Non-Linear Vibration
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Module - 5
Numerical Techniques
Lecture - 3
Frequency Response Curves

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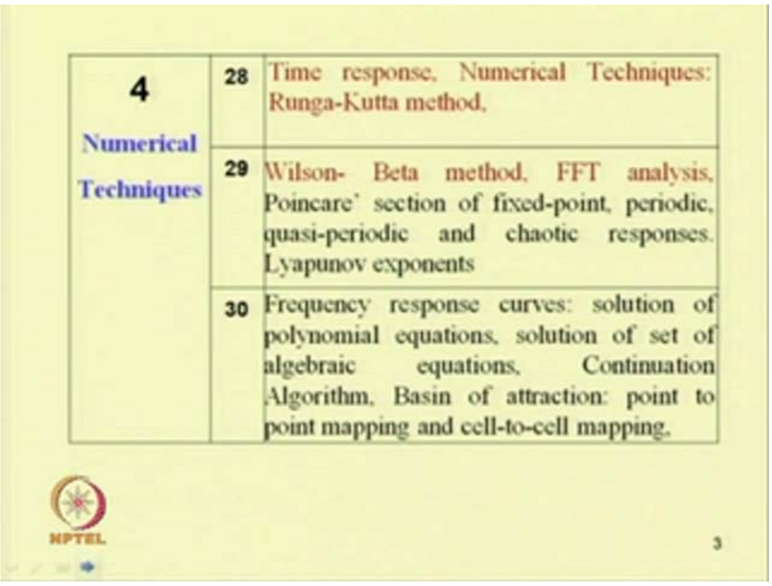

So welcome to today class of non-linear vibration. So, today class we are going to study about the frequency response curves, and then some continuation techniques, and Basin of attraction in the non-linear system.

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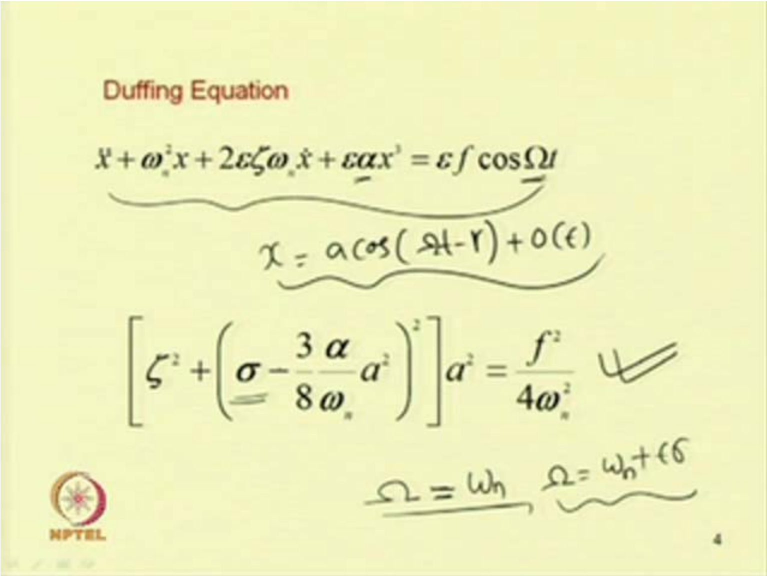

So, previous two class in this module, we have discussed about the time response of the system, so how to obtain the time response we have discussed in this last two lecture; and today lecture we are going to study, how to plot the frequency response of the system. So, in this module we are discussing about the numerical techniques, one can use to study the non-linear systems.

For example, let us take the Duffing equation; so in this case of Duffing equation one can write the equation motion in this form, so that is x double dot plus omega n square x plus 2 epsilon zeta omega n x dot plus epsilon alpha x cube equal to epsilon f cos omega t. Here x is the response amplitude, omega n is the natural frequency of systems, and zeta is the damping factor or a damping ratio, and this alpha is the coefficient of the cubic non-linear term, and epsilon is the book keeping parameter, and f is the forcing amplitude, and capital omega, so this omega is the forcing frequency of the system.

So in this case already we know, we can use different methods to find the find this solution of the system, so where the solution of the systems can be written in this form, that is x equal to a cos omega t minus gamma plus order of epsilon; so one can write the solution of this equation in this form by using different methods. So, different methods include this perturbation methods, by method of multiple scale, Lindstedt poincare method, method of averaging, or one can use this harmonic balance method also in this case.

So, the first order solution of this system can be given by this equation, x equal to a cos omega t minus gamma; so if one consider the primary resonance condition, so that is in case of primary resonance condition, that is when this external frequency is nearly equal to the frequency natural frequency of the system, so this is nearly equal to the natural frequency of the system, or one can write this omega equal to omega n plus epsilon sigma, where sigma is the detuning parameter.

So, one can write this external frequency omega equal to omega n plus epsilon sigma, where sigma is the detuning parameter, and by using method of multiple scale, so one can obtain, one can obtain a frequency amplitude relation in this form. So where the frequency amplitude relation can be given by, so this zeta square plus sigma minus 3 by 8 alpha by omega n a square whole square into a square equal to f square by 4 omega n square, here in this equation one can find the coefficient, one can find that a is of the order of 6; a square in to 2 a square square then a to the power fourth, and multiply by outside a square, so in this equation it is written in the form of a to the power six, the highest order amplitude is six, and but one can see that this detuning parameter is of the quadratic order, that is sigma square.

So this equation one can write, using either sigma square, or one can use the variable, so taking one as the unknown variable either sigma or a, one can write this equation in a quadratic form using sigma, or using a sixth order equation in terms of a. The same equation can be written either in quadratic form in terms of sigma, or sixth order equation in terms of a.

So, now one has to solve this equation to find the response of the system, or one can plot this frequency response, that is amplitude versus sigma, that is detuning parameter or amplitude versus omega, as omega equal to omega n plus epsilon sigma. So this plot which in, which we can plot a versus sigma, or a versus omega is the frequency response curve of the system. So now so in this, in this particular case, so it is easier to solve this quadratic equation in terms of sigma.

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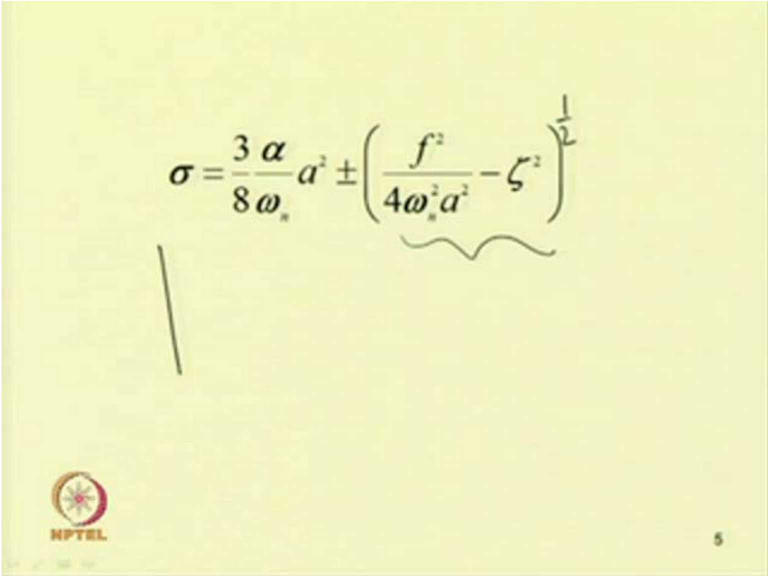

And if you can write in terms of sigma, one can write this equation in this form, so it will be so to the power 1 by 2, so it will be sigma will be equal to 3 by 8 alpha by omega n a square plus minus f square by 4 omega n square a square minus zeta square to the power 1 by 2. So in this case, first one has to find this term so when this term equal to is positive, then one can find this two roots, and if it is negative, then this will yield a imaginary term; so one has to first check for what value of this forcing parameter one gets the real parameter, now one can plot this response in terms of a and sigma.

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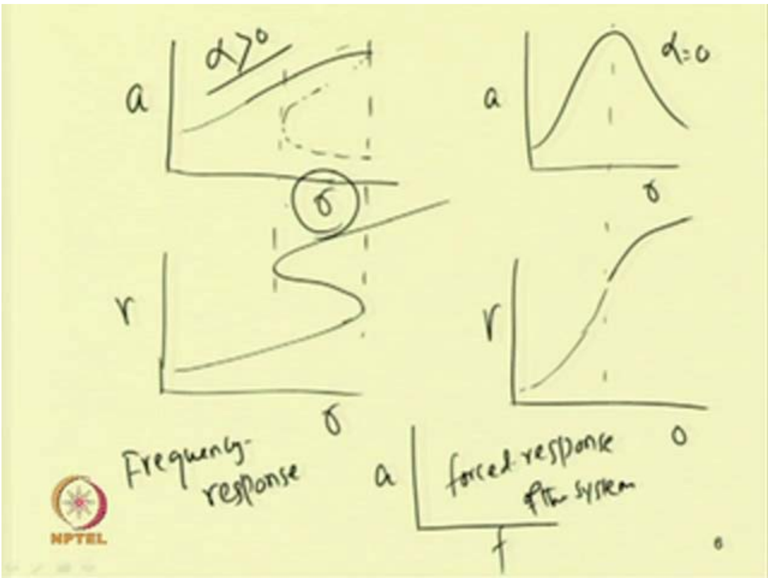

So, if one plot let us plot, so for positive value of sigma or positive value of alpha, so if one plot a versus sigma, the curve will be so one can one can plot this curve, and one can see that this curve will be similar to this, so this is a that is the amplitude, one can also plot the face of the system, that is gamma. So, gamma versus sigma also one can write so here, one can see so if it is, so one can plot this gamma versus alpha also. So, in this way one can plot this a versus gamma, a versus sigma, or gamma versus sigma, so this for alpha greater than 0.

So, one can plot also for alpha less than 0 or alpha equal to 0, alpha equal 0 is the linear system, and already we know that, for alpha equal to 0 the system response will be like this, a versus sigma, if one plot then the response will be like this. For in case of a versus for alpha equal to 0 similarly, one can plot the gamma versus sigma also, so in this case gamma versus sigma, so one can get, so the response will be like this. So gamma is the face, and a is the amplitude of the system; so this curve a versus sigma is the frequency response curve of the system.

Similarly, one can plot the force response curve by plotting this a versus sigma, or a versus so if one plot, this a amplitude versus this forcing parameter, then it will be forcing response of the system or force response of the system; and similarly, so this is the frequency response so here we are plotting frequency frequency part, this sigma is the frequency part, this is the detuning parameter, so here for alpha greater than 0, so one can get this curve.

And we can later discuss that in this case, one can have this jump off jump down phenomena, and in this class we are particularly interested to know, how we are going to find this response when we have or what are numerical techniques, we are going to use to solve this type of equation, or this type of algebraic equation or if you have a number of equations, then what is the methods we are going to solve.

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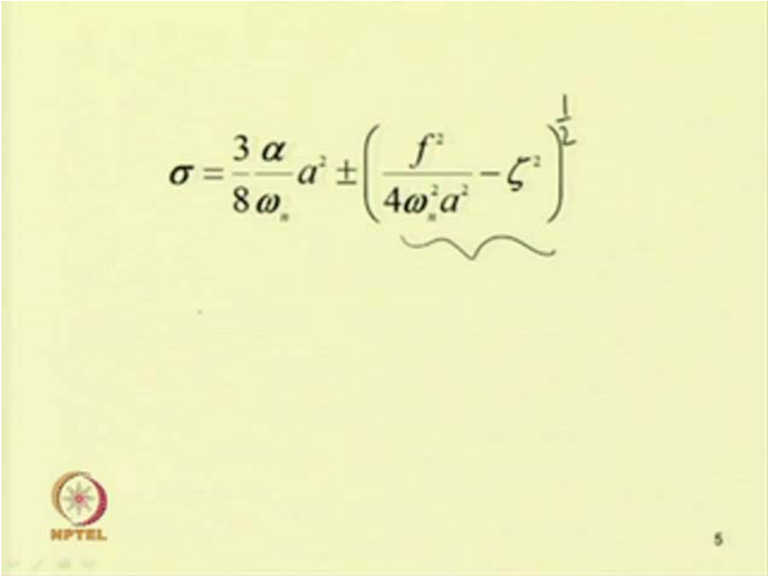

So if one interested to solve this sixth order equation instead of this quadratic equation, one can solve, also the equation with sixth order in a, and in that case one can use different methods, different numerical techniques to solve this equation; so the numerical techniques what we have already discussed are method of so already we have discussed several method; method of bisection, method of false position Newton's method, Muller method, and second methods, so one can use these methods to find the solution of this equation.

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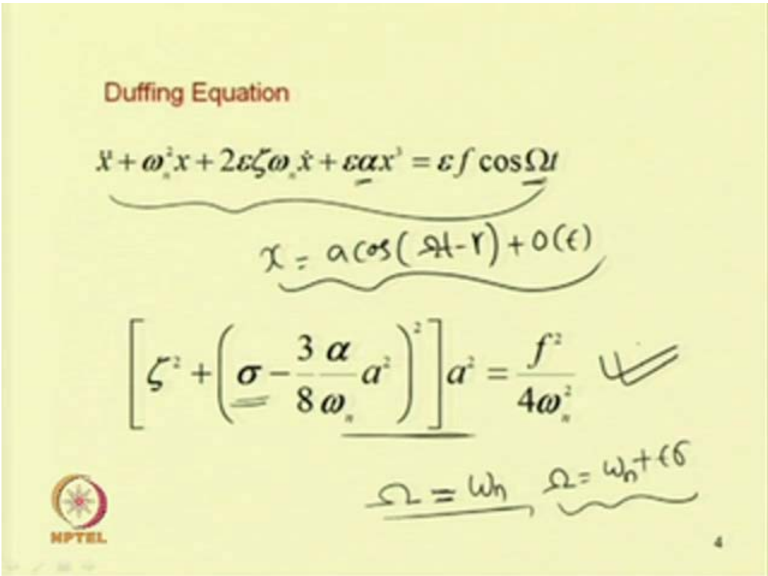

But, in this case we have seen only we have a single equation, and which we have solved to find the response of the system, but sometimes we may have more number of equations, and that time we can we have solve this equation, or we have to solve a system of equations by using Newton's method.

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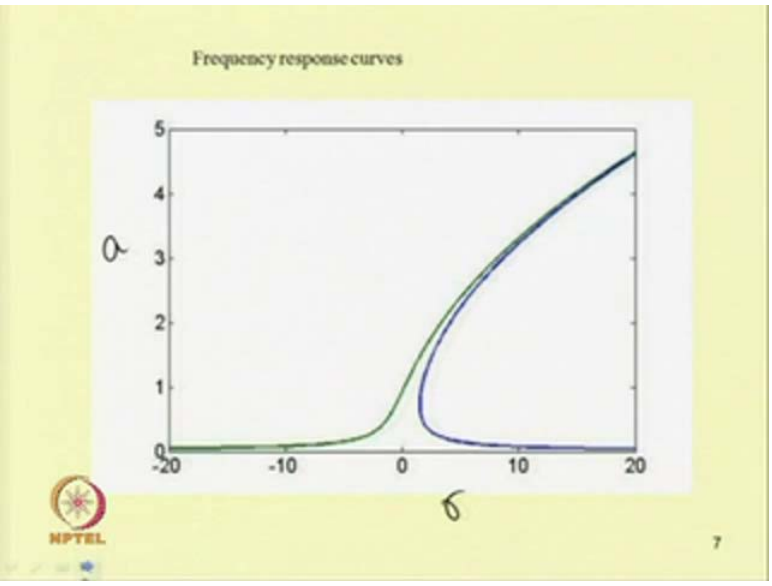

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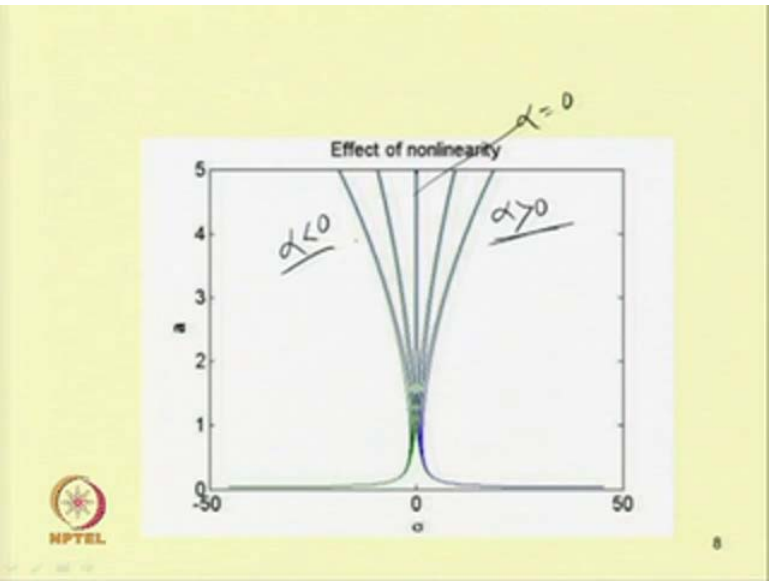

So, this is the frequency response curve. For this frequency response curve for this Duffing equation, or similar equation, equation similar to Duffing type, and so here the effect of nonlinearity is shown; so this is for alpha equal to 0, so these are for alpha less than 0, and this is for alpha greater than 0; so for alpha greater than 0, so which shows hardening type of spring the response tilts towards right, and for alpha less than 0, when this system is that of a soft spring, then the response tilt towards left. And for alpha equal to 0, this is the linear curve, one can get similar due to the presence of damping, so this response amplitude will decrease, those things in details will be covered in module six.

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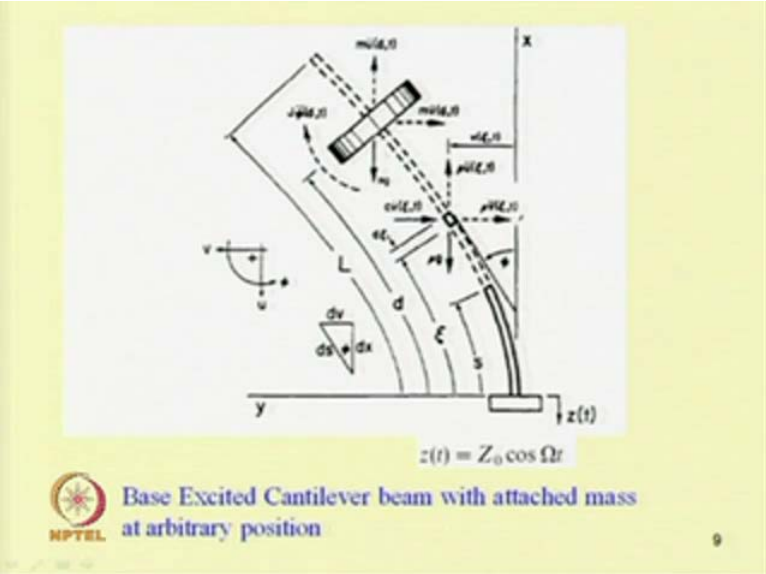

So let us take another example, so in this example, this is a base excited cantilever beam with attached mass at arbitrary position, so in this case for different position of the system, one can observe this internal resonance.

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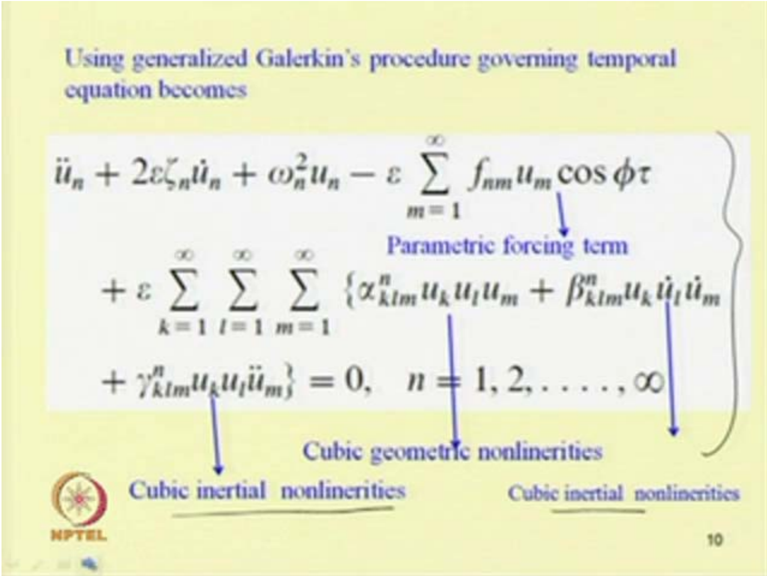

And instead of a single governing, so instead, so the governing equation of motion can be written in this form. So, this is the covering temporal equation of motion, so in this case this u n is the time modulation, zeta n is the damping parameter, omega n is the natural frequency, and f n m is the forcing term; so here for so here, these two are the cubic geometric nonlinearity, and this is the cubic inertial nonlinearity, so we have two inertial non-linear term u dot and u dot; here u dot square into u k this is the cubic non-linear inertial non-linear term. Also this is u double dot this is another cubic inertial non-linear term, so we will study about this system in detail in module six.

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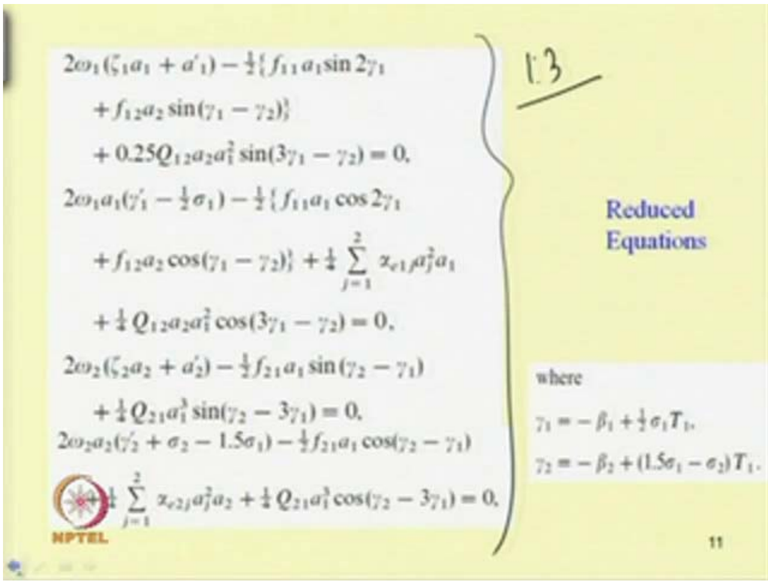

But in the present case the objective is to show, that in this case, if we are considering the internal resonance, so one can get a set of equations or one can by using this method of multiple scale. So, one can get a set of equation here considering 1 is to 3 internal resonance case, so here one can obtain four set of equations, so these are the four equations; unlike in case of this dufffing equation where one will get two reduced equations only.

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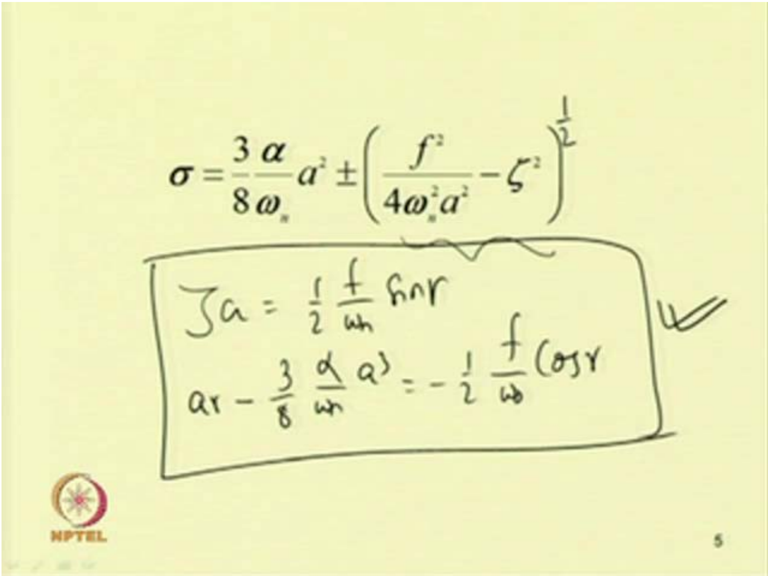

So, the reduced equation in case of Duffing equation can be written in this form, zeta a equal to 1 by 2 by omega n sin gamma, and this a gamma minus 3 by 8 alpha by omega n a cube equal to minus 1 by 2 f by omega 0 cos gamma; so this equations are written for the steady state, so one has two equations here, one can solve this two equation in case of the Duffing equation.

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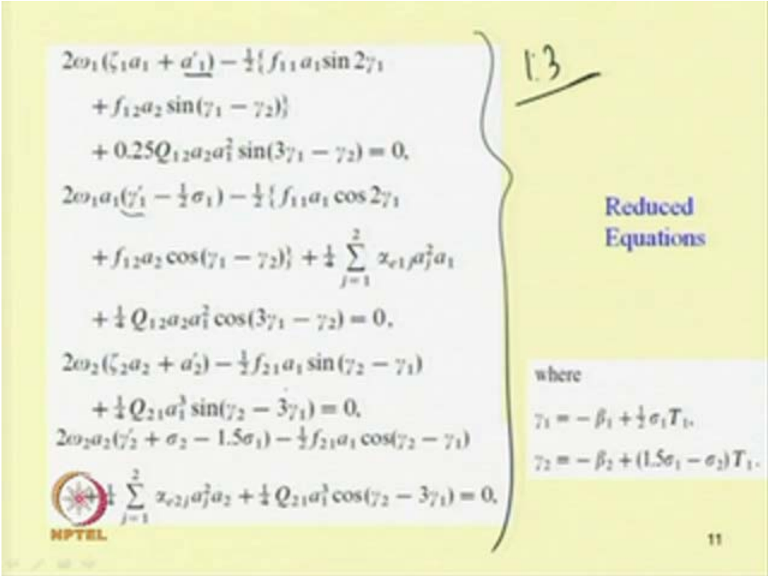

But, in the present case, so in this case, one has a set of four equations, so where this a 1 gamma 1 are for the first mod, and a 2 gamma 2 are for the second mod. So considering the two mod interaction, so where the first mod frequency is taken non dimensional, first mod frequency is taken to be 1, the second mod frequency is nearly 3 times the first mod; so here, 1 is to 3 internal resonance condition is considered, so in that case, so one can get a set of four equations.

So for steady state condition, so we can put this a 1 dash, gamma 1 dash, and a 2 dash and gamma 2 dash equal to 0, and by putting these terms equal to 0, so one we obtain four algebraic or transcendental equation, so solving those equations, one can obtain the response of the system.

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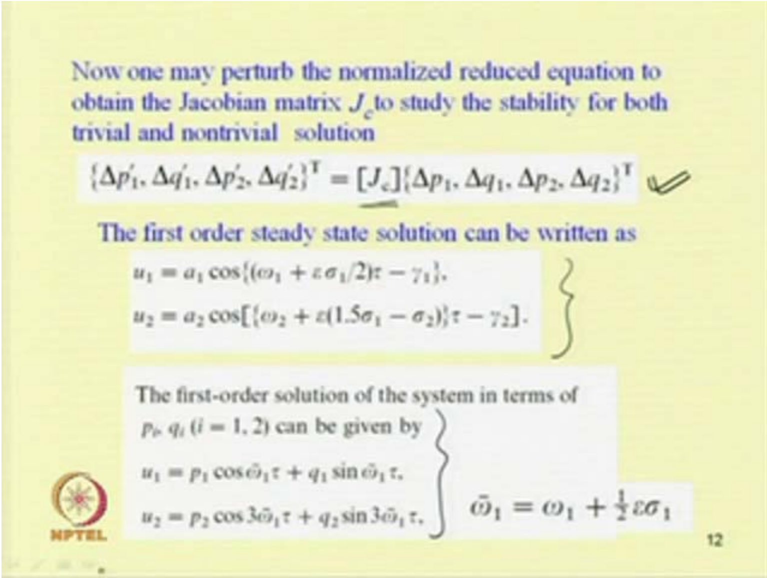

And while solving this equation, it is required also to find the stability of the system, so to plot this different branches, one should know, so which branch is stable, and which branch is unstable. So, to study the stability of the system, so in this particular case so one can make the transformation, as we have this coupled term a 1 gamma 1 dash, and a 2 gamma 2 dash, and for trivial state this perturbation will due to the presence of this coupled term, the perturbation will not be there, and to circumvent that thing so one can use some transformation in terms of p and q.

So by using that transformation, one can write the perturbed equation in this form, so here p 1 is taken equal to a 1 cos gamma 1, and p 2 equal to a 2 cos gamma 2, and q 1 equal to a 1 sin gamma 1, and q 2 equal to a 2 sin gamma 2; so this four reduced equation can by can be reduce to or can be written in terms of another four normalized reduced equation, and using those normalized equation, and perturbing those equations, one can write this equation where this is the Jacobian matrix, and by finding the Eigen value of the Jacobian matrix, one can find the or study the stability of the system.

If the real part of the Eigen values of those Jacobian matrix is negative, then the system is stable, otherwise the system is unstable. So in this case, so we are getting this u 1, so we have four equations. The objective of this lecture is to show that what are numerical techniques we will use in this case to solve these four equations.

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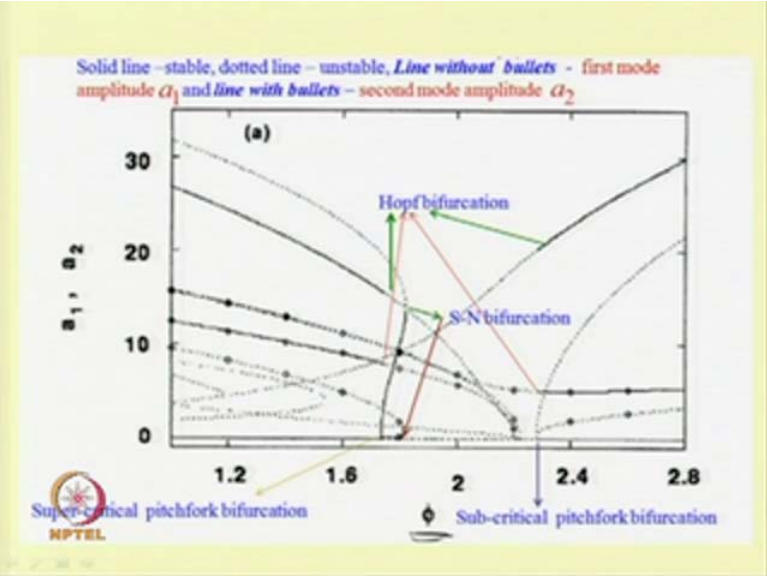

So here one can see, that one can use this Newton's method to solve this equations, so one can see this is the frequency response curve. So in this frequency response curve both a 1 and a 2 are protatech with respective to the forcing parameter, that is phi forcing frequency, that is phi, and when phi is nearly equal to 2, so we have this principle parametric resonance case, and this curve shows the response curve, when it is near to the near to twice the natural frequency of the system. So, the lines with bubbles are for second mod, and the lines without bubbles are the first mod, so one can observe many branches of the solution. So here, the question arise how to find these branches of the solution, or how in case of Duffing equation that frequency response curve what we have seen, so we can observe that for certain parameter.

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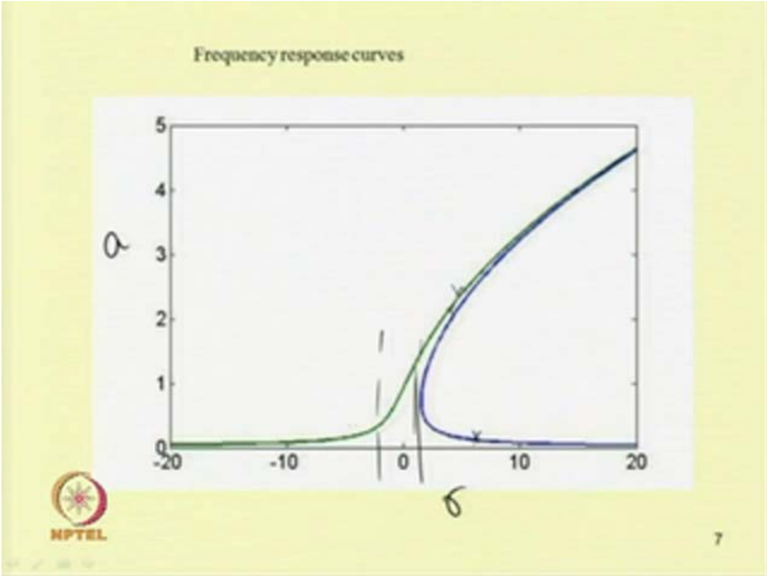

So we have only one solution, and for some other parameters so up to this, up to this, we have only single solution, and after that we have one, two, and three solutions. So after this value so we have three solutions, but or three branches. So here we have one, two, and three branch, and before that we have one branch of solution, so the question now arise, how to obtain this multi branch solution in this case of non-linear systems.

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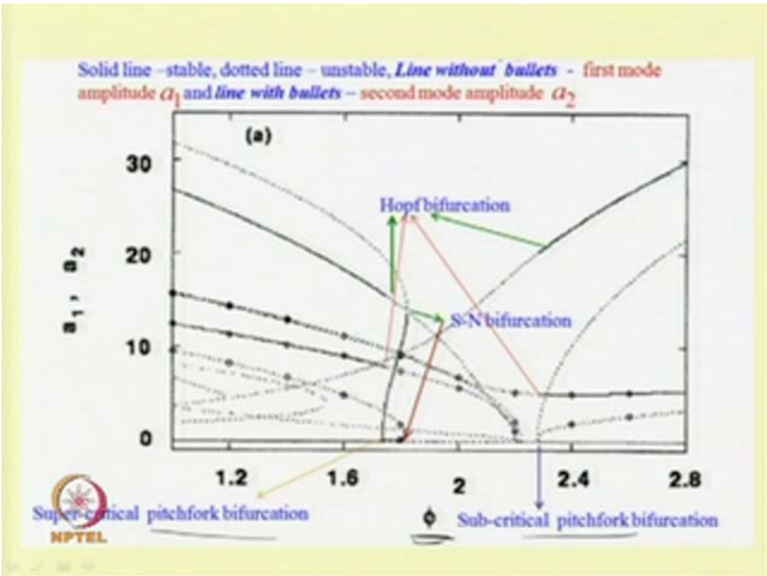

So, here one can use some continuation techniques, and continuation techniques to find the multi branch solution. So about this continuation techniques, we will discuss to in today’s class, so while studying this branching, also one has to take care of different bifurcation points, so these for example, in this case we have this super critical pitch fork bifurcation; so these points are super critical pitch fork bifurcation, soft critical pitch fork bifurcation here, the saddle node bifurcation, and this Hopf bifurcation, so many different bifurcations occur at these points. And also we know at these bifurcation points, this field vector becomes 0. So one has to take special care at this bifurcation point to know, how the branch will continue at that point.

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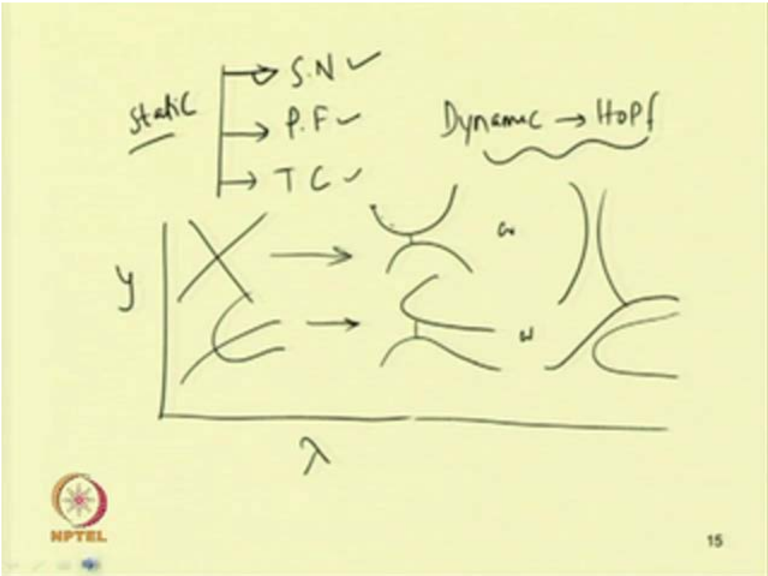

It may be noted that for one parameter, one parameter controls space problem, one parameter controls space problem, so we can have two different types of bifurcation of the fixed points, so one is the static bifurcation. So in case of the static bifurcation, so in case of static bifurcation, already we have discussed in module four.

There are saddle node bifurcation, so we can have this pitch fork bifurcation, also we have this trans critical bifurcation; so this is saddle node, pitch fork, and trans critical bifurcation. And in case of the dynamic bifurcation, so we have Hopf bifurcation, so in this static and dynamic bifurcation of the fixed point response so we know that this saddle node bifurcation and Hopf bifurcation only are the generic bifurcations, which will not change in by changing the control space.

But, the other type of bifurcation for example, so let us take this other bifurcation, so y versus lambda is the bifurcation parameter, so if we have a trans critical type of bifurcation, so it may happen that the branching, so while we will change this parameter lambda slightly this may the curve instead of becoming this trans critical, so it may become like this or it may be like this.

Similarly, in case of the pitch fork bifurcation, so the curve may becomes like so one branch so this is one branch so this becomes one branch, and the other branch bifurcate like this, or it may be a single branch. So this upper one, and then one can have this. So, there is some gap between so by changing, by changing slightly the parameter, by changing slightly the critical parameter lambda, so one can see there will be some gap generated between different branch; so except the saddle node bifurcation or Hopf bifurcation other in other bifurcation, so we can see by changing the control parameter, one can obtain this gap or this will deform this bifurcation points will deform like this.

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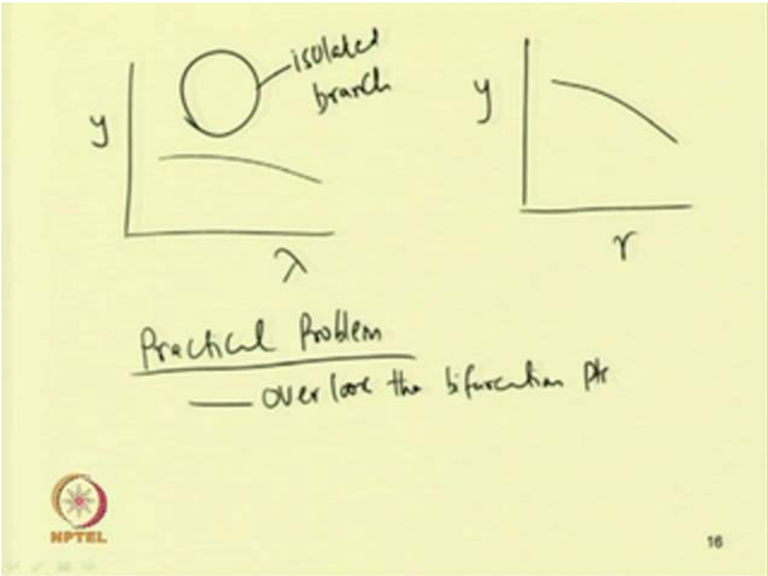

So also we may have some sometimes, we may have in case of two parameter system two parameter control parameter. So, in two parameter one can we may observe that sometimes, we may have some isolated branch; so this is an isolated branch, so this is displacement versus or parameter versus this control parameter.

So this is isolated branch, so we may have some isolated branch, so by changing that parameter so let us take some other parameter, so by changing another parameter, the curve may change. So, one has to take care or one has to note, which control parameter has to be taken for plotting the frequency response type of curve. So the practical problem one can observe, by if I am plotting this type of curves r so one may overlook the practical problems, one can so this may be, one may overlook the bifurcation point, so this bifurcation point may be overlooked also, sometimes this turning point let us take this example.

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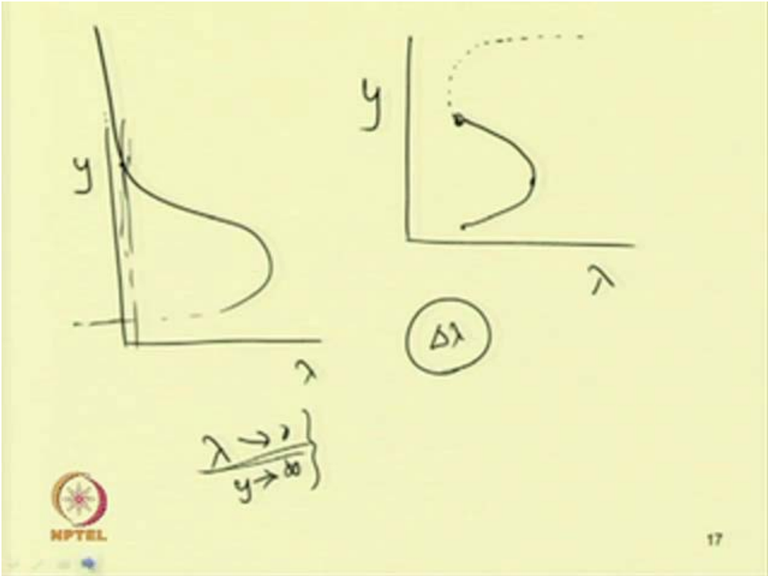

So, we may have some turning point, or some sometimes this turning point may not be the solver, may not find this turning point, so this point, and this point, may not find, or it may overlook these points. Similarly, so we may have this so what is the step size of this control parameter will take, so it may sometimes be to large or to small. So this control so we have to divide this control space into different segments in terms of this delta lambda, sometimes this delta lambda may be to small or to large, so that one has one may overlook certain points in this space.

So we have to discuss, or we should take care while, so starting from a particular point, how one has to continue this branch with change in the control parameter, sometimes it may happen, that so let us take this situation. So, in which so we have a branch, so this branch so for lambda tends to 0, so for lambda tends to 0, so one can see, this y tends to so here, y tends to infinity.

But in actual case the response will be bounded, so one has to truncated the solution, or one should take, or one should start, or one should start the solution, or one should avoid taking this value of this lambda. For which it tends to infinity, so one has to avoid this range or one has to truncated the response up to this; so in that case one will get a finite value of the solution, so one has to so we have to discuss how to continue the solution, and how to switch the solution when this bifurcation points are there.

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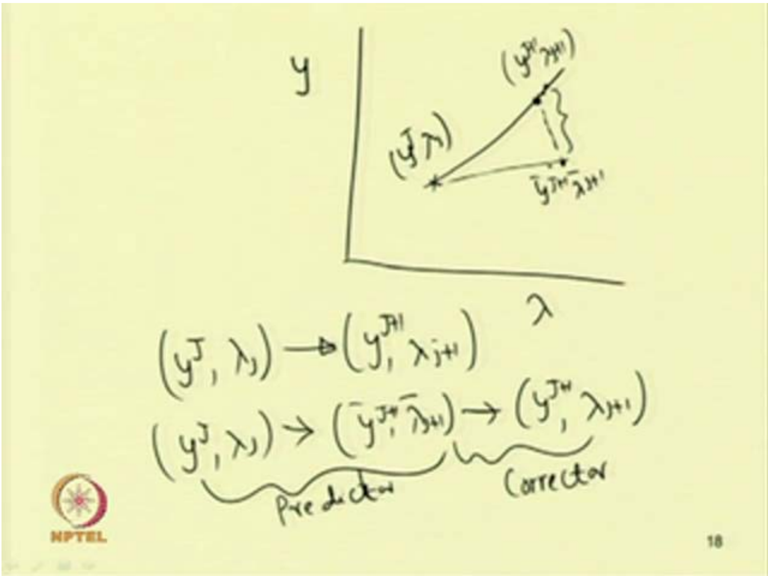

So in this case, let us take this response, this is y versus lambda, this parameter so let us see, so this is the solution at jth iteration; let us take that this is the solution y j and lambda j, for the control parameter lambda j let the solution is y j. And actually we want to find what is the solution, that is y j plus 1 j plus 1 at lambda j plus 1 so this is the solution actual solution, we are interested to find, so to find the solution we may have to use or this continuation technique. But we are going to use, should obtain this point in two state, that is one state is the predictor step, so first we have to predicated the solution, and then let us take, so this is the predictor solution.

So, y bar j plus 1 lambda bar j plus 1, so this is the predictor point, and after this predictor point, we have to correct the solution by this; so this is the corrector part, and this is the predictor part. So in this continuation from y j lambda j to so we have some the intermediate point, that is y j plus 1 lambda j plus 1 lambda is the control parameter. So this involves two parameter two process, so in the first process so we can go from y j lambda j to y j plus 1 lambda j plus 1, and then in the next step we will go, from y j plus 1 to lambda j plus 1; so this step from y j lambda j to y bar lambda bar, so is known as the predictor step, and this part is known as the corrector step.

So one can use this predictor and corrector method to find the solution, in case of this fix point response while, finding this fix point response solution, with change in control parameter, we may have a single point control parameter like your detuning parameter in case of Duffing equation, or we may have multiple parameters.

So for example, we may use this forcing parameter f or this detuning parameter sigma, in case of the Duffing equation also, in that case it will be two parameter control space, so we may have single parameter control space, we may have two parameter control space or multi parameter control space.

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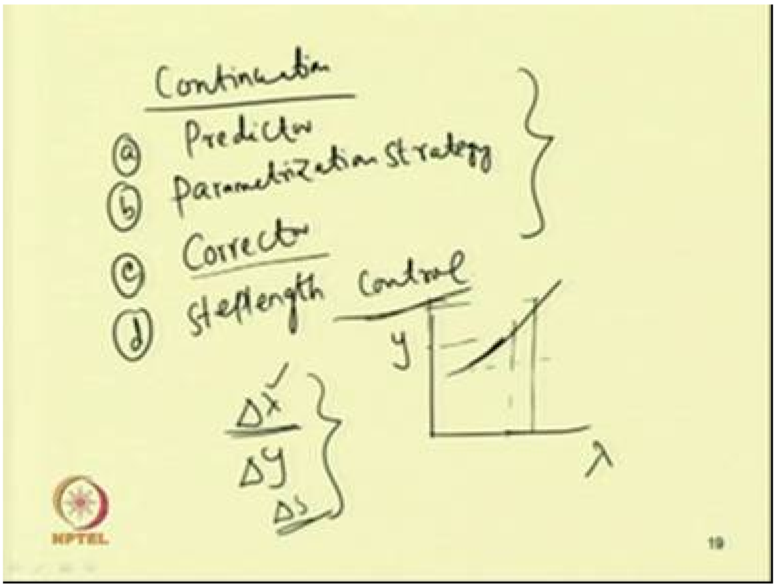

So our continuation method, so will contain a, so the continuation procedure what we are going to use, so it will have a predictor space, then which would have a parameterization strategy, how will parameterize that thing, so which would have this parameterization strategy, then we should have this corrector. And finally, we should find the step length, so step length control so which would have this step length control. So this first three we can choose independently, and this last one this step length control depends on this other three parameter. So, in this predictor method, so one can use this ordinary differential equation solver to find the predictor point, and then one can use this parameterization strategy and corrector to solve this thing.

So in case of the corrector, so either we can control so for example, let us have this, so we can control this, we have three different ways we can correct it, so either so this y versus this lambda, so either we can correct by so we have this three parameter, so correction can be done in three ways, so either we can branching by branching the parameter lambda.

So, we can change this step length lambda delta lambda by taking delta lambda, so this is delta lambda, so this corrector, it can take delta lambda, second local parameterization we can do, in case of local parameterization we can take this parameter delta y, and or we can take, or we can correct along this curve, along this curve, that is known as arc length parameterization, so delta s. So we can have three different corrector scheme, so in one case we can control this step length delta lambda, second local parameterization we can do by taking this delta y, and third we can do this arc length parameterization by taking this delta s.

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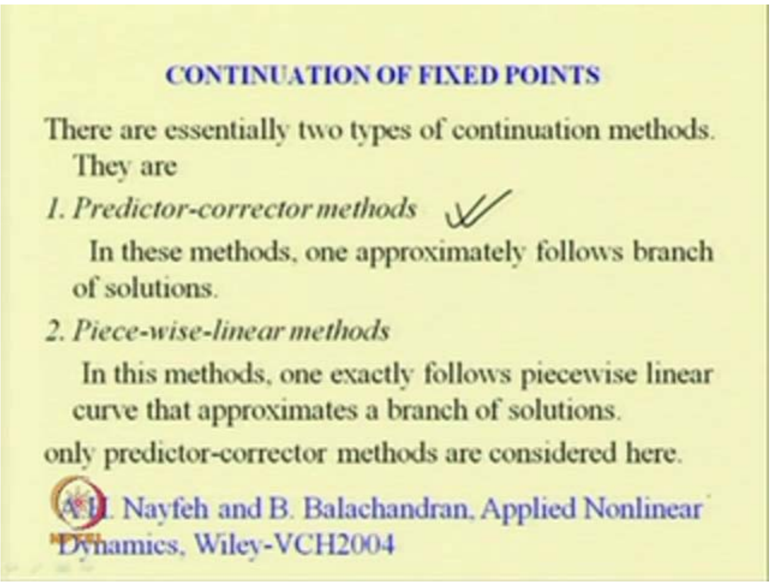

Now, we will study these different fix. For fix point response, we will see what are the different continuation schemes? So let us see, so two basically, two schemes are there, one is the predictor corrector method which we are going to discuss now, and one may see the other method also, that is piece wise linear method; so the methods what I am going to discuss are taken, or adopted from this book by A H Nayfeh and B Balachandran Applied non-linear dynamics.

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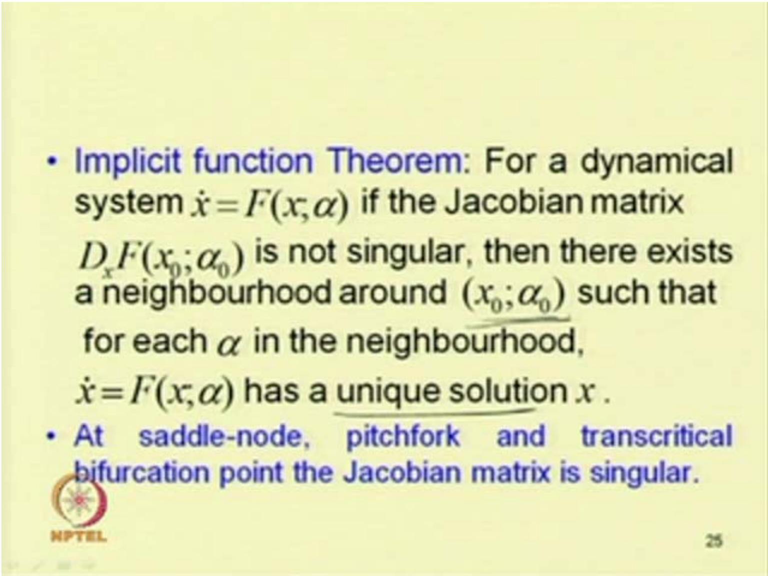

So in this case, so for the fixed point response, first we should know this implicit function theorem which is used for making this continuation. So for a dynamical system x dot equal to F of x alpha, if the Jacobian matrix D x F is not singular, then there exist a neighborhood around x 0 alpha 0, such that for each alpha in the neighborhood, so we can have a unique solution, so that means if in the neighborhood of a initial point x 0 alpha 0.

This Jacobian matrix is not singular, then we have a unique solution; so as at the saddle node, pitchfork and transcritical point, Jacobian matrix is singular. So, we have to use different method for the continuation of the branching at those points, so one can use this implicit theorem for continuing the solution, or along a branch of the system.

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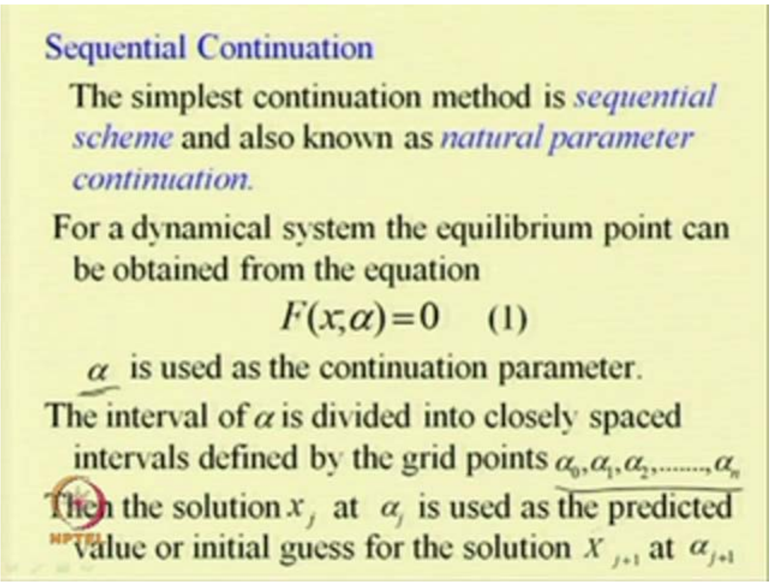

So in case of this, so we have the sequential continuation, the simplest continuation method is the sequential scheme, also it is known as this natural parameter continuation. So in for the equation x dot equal to F x alpha, so for steady state as x dot equal to 0, so we can solve this equation, x f alpha equal to 0, so were alpha is the continuation parameter. So in this case what we have do, so we have to this alpha's range, we have to divide by alpha 0, alpha 1, alpha 2, alpha n, and we have to find this x j correspond to alpha j, so first we have to predict, and then taking some initial guess we have to predict that thing and we have to find this solution in the next step.

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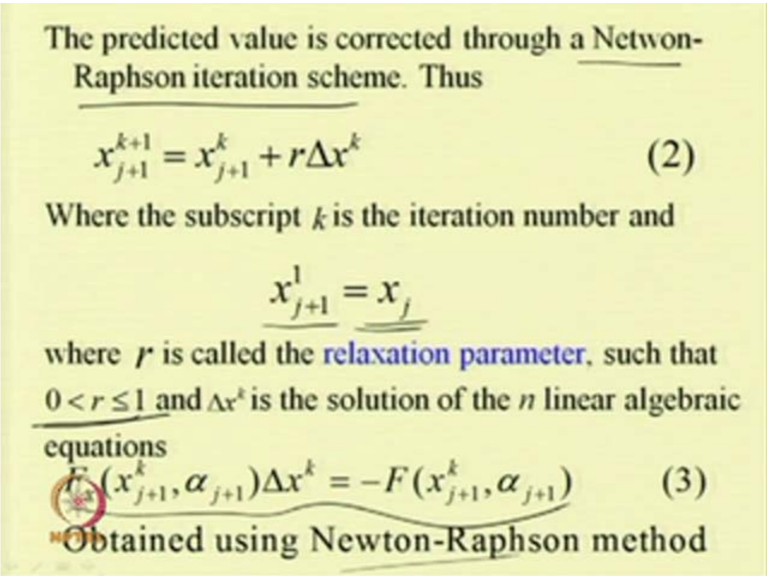

So, it can be done by using this Newton Raphson's iteration scheme, so in this Newton Raphson iteration scheme, so one can take this x j plus 1 k plus 1 equal to x j k plus r delta x k, so where the subscript k is the iteration number, and x j so at one so the first step, so this is the, so this is the, initial guess value, one can take, so here r is the relaxation parameter, such that 0 less than r equal to less than equal to 1, so this is the relaxation parameter one can take, and delta x k is the solution of the n linear algebraic equation. So, this is the equation obtained from the Newton Raphson method, so here F x that is the first derivate of F the Jacobian matrix into delta x k th iteration will be equal to minus f x j plus 1 k alpha j plus 1, so this is, this is obtained from the Newton Raphson method so by solving this equation, so one can find the solution of the system.

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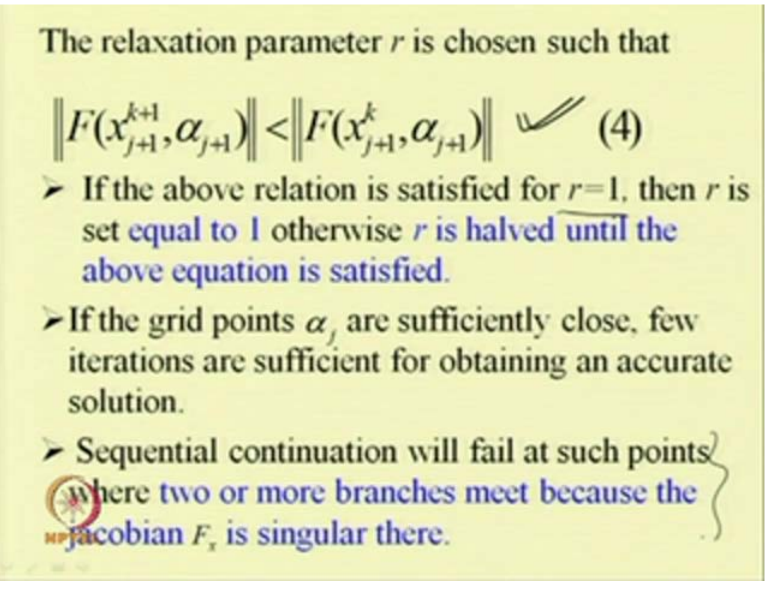

The relaxation parameter r is chosen such that, this it has to satisfy this equation, so if the above relation is satisfied, then for r equal to 1, then r is said to set equal to 1, otherwise r is halved until the above equation is satisfied, if the grid point alpha j are sufficiently close, new iterations few iterations are sufficient for obtaining the accurate solution sequential continuation will fail, at as the differentiation or the derivate is equal to 0 at the at the singular point, so it will fail at the singular point, such as the saddle node bifurcation point or pitchfork bifurcation points or transcritical bifurcation point.

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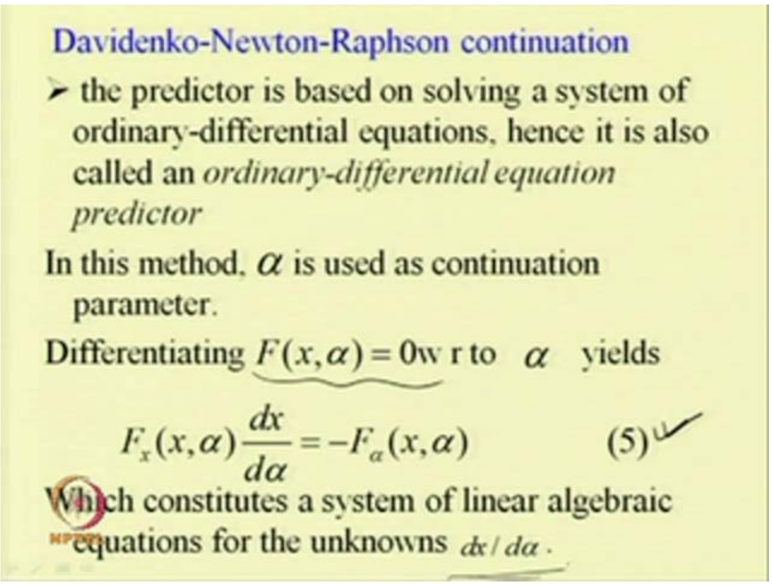

So, in this way one can use this sequential continuation technique to find so or otherwise one can use this Davidenko Newton Raphson continuation; so here the predictor is based on solving a system of ordinary differential equations, hence it is also called ordinary differential equation predictor, so in this method alpha is used as the continuation parameter. So here, differentiating this equation that is your failed equation F x alpha equal to 0 with respect to this alpha, the continuation parameter so we can write this F x x alpha d x by d alpha equal to minus F alpha x alpha, so we obtain this equation. So which constitute as we have n dimensional x x is n dimensional, so we can have n or will have a set of liner algebraic equation with unknown d x by d alpha, so where x equal to x 1, x 2, x n. So we can with these unknown we can have a set of liner algebraic equations.

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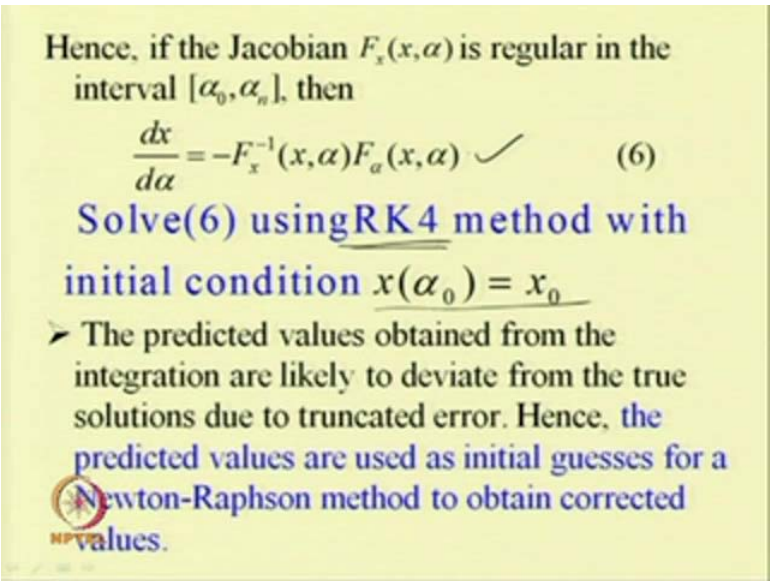

Now, this one can find this Jacobian matrix, f x alpha in this interval alpha 0 to alpha n; so one can write in this interval, this d x by d alpha equal to minus F x or one by F x into F alpha x alpha, so this is from the Newton's method, then by solving so this differential equation can be solved by using this ordinary differential equation.

For example, one can use this Runge Kutta fourth and Runge Kutta method, by taking this initial guess x alpha 0 equal to x 0, by taking this so one can now predict this value obtained from this integration, and this likely to do deviate from the true solution due to this truncated error. Hence, the predicated value are used as the initial guess for a Newton Raphson method to obtain the corrected value, so in this way so in the previous case, so that is simple continuation technique, so we have solved this differential we have solved the field equation.

But in case of this Newton Davidenko Newton's continuation technique so we have differentiated, we have differentiated the field equation with respective to this continuation parameter, and we got a set of system of liner algebraic equation, and also we have then solved this equation in term by using this ordinary differential equation to find the initial guess, and then by solving those equations we obtained the corrected value, so in this way one can get the next iteration points.

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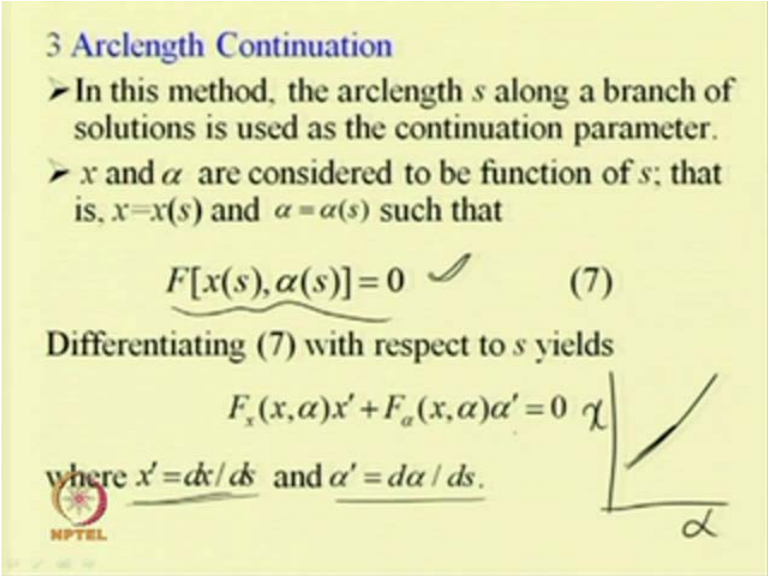

Also one can use this arc length continuation technique already, I told that we have three different methods: so in first method, we have taken this delta lambda continuation, in the second case we have parameterized, so one can parameterize that thing by taking this control parameter, and in the third so we can use this arc length continuation. so in this case so along the curve, so we have to take a arc length, and we to correct it, predict it , and correct it.

So in this case, for this arc length continuation scheme, so the arc length is along the branch, so this is along the branch, so arc length along the branch of the solution is used as a continuation parameter. x and alpha, so this is x, x is the response x and alpha, let me take this continuation parameter alpha, x and alpha are considered to be function of s, so if they are function of s, then we can write this x equal to x s s and alpha equal to alpha s, so that our field equation can be written in this form F x s alpha s equal to 0. Now differentiating this equation with respective s, we can write this F x x alpha x dash plus F alpha x alpha alpha dash equal to 0, where F x is the where x dash is d x by d s and alpha dash equal to d alpha by d s.

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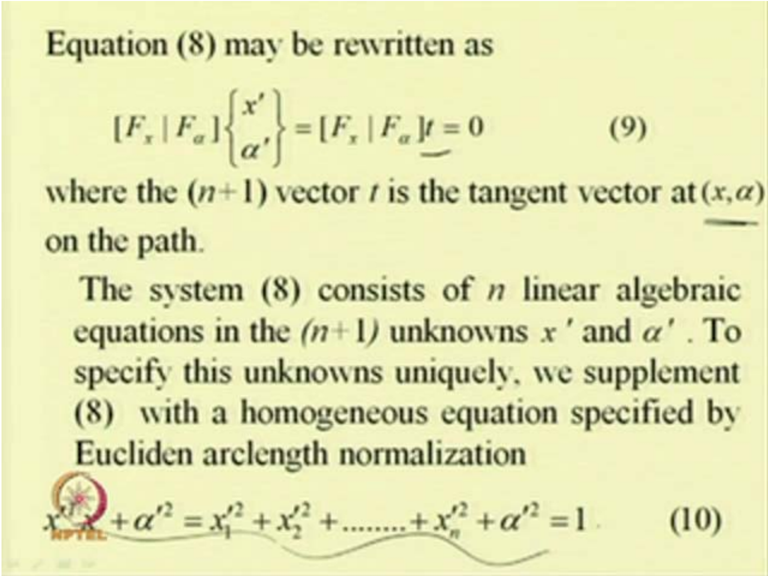

Now this equation, now from this equation, so we can write this F x F alpha x dash alpha dash equal to F x F alpha t equal to 0, where t so where n plus 0 vector, t is the tangent vector, at x alpha on the path, so one can find this tangent vector, so the system, so this equation the system of equation 8 consist of n liner algebraic equation, in the n plus 1 unknowns x dash and alpha dash. To specify this unknown uniquely, we supplement this equation by a homogenous equation specified by this Euclid Eculiden arc length normalization. So, one can do this normalization by using this equation that is x dash transpose into x dash plus alpha dash square equal to, so by taking n term in x, so one can write this is x 1 square x 1 dash square plus x 2 dash square plus x n dash square plus alpha dash square equal to 1.

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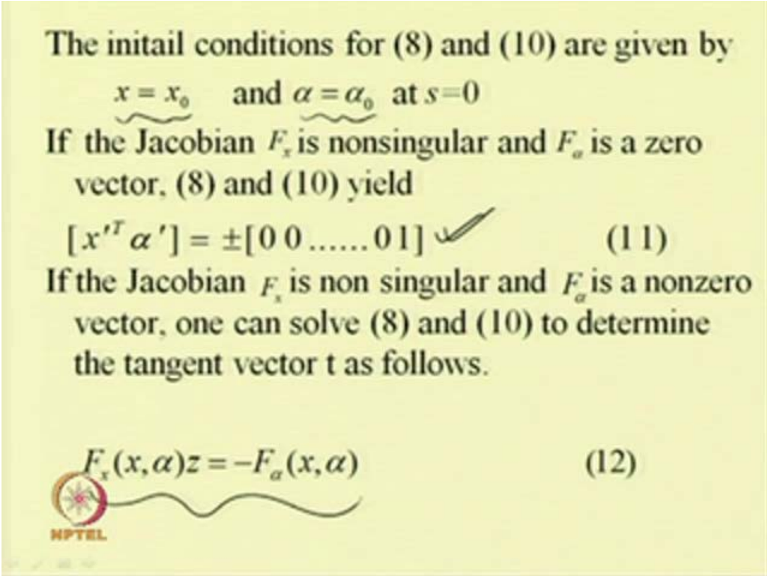

So, then the initial condition for 8 and 10 are taken, by so initial condition x,, we can take x 0 and alpha equal alpha 0 at s equal to 0, so if the Jacobian matrix F x is nonsingular and F alpha is a 0 vector, then we can have this equation, that is x dash transpose alpha dash equal to plus minus 0 0 0 1, and if the Jacobian F x is nonsingular and F alpha is nonzero vector, one can solve equation 8 and 10 to determine the tangent vector t, so after determining so one can determine the tangent vector t from this equation, that is F x x alpha z equal to minus F alpha x alpha.

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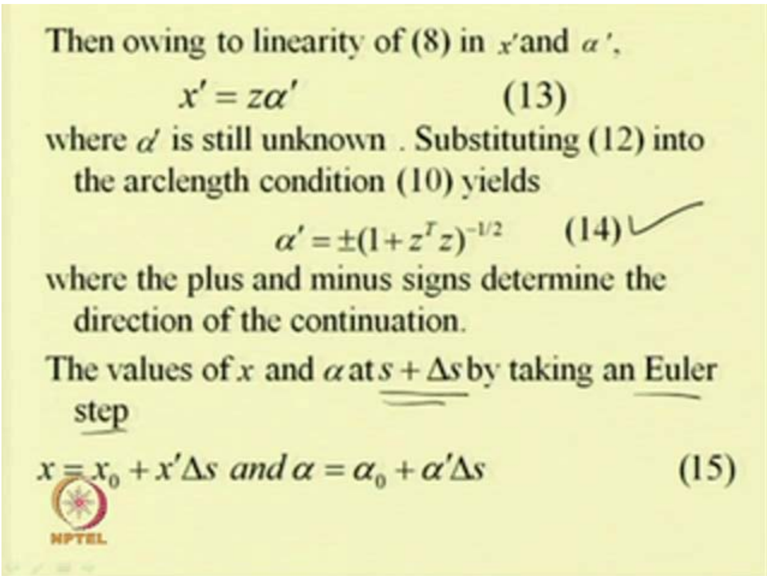

So, then we can have this x dash equal to linearity, so taking this linearity condition, that is x dash equal to z alpha dash, so where alpha dash still unknown. So we can substitute this equation 12 equation 12, so in to the arc length condition 10, so that we can obtain this alpha dash equation, so where this alpha dash equal to plus minus 1 plus z transpose z to the power 1 by 2, where the plus and minus sign determines the direction of the continuation, the values of x and alpha at s plus delta s can be the values of x and alpha at s plus delta s, by taking an Euler step determines, where x equal x 0 plus x dash delta s, and alpha equal to alpha 0 plus alpha dash delta s.

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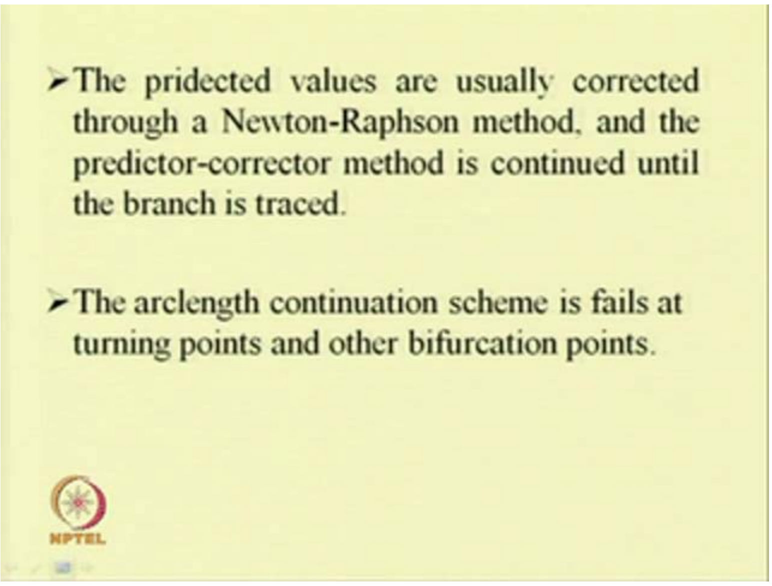

So next iteration value, we can obtain in this way and by using this arc length technique one can find the continuation. The predicted values are usually corrected through a Newton-Raphson method, and the predictor corrector method is continued until the branches are traced, the arc length continuation scheme failed at the truing point and other bifurcation points.

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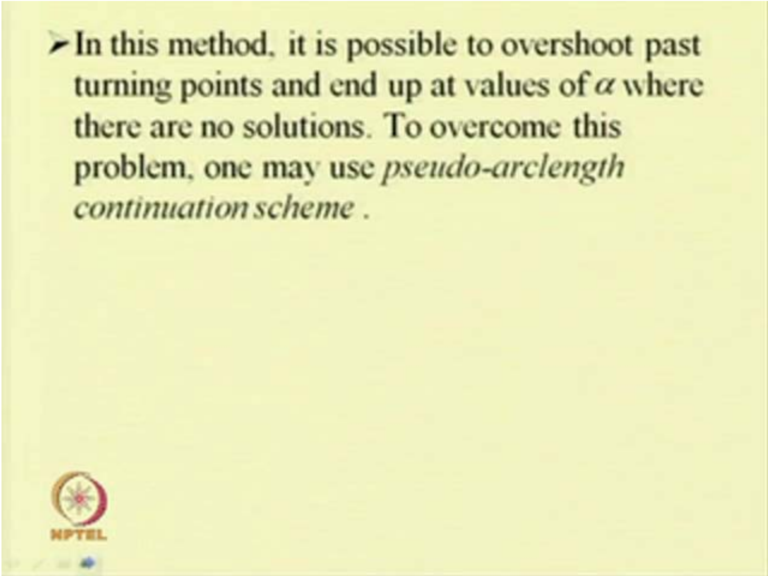

So in this method, it is possible to overshoot past turning points and end off at values of alpha where there is no solution. To overcome this problem, so one may use this Pseudo arc length continuation scheme.

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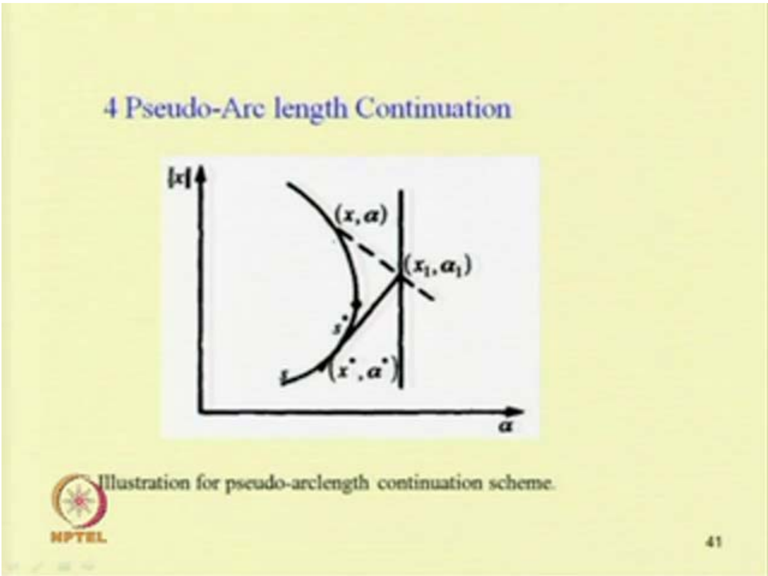

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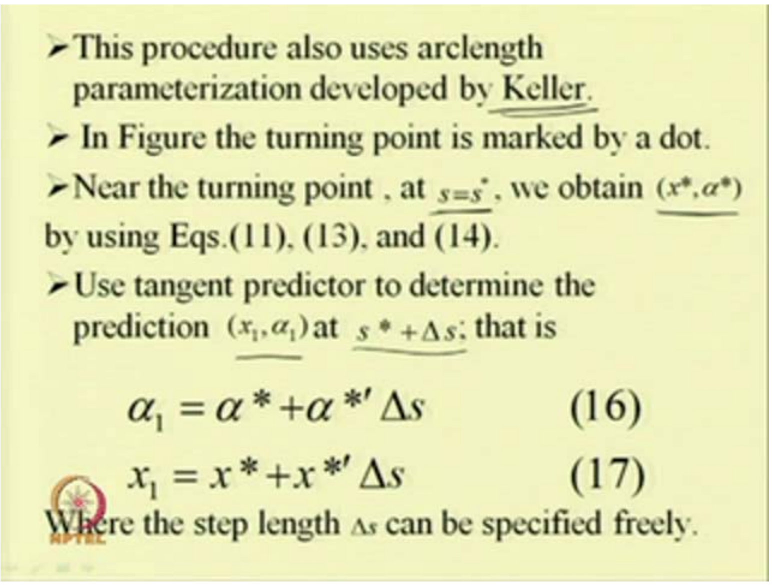

So one can use a Pseudo arc length instead of so this is the arc length continuation we are taking. So, let we have this saddle node bifurcation point noted by s star, so using this arc length technique, so one can find the predictor point, and then one can correct it. And let us see this Pseudo arc length continuation, in which we have to overcome this truing point continuation.

So, this procedure also uses the arc length parameterization, so here we are discussed the method developed by Keller, so that reference I will show few minutes, so already I told that at s equal to a star, we have this turning point. Now, one can get the near the turning point at s equal to s star, we can obtain this x star alpha star by using our previous equations, so then use the tangent predictor to determine the prediction x 1 alpha 1 at x star plus delta s, so we can get the tangent.

At x 1 alpha 1, so at x 1 alpha 1, so this is the point x 1 alpha 1 so use tangent predictor to determine the predication x 1 alpha 1 at a star plus delta s, that is alpha 1 equal to alpha star plus alpha star dash into delta s, and x 1 equal x star plus x star dash into delta x, where the step length delta s can be specified freely.

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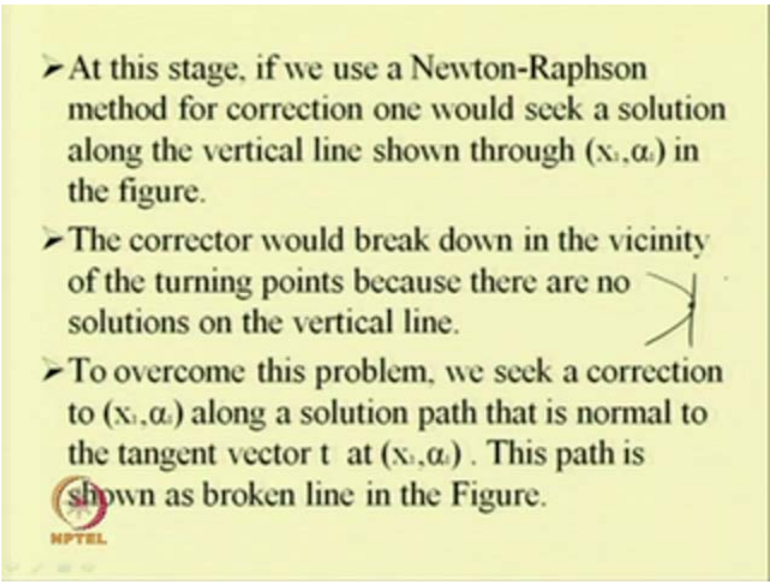

So at this stage, if we use a Newton-Raphson method for correction, one would seek a solution along the vertical line, so one can seek a so this vertical line, along this vertical line one can so the corrector would break down in the vicinity of the turning point, because there are a no solutions on the vertical line. So along the vertical line, as we have the saddle node bifurcation point, so along this vertical line, there will be no solution, so this will break down.

So to overcome this problem, so we seek a solution to x 1 alpha 1 along a solution part, that is normal to the tangent vector t at x 1 alpha 1; so these path is shown here, so this is the path, so we can take a normal to these, so this is the, so along this path we can find the solution.

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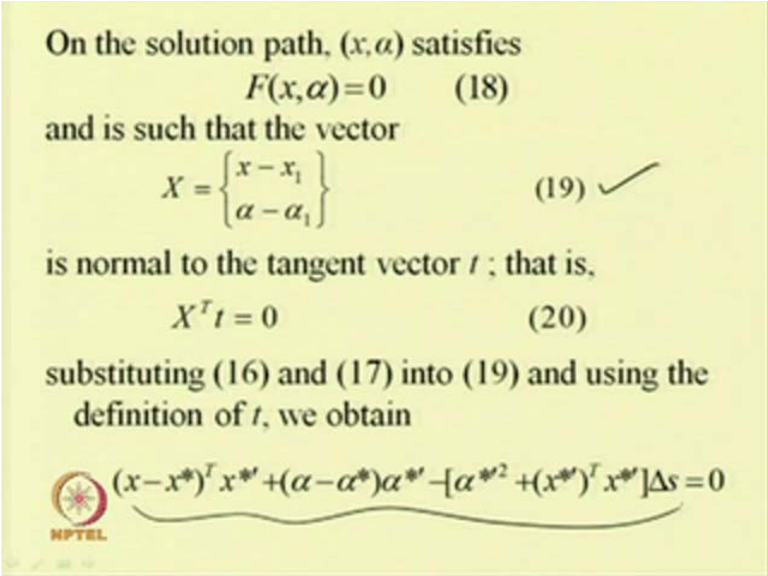

So on the solution path, x alpha satisfy this kill equation, so that is F x alpha equal to 0 and is such that, this vector x equal to x minus x 1 alpha minus alpha 1 is normal to the tangent vector t. that is x transpose t equal to 0 substituting this equations 16, so this equation 16 substituting this equation 16 and 17 in this equation, and using the definition of t, so one can obtain this equation, so that is x minus x star transpose into x star tarn dash plus alpha minus alpha star in to alpha star dash minus alpha star dash square plus alpha star dash star transpose into x star dash in to delta s equal to 0.

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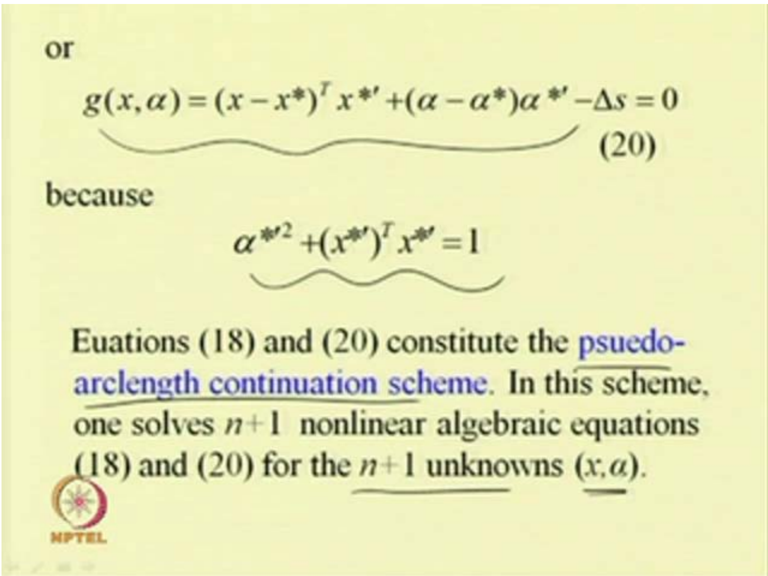

So from this or one can get this equation, so as we have this equal to 1 due to this condition, we have taken, so we have this equation, so equation 18 and 20, so this equation 18, so this equation and this x transpose t equal to 0, so they constitute this pseudo arc length continuation scheme, so in this scheme one solved n plus 1, non-linear algebraic equations 18 and 20, so for n plus one unknowns x an alpha, so after find that thing, then on each continue this scheme.

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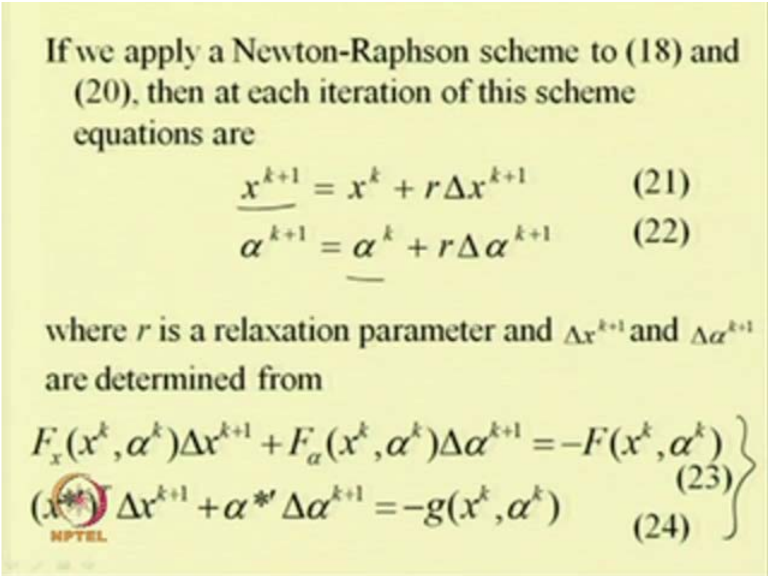

So, if we apply a Newton-Raphson scheme to 18 and 20, then each iteration of the scheme x k plus 1 will be x k plus r delta x k plus 1, alpha k plus 1 will be alpha k plus r delta alpha x plus 1, where r is the relaxation parameter, delta x k plus 1 and delta alpha k plus 1 are determined from this equation. So this equations are obtained from this Newton-Raphson method, so by solving so if a x is nonsingular, one can solve 23 equation 23, so this equation.

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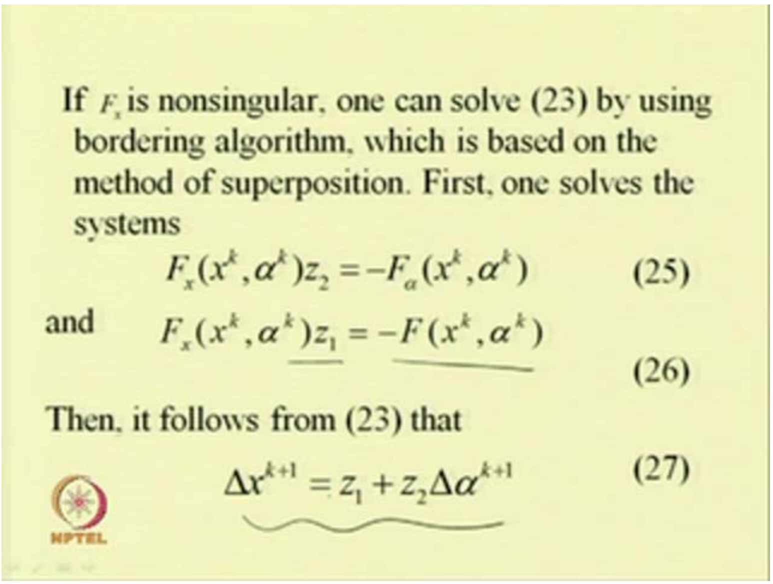

One can solve this equation by using this bordering algorithm which is based on the method of super position, first one solve this system F x k F x z 2 equal to minus F alpha x k alpha k, and F x x k alpha k z one equal to minus F x k alpha k; so by solving this then follow, then it follows from equation 23 that x k plus 1 will be equal to z 1 plus z 2 delta alpha k plus 1.

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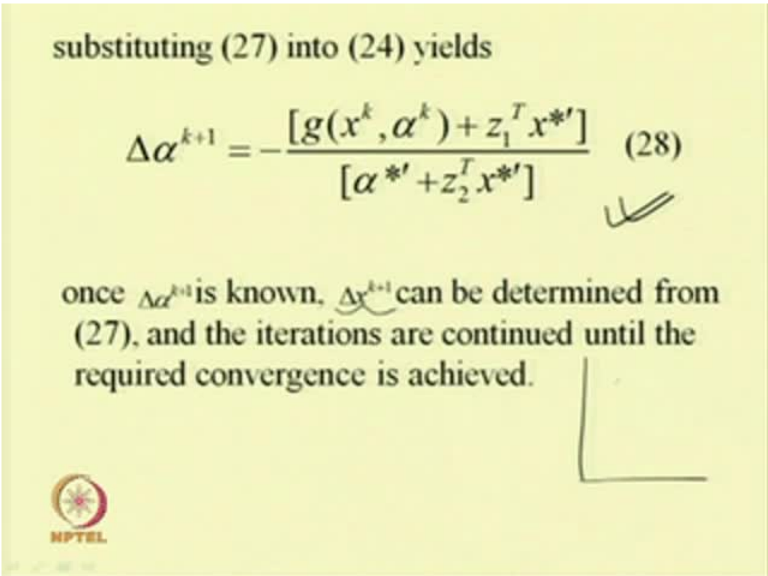

So substituting this equation, so one obtain this equation, so after delta alpha k plus 1, is known, then delta k x plus 1 can be determined, and then this iteration are continued until one converse the solution. So, in this way by using different continuation scheme, so one can obtain this solutions, or one can continue the, or one can plot different branches of the solution, different branches of a multi parameter, or single parameter multi solution, or the response of a multi valued system. So, in the system when we have multiple solutions, where we can have different branches, and by using different continuation schemes, so one can obtain different branches of the solution, particularly by using this pseudo arc length continuation scheme, we can overcome this turning point. So, for more detail study on this continuation scheme, one can refer different books or the book by this Nayfeh and Balachandran, so one can use this for periodic response one can use this shooting technique or this continuation arc length continuation technique, or pseudo arc length continuation technique.

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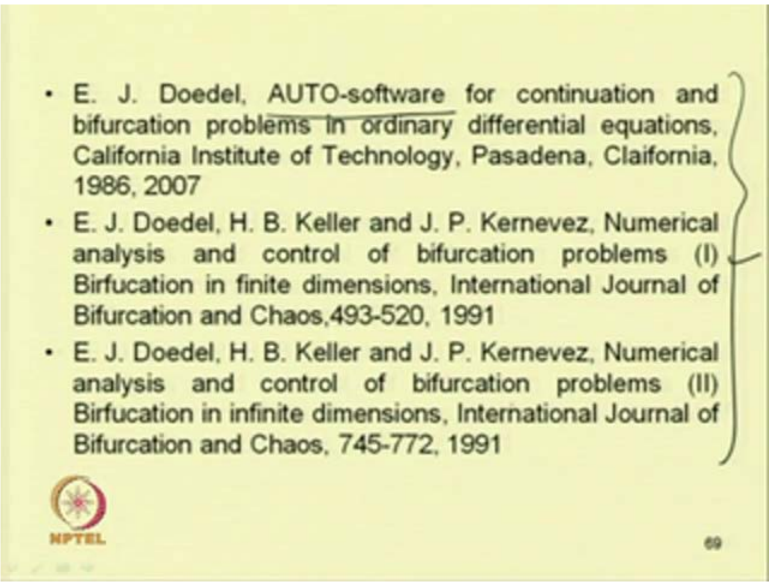

So this schemes are described in detail in the book by non-linear dynamics by Nayfeh and Balachandran, so one can refer these papers also, by Doedel, so this is the software developed by E J Doedel for plotting different branches of the fix point response, so E J Doedel, so this paper auto, so this software auto software for continuation, and bifurcation problems in ordinary differential equations California Institute of Technology.

Also one can refer this paper numerical analysis and control of bifurcation problems, So which is published in international Journal of bifurcation, and Chaos by the same author, and another paper also by this numerical analysis, and control of bifurcation problems bifurcation in infinite dimensions, which is published in international journal of bifurcation and Chaos.

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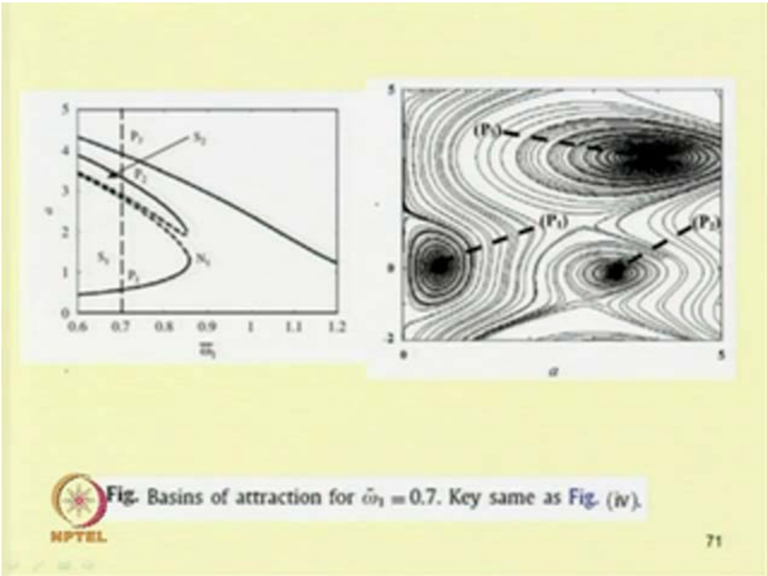

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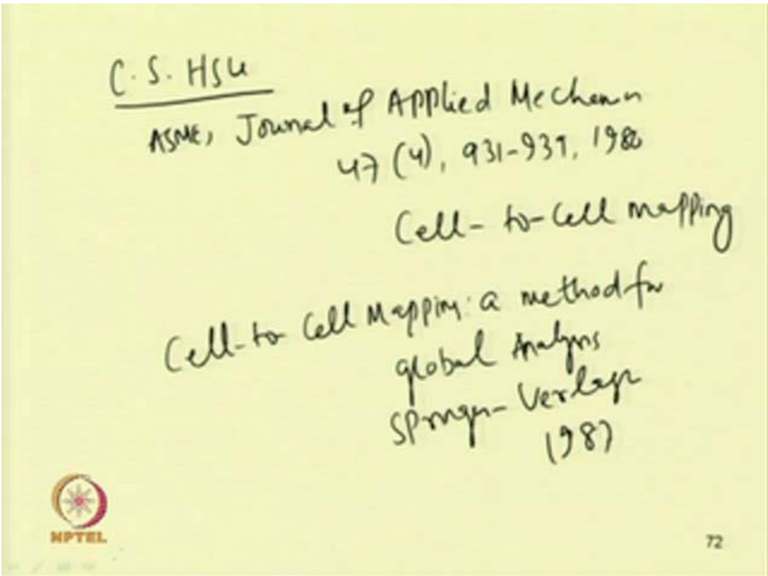

So already we have discussed that, when we have multiple solutions, one has to plot the basin of attraction; so this basin of attraction can be obtained by taking different roots, so one can one can refer the so for more information, one can refer the paper by so there are two basic method, one is cell to cell mapping. So one can refer the paper by C S Hsu which is published in this journal of journal of applied mechanics; journal of applied mechanics, that is transaction of ASME, so 47 volume 47 number 4 page number 931 to 939 in 1980.

So, this is on cell to cell mapping, so one can use this cell to cell mapping for plotting the basin of attraction, also one can refer a book by the same author C S Hsu, that is cell to cell mapping, the book name is cell to cell mapping a method for global analysis, so this is published by the Springer and 1987.

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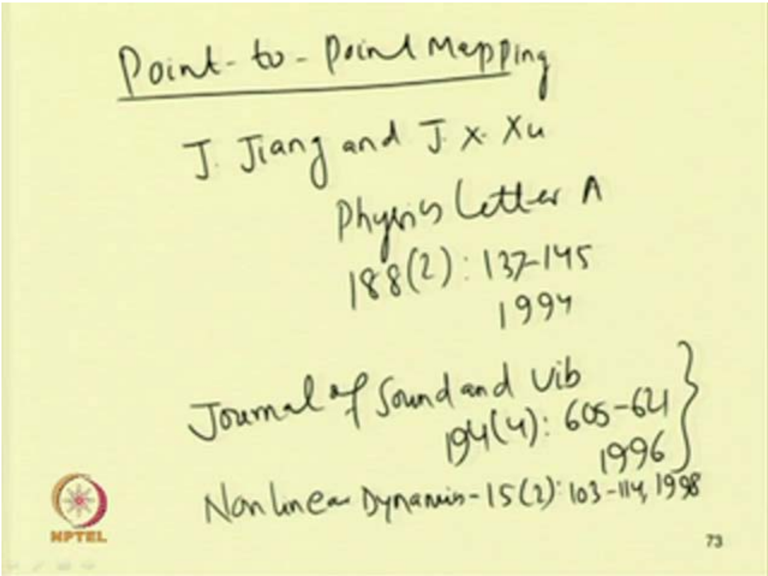

Also, one can use point to point mapping technique. Point to point mapping technique so this point to point mapping technique is developed by J Jiang and J X Xu, so this published in physics letter, physics letter A, volume 1 88 issue 2 137 to 145 in 1994. So, one can see another paper also, by this same author on this; so this is published in journal of sound and vibration, so in this paper, so this is journal of sound and vibration 194 4 so this 605 to 621 in 1996 it is published.

So this is on point to point mapping, so another paper also by the same author can be found in this journal of non-linear dynamics, non-linear dynamics so it is volume 15 issue 2, so this is 103 to 114 1998. So one can use this cell to cell mapping, or point to point mapping method for finding the basin of attraction, or one can directly use by taking different initial conditions.

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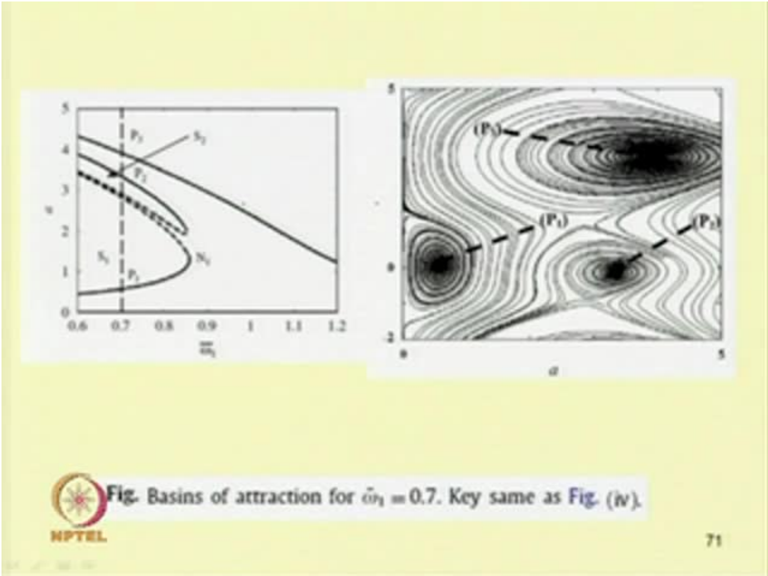

By taking different initial conditions, one can find the basin of attraction, when multiple solutions are present in the system. So, in this module, we have discussed about different numerical techniques for finding the time response, and the frequency response of the system. And in the next module, we will study about different applications of the non-linear methods, and the numerical techniques, what we have studied in this course.

Thank you. .