Non-Linear Vibration
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Module - 6
Applications
Lecture - 4
Nonlinear Forced-Vibration of
Single-Degree-of-Freedom System

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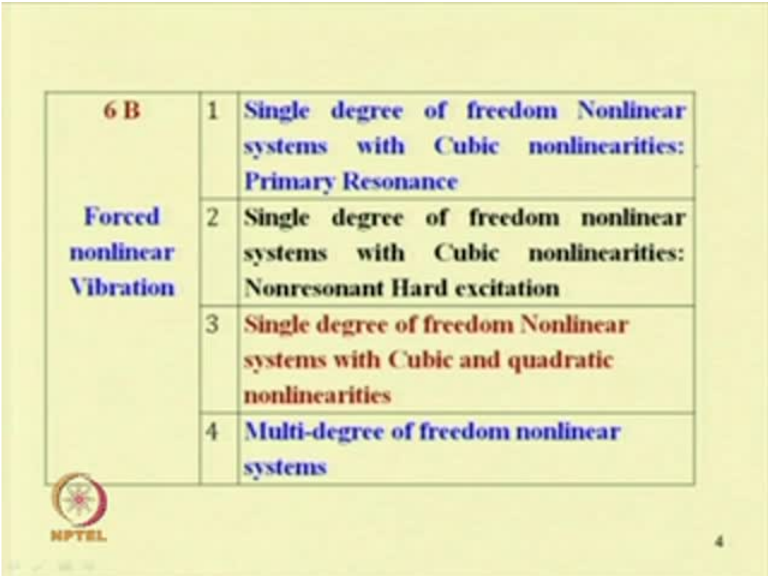

Welcome to today class of non-linear vibration. Last three classes we have studied about the non-linear free vibration of single and multi-degree of freedom systems, So today class we are going to study or start with this forced vibration of single degree of freedom systems. So in which we will study about single degree of freedom non-linear system with cubicnonlinearity. Here we will discuss about the primary resonance then will study about this cubic nonlinearity with non-resonant hard excitation. So in the first case will take soft excitation and in the third class we are going to study about the single degree of freedom system non-linear, single degree of freedom non-linear systems with cubic and quadratic nonlinearities. Very briefly will discuss about different resonance phenomena occur in this type of system, and also briefly will study about this multi degree of freedom systems. So, from this you will let to know about how we can proceed to solve this non-linear equation in case when forced response applied to a system or forced it is subjected to forced vibration.

So, unlike in case of free vibration in the presence of damping or in case of free vibration in the presence of damping, the response decreases and finally it may die out, but in case of forced vibration in linear systems we have seen that when the forcing frequency equal to the natural frequency, we have resonance condition. But in this case in this non-linear systems will see when the forcing frequency are near to the natural frequency or if it is away from the natural frequency also we have several conditions for which we will get the resonance. So, we will study those different type of resonance condition in this non-linear vibration in this case.

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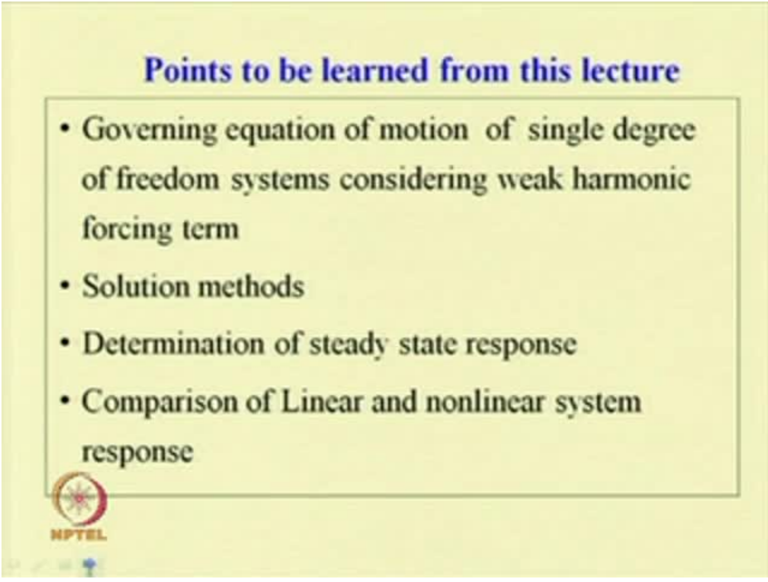

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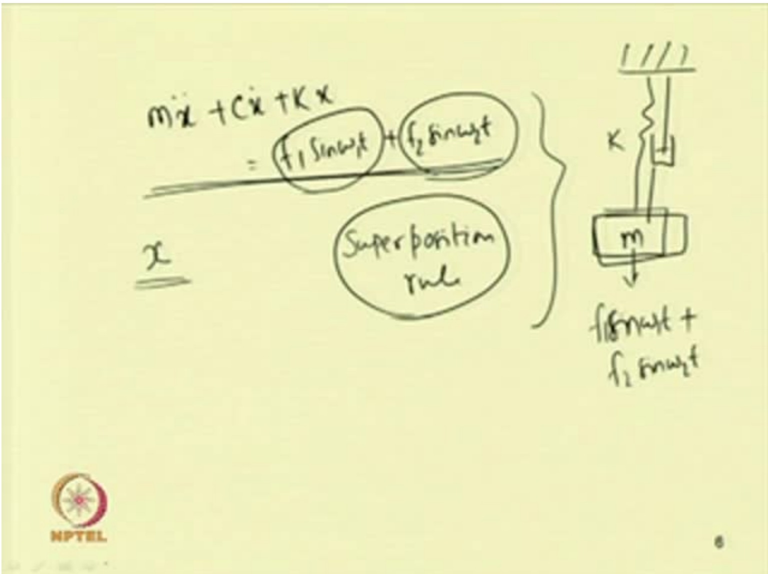

Also in case of linear systems, we know in case of linear systems we know so when more than one forcingfrequency applied to a system for example, in this case, in case of spring mass damper system, so spring; so let us take the spring this is damper and this is the mass this is m and in this case let us apply force f 1sinomega 1 tplus f 2 sin omega2 t, so in this case we know the equation motion can be written m x double dot plus c x dot plus k x equal to f 1sinomega 1 t plus f 2 Sin omega 2 t.

So in this linear case, in this linear equation motion so we know the response x can bedue to this forcing f 1 sin omega 1 t plus due to f 2 sin omega 2 t. So we can use the superposition rule, so we can use superposition rule to find to find the response of the systems. But, we know in case of the non-linear systems this superposition rule will not hold good so we have tostudy the vibration of the systems considering different resonance conditions. So unlike in case of free vibration where we have found the response total response of the systems by superposing the response due to different forcing in case of forced vibration so we have to analyse the systems in a different manner. So we have to consider the resonance conditions different resonance conditions in this case and we have to find the system response.

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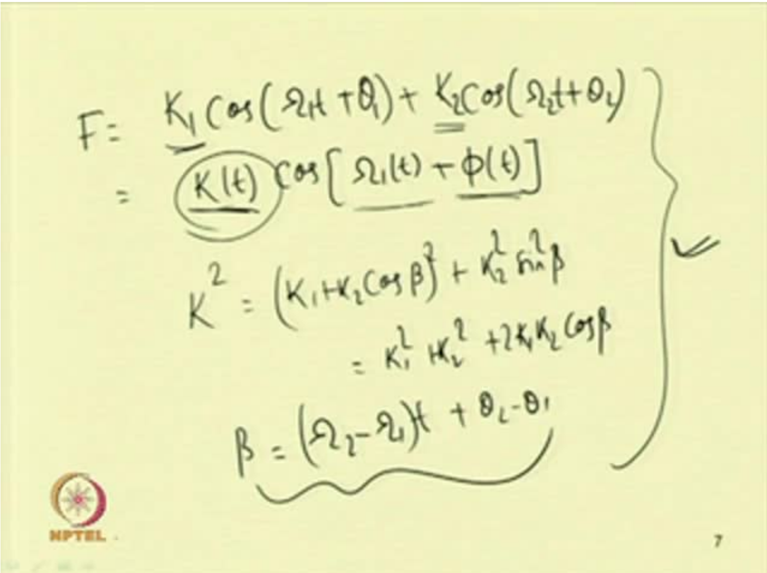

So, when two different forcing act on a system, so we can add them to sometimes we may write in this way let a force k 1 cos omega 1 t plus theta 2 or plus theta 1 plusk 2 cos omega2 t plus theta t acts on a system. So we can add these 2 force so these 2 we can add and we can write this f equal to f equal tok t, so cos omega 1 t plus omega 2 tor we can write this cos omega 1 t plus pi t. This way you can write where this previously we have seen this k 1 and k 2 are constant. But, now by adding this thing we can write this is a function of time, and this pi also is a function of time. So we have seen this pi that is phase, this phase also is a function of time and this coefficient is also a function of time. So where this k square we can write by combiningthis thing we can write this k square equal to k 1 plus k 2, so k 1 plus k 2 cos beta square plusk 2 square sin square beta. So this is equal to k 1 square plus k 2 square plus 2k 1 k 2 cos beta.

So where beta equal to, one can write this beta equal to omega 2 minus omega 1 t plus theta 2 minus theta 1. So one can add these two forces but while adding one can see this coefficient no longer become constant and we have a time varying coefficient in this case. So we can for multi deg multi frequency excitation, so when we have multi frequency excitationso we can add them to make a single frequency. But, in this case the coefficients and the phase will be a function of time, so we can analyse this type of situations also or in case of multi frequency excitation in this case. So today class mainly will focus on this cubic non-linear systems and will take simple duffing equation with cubic nonlinearity and subjected to soft excitation. Soft excitation means...

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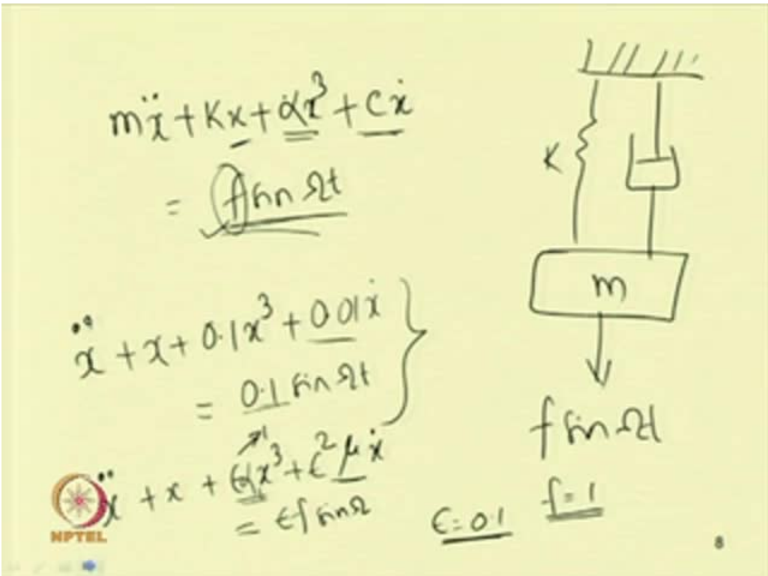

So let me write the equation motion or the spring mass system with taking a cubic nonlinearity. So this is the spring spring mass damper system, and it is subjected to a force f sin omega t or f sin or omega t or let me write this omega excitation frequency this way. So this equation when I am writing, so this is the inertia force, then the spring force I will keep it non-linear that is k x the linear spring force plus let me write l alpha x cube so this is cubic nonlinearity plus I can add this damping c x dot equal to f sin omega t. So in this case we can always take the scaling parameter and the book keeping parameter in such way that the coefficient of the response that is x for the non-linear term and for this damping term will be of one order lower than that of the linear system. Similarly, this forcing term also we can have some situation in which this forcing also will be one order lower than that of the excitation term, the coefficient of the forcing.

That means this amplitude of the forcing we can take in such way that or this amplitude of forcing will be very less then we can write for example, in this case taking some numerical value I can write this equation x double dot plus let me write x plus this is equal to 0.1x cube, 0.1x cube and let me write this is equal to also 0.01x dot. So if i will write this f equal to 0.1sin omega t then i can write this equation by using this book keeping parameter in this way, that is x double dot plus x plus epsilon x cube so here here the parameter alpha 1 will be equal to1, so epsilon let me take epsilon equal to 0.1, so if i am taking epsilon equal to 0.1, so this alpha 1 or the coefficient of x cube that is epsilon alpha equal to 0.1.

But, this alpha 1 equal to 1 which is same as that of the coefficient of x so epsilon alpha 1 x cube plus. So this term epsilon 0.01, so as it is 0.01 so I one can put this is of the order of epsilon square x epsilon square mu x dot. So here mu will be equal to 1 which is same as that of the coefficient of the linear term then so this will be equal to 0.1, so this 0.1I can again write epsilon into f sin omega t, so here f equal to 1 so I can have this f equal to 1 so epsilon equal to 0.1 which will give rise to total 0.1, so in this way we can use the scaling parameter and book keeping parameter. Here only i have used the book keeping parameter epsilon to write this equation and make them of the same order so if we have some higher non-linear terms.

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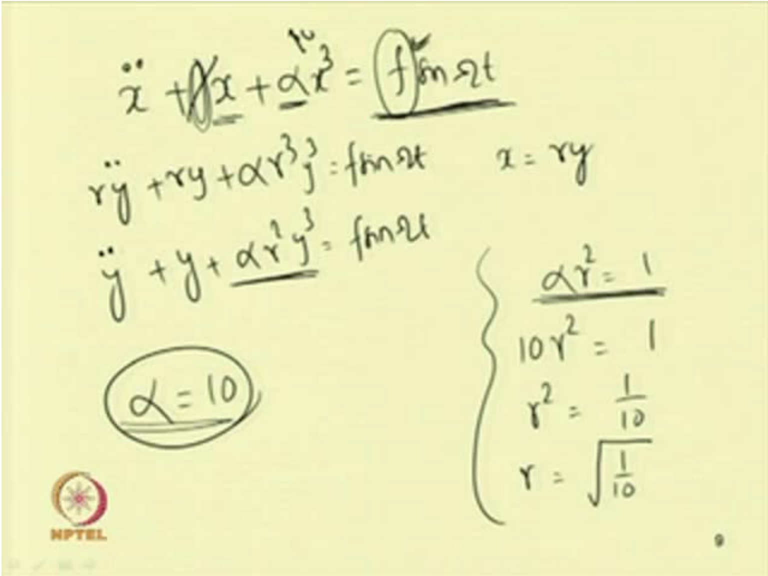

For example, let us take this equation so let it is x double dot plus let us write xplus let we have a non-linear term that is alpha x cube, so this is equal to f sin omega t. Now so let this, so if I will take this x equal to r y so this equation will become so r y double dot plus r y plus alpha, so r cube y cube so this is equal to f sin omega t.

So this way I can write this y double dot plus y plus alpha r square y cube equal to f sin omega t. So now to make this cubic non-linear term weak, so we can put thisthing that means this alpha r square so alpha r square coefficient of alpha r square. I will put equal to 1so for a given value of alpha, so let this alpha equal to 5, so if we are taking this alpha equal to 5 or 10, if you take this alpha equal to 10 in this case in the first equation if I take this alpha equal to 10; then this coefficient is very much higher than the coefficient of the linear term one, so we cannot tell this weak nonlinearity. So in this case the nonlinearity is strong, so to make it weak so I can put now so this alpha equal to 10. So r square, so this is equal to 1, so I can put this r square equal to 1 by 10, so or r I can find equal to 1 by 10 root over.

So this way one can find by using the scaling parameter, so one can reduce this non-linear term and make it a weak non-linear so but this external excitation term if the coefficient of the external excitation term is very small or 1 order less than or 1 or more order less than the linear term that is the coefficient of this linear term. Then we can tell this to be weakly nonweak non-linear or weak forcing, so we will take this weak forcing if the coefficient that is f is very very less than the coefficient of x. So in case of hard excitation, so this term will no longer be small or it will be very large in comparison to the linear term coefficient of the linear term that is coefficient of x.

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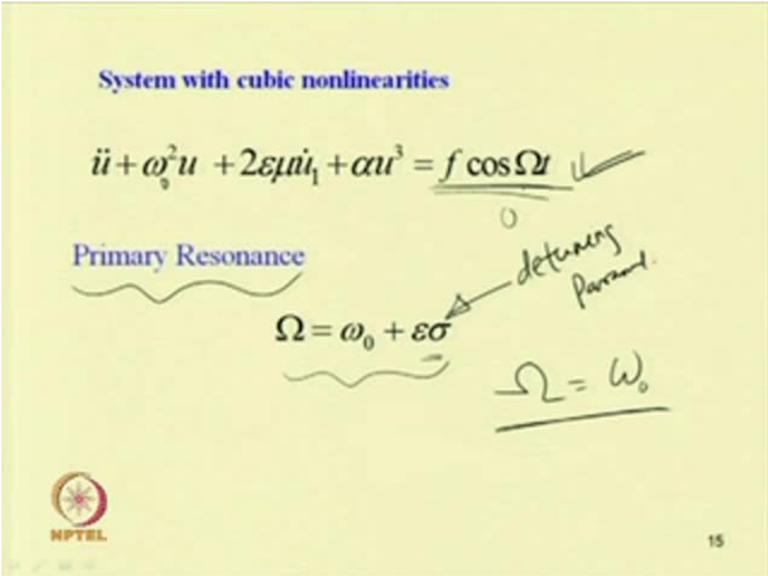

So, today class we are going to study when this term is very small, so when this term is very small so we can use; let us write the equation motion, so this is the equation motion duffingequation motion we have written and today class we are interested to study only the primary resonance case. Already we have studied the free vibration system of this system by putting this equal to 0, so in case of forced vibration let us take the situation because in case of linear systems we know so when this omega nearly equal to omega, so let me put omega 0 so omega equal to omega 0, so we will have resonance. So in non-linear case also we will have similar situation, so if omega equal to omega 0 will have this primary resonance.

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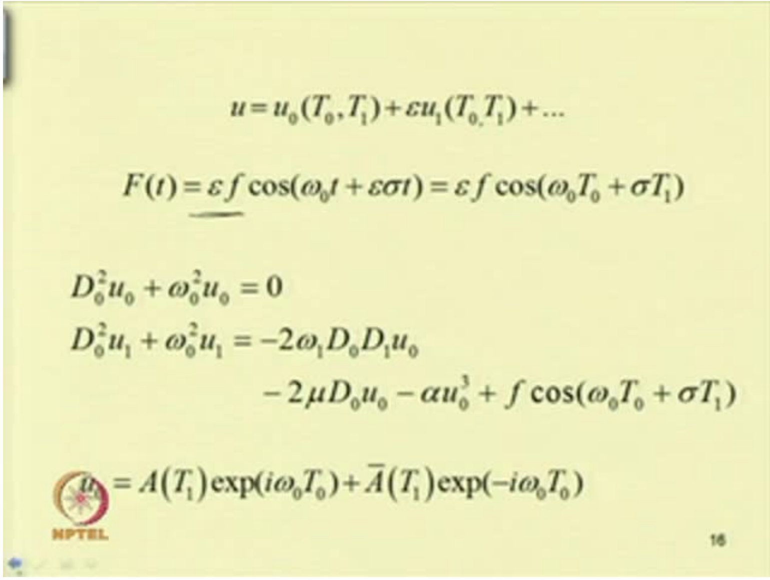

To take into account the nearness what will happen to the nearer to the frequency nearer to the natural frequency where taking a detuning parameter sigma, so will study in the neighbourhood of this omega 0, that is near the natural frequency. That meansif the excitation frequency is near to the natural frequency, we are going to study the system behaviour. So that means we have taken this omega capital omega equal to omega 0 plusepsilon sigma, here sigma is the detuning parameterdetuning parameter. So one can study this equation by directly solvingthis equation by using some numerical techniques like Runge-Kutta method, or one can study the system by using some perturbation method. So as we are considering weak non-linear system, let us take first u equal to u 0 plus epsilon u 1 t 0T1, so here f T.

We have taken this forcing function we have taken as it is non-linear, it is equal to epsilon f cos omega 0 t plus epsilon sigma t, and we are taking as we are consideringmethod of multiple scale for our analysis purpose. So this T n that is time scale it is taken to be T n equal to epsilon to the power n t where epsilon is the book keeping parameter and t is the original time so we have different time scales like T 0, T 1, T 2 and higher order terms. Now taking these terms, so we can write this u equal to or u equal to u 0 plus epsilon u 1 Ft equal to epsilon f cos omega 0t plus epsilon sigma t that is equal to epsilon f cos omega 0T 0 plus sigma T 1, soas epsilon T equal to t equal to T0 and epsilon t 0 or epsilon t equal to T 1.

So we can write this forcing term equal to epsilon f cos omega 0 T 0 plus sigma T 1, now substituting this equation in thisoriginal equation and substituting d by D t equal to D0 plus epsilon T 1 and and the higher order derivatives also can be written in terms of T 0 and T 1. So writing those terms and putting this thing in this original equation and separating of the order epsilon, so one can get this equation so this is epsilon to the power 0, so those equations becomes D0 square u 0 plus omega 0 square u 0 equal to 0.

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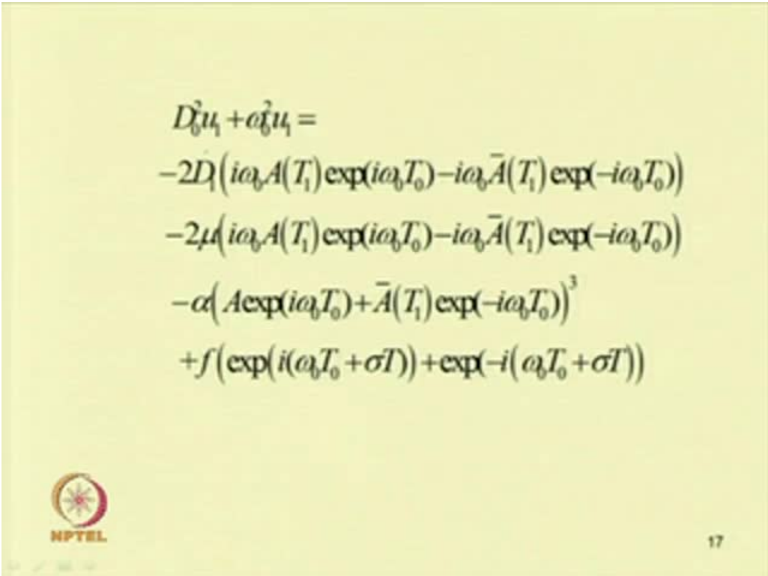

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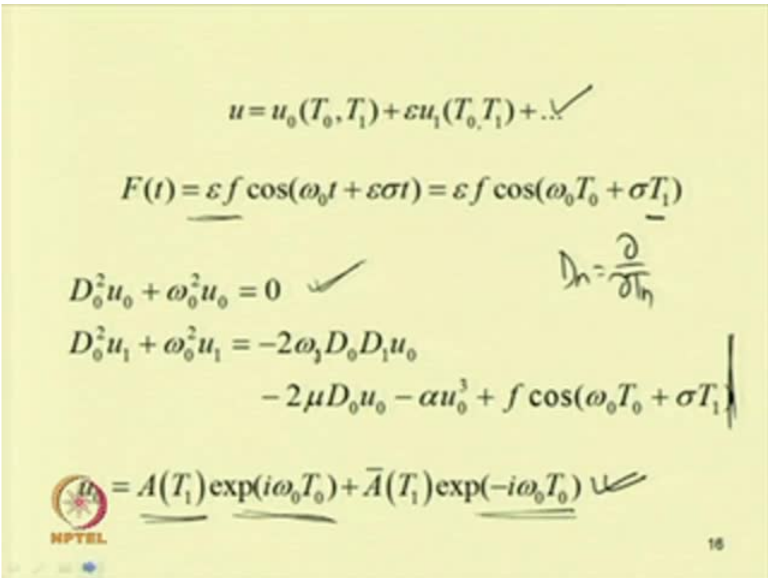

And the second equation becomes epsilon to the power 1, so D 0 square u 1 plus omega 0 square u 1 equal to minus 2 omega 1D 0 D 1 minus 2 so minus 2 omega 0 D0D1 u 0 minus 2 mu D 0u 0 minus alpha u cube plus f cos omega 0 T0 plus sigma T1, and then so for this equation D 0 square u 0 plus omega 0 square u 0 equal to 0. So the solution becomes a, so the constant should not be a function of T 0. So this is a function so you can write this is a function of T 1, T 2 or higher order terms only taking up to T 1. So you can write this u 0 equal to a T 1, e to the power i omega 0 T 0 plus a bar T 1, e to the power minus i omega 0 t0; so substituting this equation in this in this equation so you can write these terms, D 0 square D 0 square u 1 plus omega 0 square u 1 equal to minus 2D 1. So as this is a function of, this is a function ofT 0, so you can differentiate these terms. D 0, D 0 means D, Dn equal to del by del n, so differentiating withdel n by del T n, del n by del T n, so D0 will be equal to del by del T 0.

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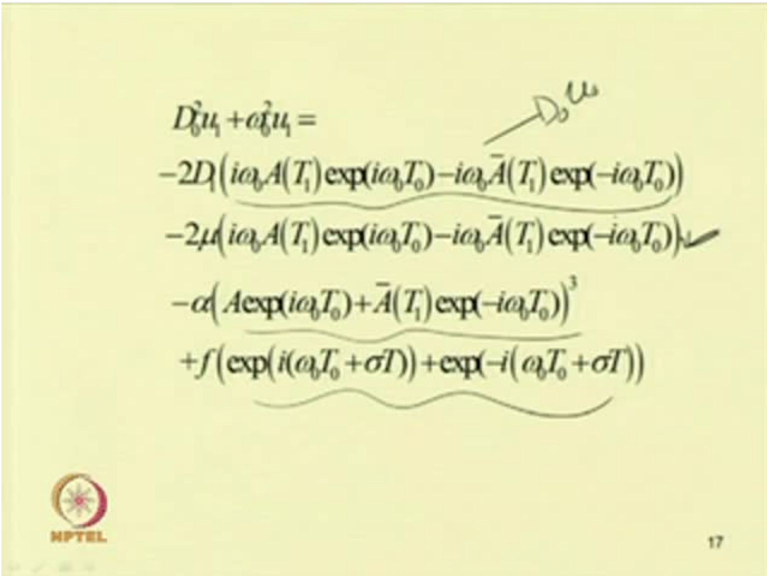

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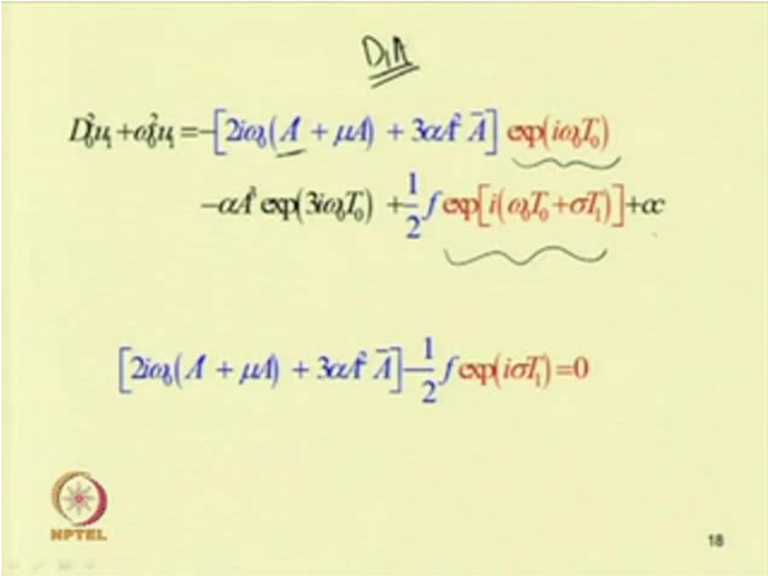

So, differentiating these terms with respect to T 0, so we can write these D0 square u 1 plus omega 0 square u 1 equal to minus 2 D 1 i omega 0 a T 1 e to the power i omega 0 T 0minus i omega 0 a bare to the power minus i omega 0 T 0 and so this is D0, so this term is nothing but our D0 u 0 so this term becomes minus twoomega 0 D0 D 1 u 0. So for D 1 u 0 we can substitute this term and then for the third term that is minus 2 mu D0 u 0, so you can writethis is the expression and then for alpha u cube. Now expanding this or a plus b whole cube in that formula if you apply then it will be a cube plus b cube plus 3 a square b into a b square, so by applying that thing, so we can write or we can expand this term and then this is the forcing term.

Now, expanding these terms and killing the secular term, so like previous case we should kill the secular terms or before that we can write this D0 square u 1 plus omega 0 square u 1 will be equal to minus 2 i omega 0 a dash a dash is D, D 1 a, so D 1 a is written as a dash so 2 i omega 0a dash plus mu a plus 3 alpha a square a bar e to the power i omega 0 t 0 minus alpha a cubee to the power 3 i omega 0 T0 plus halff e to the power i omega 0 t 0 plus sigma T1 plus its complex conjugate. So this forcing term if one see this forcing term which is written in terms of cos, so this forcing term is written in terms of so actual forcing term a epsilon f cos omega 0t 0plus sigma T 1, so taking this as theta so epsilon f cos theta cos theta equal to half e to the power i theta plus e to the power minus i theta by 2 so cos theta equal to e to the power i theta.

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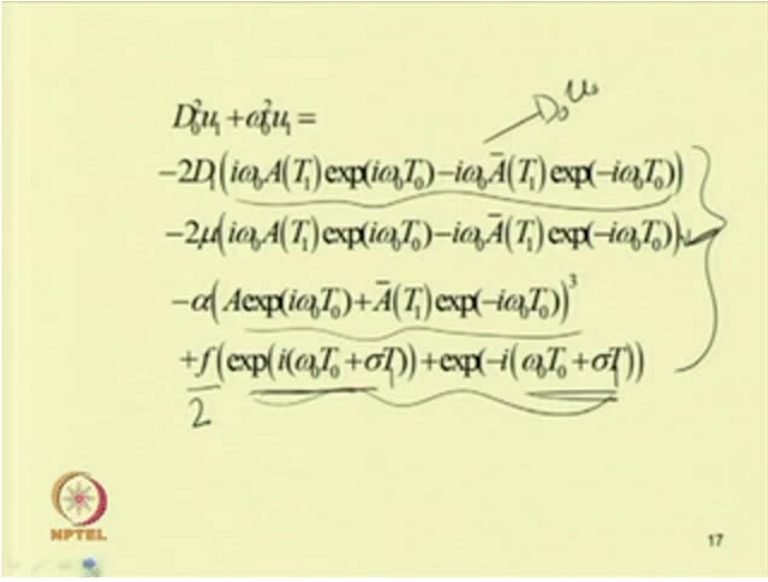

Because e to the power i theta equal to cos theta plus i sin theta. So by substituting that expression, so we can write this forcing term also in terms of exponential function that is f e to the power i omega 0 T 0plusso it should be f by 2, soe to the power i omega 0 T0plus sigma T plus e to the power. So this is T 1 sigma T 1 and sigma T 2, so we can have this expression so complex conjugate.

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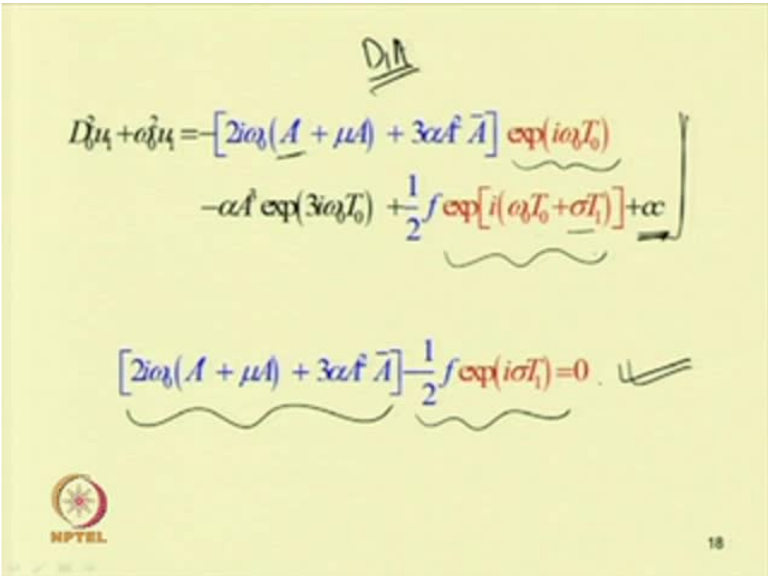

So, one can see that this term is the complex conjugateof this one similarly, so we have a pair of terms, so taking only one of the terms and keeping other terms in its complex conjugate, we can write this equation in this form; nowone can see that the coefficient of e to the power i omega 0 T 0, the terms containing the coefficient of e to the power i omega 0 T 0 should be eliminated as this term is a secular term. Because the presence of these terms will lead to infinite response of the systems and as the actual response of the system is bounded.

So we should we should eliminate the coefficient of these terms, so eliminating the coefficient of this term that is 2 i omega 0into a dash plus mu a plus 3 alpha a square a bar minus half a e to the power, so this term we can put, so taking common e to the power i omega 0 T0, so you can write this; so this equal to 0, so eliminate now one can see that this is the secular terms. But, this leads to mixed secular term, because this is this is exactly not so when thissigma T 1 tends to 0, that means when this sigma tends to 0only these terms will tends to omega 0 T0and it will be a secular term. So if sigma is near to 0, then only these terms will be equal to 0and it will lead to secular term.

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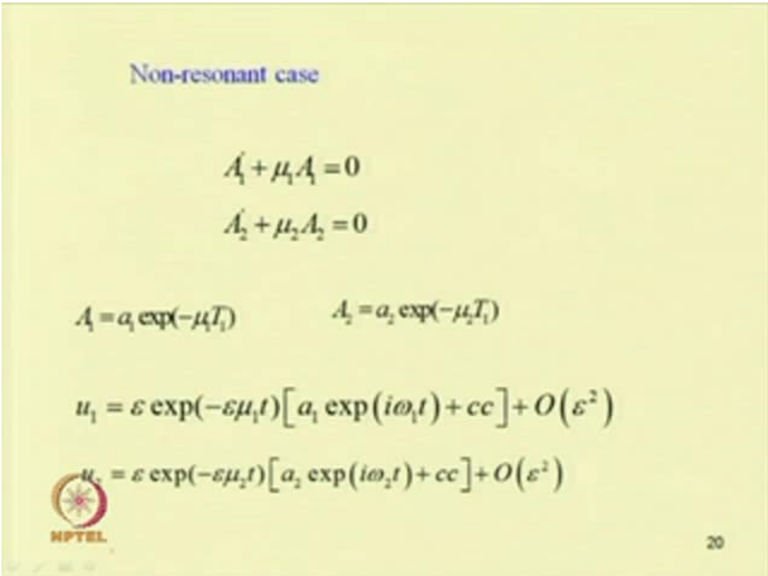

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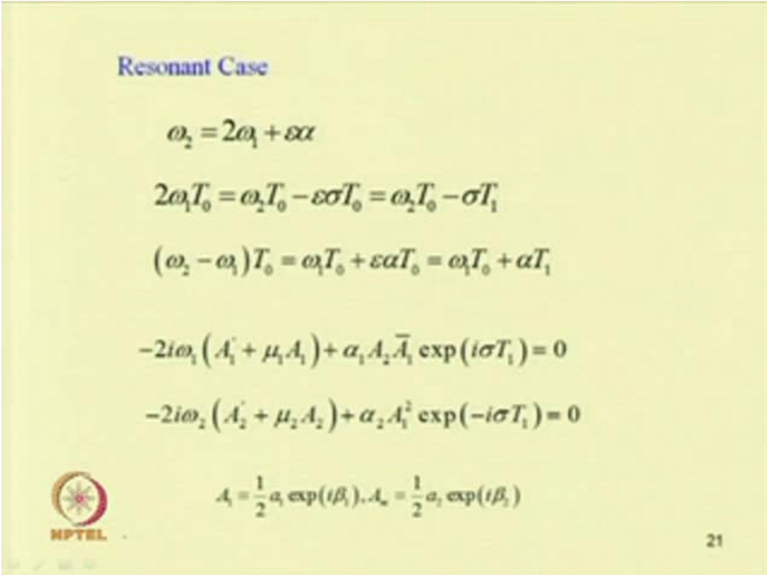

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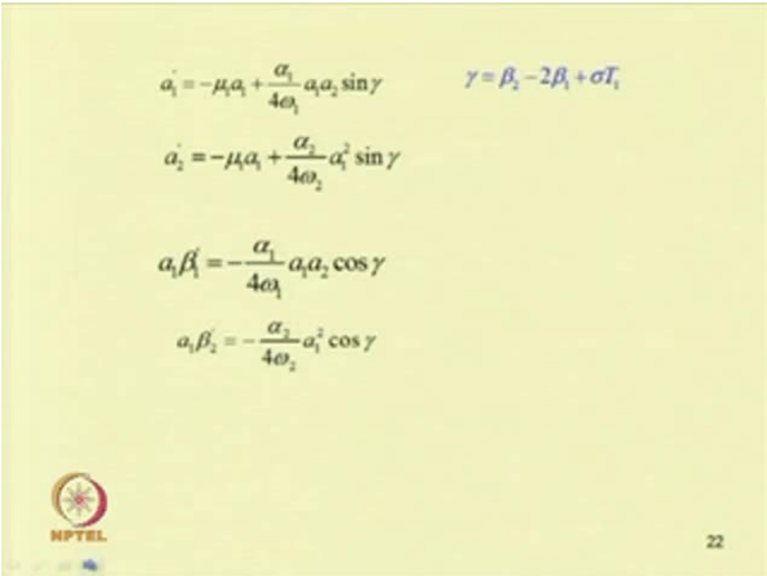

That is why it is known as mixed secular term and this detuning parameter has to be taken in such way that it willit will be in the neighbourhood of this omega 0. That means in the neighbourhood of the natural frequency, so if the forcing frequency is very close to this natural frequency then only we can take this term to be the secular terms or and now by adding these two terms; so we should kill these terms or we should eliminate these terms so that we can have a bounded solution. So to eliminate this thing now taking this a in its polar form, so we can take so we can have two different conditions, so taking in its polar form we can write the solution of the system.

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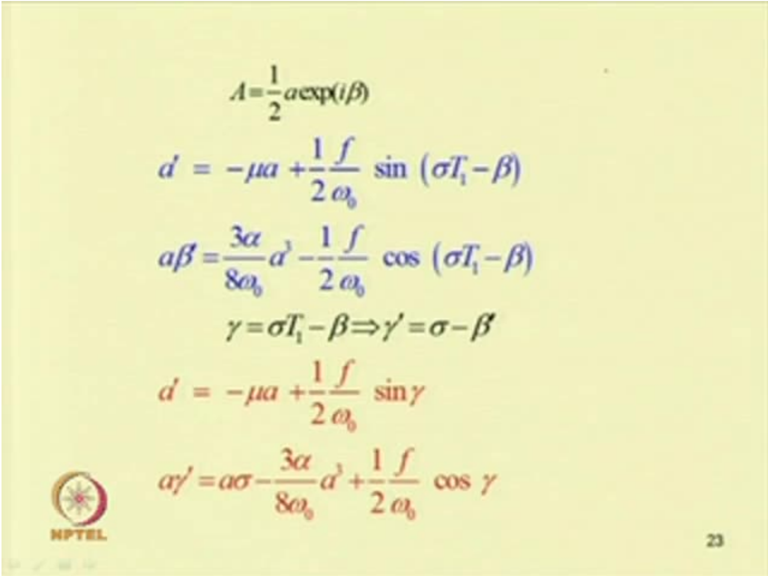

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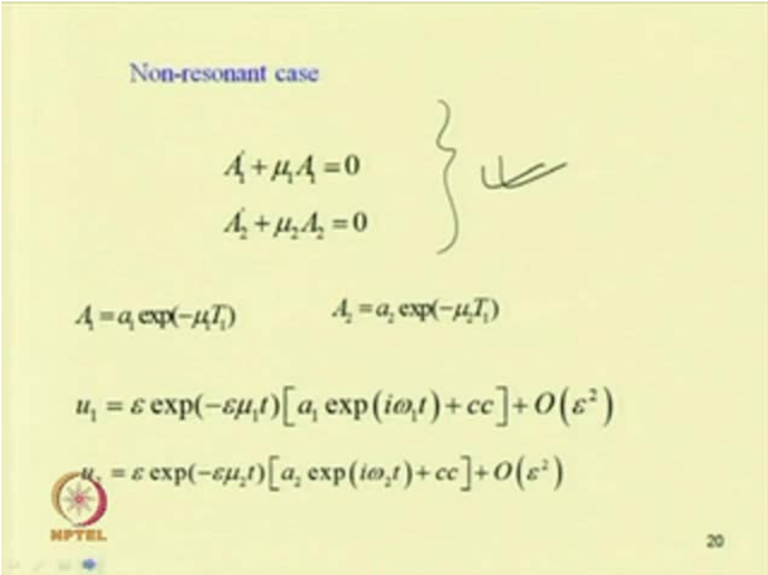

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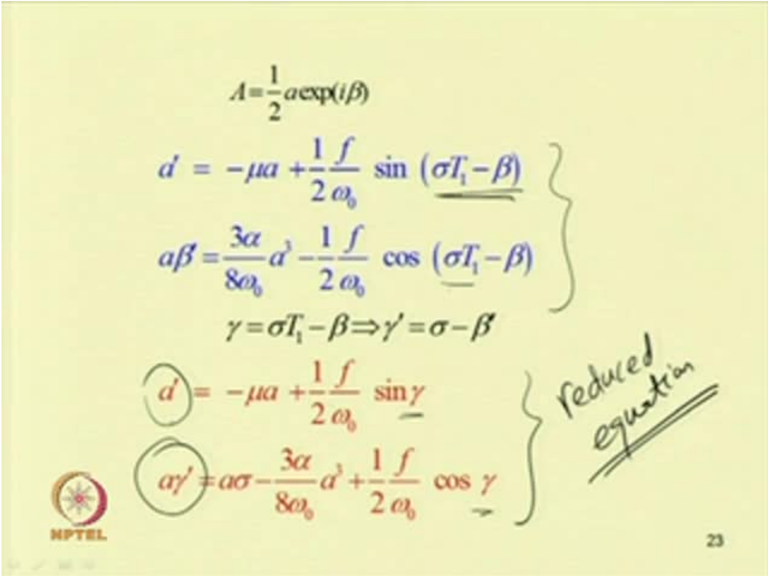

So this is for the free vibration case. One can already, we have studied for the free vibration this non resonant and resonant case when internal resonance were present, so in case of this force vibration, now taking a equal to half a e to the power i beta. So we can write that equation a dash equal to minus mu a plus half f by omega 0sin sigma T 1 minus beta, a beta dash equal to 3 alpha by 8 omega 0 a cube minus half f by omega 0 cos sigma T 1 minus beta. Now for the steady state solution, so by so as these terms are written in terms of time so you can write them in terms of autonomous systems by putting this parameter equal to gamma. So taking gamma equal to sigma T 1 minus beta or gamma dash equal tosigma minus beta dash, so we can write these two equations in this form where the, where it is not explicitly function of time, so it is written in terms of gamma.

So, a dash equal to, so in this case a dash equal to minus mu a plus half f by omega 0sin gamma and a gamma dash equal to sigma a sigma minus 3 alpha by 8 omega 0a cube plus half f by omega 0 cos gamma. Now for steady state solution, so this will not be a function of time, this a dash and a gamma dash or gamma dash should not be function of time that means a dash should be equal to 0and gamma dash should be equal to 0. So by putting this a dash equal to 0and gamma dash equal to 0we can get the condition for the steady state. So these equationthese two first order equation which are obtained by applying method of multiple scale on the original second order differential equation, so are known as the reduced equation. So these are the reduced equation.

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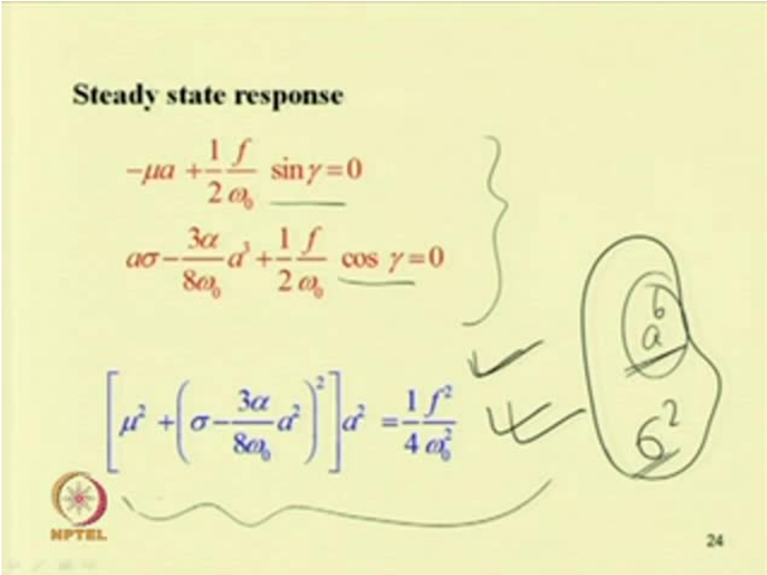

So, in this reduced equation to obtain the steady state solution we have substituted this a dash equal to 0and gamma dash equal to 0, so then we got these two conditions, so either one can solve these two equations by numerically to find the steady state solution or analytically to find the solution now one can eliminate this gamma term. So eliminating this gamma term to eliminate this gamma term, so you can write minus mu a equal tominus half a by omega 0 sin gamma and a sigma minus 3 alpha by 8 omega 0a cube equal tominus half f by omega 0 cos gamma. Now squaring the right hand side and adding, so we get this expression which is mu square plus sigma minus 3 alpha by eight omega 0a square whole square into a square equal to 1 by 4 f square by omega 0square now to obtain, so this one can observe that this equation in this equation so it appear so a is in a is of the order of sixth, a sixth.

So a to the power 4 and multiplied by the square, so it is of the order of6. So in this equation a is of the order of 6 but sigma is of the order of 2 quadratic. So one can write a sixth order equation in terms of a by expanding this equation, one can write this equation in the form of sixth power of a or quadratic equation in terms of sigma.

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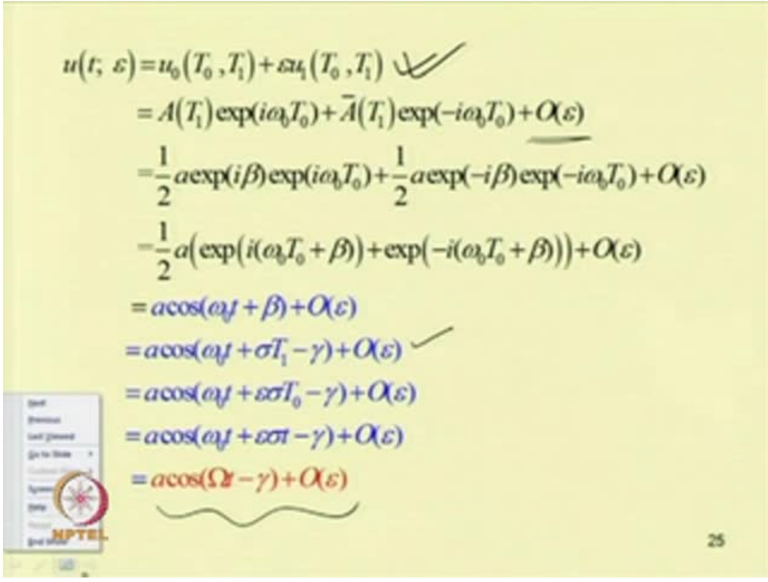

So it is easier to solveby taking a quadratic equation in terms of sigma, so by taking a quadratic equation in terms of sigma one can write this equation and then solve to get the response of the system. So from this one can find the frequency response of the systems. We will see this frequency response of the systems by plotting this frequencyresponse plot from this diagram. Now by knowing this a and sigma so one can find the response of the system that is u, so u which is a function of T and parameter epsilon, so u can be written as u 0plus epsilon u 1, we have assumed this thing before so taking only up toorder of epsilon, so already we know this a u 0can be written in this form a T 1 e to the power i omega 0 T zero.

Plus a bar T 1 e to the power minus i omega 0 T0, then we can write this a T 1, we can write this a T 1 equal to half a, so we can write this a to 1 equal to half a e to the power i beta into e to the power i omega 0 T0, then a bar will be equal to half e a e to the power minus i beta into e to the power minus i omega 0 T 0. So by adding these two terms we can write this equal to a cos omega 0 T plus beta. As already we have considered this beta in terms of gamma, so we can write this equation in terms of a cos omega 0 T plus sigma T 1 minus gamma plus order of epsilon. So this equation finally, can be written or the steady state solution or the solution of the systems can be written in the form a cos omega T minus gamma plus order of epsilon, so where this a is a function of frequency.

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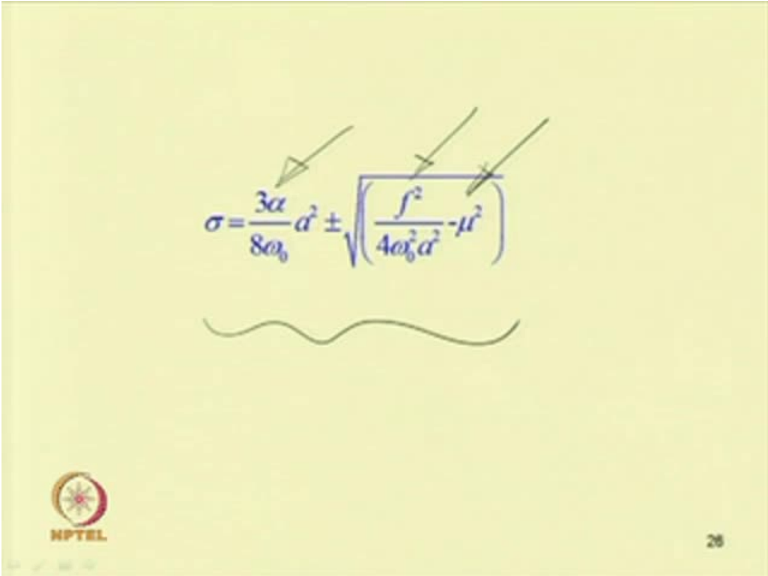

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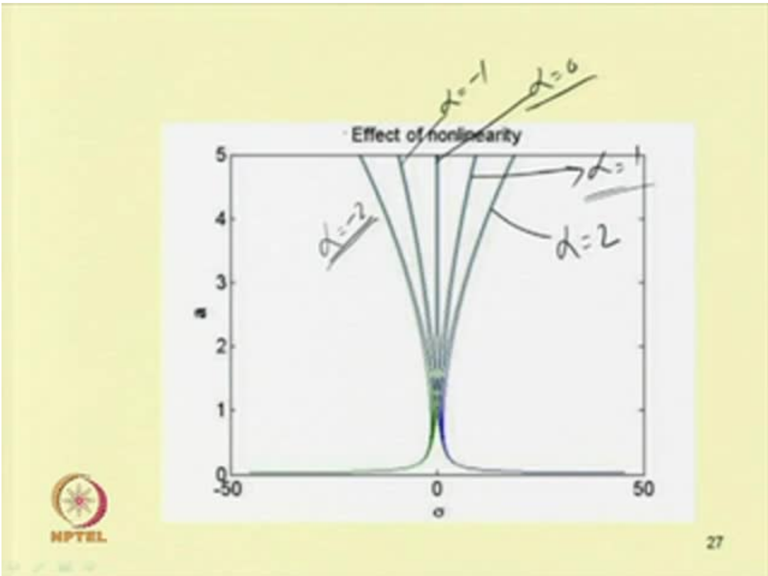

So one can plot this a versus gamma plot. So by solving that quadratic equation so one can find sigma equal to 3 sigma equal to 3 alpha by 8 omega 0a square plus minus root over f square by 4 omega 0 square a square minus mu square. So for different value of this alpha, so which is the coefficient of cubic non-linear term this f that is the forcing term and this mu that is the damping term. So one can find the response of the system so here some typical curves have been plotted.

So this is alpha equal to minus 2 alpha equal to 1 minus 1alpha equal to 0. So this correspond to alpha equal to 1, and this correspond to alpha equal to 2, so for 5 different curves it has been plotted. So it can be noted that for alpha equal to 0, so it correspond to the linear systems that is the linear spring mass systemsand for alpha equal to this minus term, so this which is soft so which behave as a soft spring or alpha equal to this positive value which behave as a hard spring.

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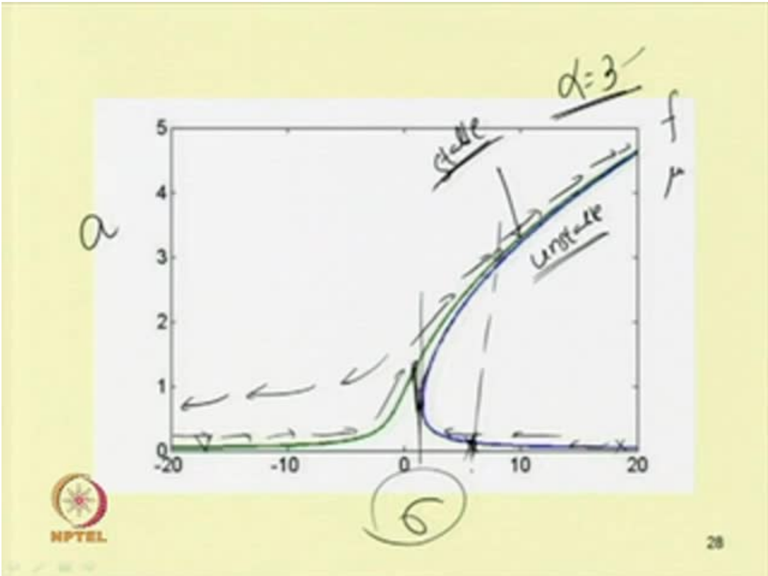

So we have this different type of response. So one can study the stability of the system also stability of the systems, and from this one cantaking a discreet case, or taking one one case, that is alpha equal to, so here in this case alpha equal to3 taking alpha equal to 3, this plot has been this response has beenhas been plotted. So in this case one can see that by increasing initially, by increasing this frequency this is detuning parameter and this is amplitude. So by increasing detuning parameter the system response follow this path and so it will follow this pathand finally, it may or it will goes on increasing and if one decrease sigma; so let one start from this position, so if one decrease sigma it will follow this path. And if one study the stability, it can be shown that up to this the system response or the system response stable and this branch isunstable. So as this branch is unstable so at this point at this value of sigma, so the system will have a tendency to jump up that means it will jump up to the upper branch and then it will follow this path to or it will follow this path. So when we are sweeping of the frequency that means we are taking this sigma, we are increasing sigma from this value, so the system response goes on increasing as the upper branch stable. So one can check the stability of the systems by using the methods we have studied before.

So this branch is stable, so this is a stable branch and this lower branch is so lower branch from this to this is unstable. So this is a stable branch and this branch is also stable, so before this thing that means before this point the systemhas only a single response. It has a single response curve but after this value the systems has 3 equilibrium positions corresponding to the upper branch it has 1 equilibrium position and corresponding to this lower branch it has 1 equilibrium position and this is the unstable branch. So this unstable branch represent the saddle points, so if one plots the phase portrait so at a particular value of sigma, so for a particular value sigma if one find the response, so for different initial conditions either it will be in this state or it will be in this state. Because the middle state which is unstable the system cannot reach to an unstable state, so either it will be in the lower stable state or it will be to the in the upper stable state.

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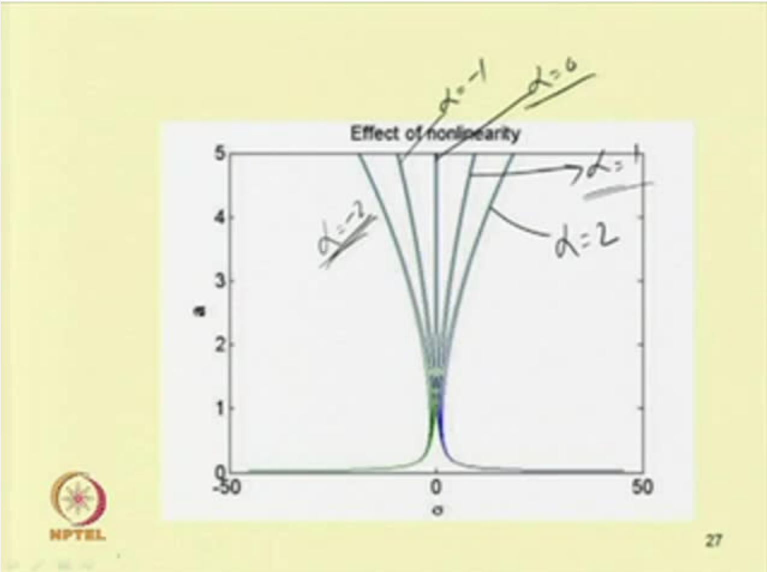

So in this case one can observe this jump of phenomena, so by sweeping down this frequency at this detuning parameter the system suffers a the system has jump from this position to this position similarly, for different value for so this is for a typical value of alpha equal to 3 mu equal to so for small value of mu it has plotted. So you have a matlab port by running which you can find the different type of response for different value of this alpha. Then different value of the forcing function f, and different value of mu so one can see the effect mu. So in this case one can observe that,

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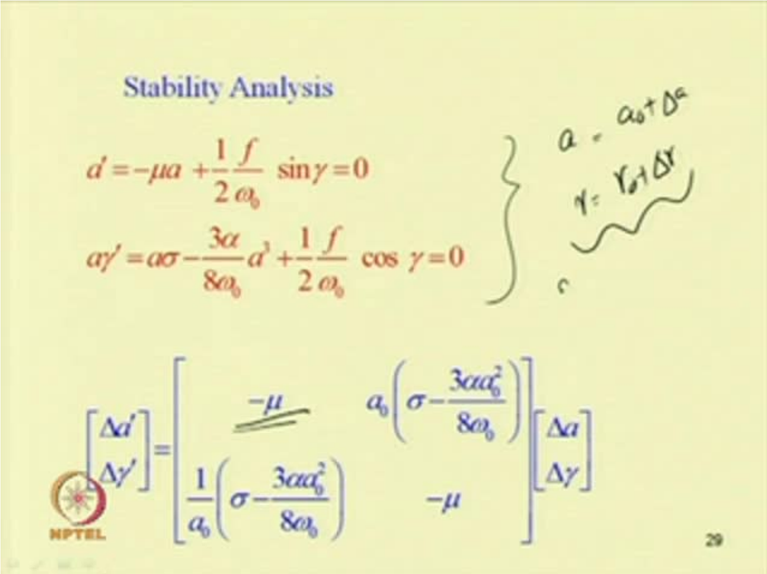

One can observe that for hard spring, so it tilts towards right and for soft spring it tilts towards left and for when there is no nonlinearity it is same as that of the linear system. The response is same as that of the linear system, so one can plot the forcing also, so one can plot for different value of, so let us so for different so for doing the stability analysis one can perturb this equation. One can have this dash and gamma dash equation now taking this a equal to a 0plus delta a and gamma equal to gamma 0 plus delta gamma where a 0 and a 0 and gamma 0 are equilibrium position so by perturbing this equation.

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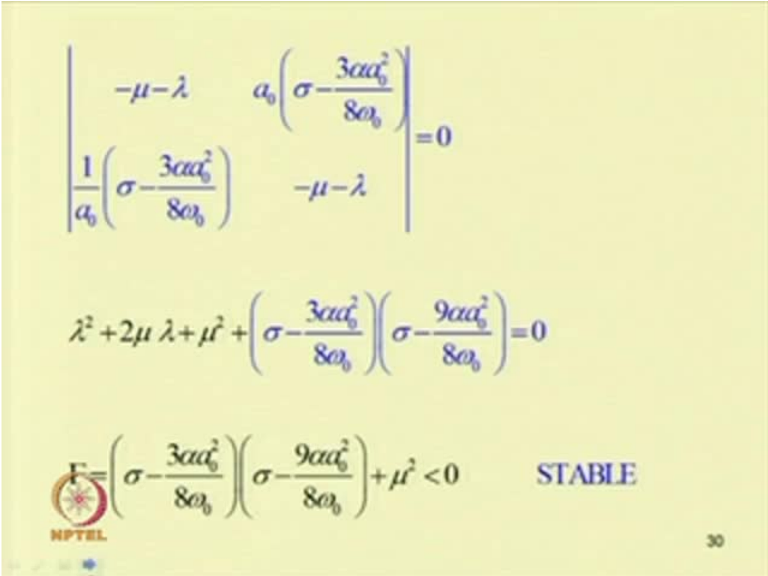

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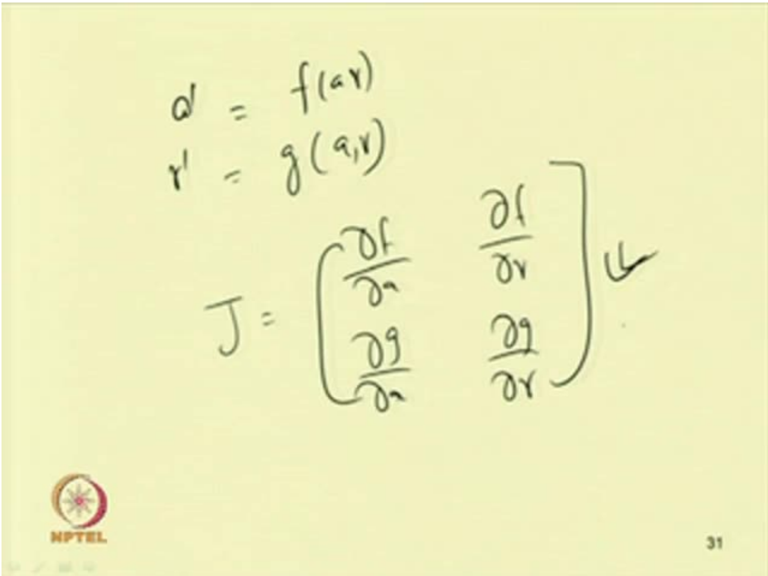

One can write this equation in this form delta a dash delta gamma dash equal to, so this term can be obtained so let us take this equation in this form that is a dash a dash and gamma dash so this is equal to function of a and function of, so this is the function of a and gamma so this is function of a and gamma so this Jacobian matrix can be obtained by del f by del a and del f by del gamma and this is del g by del a and del g by del gamma, so this Jacobian matrix can be obtained by this.

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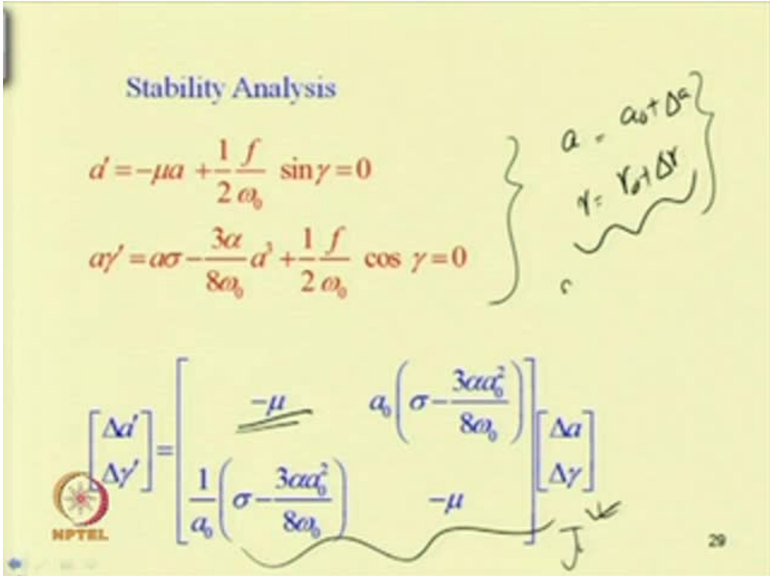

So here similarly, so we have this reduced equation and by substituting this a equal to a 0 plus delta a, and gamma equal to gamma 0 plus delta gamma so we can obtain this jacobian matrix. So this is the jacobian matrix.

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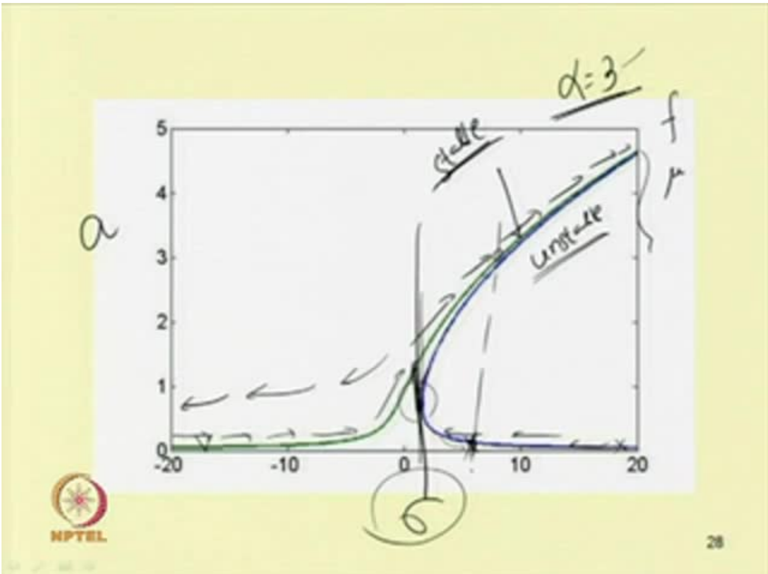

Now for different response what we have plotted, so at this response or for this one value of a and gamma so by substitute this a 0and gamma 0 in this equation, so we can find the Jacobianmatrix and by finding the eigenvalues of this Jacobian matrix we can study the stability of the system. So when the real part of the eigenvalues of this Jacobian matrix is negative then the system is stable and when it is positive the system is unstable. So in this way one can study the stability of the system and by studying the stability of the systems one can observe that this upper branch is stable and this lower branch is also stable.

So in this multi stable region, that is after this value this is multi stable region so here we have two stable zones. So this is known as by stable zone so in this by stable zone so one can find two stable positionsand one can have a jump of phenomena at this position. And one can plot this basin of attraction to know to which state the system response will go or the system. To know the system response one can, one may plot the basin of attraction and find for what initial conditions the systems response will be in this trivial state; or for what value of.

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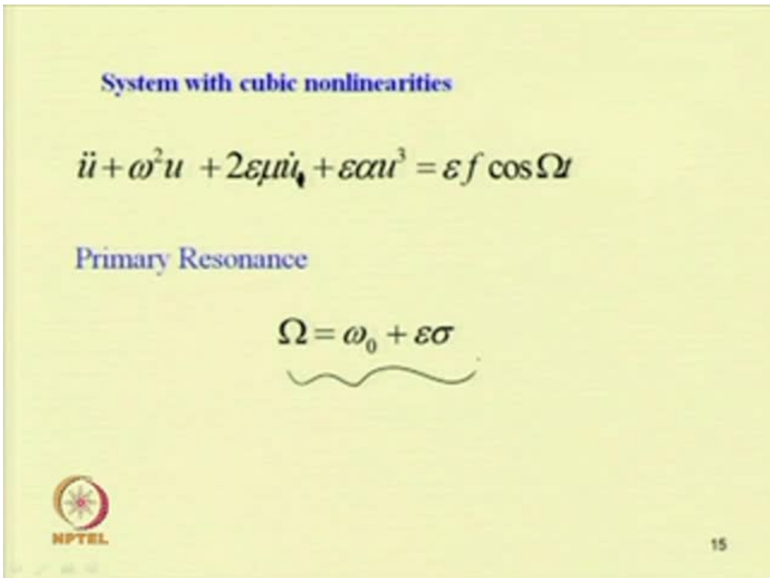

This system response will be the non-trivial state. So by taking different system parameter in this way one can find the response of the systems so let us continue this module 6 lecture 4, so in which we are discussing about the system with a non-linear system with cubic nonlinearity. So in this case so we are discussing about the system with duffing equation so which is given by this u double dot plus omega 0square u plus 2 epsilon mu u 1 dotor u dot plus epsilon alpha u cube equal to epsilon f cos omega t. so when you are discussing about this primary resonance condition that is omega is nearly equal to omega 0plus epsilon sigma where sigma is the detuning parameter.

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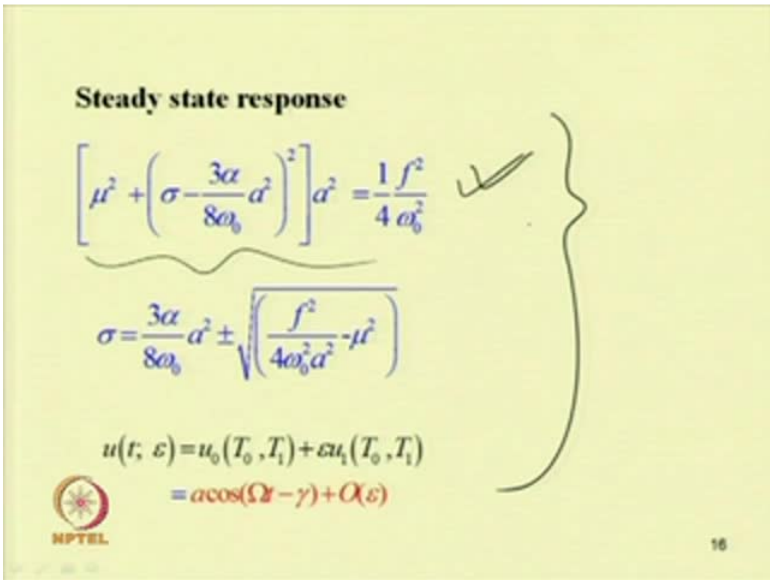

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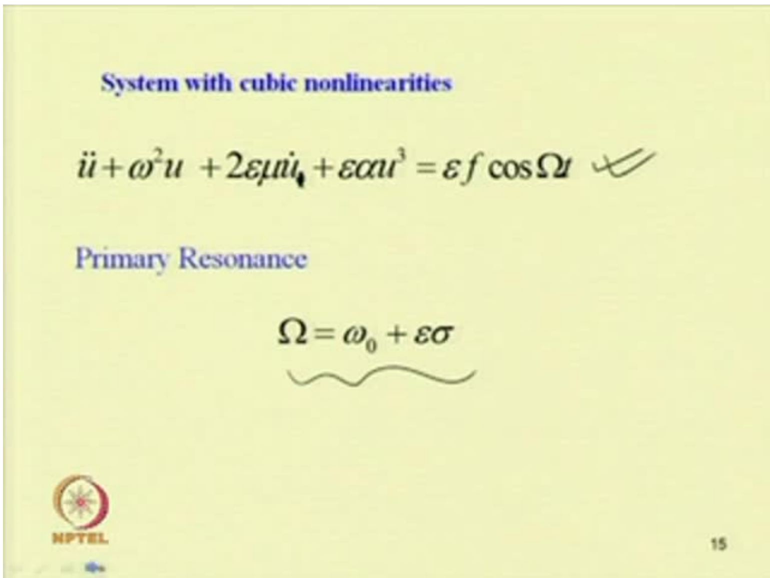

Already we know for the steady state response we have to by applying method of multiple scale, so we have to we obtain this equation which has to be solved to find the frequency response curve. So in this case the steady state response is given by this equation where mu is the damping parameter mu is the damping parameter, sigma is the detuning parameter, alpha is the coefficient of cubic non-linear term, omega 0is the non-linear or this is the frequency term natural frequency of the system and a is the amplitude of thesystem. So one has to solve this equation which is sixth order in terms of a or quadratic in terms of sigma to obtain the response of the system. So let us develop a met lab code to find the solution of this equation.

Also we can directly solve this equation by using this RK 4 method or Runge-Kutta method to solve this equation to find the response of the system. So here we have to write 2 first order equations instead of this second order differential equation, so we have to write to first order differential equation to find the response of the system.

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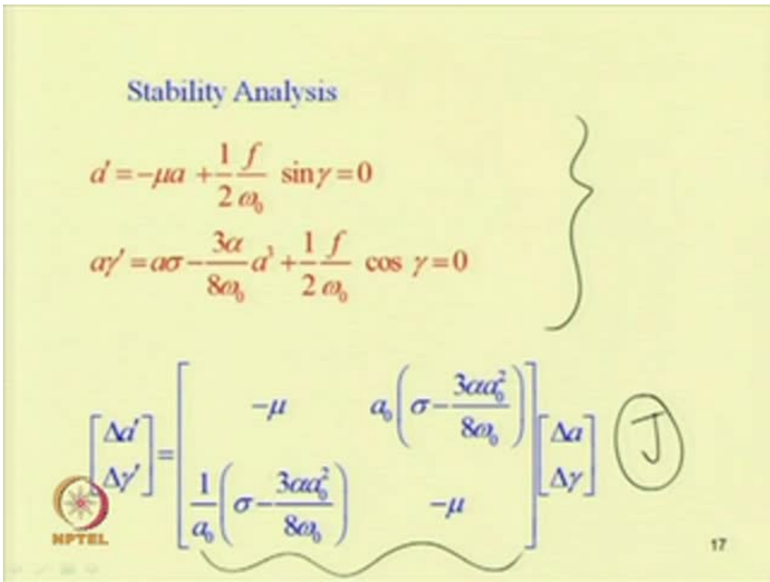

Now with the let us develop some met lab code for finding the response, so first we will solve this equation so for solving this equation. So it is easier to solve this equation from this quadratic equation, Or writing this equation in the quadratic form in terms of sigma that is the detuning parameter where we can obtain this sigma equal to 3 by 8, then alpha by omega 0 into a square plus minus f square by 4 omega 0square a square minus mu square. So already we have discussed about different resonance condition, so in this resonance condition how the jump up phenomena or jump down phenomena will occur in the system.

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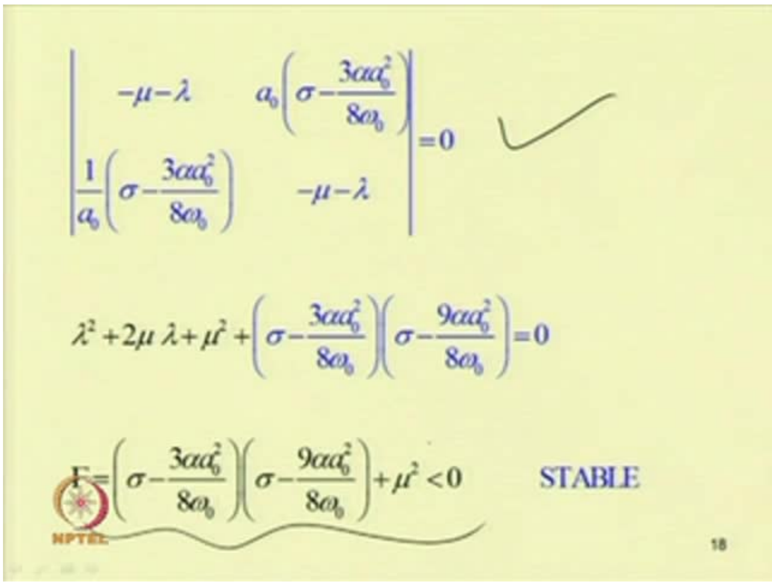

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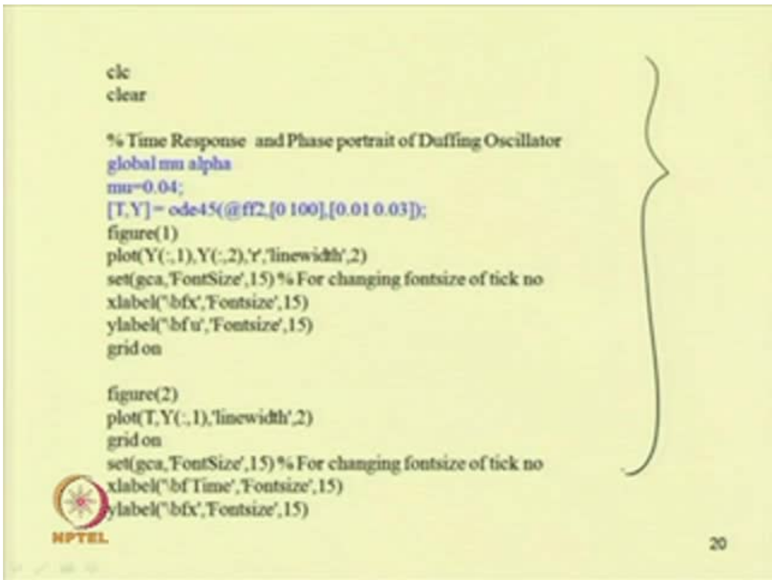

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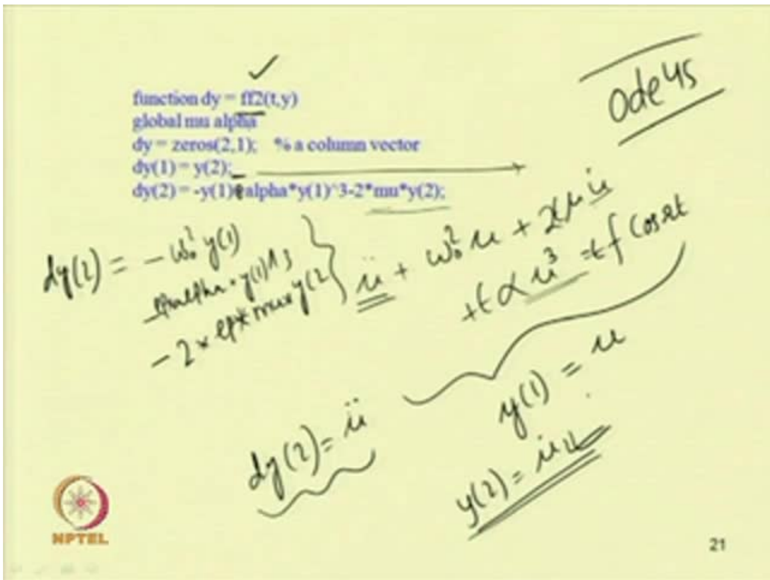

So now let us discuss what will happen or the effect of damping and effect of this forcing parameter so before that, so already we have discussed about the stability of stability analysis of the system. This stability analysis can be carried out by perturbing these equations and this is the Jacobian matrix, so finding the eigenvalue of the Jacobian matrix so one can find the solutionof the system. Or otherwise one can find so in this Jacobian matrix one can obtain this equation, so if this is less than 0the system is stable so the system is stable if it is less than 0 and it is unstable if it is greater than 0. So the effect so already we have seen, or let us so this is the met lab code written for finding the response and phase portrait.

So, in this case first we have to write 2 first order equation so this is the first order equation, so as our equation is u double dot plus omega 0 square u plus; so let us take so you can take damping also, if you take damping then this is 2 mu u dot so this will be equal to or by adding this alphau cube, so this is equal to f cos omega t. Now writing this in terms of the epsilon so you can write this equation or you can introduce this epsilon here so 2 epsilon mu u dot then epsilon alpha u cube equal to epsilon f cos omega t, for which we have studied the principle parametric resonance. So in this case so it can be reduced to 2 first order equation by putting this y 1 let let us put y 1 equal to u, so then u dot equal to y 2 so let us put y 2 equal to u dot so we can have 2 equations, 2 first order equations the first first order equation is u dot equal to y 2 so in this RK 4 method or by using this o d e 4 5 in met lab, so we can use this o d e 4 5 functionin met lab to find the response of the system.

So, in that case the first equation that is d y 1equals to y 2, so this is the equation so u dot equal to y 2 can be written in this form. That is d y 1 equal to y 2, the second equation which is u double dot so u double dot equal to d y 2 so d y 2 represent u double dot. So that can be written as equal to epsilon f cos omega t minus omega 0square u that is for u we can write u y 1, then minus 2 epsilon mu u dot so for u dot we can put this is equal to y 2 then minus epsilon alpha y cube u cube. So this u can be written in terms of y 1 so this is the expression written for d y 2 so d y 2 equal to minus y 1 plus or we can put plus alpha 1 cubethen minus 2 mu y 2, so this this we can put minus also, my and here also we can so this d y 2 we can modify here omega is put to u 1.

So if you put this actual value of omega so then you can write this is equal to minus omega 0square y 1 minus alpha into y 1 cube then minus 2 into so here epsilon also you can multiply epsilon to epsilon let e p for epsilon multiplied by mu into y 2. So this way by writing these 2 function, so this is the function file so here the function file name is f f2 is put f f2 t y so the t is the time y is the response we are getting and this mu and alpha value can be obtained from the other equation.

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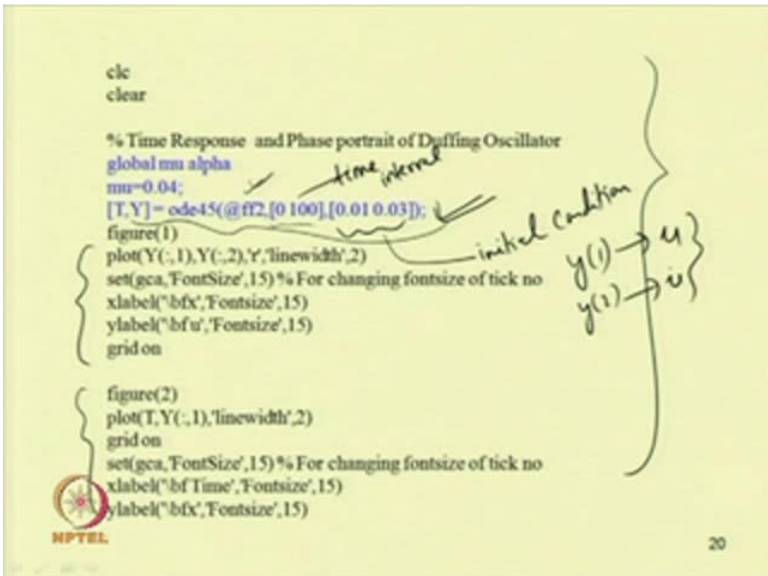

Other program main program so this is the main program where we have defined this value of mu and alpha, so by defining the value of mu and alpha, so you can use this function. So this is the command for using this o d e 45 , so here it is written T,Y equal to o d e 45, so this is the file we are calling function file f f 2, so this is the time interval time interval 0to 100 time interval. So this represent time interval and this is the initial condition, so initial condition for y 1 and y 2, so initial condition for y 1 and y 2, so y 1 is and y 2 is u dot, so given this initial condition, so we can find this time and responseso these are for plotting this thing.

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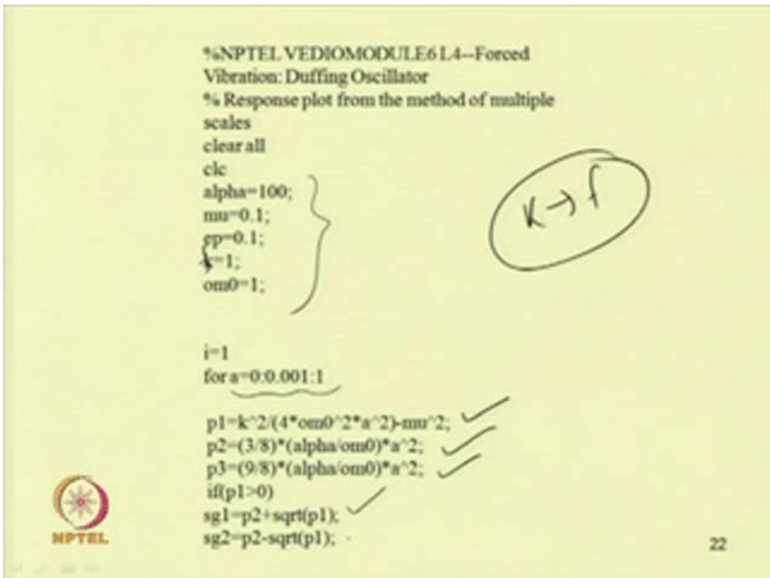

So, will draw the met lab port to show the response of the system, so for finding the, so this is the code for finding the response. Now let us see another code is written for finding the response obtained from or using this method of multiple scales, so already we have this expression. Already we got this expression for finding the response of the system, so here let me put this parameter as P1 and this parameter as P 2, so this is P 1 and this is P 2, so then sigma will be equal to P 1plus minus root over P 2 plus minus root over P 1, so for finding the response, so one can write the, so these are the initial conditions alpha mu epsilon k or fso here in this case we are using f, so f and omega 0has to be defined, so after defining this value.

So let us vary this a, so you can vary this a from 0to 1 with an increment of point 001 or you can take different increment then this is the P 1 value then P 2 and we can define this

P 3 for finding this stability. So P 1 equal to k square by 4 omega square, so in our code we have written k, so this is fso if p greater than, if P 1 greater than 0. So as our, so if P 1 is less than 0, then we will have imaginary roots so the solution is not feasible so we have only feasible solution when P 1 is greater than 0.

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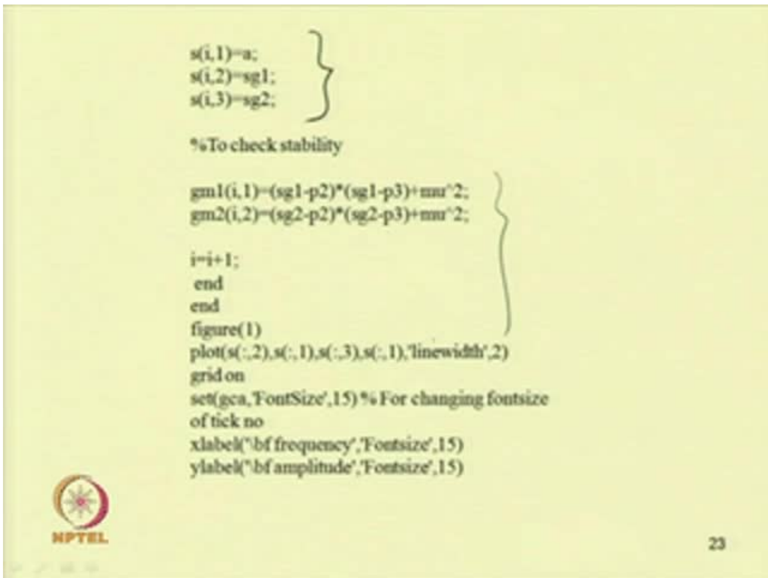

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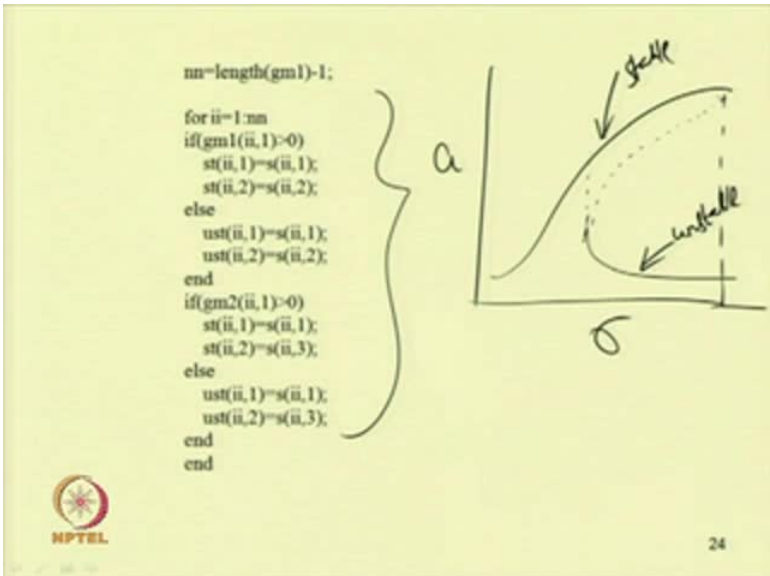

So for finding P 1 or you have to find or we have to take the value of a for which P 1 greater than 0. Now taking P 1 greater than 0, so you can find this sigma 1 by taking this P 2 plus square root of P 1 and sigma 2 by taking P 2 minus square root of 1 and then we can assign this value for plotting the response of the plot. Response of the system to check the stability again we can use this program. So here we can find this sigma 1 minus P 2 into sigma 1 minus P 3 plus mu square that is gamma 1 and gamma 2, so for corresponding to sigma 1 we can find this gamma. So this is followed from this equation from this equation, from this, so this is sigma so this term is taken as so this is P 1, this term is taken P 1 and this isP 2, so P 1 into P 2 plus mu square should be less than 0. So if we use that expression then we can, so this will be this will be modified by this is P 1 and this is P 2, so P 1 into P 3 plus mu square, so it should be gamma 1 and here P 2 into, so in that way you can find by checking this condition. So you can write the stable and unstable part of the response and then we can plot. So already we have seen, so if you plot the so this is a versus gamma so if you plot a versus gamma then we have seen a versus sigma that is our detuning parameter, so we have already observed that for positive value of gamma, for positive value of alpha so one can obtain the response to like this. So this partis stable and this is unstable and one can have a jump up phenomena here or jump down phenomena here depending on the initial condition.

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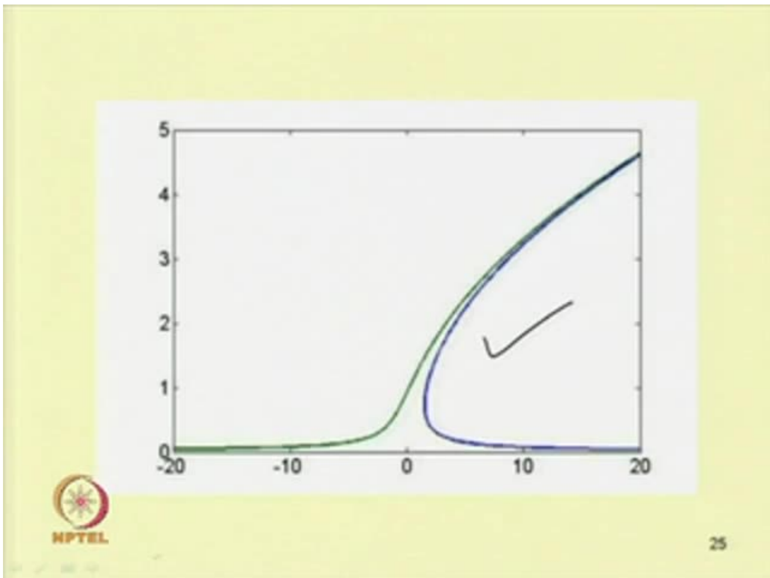

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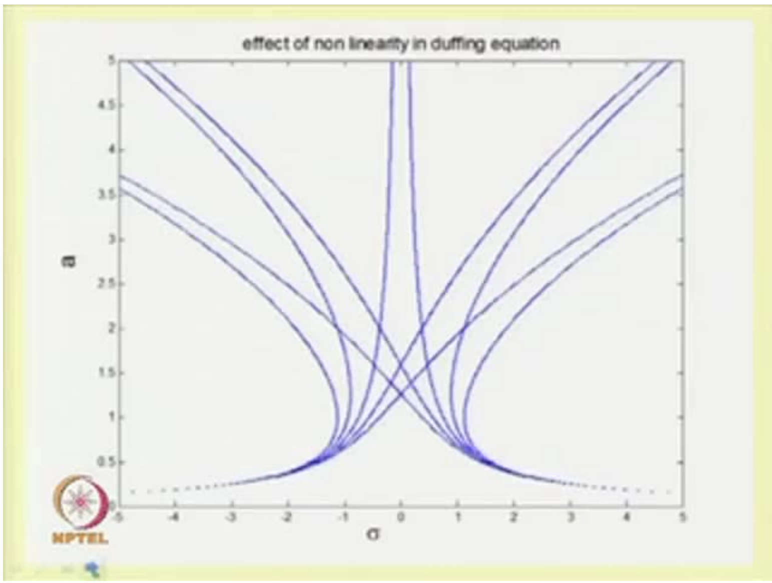

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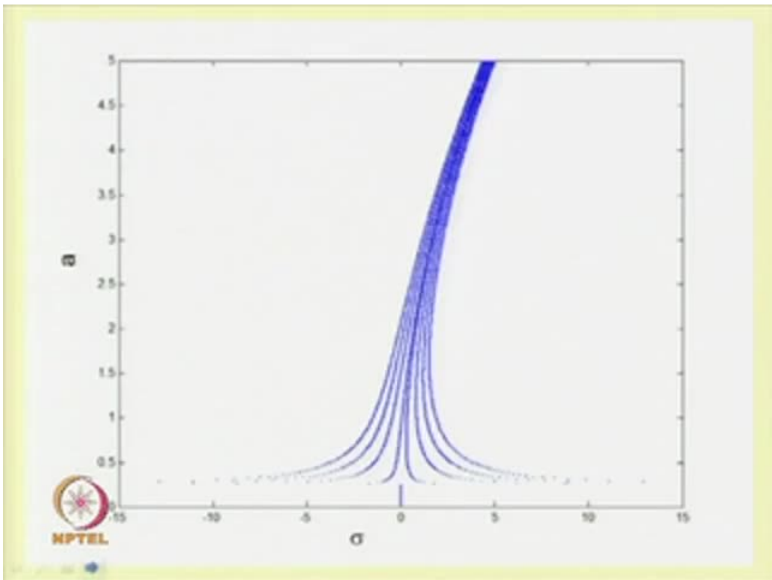

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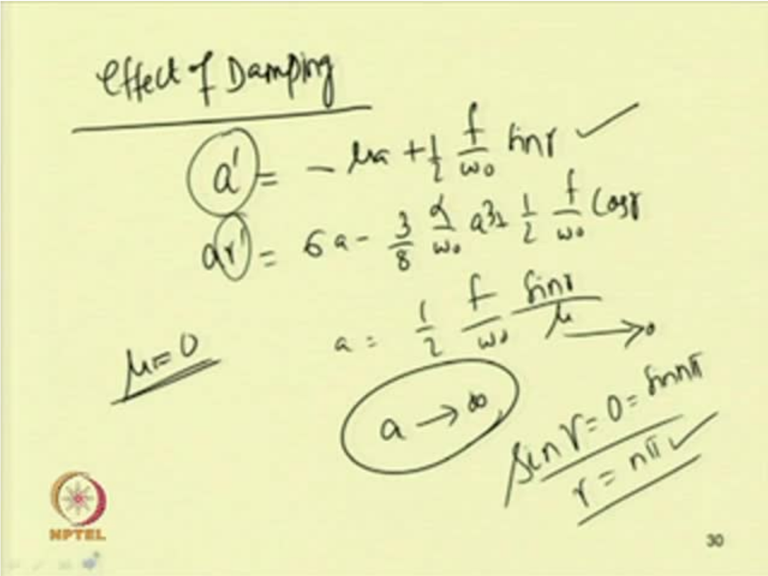

So let us see, so already we have discussed about this curve andwe can let us see the effect of damping effect of damping, so from our equation reduced equation, so we know our reduced equation is a dash equal to minus mu a plus half f by omega 0sin gamma and then a gamma dash equal to sigma a minus 3 by8 alpha by omega 0a cube plus half f by omega 0 cos gamma. So ifwe are considering mu equal to 0, so if mu equal to 0that is un-damped system, so from this equation for the steady state for the steady state a dash and gamma dash are 0.

So for the steady state one can see this mu a equal to half f by omega 0 sin gamma, so a is equal to f by mu. So if mu tends to 0; so this a tends to infinite so this so for un-damped system like linear system, so here the amplitude tends to infinity when there is no damping present in the system. So the steady state response for an un-damped system that is the peak response amplitude tends to infinity and when damping is present then one can get the finite value as already we have seen. So here again for mu equal to 0the sin gamma.

So sin gamma equal to 0, so this is equal to sin n pi. So in this case so one can get this gamma equal to n pi, so for un-damped system one can see n equal to 1 2 integer number so one can see for un-damped system, so the system will be either in phase or out of phase. So in case of n equal to 1 so this is equal to pi gamma equal to pi then the system is out of phase, if n equal to 2 then this is equal to 2 pi or 0, so then the system will be in phase with the excitation frequency.

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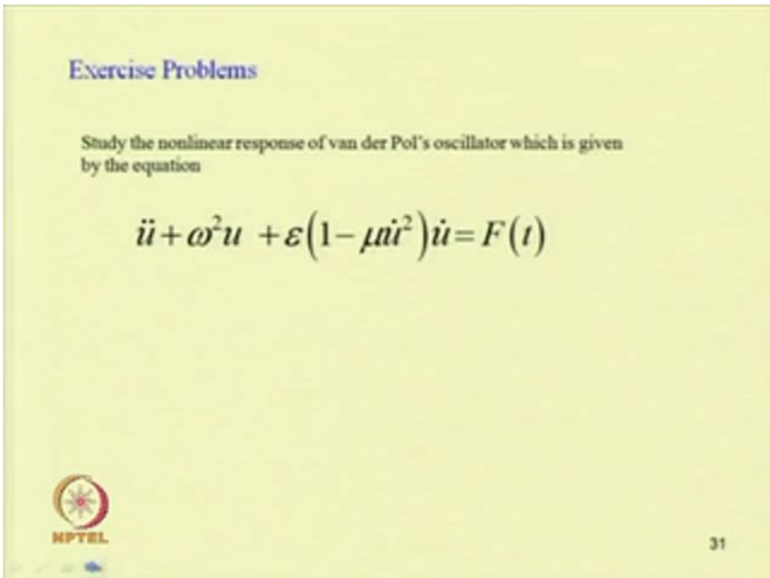

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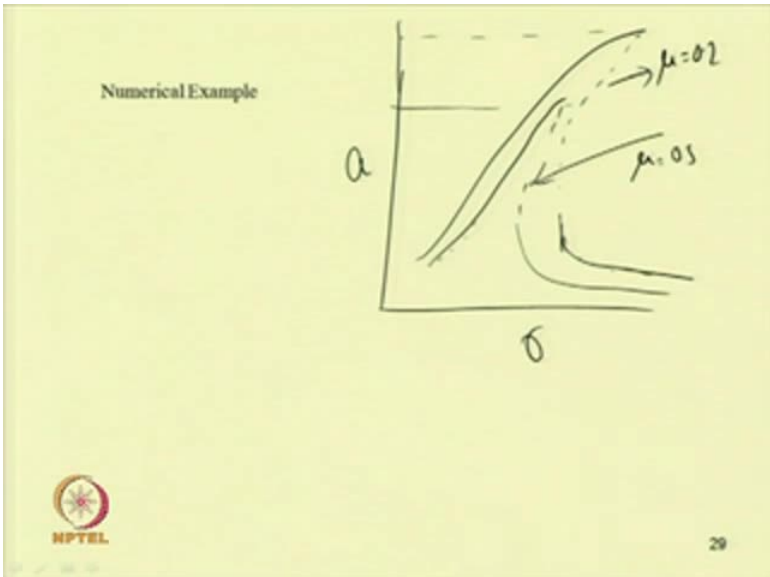

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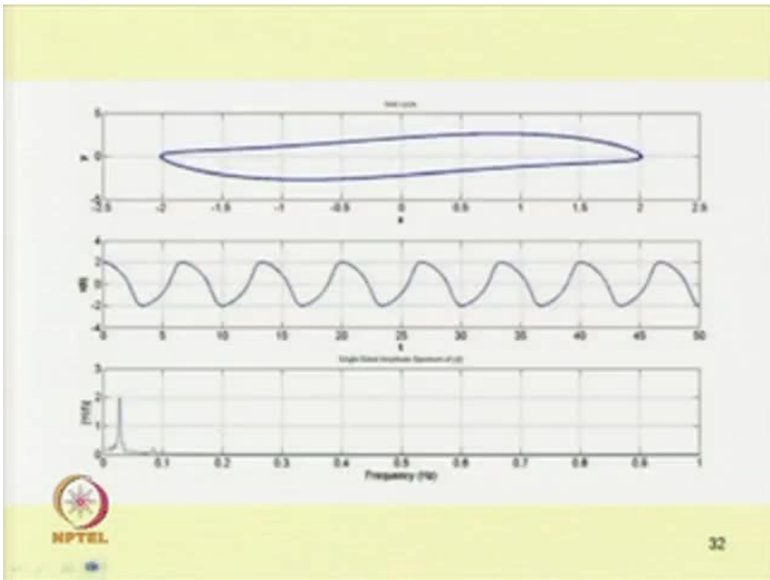

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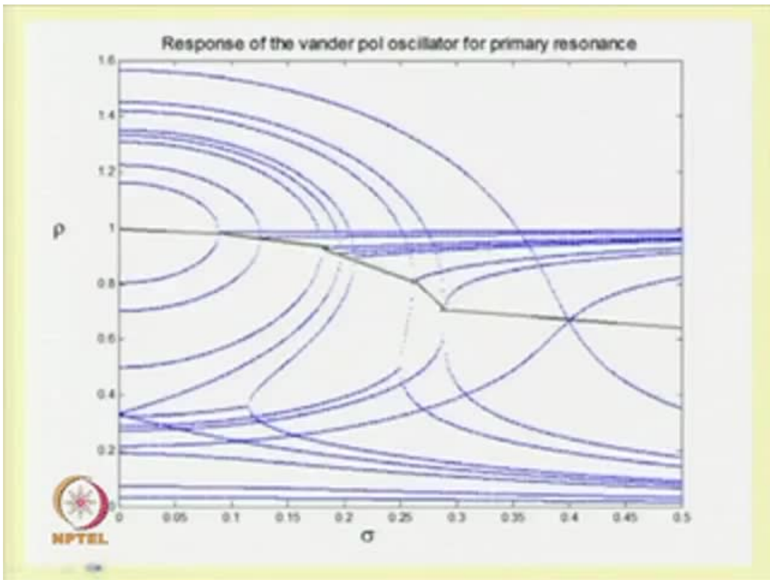

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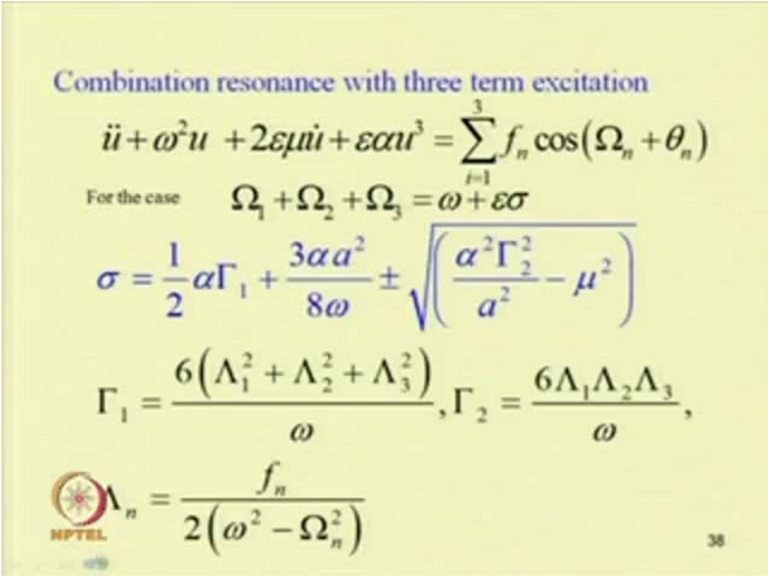

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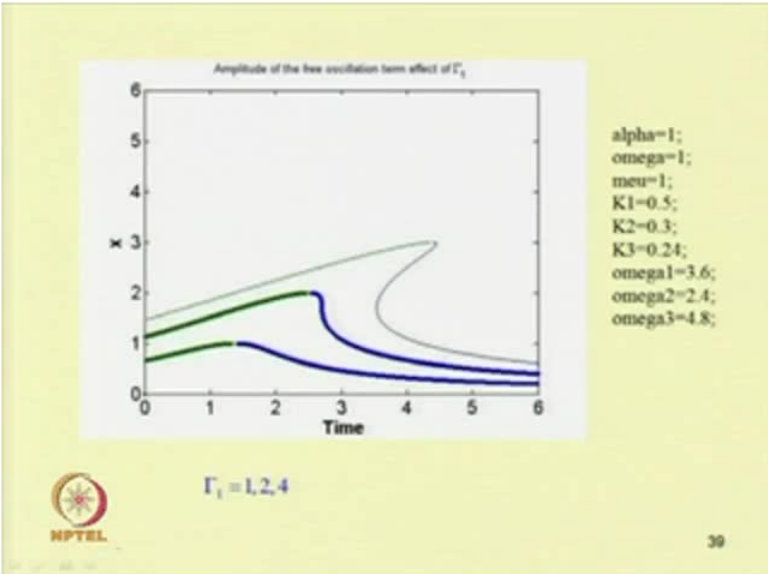

So the system will be in phase or out of phase depending on value of n. So in case of damping, so in un-damped system we have seen the response is infinite but in case of in the presence of damping. So if one plots the response of thesystem, so let us take the same numerical value. So if one plot a versus sigma one can see the response curve will look like this, so this is for let mu equal to 0.2 so by increasing this muone can get the response curve like this. So this part is stable, so the response curve the peak amplitude will decrease with increase in thedamping parameter. So this is for mu equal to 2 and this will be mu equal 0.5. Similarly, if we plot so let us plot theresponse curve for, so if we plot the response curve or let us see what will happen if you go on increasing the value of forcing parameter.

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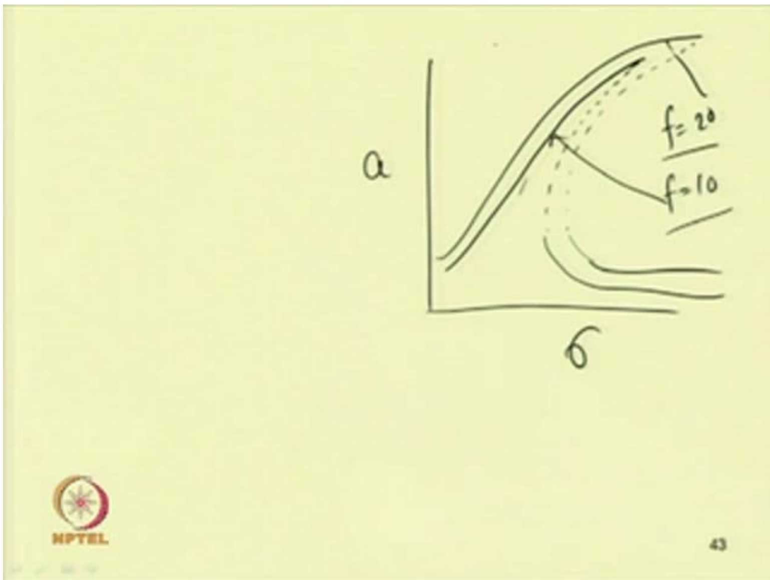

So, if one plot the forced response curve a versus forcing for different value up so for different value up or a versus sigma for different value up let us first plot a versus sigma for different value of f, so one can see for higher value of forcing the responsewill be like this, go on. So if weincrease or decrease the value of forcing parameter, so let us decrease the value of forcing parameter. For example, if f equal to 20, so this is the curve one can get and this correspond to f equal to10, so if you go on decreasing theforcing amplitude the response curve also the peak amplitude peak response amplitude also will decrease, so...

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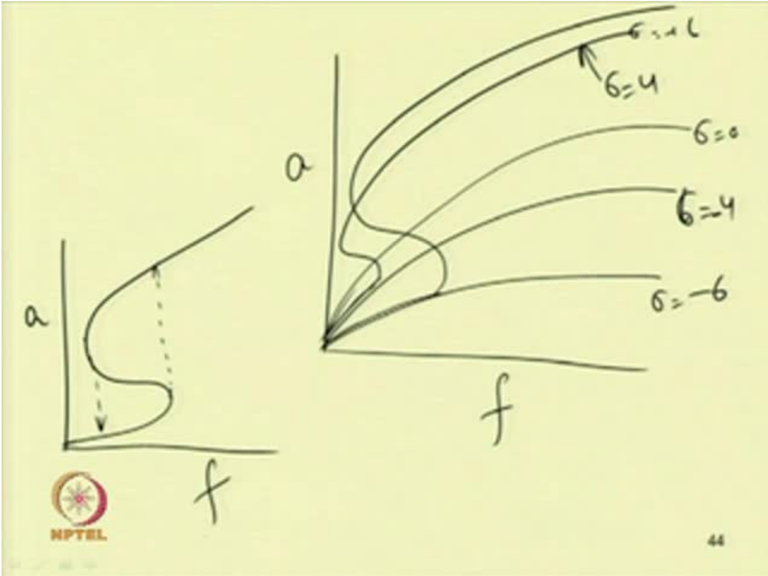

But if one can plots the force response curve that is for different value of so a versus force for different value of sigma, so this is for sigma equal to, let this is for sigma equal to minus 6. So let us take this is sigma equal to sigma equal to 4, so sigma equal to, let this is sigma equal to 0. So then this is minus 4 sigma equal to minus 6 minus 4. This 0so for plus 6, the response curve will be will be it will look like this so it will follow this path and then so this is for sigma equal to plus 6 and for sigma equal to plus 4. So the curve will follow some part like this and this will be like this and sigma so this is correspond to sigma equal to plus four so for positive value of sigma.

So, one can see the curve forcing response curve will look like this so in this case one can observe a jump of phenomena clearly one can observe a jump of phenomena, so this is forcing parameter and a so clearly one can observe the jump of phenomena here and jump down phenomena here, so sweeping of the frequency amplitude, so this isforcing amplitude so while sweeping of the forcing amplitude or by increasing the forcing amplitude one can observe a jump of phenomena here and a jump down phenomena at this place. So let us take some exercise problem in this case and after that I will show the metlab code ormetlab code for finding the response of the system.

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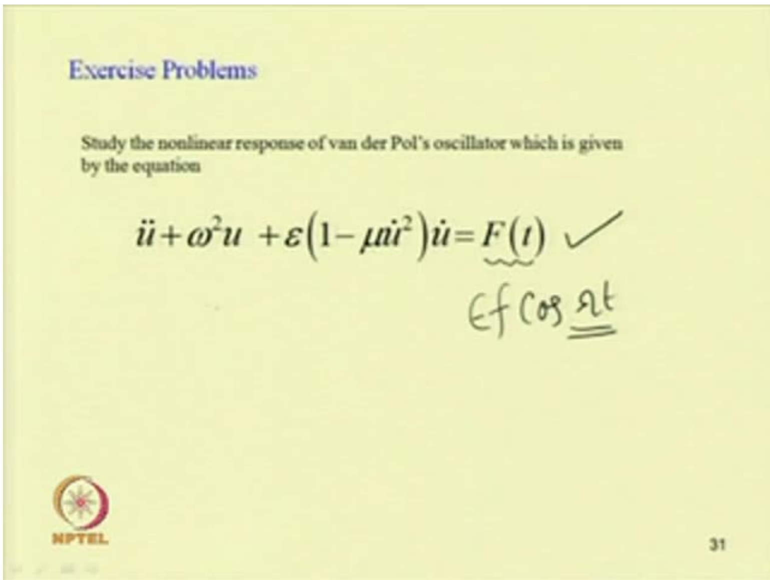

So let us take 2 exam 2 exercise problem, so in the first exercise problem let us take the Van der Pol equation, so in case of the Van der Pol equation, forced Van der Pol equation the equation can be written in this form that is u double dot plus omega square u plus epsilon into 1 minus mu u dot square u dot equal to forcing function, so this forcing function 1 may take equal to epsilon f cos omega t. So here similar to this case what we have studied before here also one can observe many resonance conditions primary resonance conditions for harmonic and sub harmonic resonance condition.

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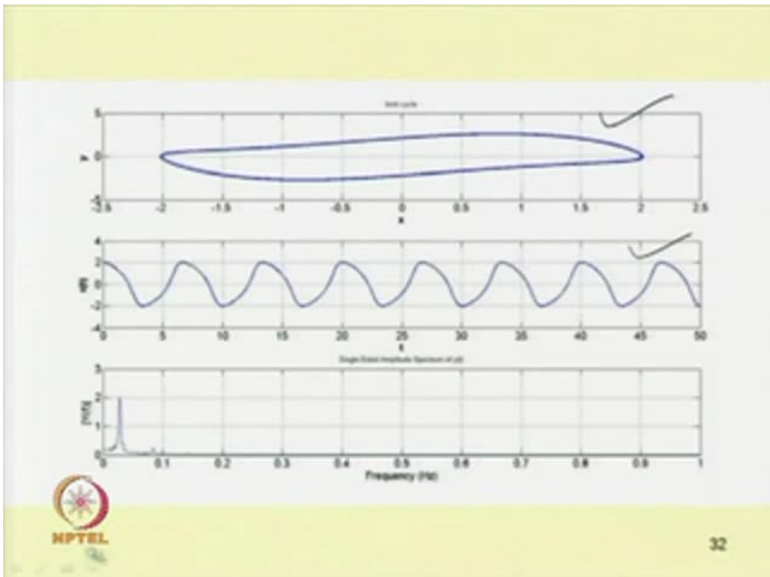

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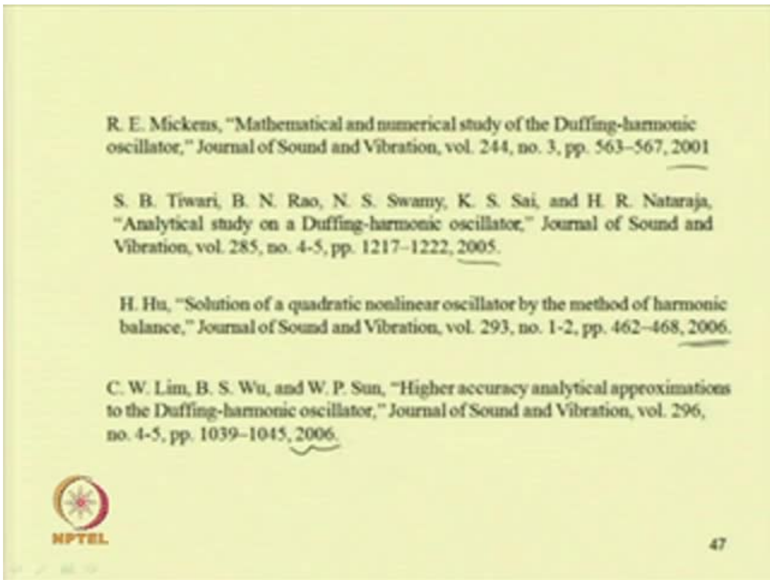

And one can find the response of the curve by response of the system by directly solving this equation using RK 4 method or Runga Kutta method or bysolve using this method of multiple scales and one can study for different system parameter. So this is one of the exercise problem to find the response curve, so by using o d e 45 one can check that one can get a limit cycle, so this is the time response this is the phase portrait corresponding to that. So in this exercise problem find the periodic response or find the phase portrait periodic response and power spectra for the system. So, to know more about this duffing equation, so one can refer this papers by R E Mickens, mathematical and numerical study of duffing harmonic oscillator which is published in journal of sound and vibration in 2001.

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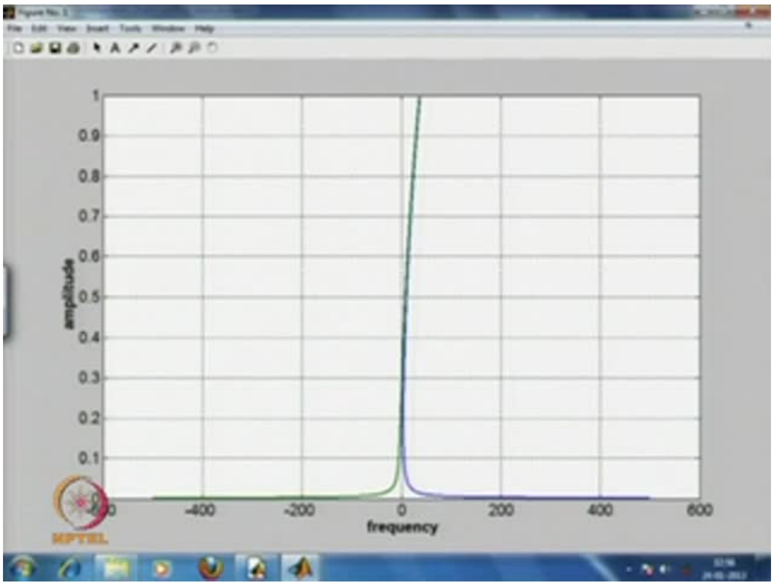

Or other papers like "Analytical study on a Duffing harmonic oscillator", so which is published in 2005, then "Solution of a quadratic non-linear oscillator by the method of harmonic balance" so which is published in journal of sound and vibration in 2006, and another paper by Lim and Sun "Higher accuracy analytical approximation to the duffing harmonic oscillator" which is published in same year 2006 by journal of sound and vibration. Also one may see this reference "Determination of frequency amplitude relation or a duffing harmonic oscillator by energy balance method" which was published in Computers and Mathematics with applications in 2007. So there are many other papers also available on on this duffing equation, so one can study those papers to know more about the applications of duffing equations. And let us now see themetlab port for finding the response of thesystem, so this is the code for finding theforced response of duffing oscillator. So here the value of alpha is taken to be 100 mu equal to 0.1 epsilon equal to 0.1 and f equal to 1omega 0also its taken to be 1, So one draws this program.

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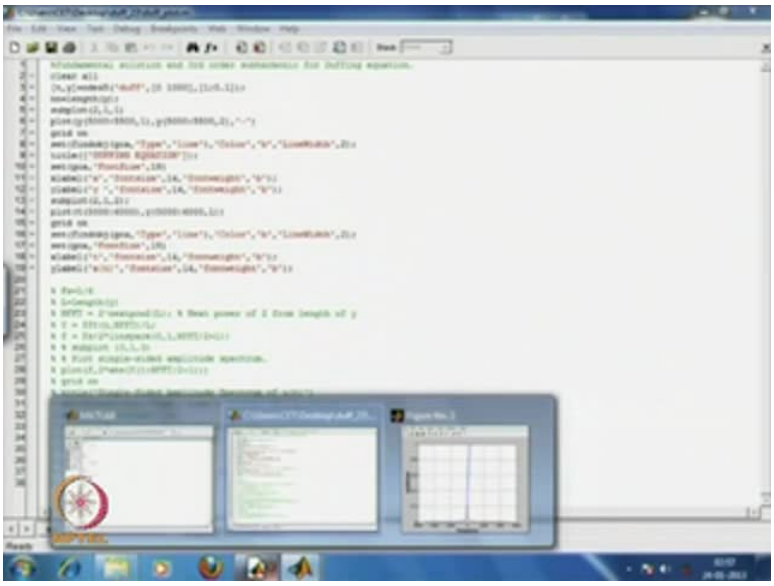

One can find the response amplitude; so the response curve is plotted. So this is the response curve one can see so for which this branch is stable and this is also stable branch and in between one can find one unstable branch. So by changing the system parameter so one can plot this amplitude versus frequency curve.

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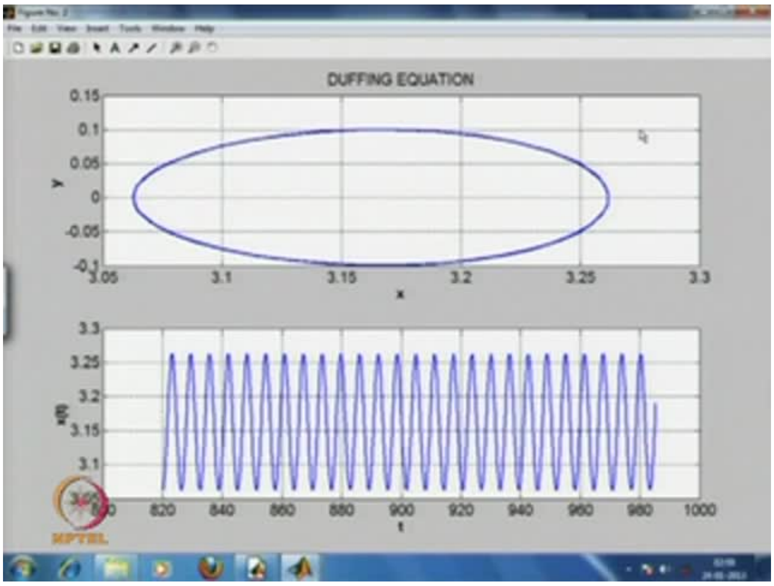

For a different system parameter so one can use this metlab port for finding response of the system for different system parameter. So let us see or let us use the otherprogram it is the RK 4, using this RK 4 method. So, this is the code written for finding the response using RK 4 method. So by running this code, so let us find the response. Soit’s running; so by running this code so one can obtain this phase portrait. So this phase portrait clearly shows that the response is periodic and this is the time response. So this isthe time response of the time response, periodic time response obtained in this case and this is the phase portrait obtained in this case.

So, next class we are going to study about the duffing equation with hard excitation term. So in the present case we have taken the duffing equation with soft excitation term, that is that forcing term we have taken epsilon f and next class we are going to study the duffing equation when this f term cannot be neglected or when this f term that is this forcing amplitude is not small. So in case of forcing amplitude is high then we can observe many different resonance conditions like super harmonic, soft harmonic resonance condition along with this principle for or along with this parametric, along with the primary resonance condition.

Thank you.