Non-Linear Vibration
Prof. S. K. Dwivedy
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 6
Applications
Lecture – 3
Free Nonlinear Vibration of
Multi Degree of Freedom System

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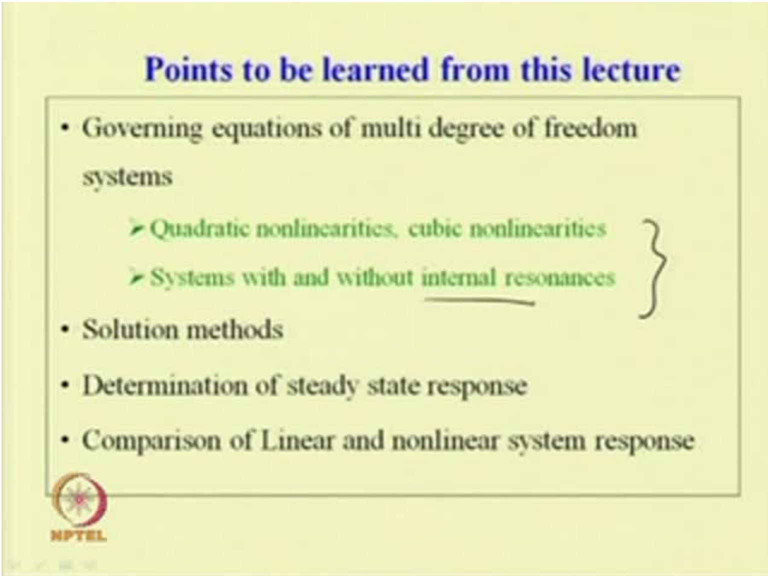

So, welcome to today class of non-linear vibration, so in this module, we are discussing about the applications of non-linear systems, where we are discussing about the free vibration force vibration and then parametrically excited systems, so last two classes we are discussing about the free vibration of single degree of freedom system and today class we are going to discuss about non-linear free vibration of multi degree of freedom system, so in last two classes, so we have seen, so how this single degree of freedom systems non-linear behaviour can be can be obtained from this free vibration analysis, so difference between free vibration of linear and non-linear systems we have discussed, so we have seen in case of linear system, so there is the frequency is not dependent on the amplitude but, in case of non-linear system the frequency of oscillation depends on the amplitude of the non-linear system, and in case of multi degree of freedom system today we are going to study how to determine the governing equation of motion in case of the multi degree of freedom system.

So, here we will discuss about this quadratic and cubic nonlinearities systems with and without internal resonance, and then the solution methods and then we will study or we will discuss about the determination of steady state response stability, and then we will compare the response of a linear and non-linear system.

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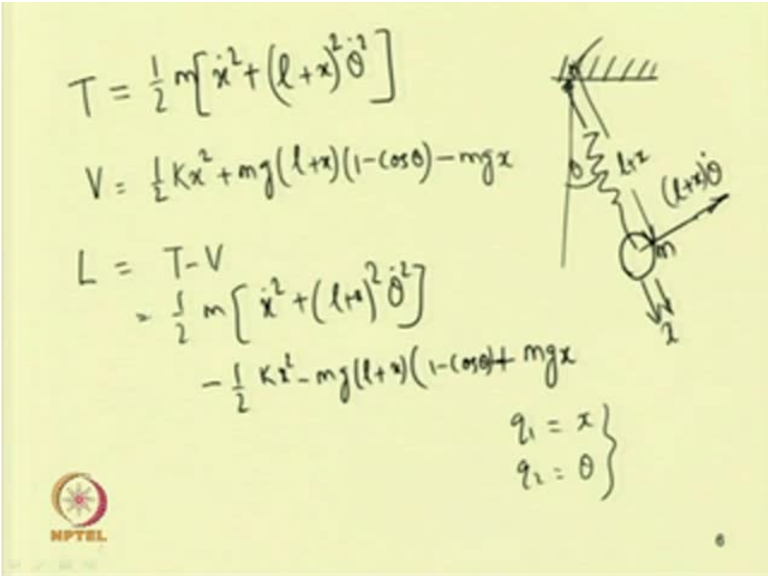

So, let us take a simple example of a spring mass system, so unlike the spring mass systems we have studied before let it is spring mass, where so the spring can extend with this mass m, so let the original length of the spring is l now with extension, so it has extended in this direction by an amount x, so this extension extended length becomes l plus x original length was l, now the length change to l plus x, so to derive the equation motion, so one can use different methods but, particularly for multi degree of freedom systems it is convenient to use Lagrange equation, so here to use this Lagrange equation one should know, the velocity of the system, so the velocity in this direction that is in this direction it is x dot and the velocity in this direction, so as it is moving by an amount theta or rotating by an amount theta.

So, the velocity here will be l plus x that is the distance into theta dot, so the kinetic energy one can write the kinetic energy and potential energy of the system the kinetic energy of the system k or T kinetic energy T can be written as half m into x dot square plus velocity in this direction, so plus velocity in this direction also, it will be equal to l plus x square and theta dot square, so the potential energy u can or v can be written as half, so for the spring.

So, let us assume this spring as a constant spring constant k, so then this potential becomes half k into x square plus m g into l plus x into 1 minus cos theta, so this angle is theta 1 minus cos theta minus m g x, so the potential energy equal to half k x square that is this extension of the spring, if you are assuming the extensions of the spring is x then it is equal to half k x square then due to change in position of the mass, so the change in position of the mass becomes m g into l plus x into 1 minus cos x cos theta 1 minus cos theta minus m g x, so now one can write this L equal to T minus V.

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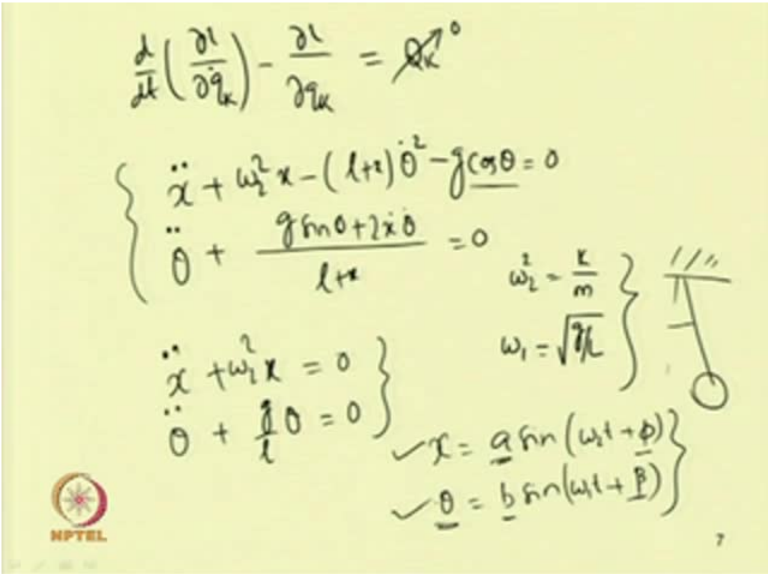

So, this will be equal to half m x dot square plus l plus x whole square theta dot square minus half k x square minus m g into l plus x into 1 minus cos theta, and minus minus plus, so this becomes m g into x, so this is the Lagrangian of the systems, so after getting the Lagrangian of system, so as we have two generalized coordinate, the generalized coordinate q 1 equal to we can take x and q 2 equal to theta, so if you take these two generalized coordinate then the equation motion can be written in this form that is d by d t of del l by del q k dot minus del l by del q k, so this will be equal to capital Q k as there is no force acting on the systems as we are considering only the free vibration in this case, so this Q k equal to 0, so now taking this q or q 1 equal to x and q 2 equal to theta.

So, we can write the equation motion in this form, that is x double dot plus omega 2 square x minus l plus x theta dot square minus g cos theta equal to 0, and second equation becomes theta double dot plus g sin theta plus 2 x dot theta dot by l plus x equal to 0, so here omega 2 square equal to k by m and this omega 1, we can write equal to root over g by l, so which is the we have not used this omega 1 term till now, so omega 1 you can write the oscillation of the frequency of the pendulum, so if when we are considering the spring.

So, if we have a simple pendulum of length l, we know the frequency equal to root over g by l, so we got these two equations, so in this two equation, so these are the non-linear terms that is theta dot terms containing theta dot square and this cos theta term also, we can expand this cos theta, if theta is not small, so then we can expand cos theta, so that is also a non-linear term. Similarly, in the second equation, so we have this sin theta term which is non-linear and 2 x dot theta dot term also non-linear terms, so if we neglect these non-linear terms assuming small amplitude of oscillation then this equation reduces to x double dot x double dot plus omega 2 square x equal to 0, and second equation reduce to theta double dot plus g sin theta g by l sin theta, so this is equal to or g by l sin theta, and this sin theta taking, sin theta equal to theta for theta to be small, so this equation becomes this, so these two are two uncoupled equations and one can obtain the solution of this uncoupled equation like x equal to a sin.

So, x can be a sin omega 2 t plus phi, and this theta can be b sin omega 1 t plus beta, so here this a and phi are the initial, so a and phi can be obtained from the initial condition similarly, beta b and beta can be obtained from the initial condition, so in this case we have two uncoupled equation, so if theta is small, so we have two uncoupled equations and the response are uncoupled; that means, the extension of this spring that is by x does not depends on theta that is the rotation of the pendulum, so these two are two different modes of oscillation but, it has been shown.

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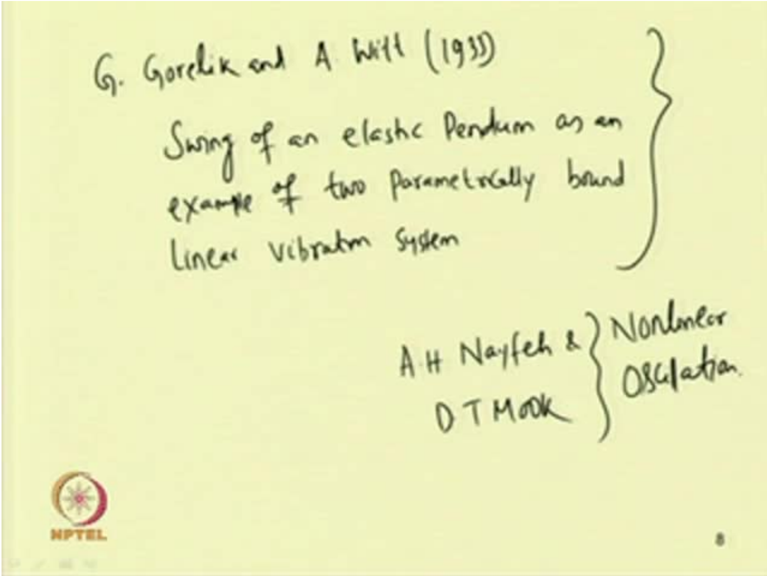

So, it has been shown by Gorelik and A. Witt, G. Gorelik g o Gorelik and A Witt, so it has been shown by A Gorelik and Witt in 1933, so this is published in swing in published in or the title of the paper is swing of an elastic pendulum, swing of an elastic pendulum as an example of two parametrically, two parametrically bound linear vibration system also one may refer the book by A.H Nayfeh and, and D T Mook for detailed discussion on this non-linear oscillations.

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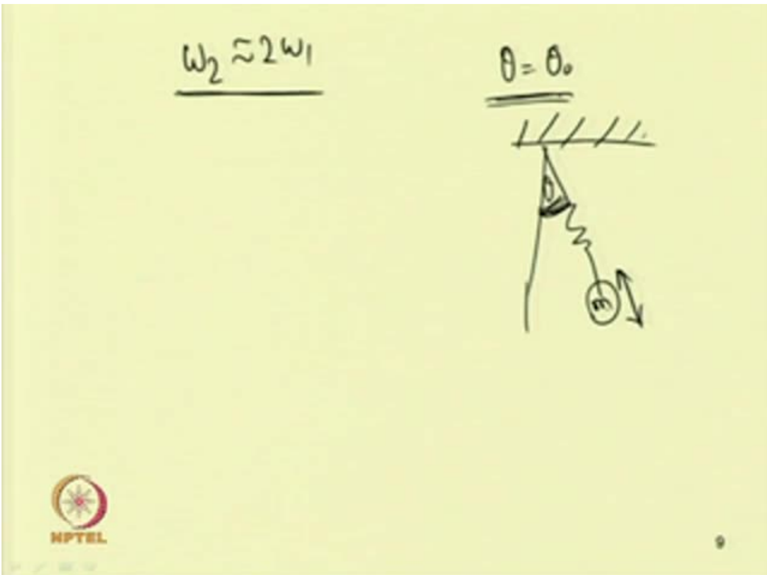

So, here it has been shown that if there is relation between the second mode or there is relation between this omega 2 and omega 1, for example, if one take this omega 2 nearly equal to 2 omega 1, then this though the amplitude of oscillation be small, so in that case, so by extending or by taking this motion, initial motion theta equal to theta 0, where theta 0 is not equal to 0 then there is, so, this is the spring, so by giving the small oscillation theta, theta equal to theta 0 and released, so if it is released then it is observed that by pulling this mass m slightly down, so there is energy transfer between these two modes; that means, so there is alternate, there is alternate rotation of this pendulum or there is there is oscillation displacement in this direction and also rotation about this direction.

So, if one starts the motion when theta equal to theta 0, where theta 0 equal to 0 but, a very small by pulling the mass m down, one finds that the mass oscillate up and down first, so the mass oscillates up and down first and then a pendulum type of component of motion develops and grows at the expense of spring type motion after a while the pendulum type motion starts to decrease and the spring type motion starts to grow, so thus energy is transferred from one mode to other mode, so this shows the internal resonance, so though there is no external force acting on the systems as omega 2 is taken equal to twice omega 1, so there is transfer of energy between these two modes of vibration, so we have seen or it is shown that though the amplitude is small, in case of or considering the system to be non-linear, so there is energy transfer between two modes of vibration.

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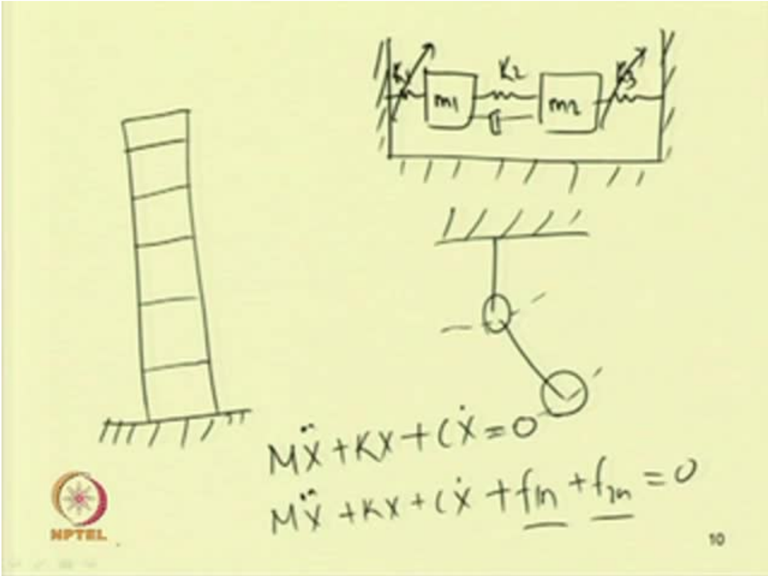

So, like these systems one can take several other type of systems also to derive the equation motion non-linear equation motion of the system for example, one can take this simple spring mass system, so where let us take only two one can take more also more number of spring mass system, and let us add this damper and this spring, so only damper let us put, we can put the damper or we may have a damper in the middle, so this is k 1 this is k 2 and this is k three, so in these systems by considering this spring to be non-linear, so let this k 1 and k 3 are non-linear spring then one can find the governing equation motion, governing non-linear equation motion similarly, one can take another example of the double pendulum, so in case of the double pendulum, so as the oscillation, so taking the oscillation not to be very small then one can find a coupled equation motion in this case, so in case of the double pendulum similarly many other systems.

For example, let us take a multi stored building, multi stored building subjected to earthquake excitation, so in that case after the excitation earthquake excitation, the systems will be reduced to that of a non-linear multi degree of freedom system, similarly, many other systems can be modelled, so many other systems can be modelled as multi degree of freedom systems and one can study the equation motion in those cases, so for the linear multi degree of freedom systems, the equation becomes M x double dot plus K x plus C x dot equal to 0. But, for non-linear, so we can add the non-linear terms, so we can write M x double dot plus K x plus C x dot, and here the non-linear terms can be added, so non-linear, so this is this is f 1 n f two n, so one can go on adding these non-linear terms which will be equal to 0, so now, you may study or we can study a simple case that is the non-linear systems with quadratic nonlinearity in detail and then and briefly will study about a system with cubic nonlinearity.

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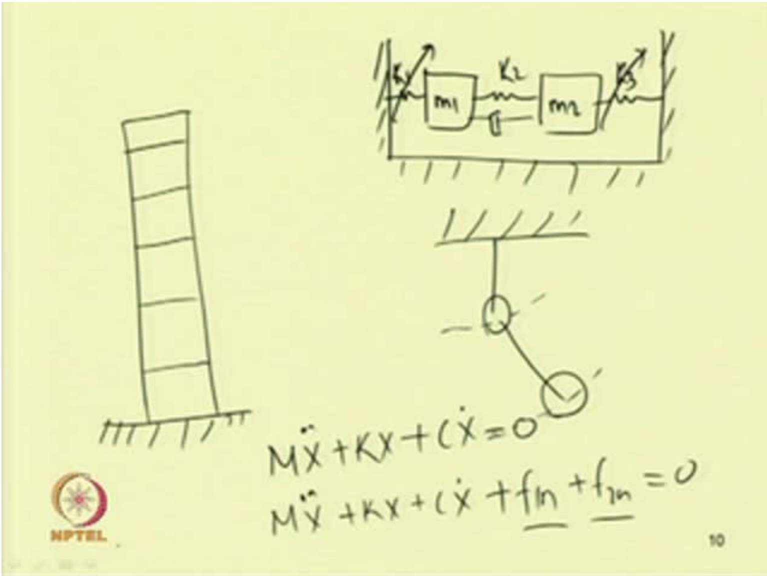

So it has been shown by in this book of Nayfeh and Mook and by other researcher that the non-linear systems with quadratic damping with quadratic nonlinearity, so will behave as a linear systems, so let us find the equation motion in that case.

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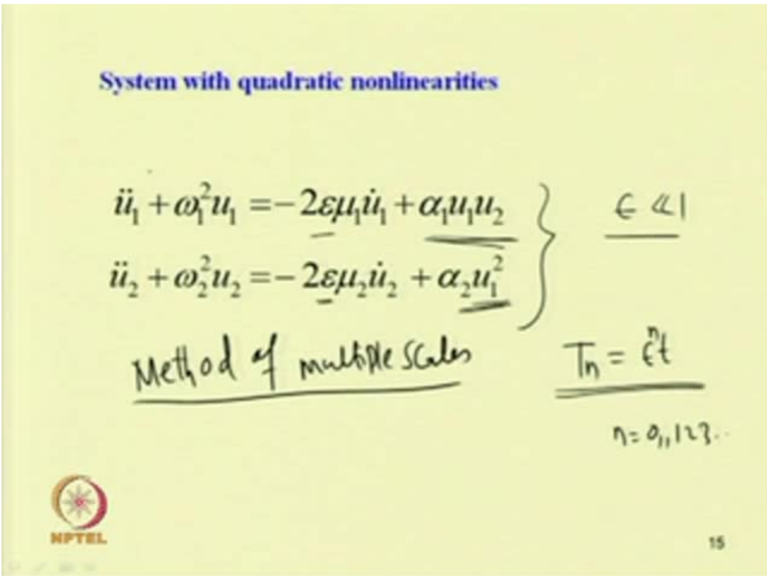

So let us take this non-linear systems with quadratic nonlinearity, so here we can write the equation two, we have taken a two degree of freedom system, where the equation is written u 1 double dot plus omega 1 square u 1 minus 2 epsilon mu 1 u 1 dot plus alpha 1 u 1 into u 2 then u 2 double dot equal to or u 2 double dot plus omega square u 2 equal to minus 2 epsilon mu 2 u 2 dot plus alpha 2 u 1 square.

So here this epsilon term is used as a book keeping parameter to make this damping term of the same order as that of the non-linear terms, so this book keeping parameter is generally taken very very less than 1 and by taking this non-linear term, so here we have a coupled non-linear term that is u 1 into u 2, and here also in this second equation u 2 equation we have taken this u 1 square, so as we are taking this coupled equation though there is no external force acting on the systems, so one may think that there will be energy transfer between this u 1 and u 2, so we can consider two different type of case, so in one case one is the resonant case, so this resonant case will come due to the internal resonance in the systems, and the second case you may consider non resonant case.

So one can find the solution of these type systems by using different numerical technique such as Runge kutta method, central difference method or one can use this Newmark beta method or Wilson theta method to find the non-linear response of the system but, to get the frequency amplitude relation, one can use this perturbation methods, so already we have studied the different types of perturbation method and today class we will take the method of multiple scale to solve this equation otherwise one may use this Lindstedt poincare technique, modified Lindstedt poincare technique or this harmonic balance method, and one can take also other different types of harmonic balance method for example, one can take intrinsic harmonic balance method or combination of harmonic balance method and method of multiple scales method of averaging, and different other types of averaging methods.

Also one can take to solve these type of equations and in this equation, so to use method of multiple scale, so let us use method of multiple scale to solve this equation, so to use method of multiple scale, so we have to take different time scales that is T n, so let T n n, n equal to 0 1 2 3 , so T n equal to epsilon to the power n t so; that means, T 0 equal to T, T 1 equal to epsilon T and T 2 equal to epsilon square T.

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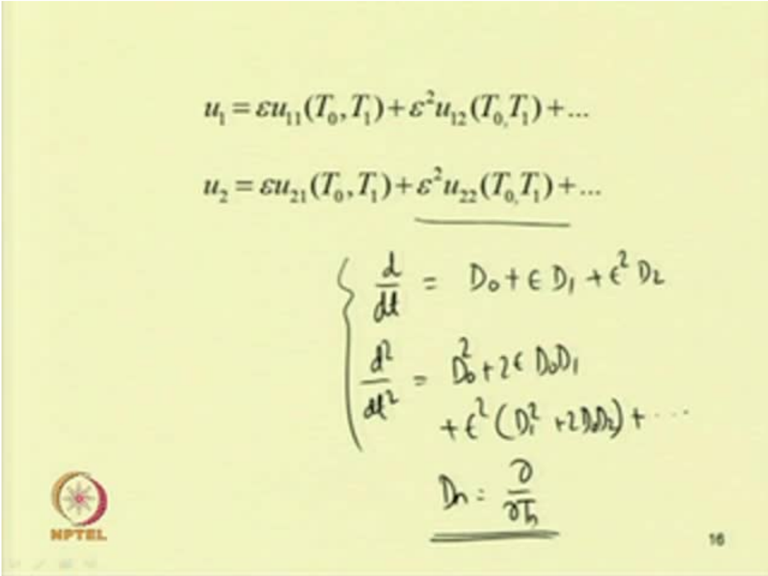

Also we can write this u 1 and u 2 using different time scales for example, we have written this u 1 equal to epsilon u 1 1 T 0 T 1 epsilon square u 1 2 T 0 T 1 and u 2 equal to epsilon u 2 1, which is a function of T 0 T 1 plus epsilon square u 2 to T 0 T 1, so one can take also higher order terms in this case but, in this case we have taken only two terms or upto epsilon square now taking this d by d t, so as in the first equation, so we have u double dot; that means, d square u 1 d t square so, so you can write this d by d t equal to taking only single term, so this is D 0 plus epsilon D 1 similarly, one can write this d square by d t square equal to D 0 square plus 2 epsilon D 0 D 1 plus epsilon square epsilon square D 1 square plus 2 D 0 D 2 plus higher order terms, one can take here also one can take epsilon square D 2, so taking this d by d t equal to D 0, so where D 0 equal to or D n is nothing but, del by del n.

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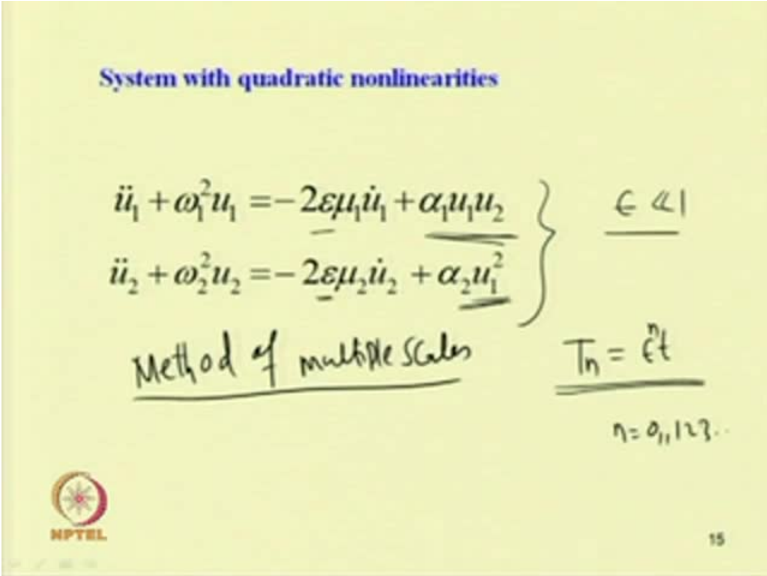

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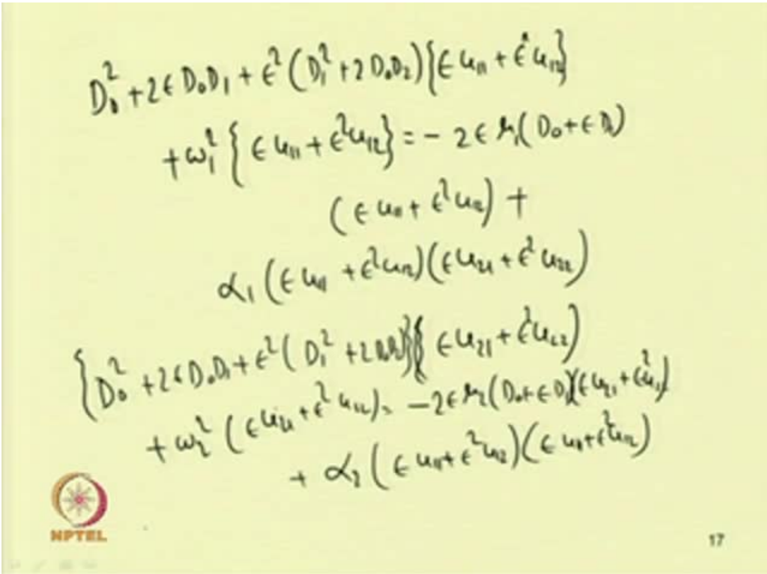

So del by del T n, so taking this equation this and this equation in equation original equation, so one can write D 0 square now expanding this thing, one can write this D 0 square plus 2 epsilon D 0 D 1 plus epsilon square D 1 square plus 2 D 0 D 2 into, so for u 1 we can substitute this epsilon u 1 1 plus epsilon square u 1 2 plus omega 1 square epsilon u 1 1 plus epsilon square u 1 2 equal to minus 2 epsilon mu 1 D 0 plus epsilon D 1 into epsilon u 1 1 plus epsilon square u 1 2 plus alpha 1 into epsilon u 1 1 plus epsilon square u 1 2 and into epsilon u 2 1 plus epsilon square u 2 2. So now, the second equation one can write D 0 square plus 2 epsilon D 0 D 1 plus epsilon square into D 1 square plus 2 D 0 D 2, so this operated on u epsilon u 2 1 plus epsilon square u 2 2 plus omega 2 square into epsilon u 2 1 plus epsilon square u 2 2, so this will be equal to minus 2 epsilon mu 2 into D 0 plus epsilon D 1 into epsilon u 2 1 plus epsilon square u 2 2 plus alpha 2 into epsilon u 1 1 plus epsilon square u 1 2 into epsilon u 1 1 plus epsilon square u 1 2.

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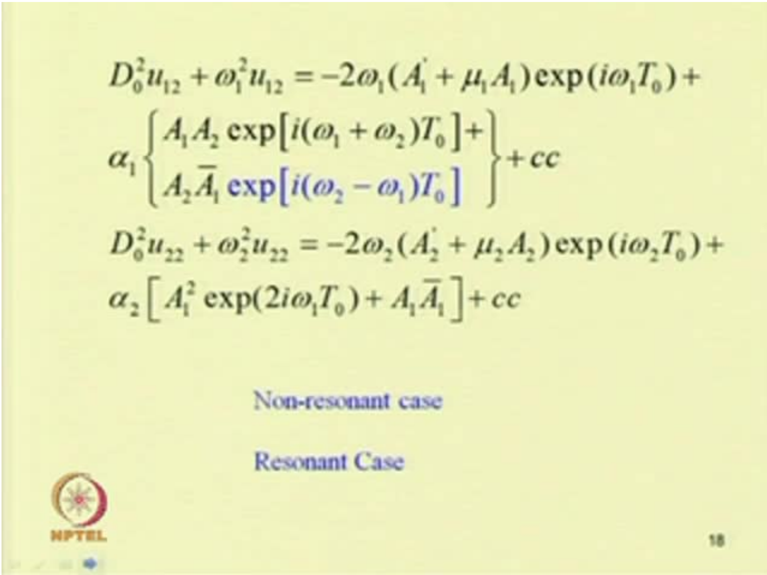

Now we can collect the term of the order of epsilon and epsilon square and we can write these equations, so by taking.

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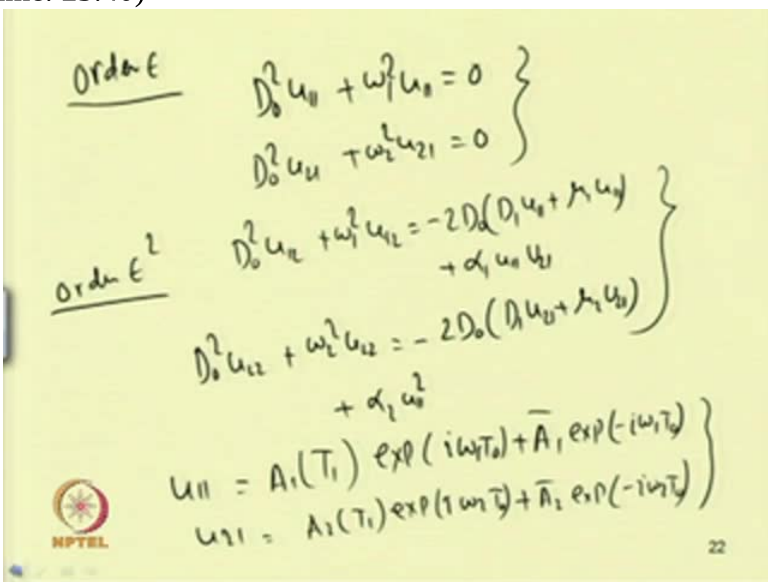

Let us take so order or epsilon, so you can write this order of epsilon, so the equation of the order epsilon equal to D 0 square u 1 1 plus omega 1 square u 1 1 equal to 0 and D 0 square u 2 1 plus omega 2 square u 2 1 equal to 0. Similarly, order of epsilon square, so this becomes D 0 square u 1 2 plus omega 1 square u 1 2 equal to minus 2 D 0 D 1 u 1 1 plus mu 1 u 1 1 plus alpha 1 u 1 1 u 2 1. Similarly, D 0 square u 2 2 plus omega 2 square u 2 2. So, it will be equal to minus 2 D 0 into D 1 u 2 1 plus mu 2 u 2 1 plus alpha 2 u 1 1 square, so in this order of epsilon, now one can write the solution of u 1 1 from this equation equal to A 1, so A 1 should not be a function of T 0, so A 1 should be a function T 1 T 2 or higher order terms, so u 1 1 equal to A 1 e to the power i omega 1 T 0 plus its complex conjugate that is A 1 bar e to the power minus i omega 1 T 0 and u 2 1 equal to A 2. A 2 also be a function of T 1 T 2, and should not be a function of T 0 e to the power i omega 2 T 0 plus A 2 bar e to the power minus i omega 2 T 0. Now substituting these two equation that is u 1 1 and u 2 1 these two equations, so one can get the equation in this form.

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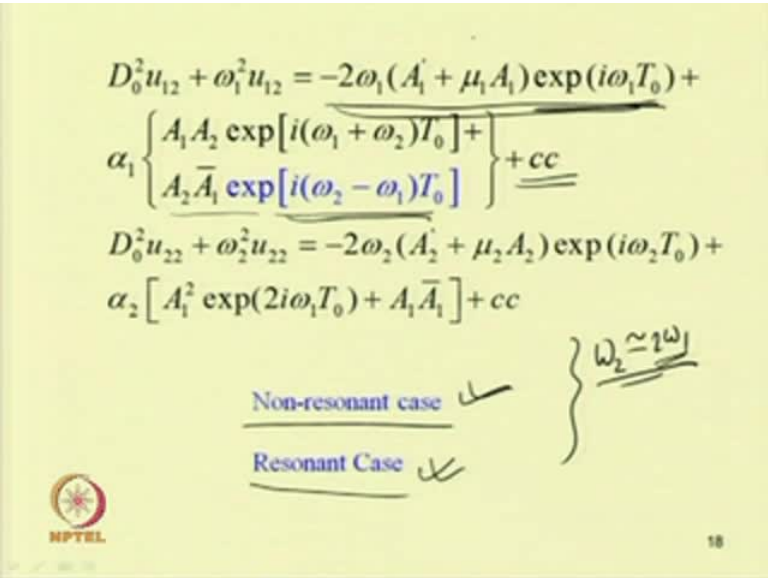

So D 0 square u 1 2 plus omega 1 square u 1 2 equal to minus 2 omega 1 A 1 dash plus mu 1 A 1 e to the power i omega 1 T 0 plus alpha 1 into A 1 A 2 e to the power i omega 1 plus omega 2 T 0 plus A 2 A 1 bar e to the power i omega 2 minus omega 1 T 0 plus its complex conjugate, and the second equation reduces to D 0 square u 2 2 plus omega 2 square u 2 2 equal to minus 2 omega 2 A 2 dash plus mu 2 A 2 e to the power i omega 2 T 0 plus alpha 2 into A 1 square e to the power 2 i omega 1 T 0 plus A 1 A 1 bar plus its complex conjugate, so here one may see in the right hand side, so we can consider two case one is non resonant case.

So as there is no external force acting on the systems, so if we take the frequency, so we can consider this non resonant case, so if we are not considering this omega 2 omega 2, so if omega 2 is nearly equal to 2 omega 1 then this term that is omega 2 minus omega 1, so this becomes nearly equal to omega 1, and this will be a secular term in addition to this term, so already this term is the secular term that is minus 2 omega 1 A 1 dash plus mu 1 A 1, and this term will also be a secular term, so if you are considering omega 2 nearly equal to twice omega 1, so if you are considering this is away from 2 omega 1 then the case will be a non resonant case; that means, there will be no resonance in the system, and if we consider this omega 2 nearly equal to 2 omega 1, so there will be internal energy flow from second system to the first system, and there will be resonant in the system.

So we can consider two cases in one case we will consider omega 2 not nearly equal to 2 omega 1 and in the second case will consider the resonant case in which we will take this omega 2 nearly equal to twice omega 1, so if we consider this non resonant case first, so then we can write the secular term, so this is the only secular term, so we can write the secular term minus 2 omega 1 A 1 dash plus mu 1 A 1 and as the solution is bounded, so we can put this term equal to 0 so; that means, this minus 2 omega 1 A 1 dash plus mu 1 A 1 equal to 0.

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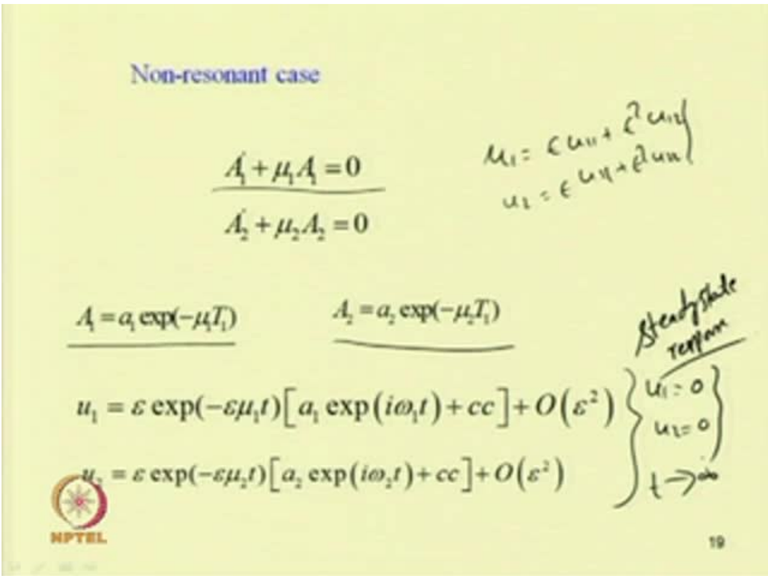

So we can substitute that thing and in the second equation also, this is the secular term in the second equation this term is the secular term and as this is the coefficient of i omega 2, so as here we have this omega 2 square, so this is the secular term, this will leads to infinite as t tends to infinity but, as the solution is bounded, so we should kill this term, so we have two terms to kill or two terms to eliminate, so first term is A 1 dash plus mu 1 A 1 equal to 0, and second term is A 2 dash plus mu 2 A 2 equal to 0. So now, the solution of this A 1 dash that means, d a 1 by d t 1 plus mu 1 A 1 equal to 0. Similarly, d A 2 by d t 2 plus mu 2 A 2 equal to 0, so the solution A 1 equal to, solution A 1 equal to A 1 e to the power minus mu 1 t 1, and similarly, A 2 equal to A 2 e to the power minus mu 2 t 1 and the solution u 1 now, so the for this non resonant case one can write this u 1 as we have written this u 1 equal to epsilon u 1 1 plus epsilon square u 1 2.

Similarly, u 2 equal to epsilon u 2 2 epsilon u 2 1 plus epsilon square u 2 2, so by substituting this only, we are taking the first order then we have the, we are not taking the second order that is order of epsilon square taking only, the first order you can write this u 1 equal to epsilon e to the power minus e to the power minus mu 1 t, so as epsilon t equal to t 1, so this minus mu 1 t 1 can be written as minus epsilon mu 1 t into a 1 e to the power i omega 1 t plus its complex conjugate. Similarly, this u 2 equal to epsilon e to the power minus epsilon mu 2 t into a 2 e to the power i omega 2 t plus its complex conjugate, so one can see that as t tends to infinity, this u 1 equal to 0 and u 2 equal to 0 as t tends to infinity that is steady state condition, so in case of non resonant case, so the steady state response becomes this u 1 and u 2 becomes 0 as t tends to 0 and let us see the non resonant case.

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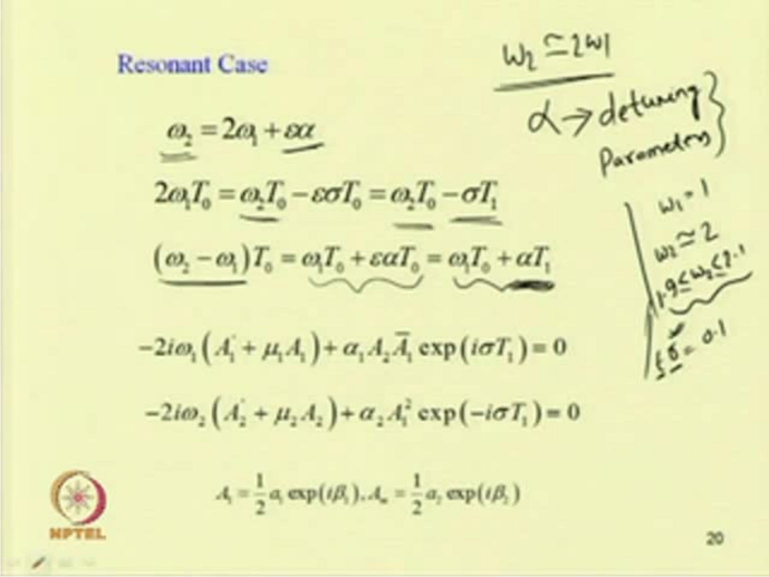

So, in resonant case, we are taking this omega 2 nearly equal to 2 omega 1, so if we are taking this omega 2 nearly equal to 2 omega 1, so we can use one detuning parameter to express the nearness of omega 2 to this twice omega 1, so two express that thing we are taking a detuning parameter alpha, so alpha is the detuning parameter, detuning parameter which shows the nearness of this twice omega 1 to omega 2, so it can take plus minus value for example, if we are taking this omega 1 equal to 1, so this omega 2 should be nearly equal to 2; that means, we can take this value this omega 2.

So, for example, we can vary this omega 2 from 1.9 to, so taking 10 percents taking, so you can vary this omega 2 to 2.1, so you can vary this omega 2, so from 1.9 less than equal to omega 2 less than equal to 2.1, so; that means, it can take this value, so if you are taking 1.9 then this epsilon sigma equal to 0.1 taking different value of epsilon, so you can find sigma so, this sigma is the detuning parameter. So, in this way you can find taking different value of omega 2, we can find sigma or for different value of sigma we can find the omega 2. Now, taking this detuning parameter, omega 2 equal to 2 omega 1 plus epsilon sigma.

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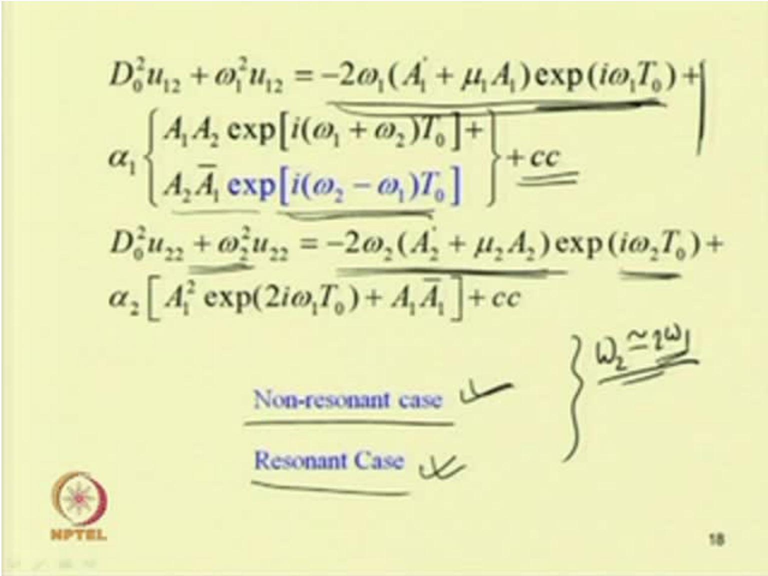

So we can write this 2 omega 1 T 0, so as this is present in our equation this equation omega 2 minus omega 1 t 0, so you can express this in terms of this i omega 1.

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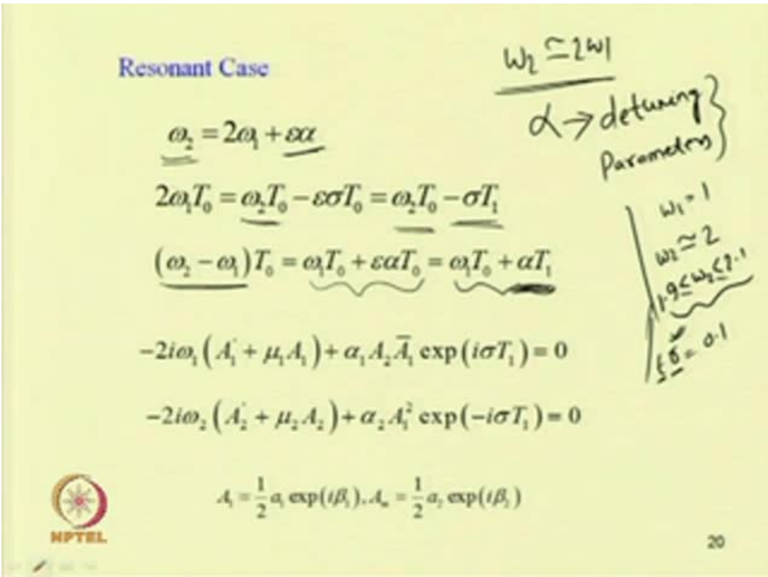

So that we can have a mixed secular or near secular term which we have to eliminate along with the secular terms to find the equation required reduced equation, so we can write this 2 omega 1 T 0 equal to omega 2 T 0, so this is equal to omega 2 T 0 by taking this thing to left hand side that is omega 2 minus epsilon alpha equal to 2 omega 1, so 2 omega 1 T 0 will be equal to omega 2 T 0 minus epsilon sigma T 0, so we can write this omega 2 T 0 minus this epsilon T 0 equal to T 1, so you can write this is equal to epsilon sigma T 1, so this 2 omega 1 T 0 equal to omega 2 T 0 minus sigma T 1.

So we can write this omega 2 minus omega 1 T 0 equal to omega 1 T 0 plus epsilon alpha T 0, so we can write this, so omega 2 T 0 minus sigma T 1, so minus omega 1 again we are putting, so we can write this is equal to omega 1 T 0 plus epsilon sigma T 0 or this is equal to omega 1 T 0 plus alpha T 1, so in this expression if you are substituting that thing that is 2 omega 2 minus omega 1 T 0 equal to omega 1 T 0 into alpha T 1, so e to the power of omega 2 minus omega 1 T 0 will be equal to expo e to the power i omega 1 T 0 plus e to the power i alpha T 1. So this term this is the additional term which shows the nearness of this 2 omega 1 to omega 2, so gives rise to this mixed secular term or nearly secular term now we have to eliminate this secular and mixed secular terms as these terms leads to infinite response while the actual system response is finite or bounded, so we have to eliminate these terms.

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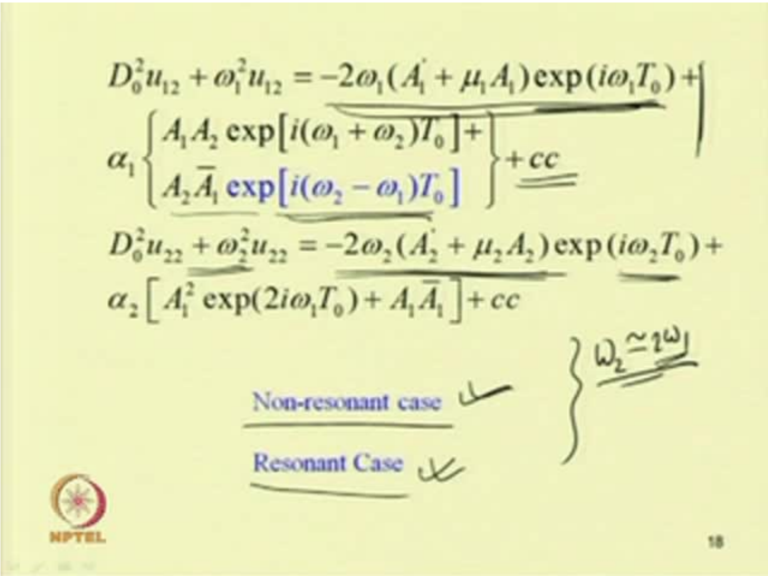

So in the first equation, so from this first equation, so we can write this minus 2 omega 1 A 1 dash plus mu 1 A 1, this should be equal to 0 and plus, so from this equation also we have this a 2 a 1 bar e to the power.

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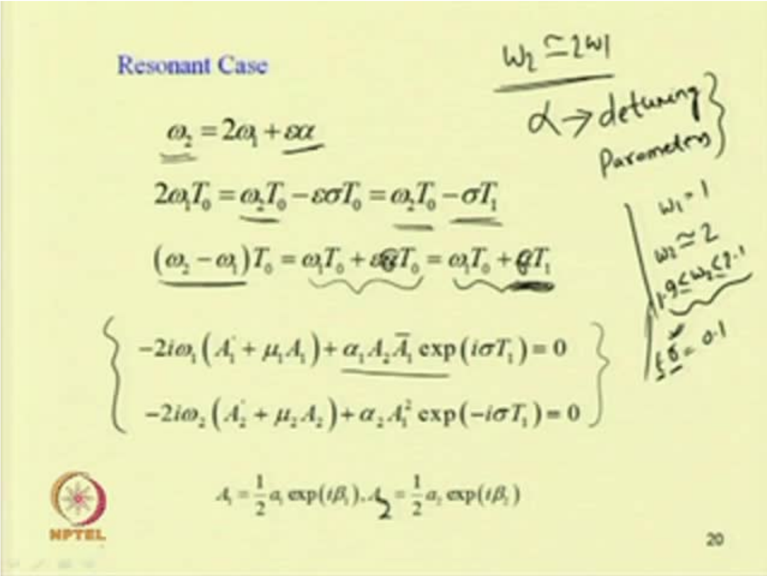

So we have this this additional term, alpha 1 A 2 A 1 bar this alpha so, for the first equation we can substitute, so this is alpha 1, so this is not alpha this is sigma T 1 this is sigma T 1, so this becomes minus 2 i omega 1 A 1 dash plus mu 1 A 1 plus alpha 1 A 2 A 1 bar e to the power i sigma T 1. Similarly, the second equation becomes minus 2 i omega 2 A 2 dash plus mu 2 A 2 plus alpha 2 into A 1 square e to the power minus i sigma T 1 equal to 0, so from these two equation now we can write, so from these two equation now writing A 1 and A 2 in polar form, so A 1 equal to half A 1 e to the power i beta 1. Similarly, A 2 equal to half A 2 e to the power i beta 2.

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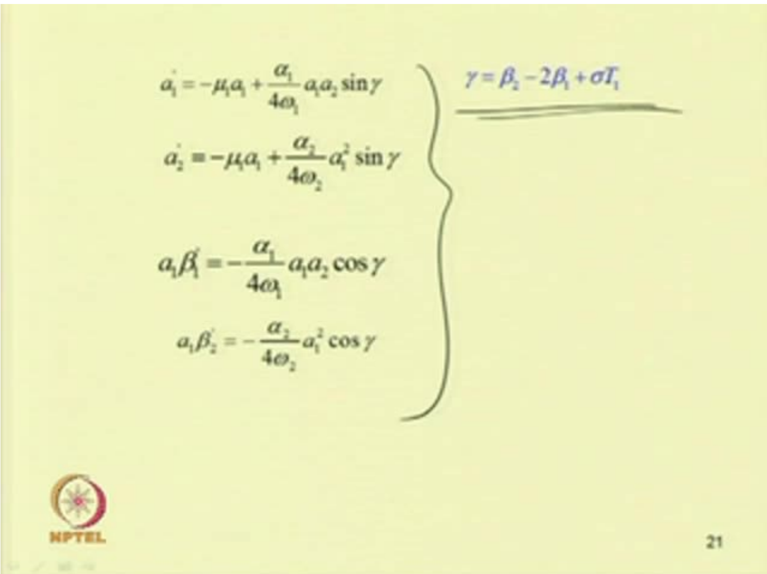

So substituting these two equation, so substituting this polar form in these two equations and then separating the real and imaginary parts, we are getting a set of reduced equations. So, in case of the single degree of freedom systems, we have got two equations and in this two degree of freedom systems, we are getting four equations, so two for the amplitude and two for the phase. So, we can write this a 1 dash equal to minus mu 1 a 1 plus alpha 1 by 4 omega 1 a 1 a 2 sin gamma, so where gamma is beta 2 minus 2 beta 1 plus sigma t 1 and similarly, a 2 dash can be written equal to minus mu 1 a 1 plus alpha 2 by 4 omega 2 a 1 square sin gamma.

So, now we to eliminate the secular term, so one can eliminate the secular term to get this expression, so by eliminating the secular term, so one can find the solution of the equation, so in this case one can observe that by eliminating this term.

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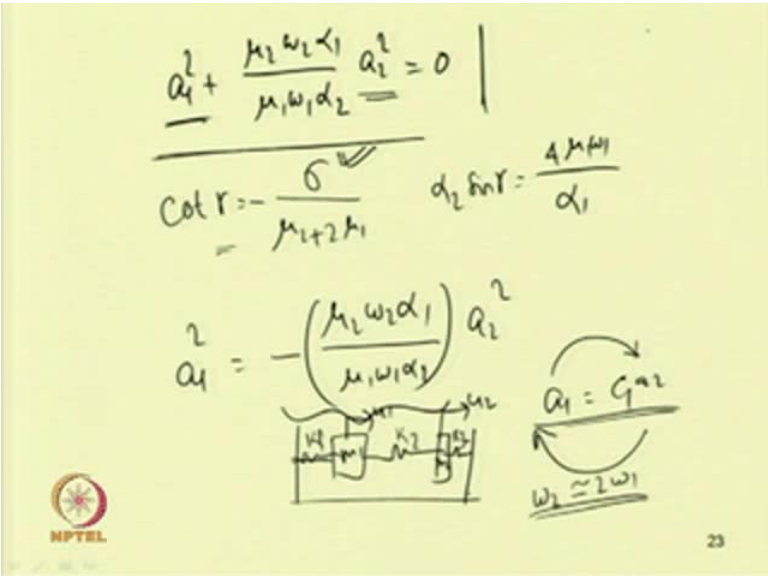

So, the equation will reduce to this form, that is one can write is a 1 square plus mu 2 omega 2 alpha 1 by mu 1 omega 1 alpha 2 a 2 square, so this is equal to 0 and one can write this cot gamma equal to minus sigma by mu 2 plus 2 mu 1, and this alpha 2 sin gamma equal to 4 mu 1 omega 1 by alpha 1. So, in this case one can see that this equation does not depends on the phase that is gamma, and here this a 1 and a 2 are related, so a 1 square plus mu 2 omega 2 alpha 1 by mu 1 omega alpha 2 a 2 square equal to 0 as mu 2 and omega 2, so they are positive numbers, so either this alpha 1 or alpha 2 to be negative, so that or of opposite sign this alpha 1 and alpha 2 should be of opposite sign, so that one can have this real value of a 1 and a 2 otherwise, so one can if they are of the same sign, so one can write this a 1 square equal to minus, so minus of mu 2 omega 2 alpha 1 by mu 1 omega 1 alpha 2 into a 2 square, so this term becomes negative, so one can get this imaginary value, so it will not yield any solution, so if alpha 1 and alpha 2 are of the same sign.

So, when this alpha 1 and alpha 2 are of opposite sign, so this term becomes positive and one can get this a 1 in terms of a 2. So, let, let us take this a 1 equal to c 1 a 2; that means, there is some energy transverse, so though there is no external force, so here the energy from a 1 or amplitude from a 1 is transferred to a 2 and vice versa. So, in this case one can first obtain this gamma term from this equation by solving the sigma by so for different value of sigma or different value of frequency, so one can get this value of gamma then by substituting this value of gamma in a 1 and a 2 equation, so one can get this a 1 and a 2, and this a 1 and a 2 are related by this expression, so in this case we have seen the though there is no external force due to this internal resonance that is when omega 2 nearly equal to omega 1, so this a 1 and a 2 are related; that means; so there is some energy transfer from the first body to the second body.

The equation what we have represented can be or can be shown by a physical system of this to mass system, so you have a two mass systems; this is mass 1 this is mass 2. So, let u 1 is the displacement of this mass and u 2 is the displacement of the second mass, then if we are taking, so this spring, so if this spring's are taken in such way that this spring and dampers are taken in such way that, this k 1 k 2 this is k 3 taken such way that, this omega 2 nearly equal to 2 omega 1, so then in that case we can obtain this, so if we start vibrating this first mass, so it will transfer energy to the second mass and in turn second mass will transfer energy to the first mass and there will be coupled motion. So, unlike the uncoupled motion we have seen in case of or we have predicted in case of the simple pendulum, so in this case the motion will be coupled as the first mass will try to move the second mass and in turn this second mass tries to move the first mass, so this way when one can study the system with quadratic nonlinearity in the presence of re internal resonance, so one can consider the secular and mixed secular terms and one can obtain the response of the system.

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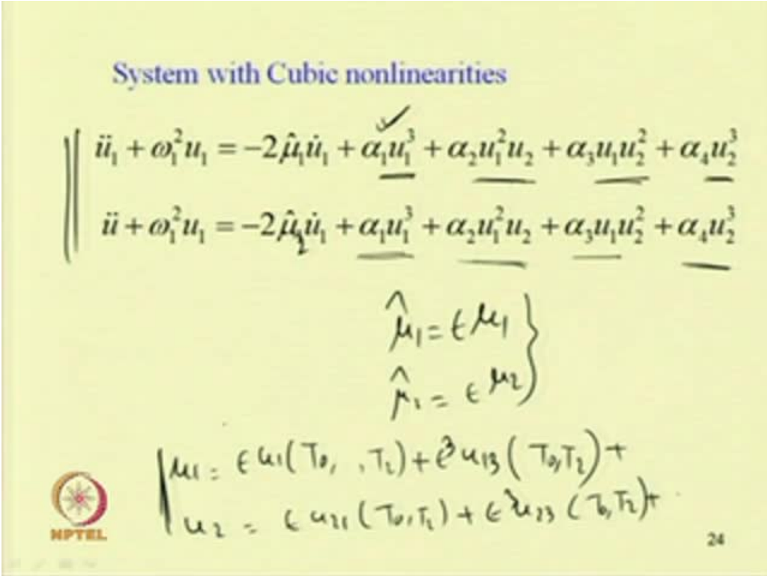

Similarly, one can study the systems with cubic nonlinearity, so here one can take, so let us take the system in this way that is u 1 double dot plus omega 1 square u 1 equal to minus 2 mu 1 u 1 dot plus alpha 1 u 1 cube alpha 2 u 1 square u 2 plus alpha 3 u 1 u 2 square plus alpha 4 u 2 cube, so here we have taken, so this is a cubic non-linear term this is also cubic, where we have taken u 1 square u 2 and this is u 1 u 1 u 2 square and this is u 2 cube, so due to this presence of this coupled and non-linear term, so these terms give rise to coupled terms and these are the non-linear terms, this u 1 cube is the non-linear term present here, so this is coupled non-linear to the coupled terms coupled and non-linear because this u 2 also affect this u behaviour of this u 1.

Similarly, in the second equation, we have this coupled equation that is alpha 1 u 1 cube alpha 1 square u u 1 square u 2, so we have all these coupled terms, so to make this non-linear and this damping term of the same order, so we can substitute this mu 1 equal to epsilon mu 1, and this mu 2 mu 1 bar and mu 2 cap equal to epsilon mu 2, and proceeding in the same way as we did in case of the quadratic nonlinearity, here also we can find the equations or we can let us take this equation in this form that is u 1 equal to we can take this u 1 equal to epsilon u 1, which is a function of T 0, T 1, T 2, T 0, T 1, T 2 plus epsilon, so in this case we can take as we are taking this cubic nonlinearity. We may not take the epsilon square term, directly let us take this epsilon u 1 plus epsilon cube u 1 3, which is a function of T 0,T 2, so this, so you can eliminate this T 1. Similarly, we can write this epsilon 2 equal to epsilon u 2 1 T 0, T 2 plus epsilon cube u

2 3 T 0, T 2, so by substituting this equation in this original equation, so we can eliminate or we can find the terms with epsilon to the power 0 epsilon to the power 1 epsilon to the power 3, so in the equation epsilon to the power 1.

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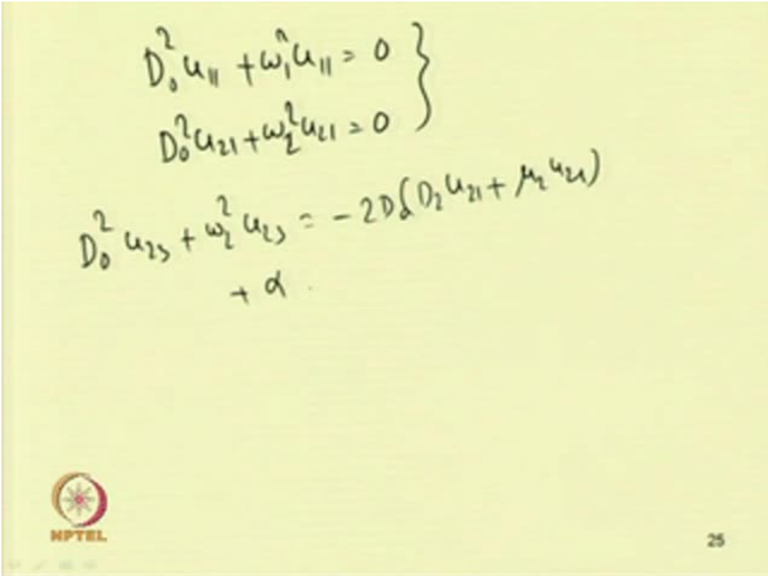

So, we can write this equation to be D 0 square u 1 1 plus omega 1 square u 1 1 equal to 0 and similarly, we can write other equation D 0 square u 2 1 plus omega 1 square u 2 1 equal to 0, so it will be 2 omega 2 square u 2 1 equal to 0. Now of the order of epsilon to the power 3, we can write D 0 D 0 square D 0 square u 2 3 plus omega 2 square u 2 3 equal to minus 2 D 0 into D 2 u 2 1 plus mu 2 u 2 1 plus alpha.

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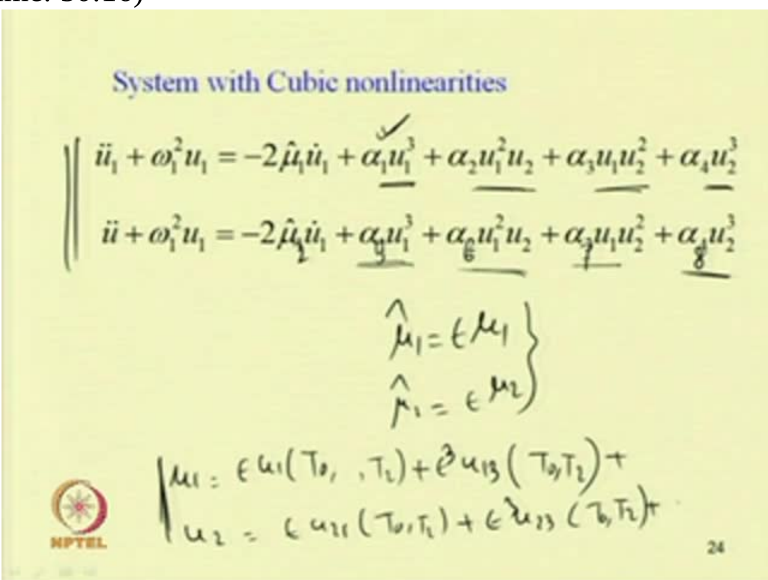

So, one can go on writing this term that is we have, so by substituting these terms we can write, so let us modify this equation slightly to, so u 1 double dot plus omega n square u 1 to epsilon u 1 dot alpha 1 u u 1 cube alpha 2 u 1 square u 2 alpha 3 alpha 3 u 1 u 2 square alpha 4 u 2 cube and the second equation let us write, so this is equal to this coefficient alpha 5, alpha 5 u 1 cube, this is alpha 6 u 1 square u 2 alpha 7, so this is alpha 7 u 1 u 2 square and alpha 8 u 2 cube. Now taking this u 1 equal to epsilon u 1 plus epsilon cube u 1 3 and u 2 equal to epsilon u 2 1 and epsilon cube u 2 3.

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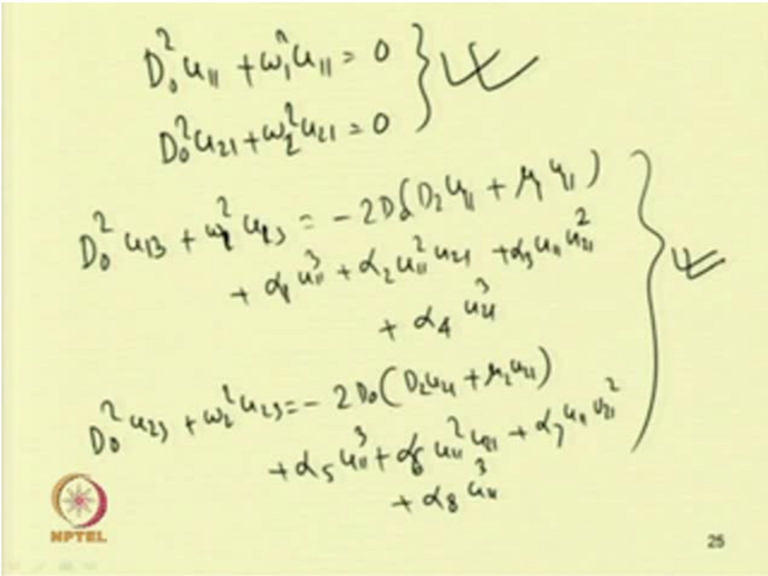

So, we can write this equation, so order of epsilon cube, we can write the equation in this form, so D 0 square u 2 3 plus omega 1 square, so this equation we can now order of epsilon cube, so you can write this is equal to D 0 square u 1 3 plus omega 1 square, first let us write first equation omega 1 square u 1 3 equal to minus 2 D 0 D 2 u 1 1, D 2 u 1 1 plus mu 1 u 1 1, mu 1 u 1 1 plus alpha 1 u 1 1 cube plus alpha 2 u 1 1 square u 2 1 plus alpha 3 u 1 1 u 2 1 square plus alpha 4 u 2 1 cube. Similarly, the second equation can be written in this form D 0 square u 2 3 plus omega 2 square u 2 3 equal to minus 2 D 0 into D 2 to u 1 plus mu 2 u 2 1 plus alpha 5 u 1 1 cube plus alpha 6 u 1 1 square u 2 1 plus alpha 7 u 1 1 u 2 1 square plus alpha 8 u 2 1 cube.

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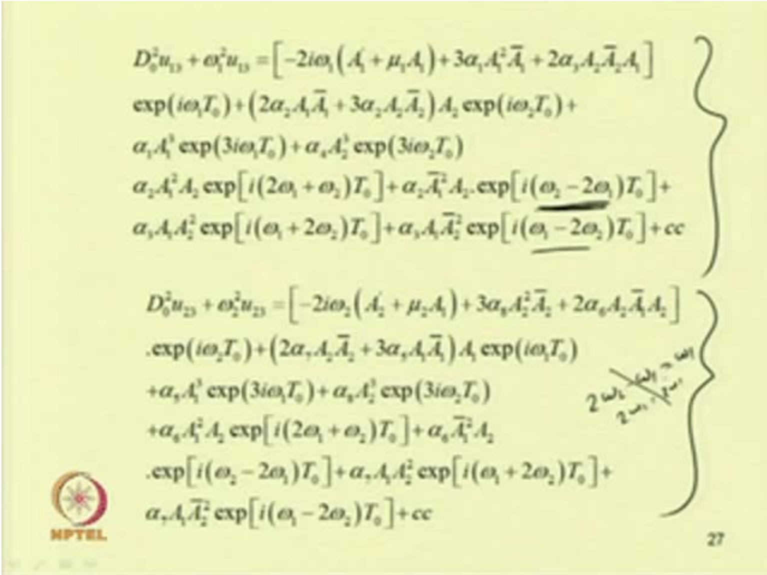

Now the solution of this equation can be written in this form, a 1 e to the power i omega 1 t 0 and u 2 1 equal to a 2 e to the power i omega 2 t 0 and substituting this equation in this equation one can get the equation in this form, so now to get the reduced equation one can take the resonant and non resonant conditions, so here one can see, one can have resonant condition, so when these omega 2, so this term may will leads to resonant condition that is if we have this omega 2 minus 2 omega 1 equal to omega 1, so we will have resonance; that means, if omega 2 is nearly equal to 3 omega 1, so we will have this resonance similarly, here omega minus 2 omega 2 same as this one omega 1 minus 2 omega 2 equal to 0, so here we can put this omega 1 minus 2 omega 2 or its complex conjugate term will gives us 2 omega 2 minus omega 1 equal to omega 1, so 2 omega 2 minus omega 1 equal to omega 1 or 2 omega 2 equal to 2 omega 1.

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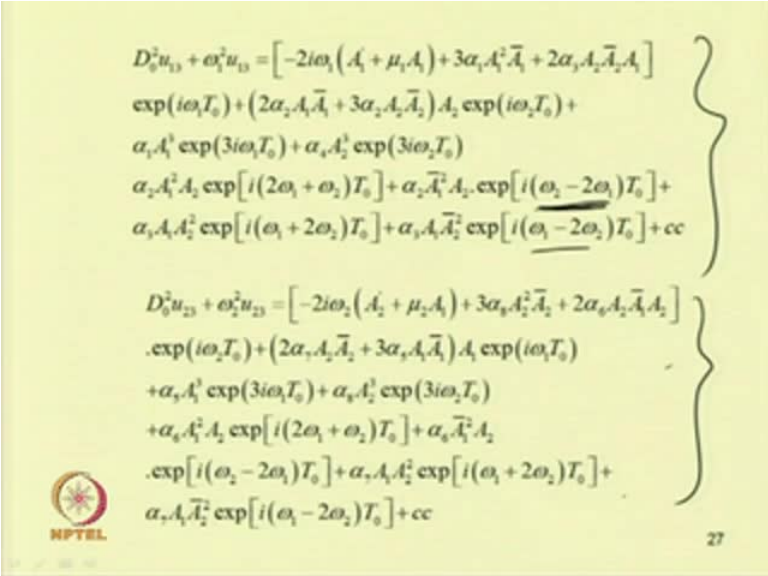

So, that means, when omega 2 equal to omega 1 also, we have resonant condition and when this omega 2 equal to 3 omega 1 also, we have resonance condition, so this is left as an exercise problem to find this response of the system for the resonant and non resonant condition, and one can see that in case of the non resonant condition the response rise down in steady state condition and in case of this resonant condition there will be energy transfer between different modes, so in the next class we will study about the force vibration of single degree of freedom system.

Thank you.