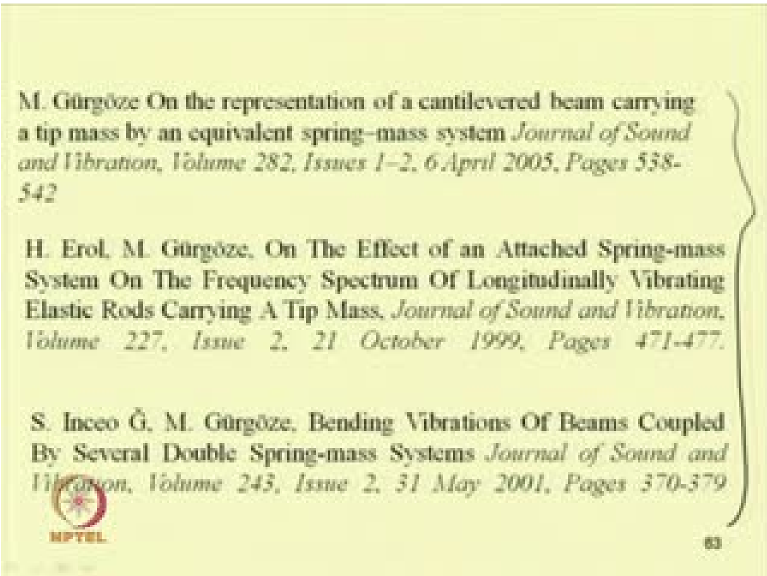

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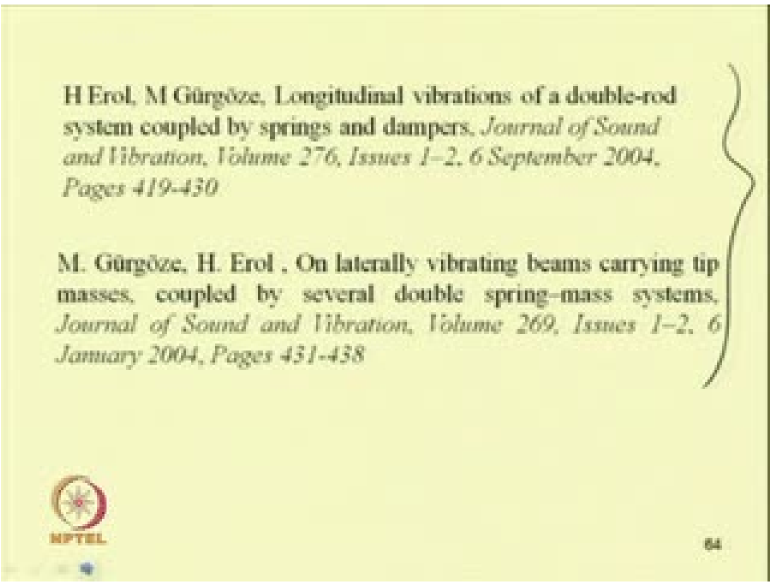

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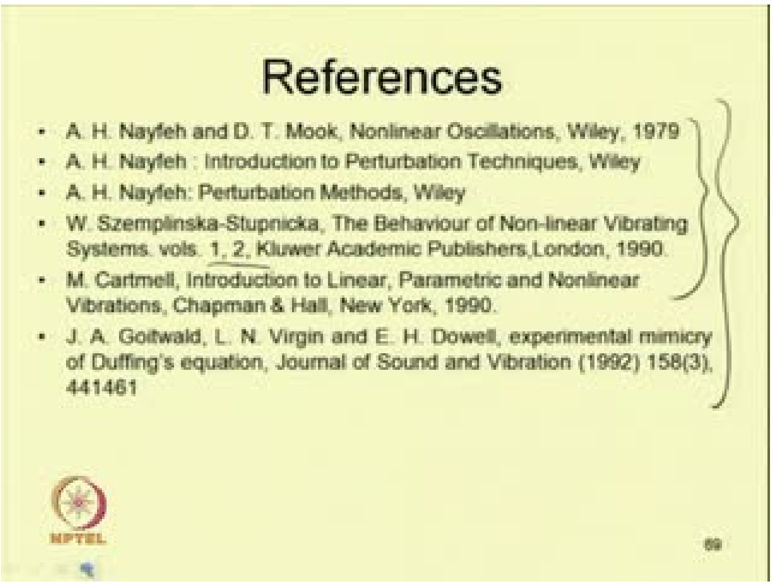

So the non-linear phenomena can be obtained by extending these cases. So for further study as this is the last class in this course, so for further study one can take or one can study these papers which are published in journal of sound and vibrations. So these three papers are by this Gurgoze and these two papers also are by Gurgoze paper and they are particularly based on this beam with tip mass.

And without tip mass also one can see most of the lecture in this course are been taken from the work or from the book by a Nayfeh and D T Mook of non-linear oscillations.

And one can refer the other books by Nayfeh that is introduction to perturbation technology, perturbation method and the book wise Szemplinska Stupnicka the behavior of non-linear vibration. So it has two volumes then the book by Cartmell introduction to linear parametric and non-linear vibrations. So these books these are very good books on this course, and in this course material these books have been referred extensively for finding the or for preparing the study material.

Also this book has also been used extensively in this course that is the book by a A H Nayfeh and B Balachandran applied non-linear dynamics. And these papers are also been referred in this work for studying the stability for example, studying the stability of the beam this paper has been used, that is K Saito H Saito and Otomi the parametric response of a horizontal beam carrying concentrated mass under gravity.

So this paper and this paper also has been referred H Saito and Koizumi parametric vibration of horizontal beam with concentrated mass at one end. So these papers can be referred and these two paper also by Zavodeny and Nayfeh can may be referred and these are some of the more references for further study in this course. So one can study these papers further paper in this course and these are also, some more references given in this work and these are some of my references or my work in the parametrically excited system, which are published from 99 to 2003 and or part of my p h d thesis which is submitted to IIT Kharagpur in 1999.

So this paper may also be referred for the for further study for 3 is to 1 is to 1 is to 3 is to 5 internal resonance or 1 is to 3 is to 9 internal resonance. Already these text books have been shown in the first class of the course, so one can refer all these things and also in addition to that one can refer the journal papers or the journals for further study, for example, one can refer this journal on this internals national journal or non-linear mechanics, non-linear dynamics, journal of sound and vibration, journal of vibration and acoustics, journal of dynamical systems measurement and control physics, the non-linear phenomena chaos soliton and fractals, international journal of non-linear science and numerical simulation, journal of computational and non-linear dynamics.

So in these journals lot of papers are they are which based on recent developments. So for further study one can refer the journals and one can have many more or one can study many different phenomena or one can know many different non-linear phenomena and non-linear phenomena associated with different systems. So this way this course is over with this module of applications of non-linear systems. I hope you have enjoyed this course.

Thank you very much.