Non-Linear Vibration

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Module - 4
Solution and Bifurcation Analysis of Nonlinear Responses
Lecture - 7
Bifurcation of Periodic Responses
Introduction to Quasi-Periodic and Chaotic Responses

Welcome to today class of non-linear vibration. Today, we are going to discuss on this Bifurcation of periodic responses, also introduction to quasi periodic and chaotic responses with the help of some examples. So, previous, last class we have studied about different type of equations, duffing equation, Van der pol equation, hill's equation and mathieu equation; and we know in these equations in addition to fixed point response, we are going to get periodic and quasi periodic or chaotic responses.

So in case of the periodic response, so the response at so if you start at a time t and it has a period of capital T, then the response is said to be periodic, then if x t plus t equal to x t; that means if it repeats, then it is known as a periodic response. So, in this case it has a minimum period of t, so it can for by closed period, closed orbit and could be treated as a fixed point in Poincare section. So, if you take the Poincare section of a periodic response, then we will get a single point. So this is the, this is a periodic response, this phase portrait is shown to be a close loop, in case of a two periodic response the so, we have two loops.

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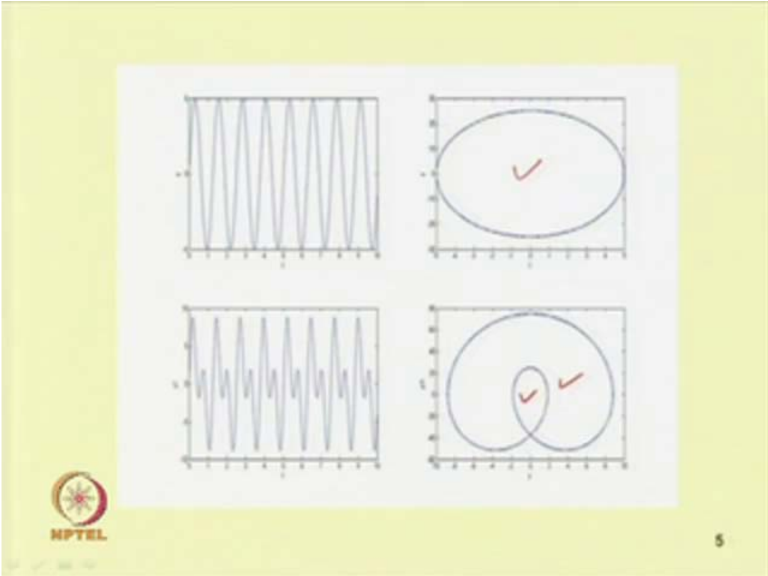

So, in case of a single periodic, a single loop is there; and in case of a two periodic, so we have two loops.

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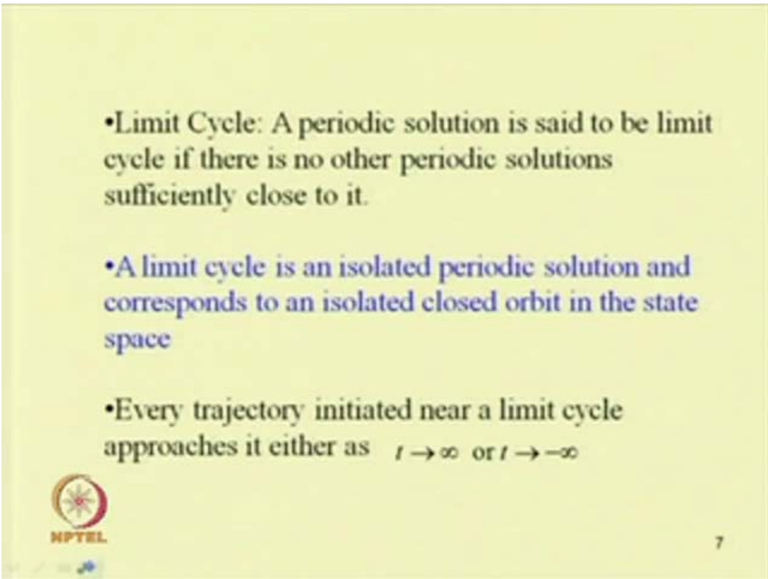

Similarly, for higher periodic, so we will have number of loops. So, we have already discussed about the limit cycles. So, a periodic solution is said to be limit cycle, if there is no other periodic solution sufficiently close to it.

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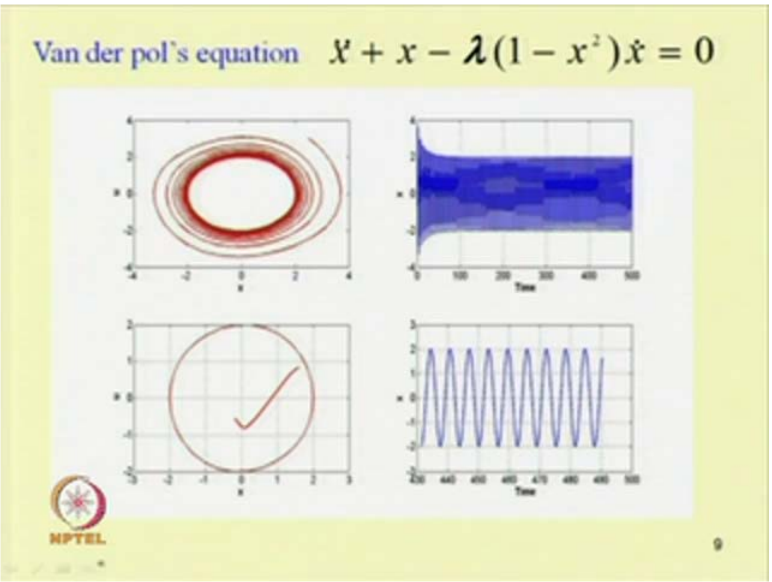

So, those things already, we have discussed and we have seen, how in case of this Van der pol equation, we are getting a periodic or limit cycle; so this is a limit cycle obtained in case of the Van der pol equation.

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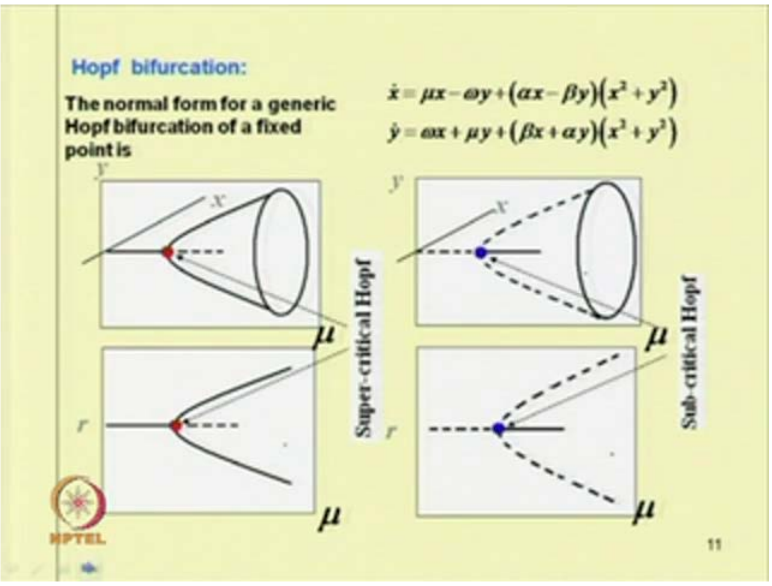

And also, we have we have discussed that, how to obtain the sub critical and super critical, sub critical and super critical Hopf Bifurcation points and you we know in case of the Hopf Bifurcation. So, a fixed point response become unstable and a periodic response emanates from that point, so here the fixed point response becomes unstable and we have a periodic response, so if the resulting periodic response, if it is stable, then we have a super critical Hopf Bifurcation and it becomes unstable, so we have a sub critical Hopf Bifurcation; so those things already, we have discussed in our previous lectures.

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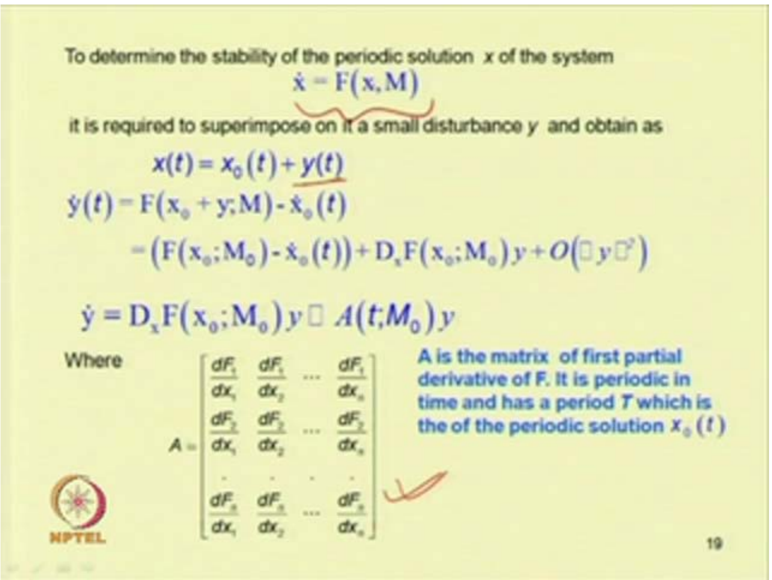

And so, today class basically we will study about, so we will study about quasi periodic responses particularly and before that, let us revise, what we have studied about the stability of periodic system. So, to determine the stability of the periodic system, so if we are taking as system x dot equal to F x M, so where M is the control parameter and x is a function, so x is a periodic solution, then we can perturb it, and after perturbing that thing, so we can get this matrix, A matrix and after getting this A matrix, which is periodic.

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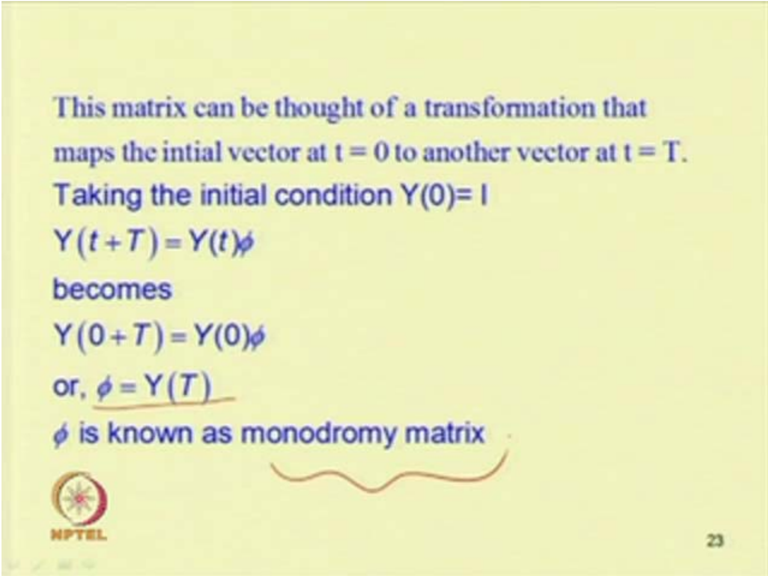

So, we can obtain the monodromy matrix, so this monodromy matrix can be obtained by taking, some initial condition so by taking a initial condition, phi initial condition and we can obtain, so we have to find this monodromy matrix phi equal to y t, taking initial condition this y 0 equal to I, that is unit vector. So, we can find this phi, so that is the monodromy matrix.

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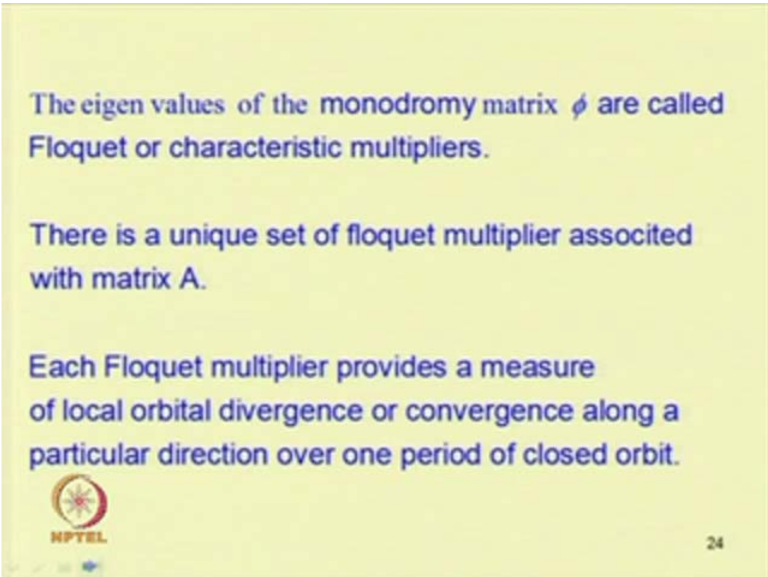

And eigen value of the monodromy matrix, will give us the will give us information regarding, whether the system is stable or unstable. So, if the eigen values of the monodromy matrix phi are called the floquet multiplier, and there is a unique set of floquet multiplier associated with matrix A, so each floquet multiplier provide a measure of local orbital divergence or convergence along a particular direction, over one period of closed orbit.

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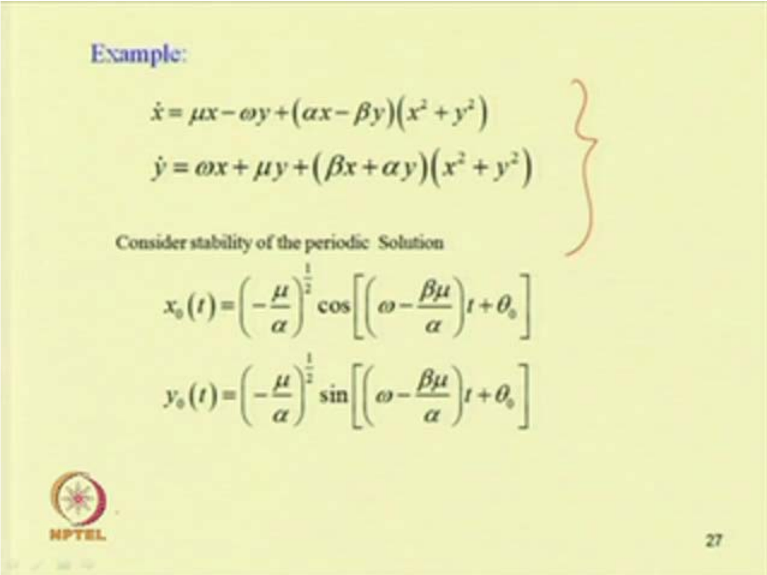

So, this thing we have discussed last class, so how to obtain this thing also, we have seen and we have also seen this example, how to find this floquet multiplier, so by getting this Floquet Multiplier.

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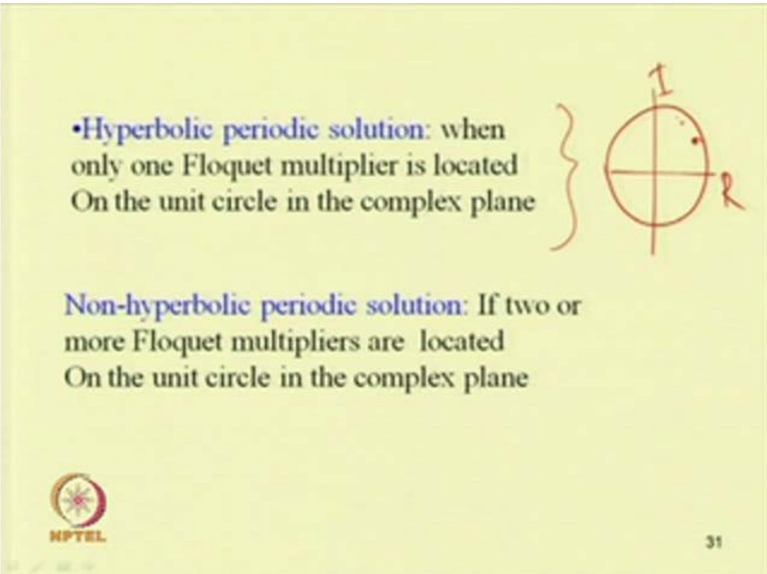

We can study this stability and Bifurcation of this thing, so we know that a hyperbolic periodic solution. We can tell a solution to be a hyperbolic periodic, when only one floquet multiplier is located on the unit circle. So, if we plot the unit circle, then this is real and imaginary part, so this is real and imaginary. So, if one root lies on the unit circle, then it is hyperbolic periodic solution. If two or more floquet multiplier are located, then we are telling this to be non hyperbolic.

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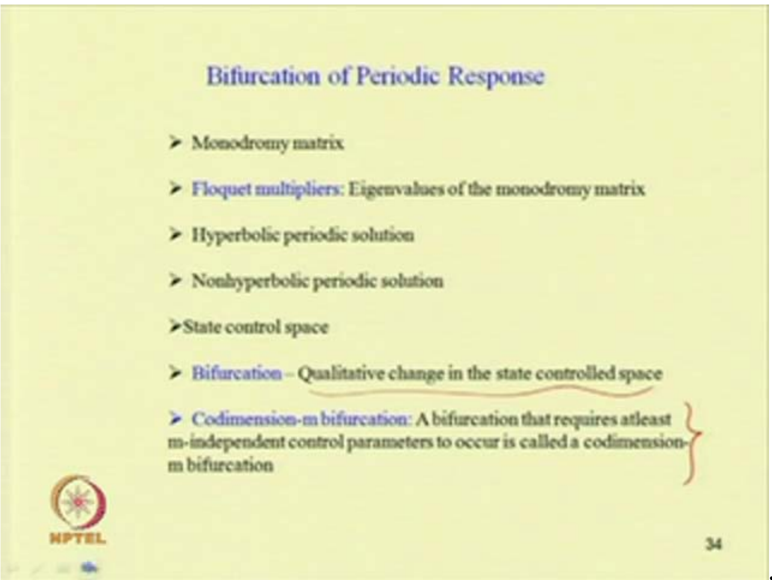

And in case of, this to study the Bifurcation of the periodic response. So, first we have to find the monodromy matrix, then we have to find the floquet multiplier, then we will check whether the solution to be hyperbolic or non hyperbolic, then in the control space we have to check whether the Bifurcation, what we are obtaining is the types of Bifurcation, we can discuss so in case of the Bifurcation, there will be qualitative or quantitative change in this trace control for space. So, a Bifurcation is said to be of four dimension m, so if it requires m independent control parameter to occur, then we tell this Bifurcation as co dimension in m. So, a Bifurcation that require at least m independent control parameters to occur is called a co dimension m Bifurcation.

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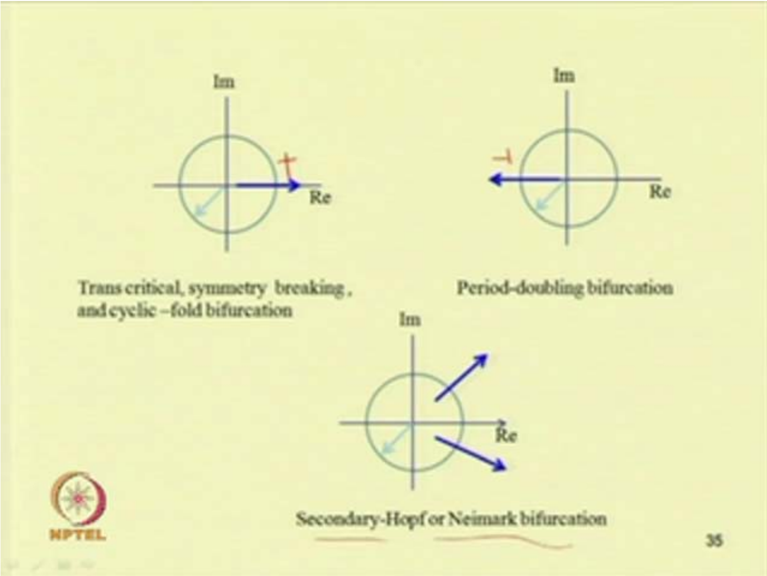

So last class also, we have discussed about trans critical symmetry breaking or cyclic fold Bifurcation will occur, if the floquet multiplier one of the floquet multiplier leave this unit circle through plus 1, so this is trans critical symmetry breaking or cyclic fold Bifurcation. And in case of the period doubling Bifurcation, so one of the floquet multiplier leave this unit circle through minus 1, and if a pair of complex conjugate leaves this unit circle, then the resulting Bifurcation will be secondary or Hopf or neimark Bifurcation s.

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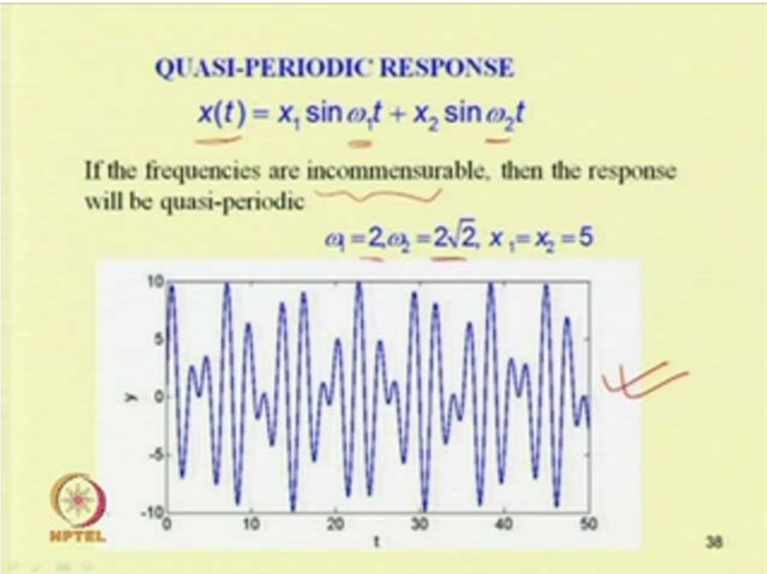

And in so, last class we have discussed about all these Bifurcation s. And today class, we will start with what you mean by quasi periodic response, a brief introduction to quasi periodic response and chaotic response will be given. And then, we will discuss one example, to show how this fixed point response, periodic response and chaotic response occur in the system and how they, how the system behaves, when these type of responses occur. And what we mean by this stability of this fixed point periodic and quasi periodic response, so in case of a quasi periodic response, the solution: Let the solution x t, so if we can write this x t equal to x 1 sin omega 1 t plus x 2 sin omega 2 t, if these frequencies omega 1 and omega 2, so if these frequencies are in commensurable, that means the ratio between these omega 2 by omega 1 is an irrational number.

So, for example, if you take omega 1 equal to 2 and omega 2 equal to root 2, then this ratio that is omega 1 omega 2 by omega 1, which is equal to root 2, so then we can tell that, these frequencies are in commensurable. So, if these frequencies ratio for example, in the previous example, we have taken omega 1 equal to 2 and omega 2 equal to 4, then this ratio 4 by 2 equal to 2, is an integer, so or we can okay, so they are commensurable; if this ratio are commensurable, then we can get periodic response but, if this ratio are incommensurable that is if the ratio is irrational number, so in that case the response what we will got what we will get, will no longer be periodic, so they will be a-periodic and these responses, so one of one such response is shown here. So, by plotting omega 1 equal to 2 omega 2 equal to 2 root 2 and x 1 equal to x 2 equal to 5, so this shows the response, so in this case one can observe that the response is not periodic.

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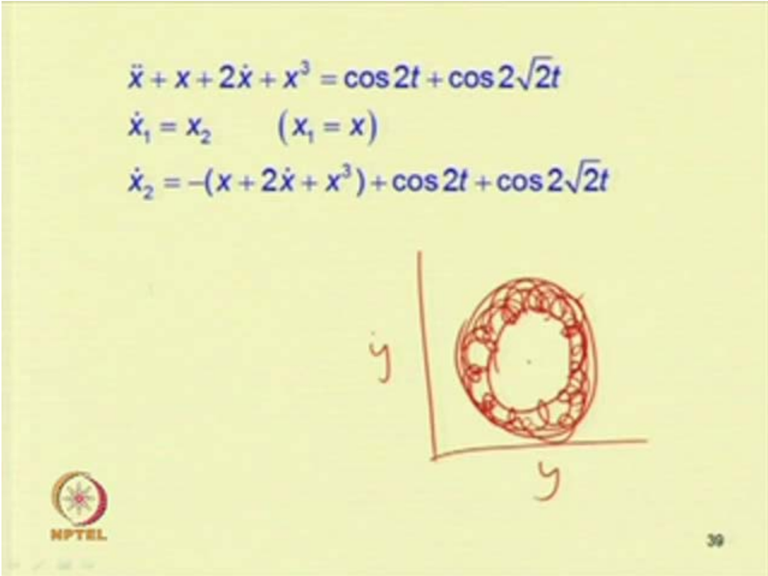

So, if we plot the phase portrait, so one can see the phase portrait will look like this, so it will look like a torus. So, as if there is a wound between these two and it will fill up this place, so it will be in between these two in between, these two curves, we can get this period or this phase portrait, so this is if we plot y versus y dot, the response the y versus y dot the phase portrait will look like this.

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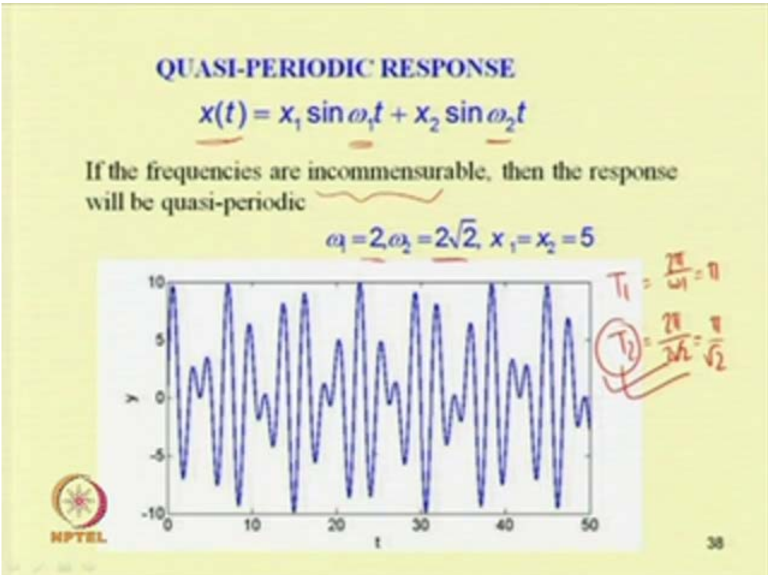

So, if we take Poincare section of this thing, so how to take Poincare section, so we have to take this Poincare section, we have to find the minimum time, so minimum time period, so in this case omega 2 equal to 2 root 2, so we have this T 1 equal to 2 phi by omega 1; so this is equal to 2 phi by 2, so this is equal to phi and T 2 equal to 2 phi by 2 root 2. So, then it becomes phi by root 2, so in this case our T 2 is the minimum time period, so by taking T 2 or if we sample this response curve, if we sample this response curve with with this T 2 time period.

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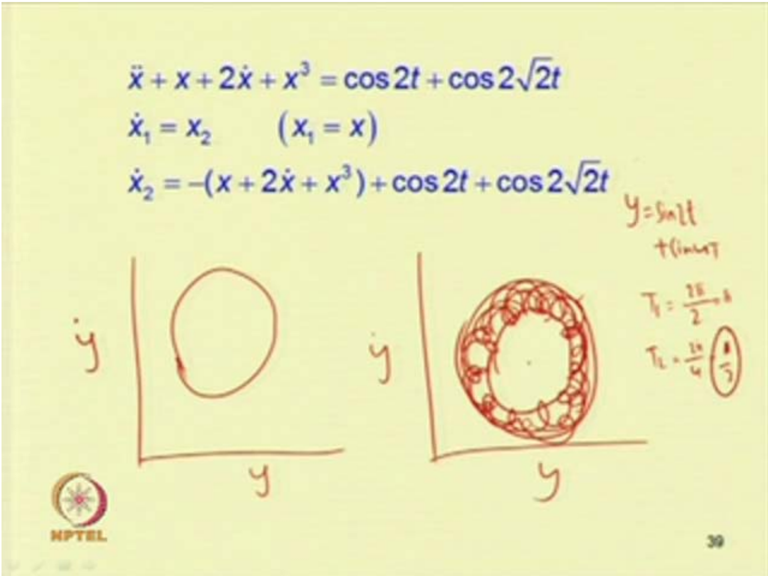

Then the resulting response, if we plot if we plot the phase portrait of the sampled time response, that means we will sample it at an interval. Let us, start with this point starting with this point, if we sample it with phi by root 2, then one can observe that the resulting curve will be a close loop, so the resulting curve will be a closed loop, so in case of the quasi periodic response.

So, one can obtain a close loop in case of drawing the Poincare section but, in case of periodic response for example, if I have periodic response, y equal to y equal to sin 2 t then or sin 2 t plus sin 4 t, so by taking this T 1 equal to, so T 1 equal to 2 phi by 2, that is phi and T 2 equal to 2 phi by 4, that is phi by 2, so if we sample it with phi by 2. So, we can see so from these two curve, if one float then one can obtain the response or one can obtain the Poincare section to be so the Poincare section.

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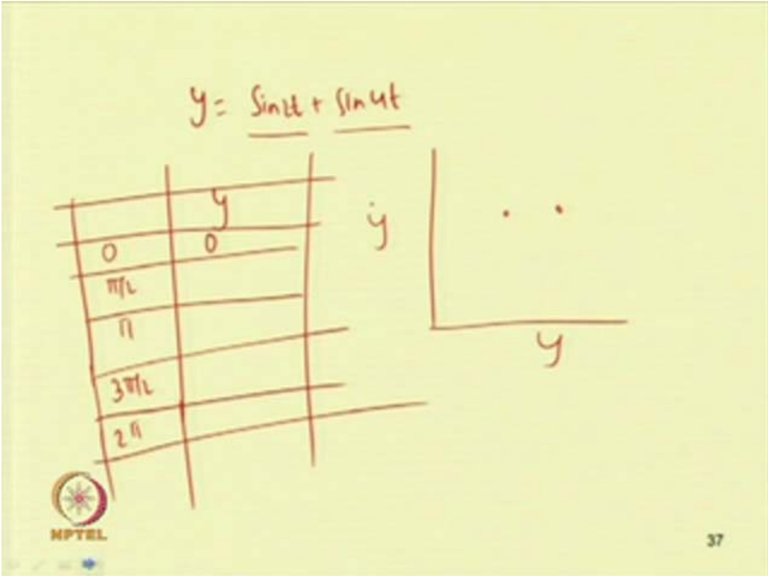

If one plot in case of le t y equal tosin two t plus sin four t so in this case the Poincare section if i want to plot this y dot versus y so this plot if i plot for sin two t and sin four t if we sample it at phi by two solet us find so we are sampling it at so we will start at zero then pi by two then pi thenso pi plus pi by two so this is three pi by two then four pi by two that is two pi and if we go on increasing this thing and find the function y so at zero so this is zero so at pi by two so at pi by two so this becomes pi so sin pi equal to zero and this sin so this becomes sin 2 pi, so this also become 0, so in this case it you take, we can find two points, so we can find two points on this curve.

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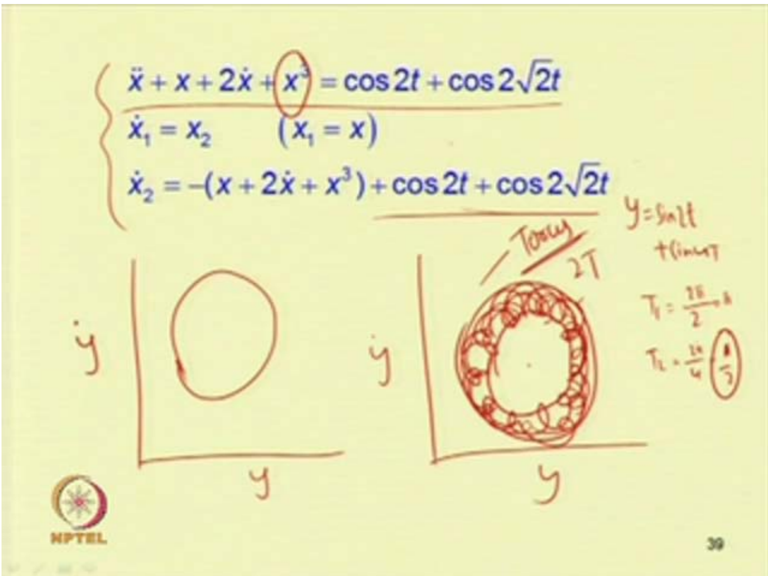

But, in case of this quasi periodic response, so instead of getting two points so we will get, so we will get a closed loop. So, in case of periodic response so depending on the frequency or number of frequency present in the response, we can get definite number of points on the Poincare section but, in case of the quasi periodic response a two period quasi periodic response is known as torus; so this is a torus or one can write this 2 T, so in torus the Poincare resulting Poincare section is a close curve.

So, this is a so one can get so for example, let us take this equation, the second order equation, so this can be part of as a string mass system with a non-linear spring along with that, so the external forcing are cos 2 t plus cos 2 root 2 t, so if one plot this so to use this or to use the numerical methods, we can we can write the second order differential equation using a set of first order differential equation; the first order differential equation first equation one can write x 1 dot equal to x 2. And second equation that is, x 2 x double dot, so that will be equal to minus x minus 2 x dot minus x cube minus cos 2 t minus 2 root 2 t, so one can write so x 2 dot will be equal to minus of x plus 2 x dot plus x cube plus, this cos 2 t plus cos 2 root 2.

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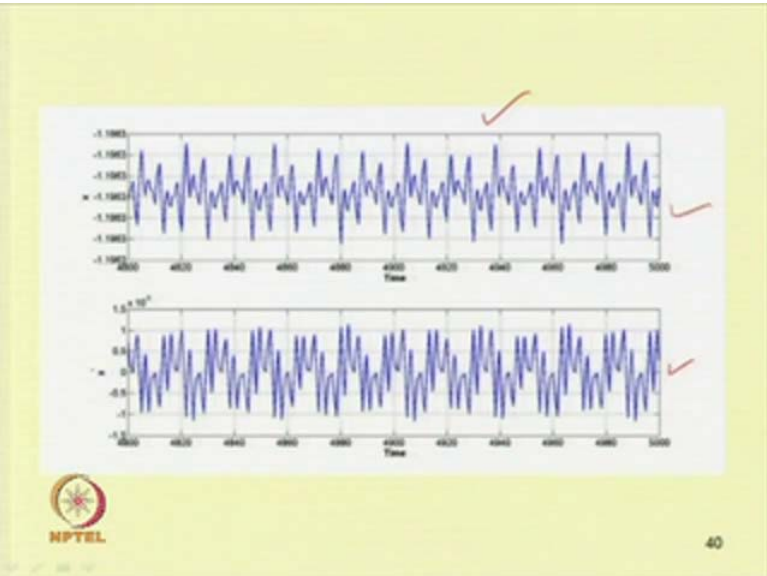

So, if one plot this curves one can find, so this shows the x versus time and this is the x dot versus time, so x versus time and x dot versus time are shown in this figure; that is x that is x 1 and x 2, so if one plot x 1 and x 2 is this is the curve and also, this is a program mat lab program, written for plotting this time response and also one can find the Poincare section in this case.

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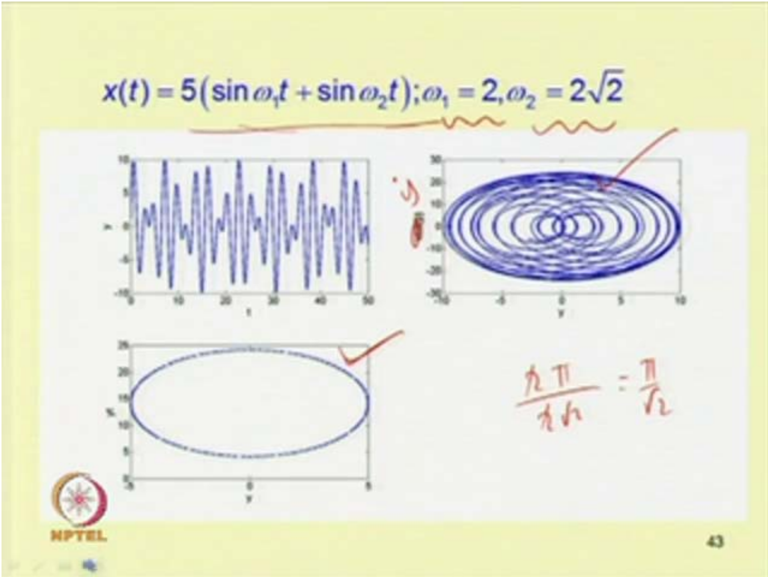

So, this is the phase portrait, so this is the phase portrait that is y dot versus y and this is for another case that is, x t equal to phi sin omega 1 t plus sin omega 2 t, so omega 1 equal to 2 and omega 2 equal to 2 root 2, so in this case this is the Poincare section, so here it is sectioned with time period pi by so 2 pi by 2 root 2, so this becomes pi by root 2 so by sampling it at pi by root 2, so one can see or one can observe of getting a close curve, so in this way one can obtain the Poincare section or one can plot the phase portrait. So, unlike in case of periodic response, where the Poincare section give definite numbers of points, so here one can obtain a close loop the close loop may not be circular, so depending on the presence of harmonics, the loops size will be different or shapes will be different, so these are some of the example.

So, here so in this example omega 1 equal to 2 and omega 2 equal to 2 root 2 and we are plotting x t equal to sin 5 sin omega 1 t plus sin omega 2 t, so this is the response and this is the Poincare section. Similarly, if you take omega 1 equal to root 2 and omega 2 equal to 2 root 5, so this is the ratio is root 5, so previously ratio was root 2 so this is the response so this is the this is the corresponding Poincare section and time response, is so there is so the time response is shown here 0 to 10 and in this case the time response also 0 to 10 but, the as the frequencies are different, so the curves loop slightly different. So, also another plot is plotted so by taking this omega 2 equal to 2 root 11, here the ratio is root 11 so here the ratio is root 5 and here it is root here it is root 2, so depending on this so one can get different type of this torus.

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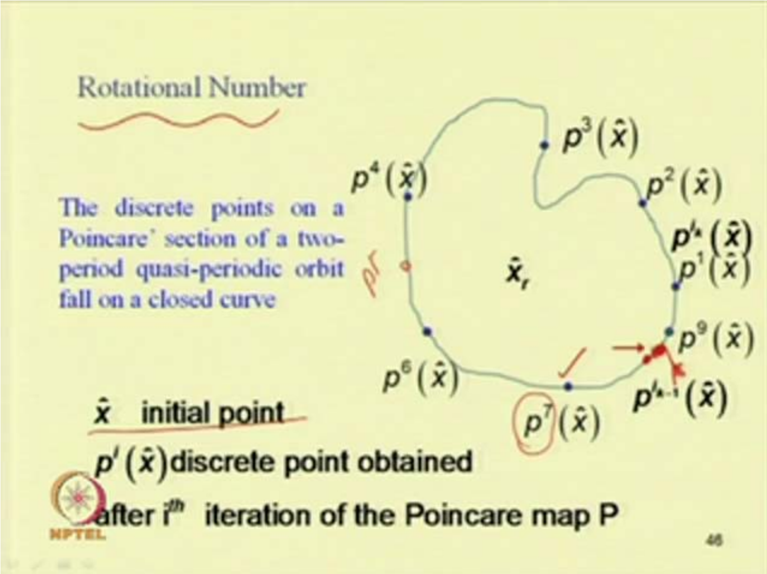

So, already we know the Poincare section, in case of this quasi periodic response and in that case, we have discuss that, one we will obtain a close loop. So, associated with this quasi periodic response, so there is a number that is known as rotational number, so when we plot this Poincare section, let us start with the Poincare section, this is p 1, so let us start as a point, so we have a start as a point, so this is the x point. So we have started at a point, this now in the first iteration, let the point this is the point in the second iteration, the point is this third iteration, the point is this, so we go on sampling this thing and so with each iteration, we got one point and so, one can obtain several points on this loop, one can get several point on this loop. If one find the Poincare section, so taking x as the initial point, so let p i x a discrete point obtained after ith iteration of the Poincare map so one can define this rotational number as that so this is a discrete point on a Poincare section of a two periodic quasi periodic or so to obtain this rotational number.

So, first we have to plot this curve and check, where, which are which iterations after what iteration, so the same point arrives that means in this case. So, if we have started at this point and we have seen in the first, when we have come across this loop, so we have seen p 7 after p 7 and p 8, so let us have this p, so this is p 6 p 7 and p 8 will be somewhere here so p 8 so, let this is the starting point, we have p 1, p 2, p 3, p 4 and let we have p 5, p 6, p 7 and then we have this p 8, so if this point lies between this 7 and 8, we can tell this rotational number to be 7.

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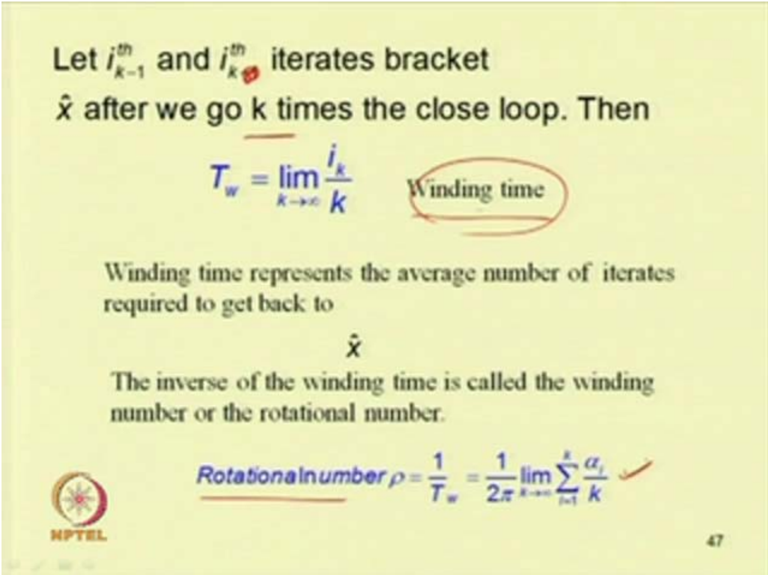

So, let i k minus 1th and i k, if iterate let k i minus 1 and let i k minus 1 and i k iterates bracket, this x after we go k times, the close loop then this winding time T w can be written as limit k tends to infinite i k by k, so if k is 1, so let us take k equal to 1, that is in the first k, so once we have come across this loop, so when once we have come across the loop, so this point or this point x is bracketed between p 7 and p 8, that is why we can tell the rotational or winding time and rotational number in this case, this way so this t w will be equal to i k by k, so i k let i k minus 1 and i k iterates bracket the x after we will go k times, so in this case k equal to 1.

So, if once we have moved this, then this i k this this k minus this becomes 7 and i k becomes 8, so we can tell this is limit k tends to infinity, so this will be then 8 by 1, so this will be 8. So, similarly we can let in second iteration, so we can find which two iterates bracket this points similarly, in third, so if we go k times, then if we average that thing that means limit k tends to infinity i k by k so we will get this winding time.

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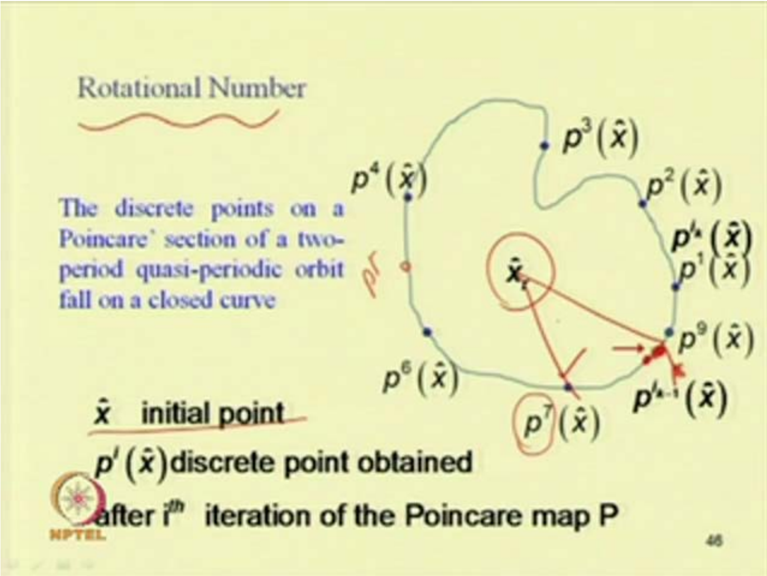

So, the reciprocal of this winding time is known as the rotational number. So, rotational number is the reciprocal of the winding time, that is 1 by T omega, so 1 by T omega, so this is equal to 1 by 2 pi limit k times to infinity i equal to 1 to k, so alpha i by k also, one can find this thing also by taking a internal point and checking, what is this angle? so angle, which bracket this and this point so by finding this angle also, one can tell about the rotational number, so this will be equal to 1 by 2 pi limit k times to infinity alpha i by k, so alpha i that angle what we have shown, so that is alpha i and one can find using this a point in the interior of this loop and find the rotational numbers.

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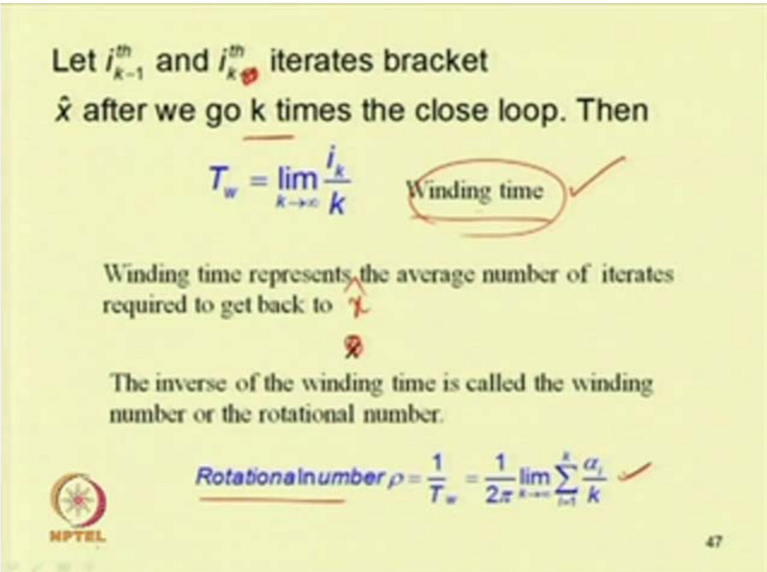

So, rotational number is reciprocal of winding number and after finding the winding number, one can find this rotational number. So, winding time represent the average number of iterates required, to get back to x ba,r so we have taken starting point x and the winding number, winding time represents the average number of iterates required to get back to x. The inverse of the winding time is called winding number, so in this way one can study or one can find the quasi periodic response and after obtaining the quasi periodic response, as we have seen the Poincare section of the quasi periodic response, is a periodic response or is a close loop. So, one can study the stability of the quasi periodic response by studying this close loop also.

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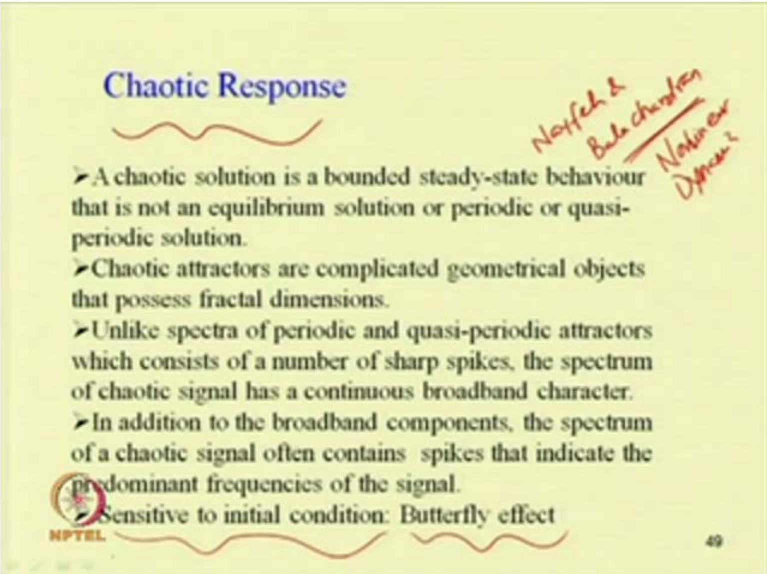

And now, let us see after studying this fixed point response, periodic response and quasi periodic response, the other remaining type of response in case of, a non-linear system is the chaotic response; so a chaotic solution is a bound rate steady state behavior, that is not an equilibrium solution or periodic solution or quasi periodic solution. So, a chaotic attractor chaotic attractors are complicated geometrical object, that possess fractal dimensions. So, unlike spectra of periodic and quasi periodic attractors, which consist of a number of sub spikes. The spectrum of chaotic signal has a continuous broad brand character, so in addition to the broad brand components, the spectrum of a chaotic signal often contain spikes, that indicate the free dominant frequencies of the signal.

So, the major difference between this periodic, quasi periodic are fixed point and this chaotic response is the sensitivity to initial conditions, in case of the chaotic response; it is very sensitive to the initial condition, in case of a fixed point or periodic response, so whatever may be the initial condition, at steady state it does not depends on the initial condition but, in case of the chaotic response, it is very sensitive to initial condition and one can observe a butterfly effect, in case of this chaotic response; that means by slight change in this initial condition will lead to another type of response, which is not at all similar to the response obtained in the previous iterations, so to know more about this chaotic response, one may refer the book by Nayfeh and Balachandran, so one may refer this book by Nayfeh and Balachandran, that is non-linear dynamics the book non-linear dynamics and so these portions have been adapted from this book by Nayfeh and Balachandran non-linear dynamics.

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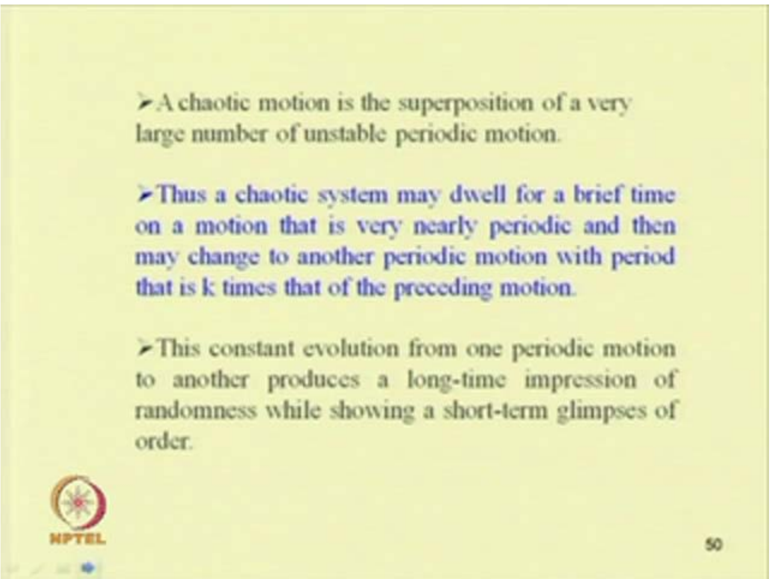

So, a chaotic motion is the super position of a very large number of unstable periodic motion, thus a chaotic system may dwell for a brief time on a motion, that is very nearly periodic and then may change to another periodic motion with period, that is k times that of the periodic preceding motion. So, this constant evolution from one periodic motion to another period produces a large time impression of the randomness, while showing a short term glimpses of the order, so that means we know that this chaotic solutions are rather fixed point response, periodic response or chaotic. Now, quasi periodic response that depend very much on the initial condition and there are several routes to this chaotic solution or chaotic response.

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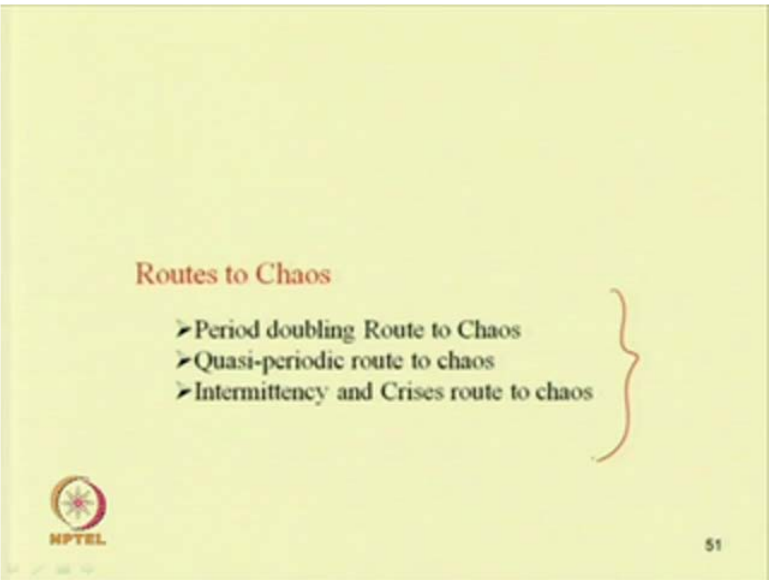

So, few of them are, so one is this period doubling route to chaos, chaotic response also obtain from this quasi periodic response, so either a torus break down, either it may be a torus break down route to chaos or it may be a torus doubling route to chaos, so like period doubling to route to chaos similarly, also torus may double up or torus may add up, to give rise to chaotic response, also it may break down to give rise to chaotic response also intermittency and crisis are other routes to chaos, so one can obtain chaos due to these routes. So, one can obtain so by changing these control parameters, sometimes a period doubling leads to chaos, so those things we will see with the help of some examples.

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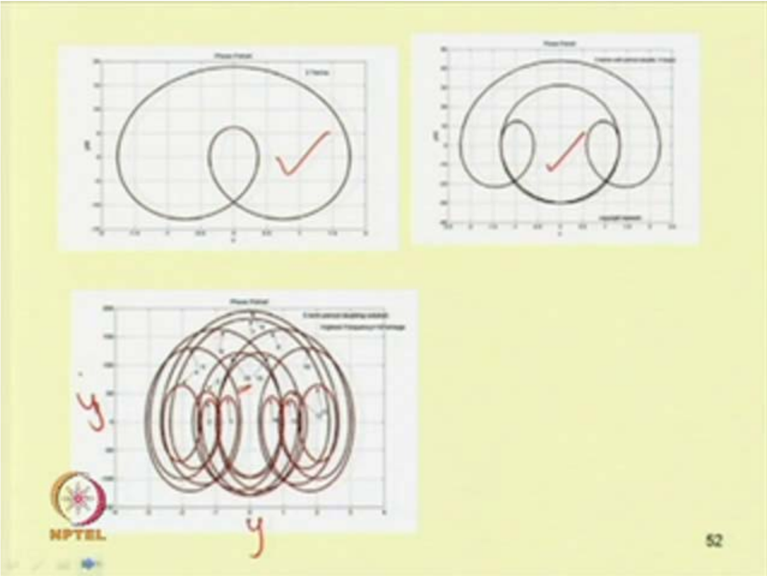

For example, let us take, so this a period to period response and now, it becomes 4 period, so then 4 will becomes 8 and 8 will becomes 16, 16 32, so in that way the period will double to obtain a chaotic response, so this is a periodic, so this is a response with several periodic. For example, so it contain 16 periodic so 16 periodic, so the orbits have been or close loops have been marked, so this is one two similarly, three so this way all the orbits are marked, so here there are 16 orbits, so these 16 orbits, so you just see the phase portrait, so that is y dot versus y looks like a chaotic response, so if we go on doubling this period by changing the system parameter or control parameter, we can land off with a chaotic response.

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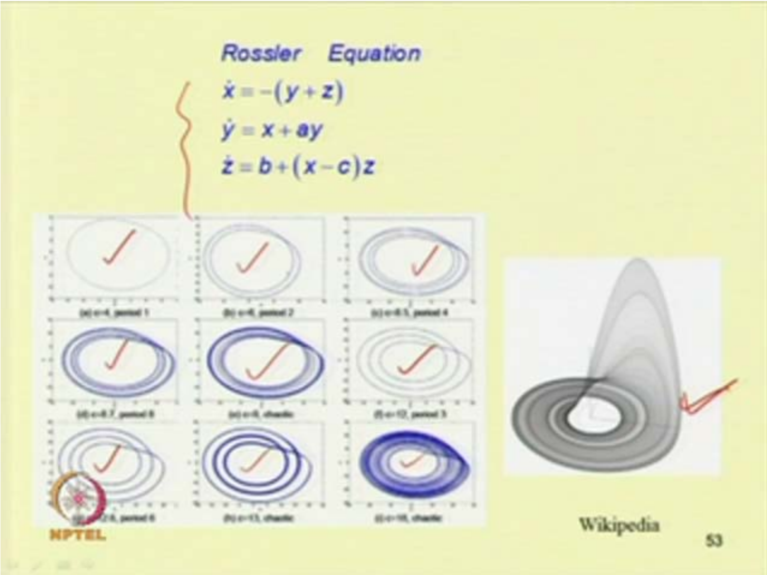

Similarly, we can we can see this is a Rossier equation, so in case of the Rossier equation so, which is these slides has been taken from this Wikipedia. So, were the equation can be written in this way, x dot equal to minus y plus z y dot equal to x plus a y and z dot equal to b plus x minus c into z, where a, b, c are constant. And if one plot this, so if one plot in x y plots a so, one can obtain initially for c equal to 4, one can obtain a period one so by changing c to 6, the c parameter to 6, one can obtain a two period and then by changing it to 8 point can obtain a period 4, so c is 8 1 5 so then c equal to, so by taking this c equal to 8.7, so this becomes period 8 and c equal to 9, one can see the response to be chaotic and again c equal to 12, so one can see this is a period three response so, then c equal 12.6 periodic become 6 c equal to 13 again it, becomes chaotic and c equal to 18, the chaotic response chaotic response continues, so this is this is the chaotic response, which so, which looks like a funnel or it is known as Rossier funnel.

So, one can obtain a chaotic response and if one plot the Poincare s section. So, for a periodic response we can obtain a single point for two periodic, so we can obtain two points that means c equal to, so if you go on increasing c equal to 4 to 6 so at c equal to 6, we obtain two points.

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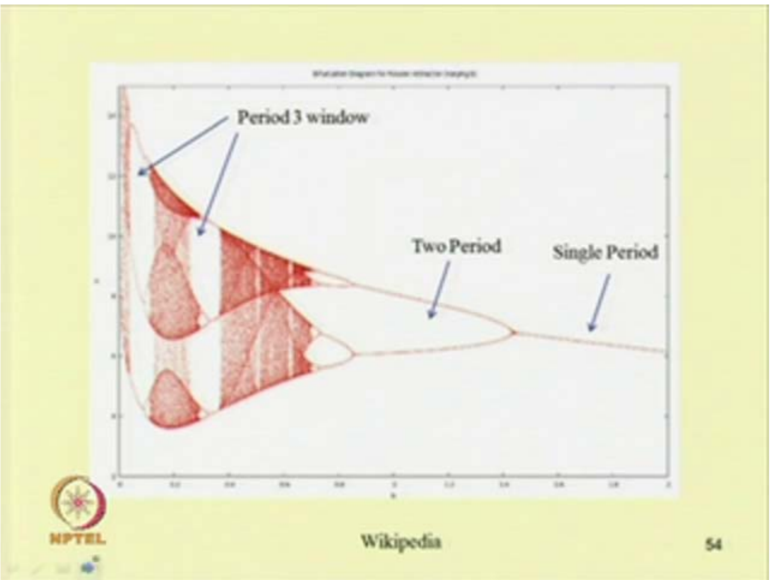

So, initially we have a single point, so it continue so from a single point to so, you go on increasing so this single point becomes two points, now these two points becomes four points, and four becomes eight, and eight becomes sixteen, to make the response chaotic and here you can see, so there are some windows, so in this case, one, two, three, four, five, six, so it is collected at these six points and in this case, you can so these are the period three windows.

So, a only at three points, it is connected so three points, we are getting so in this range we are getting only three points but, here we are getting several points, so here we have window so one, two, three, five, five windows. Here similarly, one can count this thing one, two, three, four, five so number of windows can be counted counted at these points.
And one can see between these chaotic response, one can get some windows of periodic response, so this is a period doubling route to chaos. So, here initially one has a periodic response, now by changing this parameter c in this Rossier equation, so one can obtain a two period then four period then six period, four then eight then sixteen and it continues and if one plot the Poincare section, one can see clearly the behavior to be chaotic, so unlike aquasi periodic response, the where the Poincare section is a close loop so in case of chaotic response, the Poincare section will fill up will fill up the space. So, one can distinguish the Poincare section of a periodic response, quasi periodic response and chaotic response.

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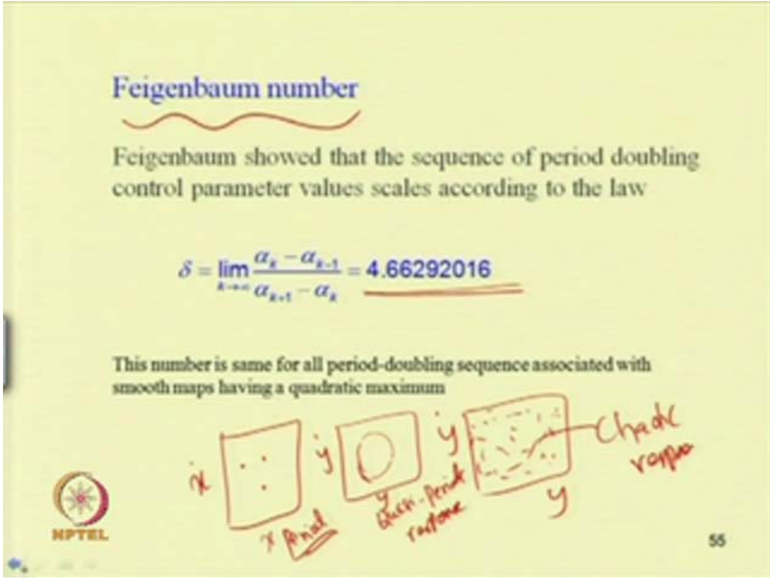

So, in case of a periodic response, it will be single period or a definite number of periodic, definite number of points and in case of quasi periodic response, it will be a close loop and in chaotic response it will fill up the, so it will fill up the space. So, if we plot the phase portrait, so x versus x dot, so let let us plot this thing for this three different thing, so for a periodic response single period one point let two period, then you will have two points for three period, there will be three points that means definite number of points will be there. So, in case of the periodic response the Poincare section will contain definite and in case of quasi periodic response, so it will be a close loop, so it will be a close loop so, quasi periodic response and in case of chaotic response, the response will fill up this space, the response will fill the whole space.

So, this is y dot versus y y dot versus y in case of or x dot versus x phase portrait, so this is a chaotic response so, in case of period doubling so a number associated with this periodic period doubling route to chaos is Feigenbaum number; this Feigenbaum Feigenbaum showed, that the sequence of period doubling control parameters value scales according to the law, delta equal to limit k tends to infinity, so alpha k minus alpha k minus 1 by alpha k plus 1 minus alpha k equal to a universal constant, which is which value is 4.66292016, so this number is known as Feigenbaum number; that means so, one can observes let alpha k, so the period so kth period is occur at a control parameter of alpha k then and k minus 1th period was at k alpha k minus 1.

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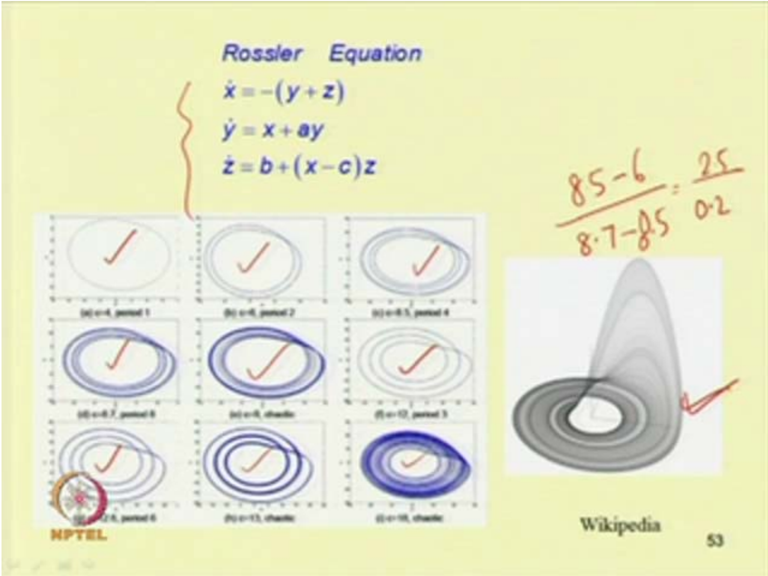

And in this case, for example, in case of this Rossier funnel, so we have this period doubling at c equal to 6 and it becomes 4th at 815 and this becomes 8 and 817, so if you take k equal to so, let us take this one so alpha k that means this 4th period ,4 it becomes 4 at 815, and this becomes so, this becomes 2, that means the previous iteration, this 6 at k equal to 6, that that means alpha k minus 1 is 6, alpha k equal to 8.5 and then alpha k plus 1 equal to 8.7, so if one find this ratio that means, 8.5 by 8.5 minus 6 by 8.7 minus 8.5, so that means 8.5 minus 6 by 8.7 minus 8. So, 8.5, so this becomes so 2.5 by, so 0.2 so, 2.5 by 0.2 so, in this way one can find this number.

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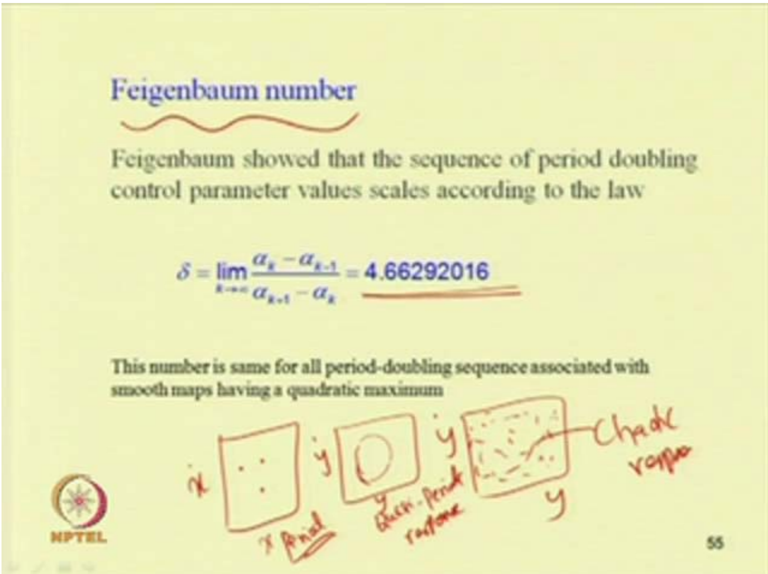

So, one can check that, this number is a universal constant; that is alpha k minus alpha k minus 1, so by alpha k plus 1 by alpha k, will be equal to 4.66292016.

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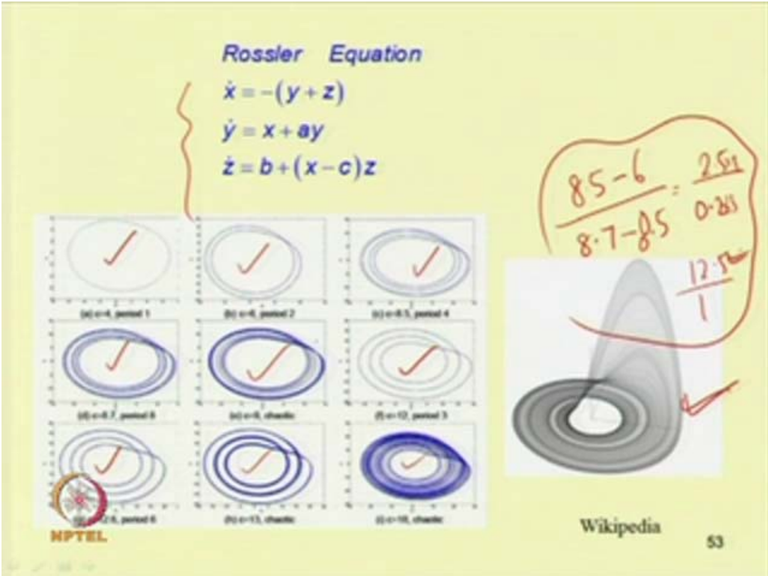

So, in this example, so as they are not the critical points so this value is coming to be so these are somethis is only some example shown so as these are not critical point so this number multiplying five here multiplying five so this becomes one and this becomesten plus two point five twelve point fiveso one can obtain the Feigenbaum number like this but, this example as we have taken this number arbitrarily so this not the Feigenbaum number.

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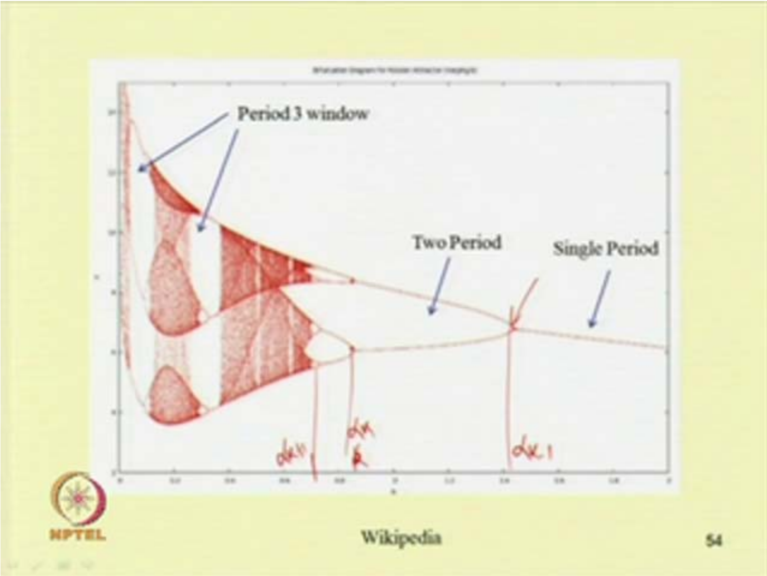

So, we are getting a ratio of 12.5, so if one can find the critical parameter, then one can obtain that number to be so critical critical parameters, will be here; so this is one critical parameter so this point now, it becomes so initially this was 0, initially it was this was single period, now this is double period and this period will be 4, now this point it becomes 4, so one has to take this value this value and now it becomes 1, 2, 3, 4, so 2 to 4 and this 4, 8 so at this frequency. So, what is this frequency, one can find so at this frequency becomes 8, so this ratio so if this is k then alpha k, so this value if it is alpha k so, this is alpha k minus 1 and this is alpha k plus 1, so the ratio alpha k minus alpha k minus 1 by alpha k plus 1 minus alpha k.

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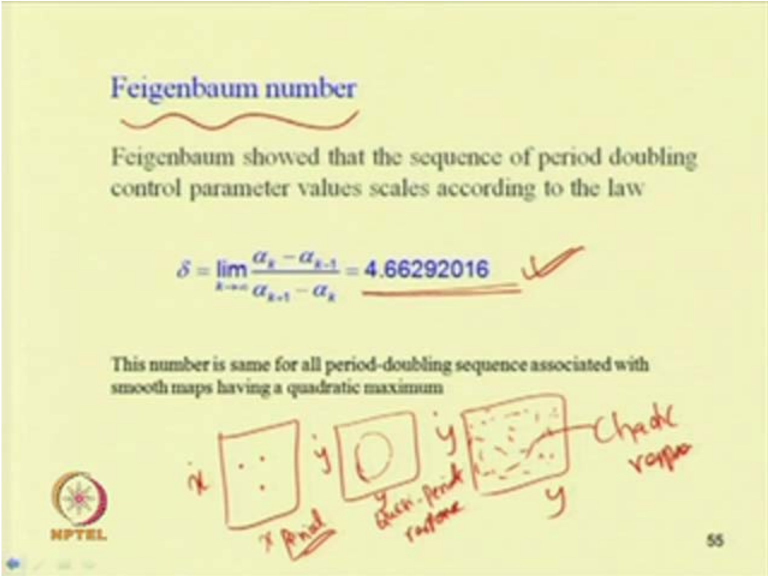

So this ratio will be a constant or is a constant, and that is known is Feigenbaum number. Or this number equal to 4.66292016, so in this way one can predict, when the next doubling will occur. So, if you know doubling occur at a particular point, then we can predict the next period doubling Bifurcation or period doubling critical parameter.

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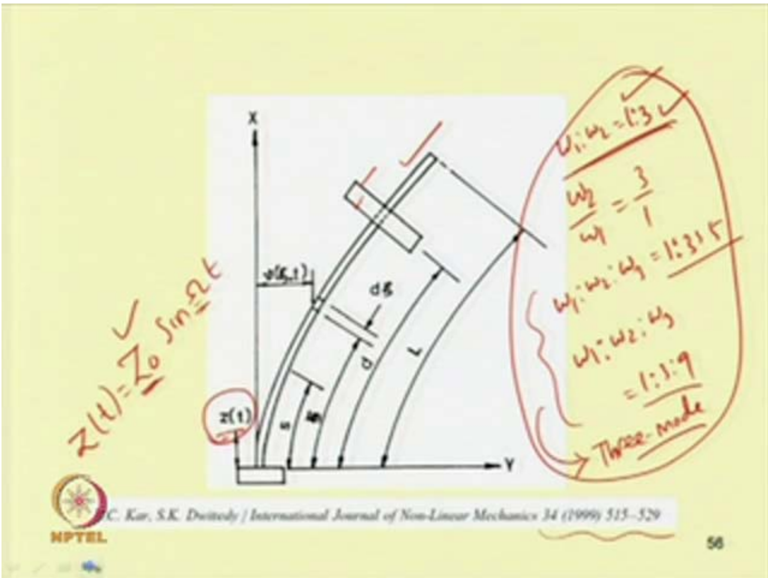

So similarly, one can find the other different routes; that is torus doubling route to chaos torus break down route to chaos and also different crisis, so through one example: Now, we will study this Bifurcation of fixed point periodic, quasi periodic and chaotic response and we will conclude this model, so this is one example taken from the work by professor R C Kar and S K Dwivedy, which was published in this international journal of non-linear mechanics in year 1999, so this is a slender beam. So, a slender beam base excited slender beam, which contain one arbitrary mass by positioning this mass at a different locations, so one can change the frequency of the system and for some values of these frequencies, so one can as this is a continuous system.

So, one can get infinite number of natural frequency of the system and this natural frequency may be distinct or sometimes they may have integer relations, that means this omega 2 by omega 1 or omega ratio of omega 1 is to omega 2 is to omega 3 will have some integer relations, for example: omega 2 by omega 1 may be equal to 3 is to 1 omega 1 is to omega 2 is to omega 3, it may be equal to so it may be equal to 1 is to 3 is to 5 or omega 1 is to omega 2 is to omega 3 m equal to 1 is to 3 is to 9, so in this case when this omega 1 is to omega 2 equal to 1 is to 3, so one can obtain one internal resonance of 1 is to 3 type, so in case or this will be a two mode interaction and if the ratio, that is omega 1 is to omega 2 is to omega 3 equal to 1 3 5 1 is to 3 is to 5 or 1 is to 3 is to 9, so in these two cases one can obtain internal resonance of third mode, so one can obtain internal resonance three mode interaction, so one can have so in this case three mode interaction and in this case one can have two mode interaction but, when these frequencies are very distinct, so there there will be no internal resonance and one can obtain this response of the system by considering single mode approximation.

So, either one can go for a single mode approximation or two mode approximation or three mode approximation depending on the ratios of the frequencies obtain in this case, so in this case by positioning this at as mass at a different locations, so it has been observed that the system can have these three type of or three type of modal interaction; that means without modal interaction second two mode interaction and in case of three mode interactions also it has been obtained, 1 3 5 internal resonance and 1 3 9 internal resonance, so this system that is a base excited cantilever beam, so it is excited periodically at this base that is z t, so z t equal to so one can write this z t equal to z 0 sin omega t, so here with the amplitude z 0 and frequency omega the base is excited.

So, when the base is excited the system move in a transverse direction for some value, of this z 0 and omega, so for some value of z 0 and omega, so it will vibrate in just it will oscillate or it will move up and down but, if these values or for some critical value of z 0and omega, when it exceed this critical value it vibrates in a transverse direction, so the so in this work the response of the system is obtained, so the response of the system is obtained for this three modal interaction and one can observe this fixed point response, periodic response and chaotic response also in this case.

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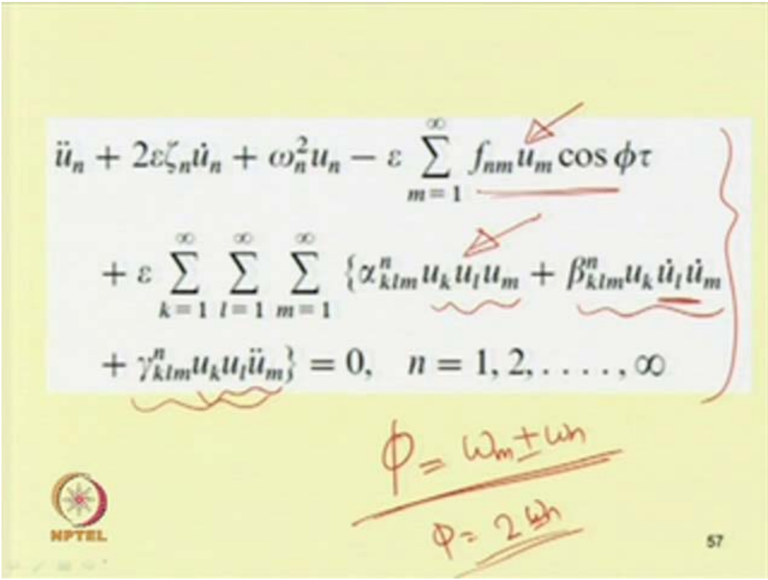

So, let us take this example so in this example, so the equation motion of this system can be written by this way, that is equal to u n double dot plus 2 epsilon zeta n u n dot plus omega n square u n minus epsilon m equal to 1 to infinity f nm u m cos phi tau plus epsilon k equal to 1 to infinity, l equal to 1 to infinity, and m equal to 1 to infinity ,alpha klm n u k u l u m plus beta n k l m u k u l u m, so beta k n k l m u k u l dot u m dot, one can note that here these are the velocity term, so in this case product of three displacement term and k l m are the modal displacement kth mode displacement lth mode displacement and mth model displacement similarly, this is gamma n k l m u k u l u m double dot, so in this case you just see so this part is the geometric non-linear term as these two parts, this is product of two velocity product of two velocity.

As it gives acceleration, so this this is also inertial nonlinearity and this is also inertial nonlinearity. So, one can observe this two inertial non-linear term and a single geometric non-linear term in this case, so here this nonlinearity is coming due to the large transverse displacement of the system. So, this is the temporal equation obtained by applying collecting method to the system, so here one can see the coefficient of so one can see the coefficient of u m that is the displacement is f nm cos phi tau, that is the periodic term; a periodic term is the coefficient of u m, that is the response that is why this type of system are known as parametrically excited system.

So, unlike fix unlike in case of the force vibration, so were the resonance occur at a frequency when it is equal to that of its natural frequency, in case of in case of the parametrically excited system, the resonance may occur when it is away from the natural frequency also, one can obtain a response, so for example, in this case by taking this phi equal to, so if I will take phi equal to omega m plus minus omega n, the resonance can occur at these cases, so if taking this m equal to 1 and n equal to 1 or m equal to n, so one can obtain phi equal to so 2 phi will be equal to 2 omega n, so if n equal to m, so phi equal to 2 omega n and this type of resonance conditions are known as principle parametric resonance condition and if they are not same, so if it is plus then it is known as combination parametric resonance, and when we are taking this negative sign, this is combination resonance of difference type, when we are taking plus then it is combination resonance of summation type some type.

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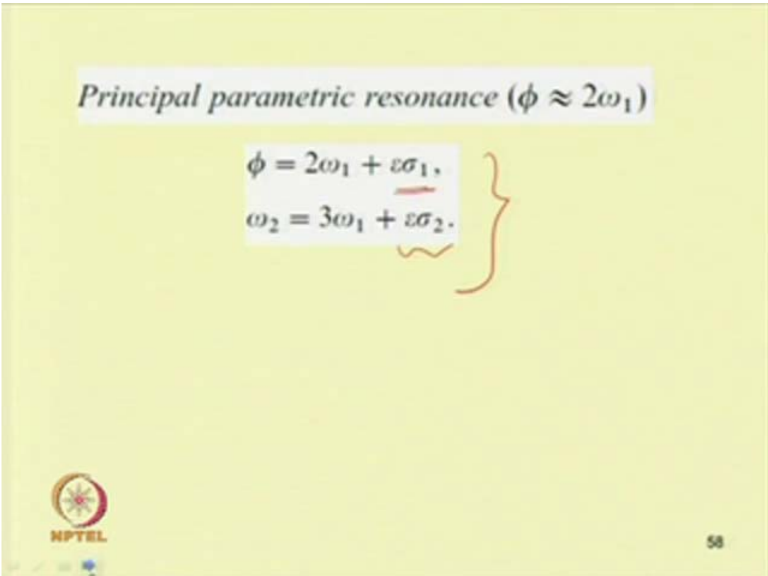

So, let us see the case of the principle parameter resonance, so in case of principle parametric resonance taking this phi equal to 2 omega 1 plus epsilon sigma, 1 were sigma 1 is the detuning parameter, and as we are considering two mode interaction, that is omega 2, nearly equal to three times omega 1; then we can also take another detuning parameter epsilon sigma 2, so sigma 2 is the detuning parameter epsilon is the book keeping parameter, so this omega 2 equal to 3 omega 1 plus epsilon sigma 2.

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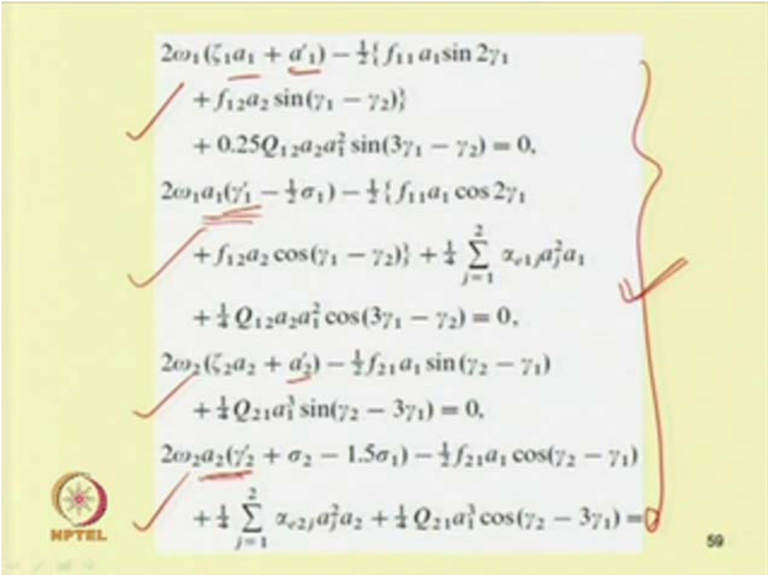

So by taking these things, so one can obtain a set of so one can obtain a set of four equations, so these are the four reduced equation one can obtain, so this is the first one, second one, third one and fourth one. So these are the four equation one can obtain using method of multiple scales, and in this equation one can see this is a dash gamma 1 dash a 2 dash, so a 2 dash, and so this is a two this is a 1 dash gamma, 1 dash and a 2 dash, and we have gamma 2 dash, so for steady state we can prove this a 1 dash a 2 dash gamma, 1 dash gamma 2 dash equal to 0, where a 1 a 2 are amplitude and gamma are phase to obtain a set of algebraic or transcendal equations, and by solving these equations one can obtain the fixed point response and by solving this equation.

This equation directly numerically one can obtain the periodic response, so initially first one has to plot the fixed point response; in this case as we have four equations, so numerically one has to solve these thing and one can see in this case, the if we perturb it the perturbation will not contain, so for these cases the perturbation will not contain as we have this a 2 gamma 2 dash term, and a 1 gamma 1 dash term, the perturbation will not contain this these terms.

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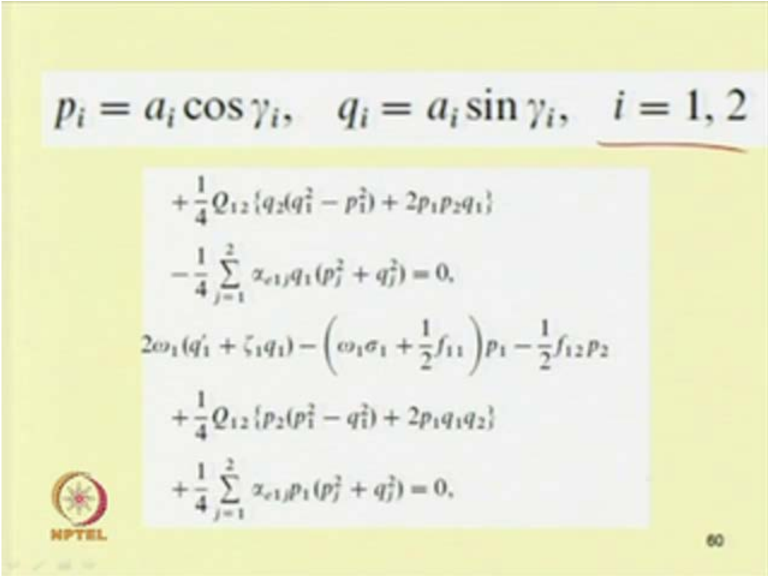

So, to avoid this thing, one can make the transformation by taking p i equal to a i cos gamma, i and q i equal to a i sin gamma I, where i equal to 1, 2.

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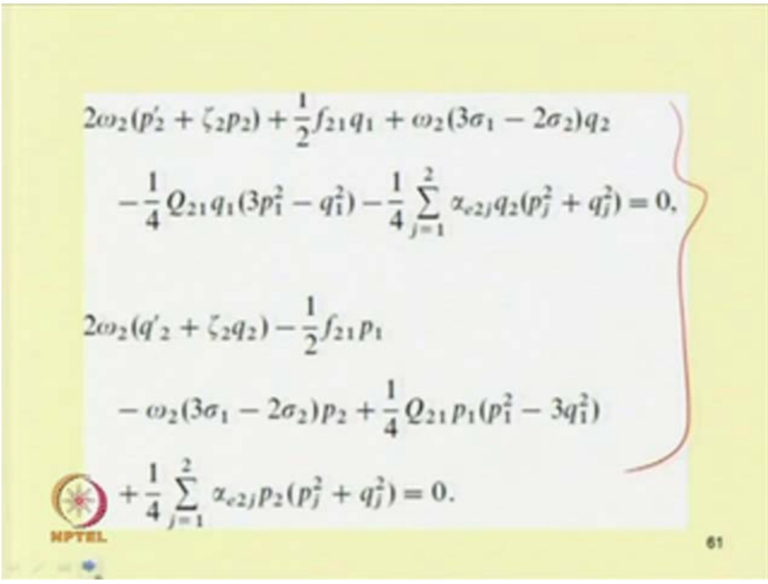

And one can write these two, these four equations by using p and q, so instead of writing in terms of a, and gamma, one can write this equation in terms of p and q, so one can have a set of equations, four equations in terms of p and q. Now by solving, these equations one can study its stability by perturbing.

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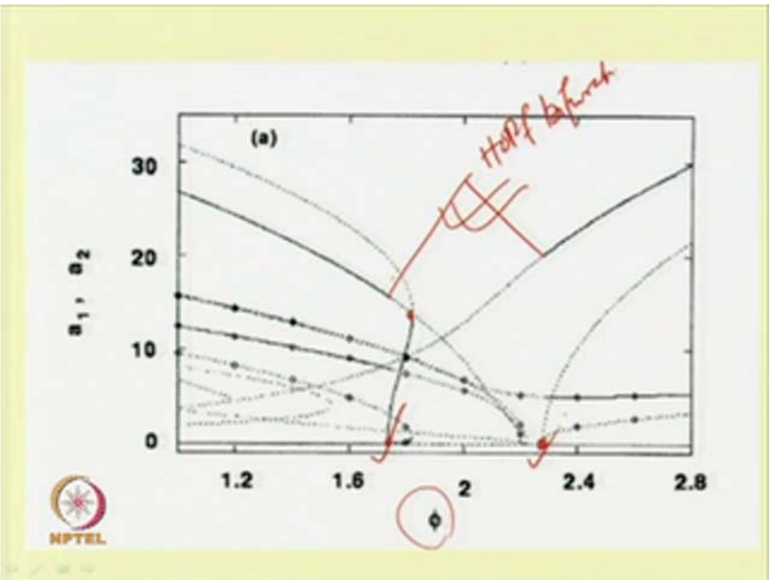

These equations one can study, it is stability and solving those algebraic equation or transcendal equation, one can get the response, so this is a typical response shown, so one can see several branches of the response is there, so unlike in case of the duffing equation, one can get with cubic nonlinearity one can get only two branches, here several branches have been obtained and several Bifurcation s points are also shown.

So for example, this point and this points are Hopf Bifurcation point, Hopf Bifurcation point and in this Hopf Bifurcation point one can obtain the periodic response so here one can see at this point if by taking this parameter control parameter phi so if one find the response one can see the periodic response similarly, this is a saddle node Bifurcation point and these points are pitch fork Bifurcation point so here uses what one one canobtain or one can observe that one has a super critical pitch fork Bifurcation up to this and at this point one has a saddle node Bifurcation and at this point one can has a so one can have so this is stable and this becomes unstable so one has a sub critical pitchfork Bifurcation while this is super critical pitchfork Bifurcation this is the sub critical pitch fork Bifurcation.

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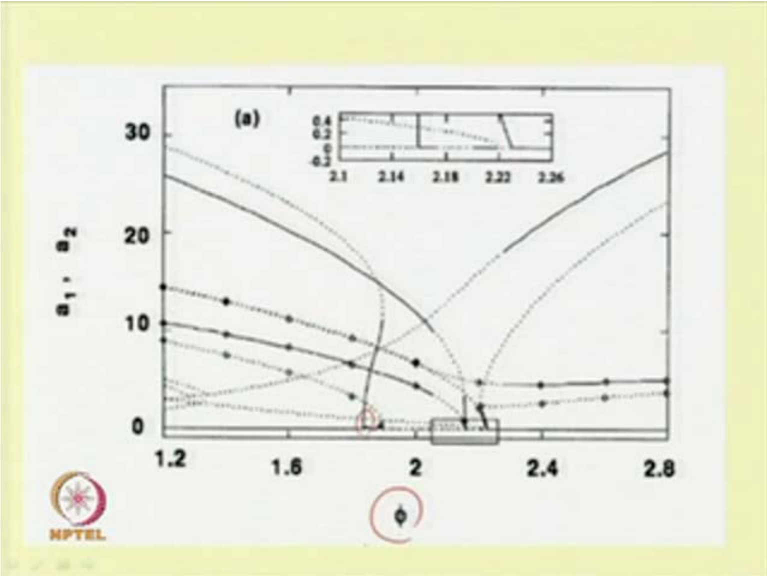

Similarly, so this point has been, so the previous point has been expanded, so this is the point expanded and to show clearly, what is happening in this trivial state, so this is the trivial state, so trivial state becomes unstable here, and again it has a stable and unstable range, so unlike in case of with single mode interaction, one can see, so here a set of stable and unstable response trivial response obtained in case of this two mode interaction.

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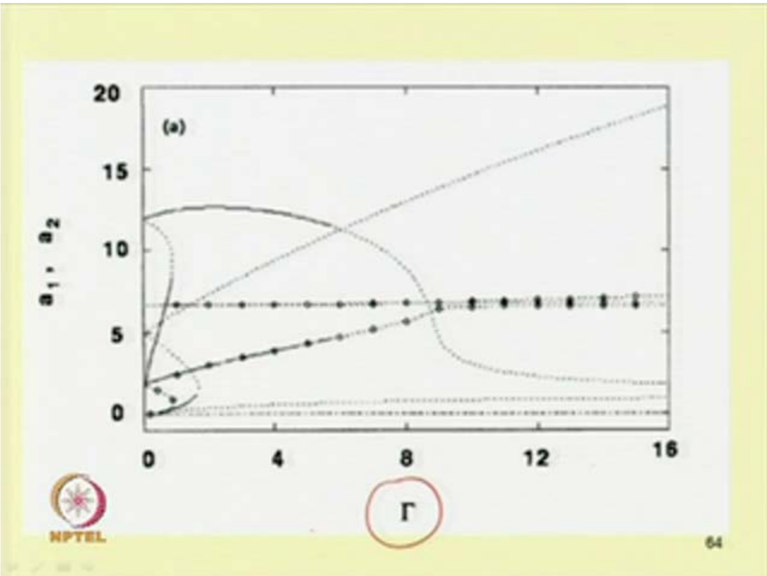

So this is with gamma, gamma is the forcing function, so these are with the frequency, this is forcing function one can obtain this response.

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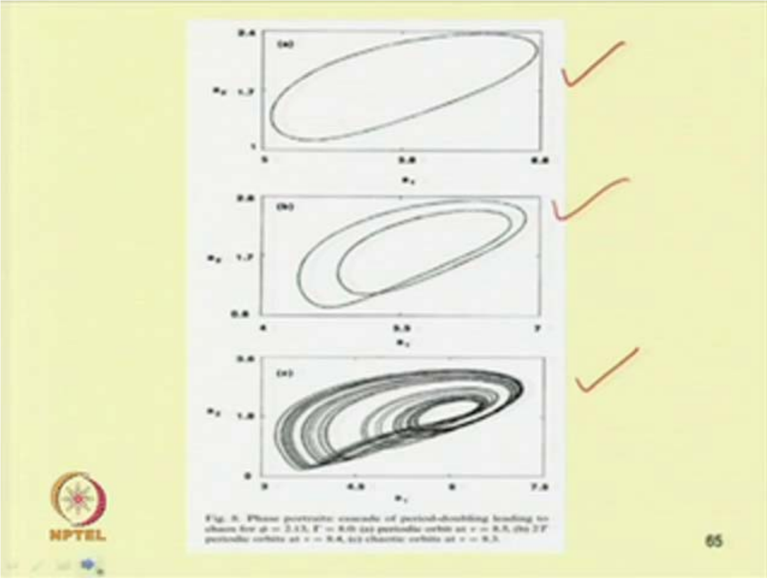

And here at this Hopf Bifurcation point, one can see the period doubling root two chaos, so this is single period this is two period and this is chaotic response.

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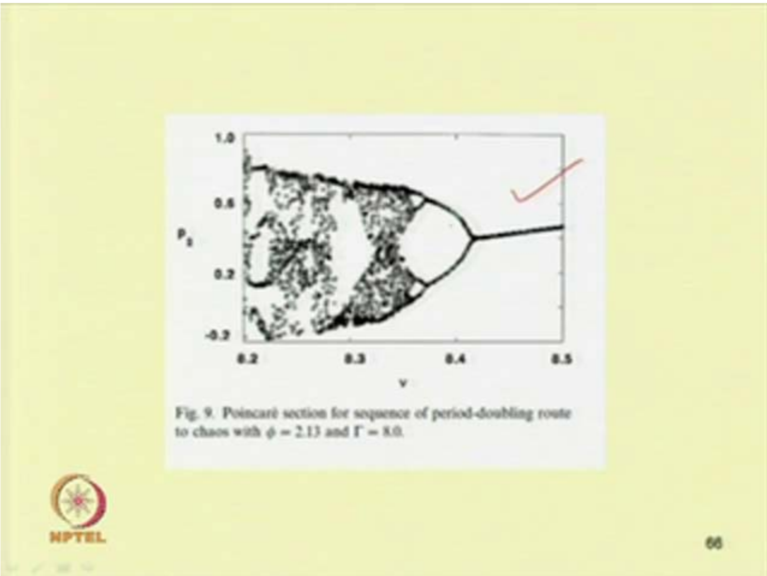

Similarly, so this is this is the Poincare section, showing a Poincare section showing the sequence of period doubling route to chaos and these responses are clearly chaotic here.

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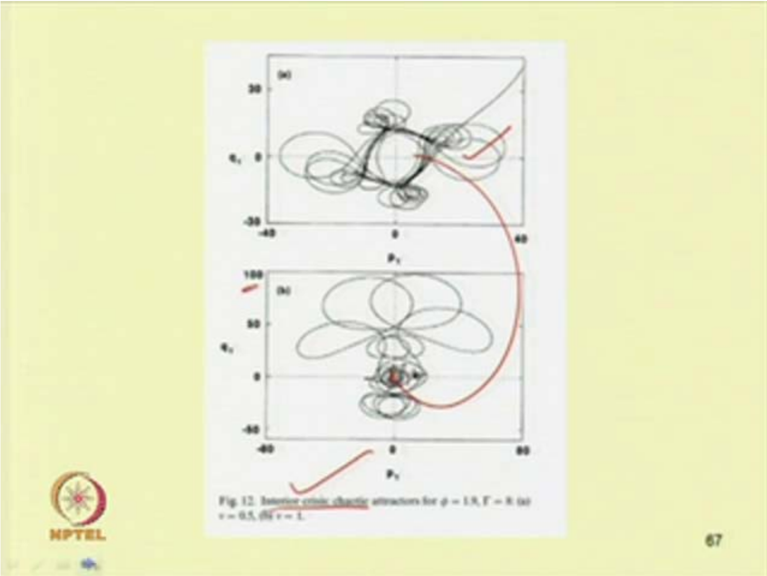

So, one can observe this chaotic, this is a chaotic response, so this chaotic response after changing this control parameter comes in contact with a unstable fixed point response, and one can observe, here the maximum value is 30, here the maximum value comes to be 100, so when it comes in contact with and unstable fixed point, then it explodes and one can obtain a bigger attractor, so these chaotic response, now is inside in this bigger chaotic response, and this type of this type of breaking of this chaotic response or this or this type of chaotic behavior is known as crisis, so when this chaotic response form in contact with an unstable fixed point response or periodic response, it explodes and one can obtain these interior crisis so this type, so this these previous chaotic attractor is in the interior part of this bigger chaotic attractor, so this is interior crisis.

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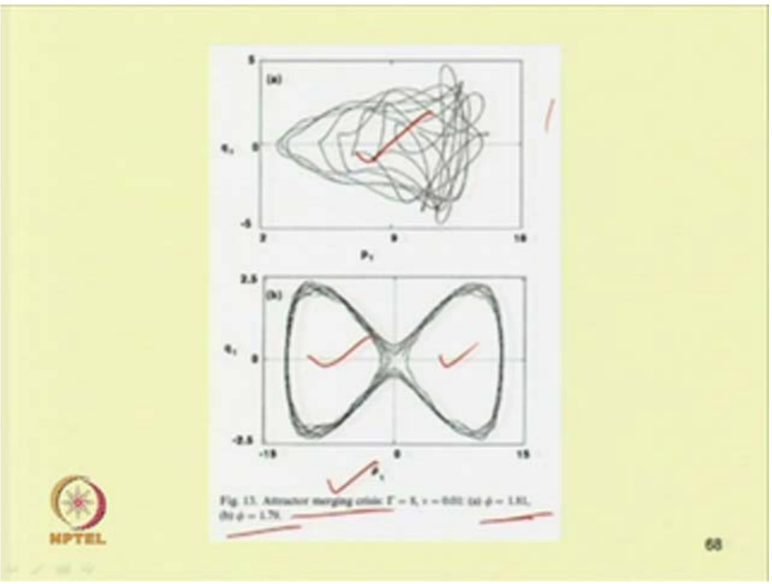

Similarly, so before Bifurcation , this is one chaotic attractor so by changing this, so let at 1.815 equal to 1.81, so one can observe observe two sets of chaotic attractor; this is one set and this side also will be another set and by changing this phi to 1.79, one can see that these two chaotic attractor merge to form a attractor merging crisis.

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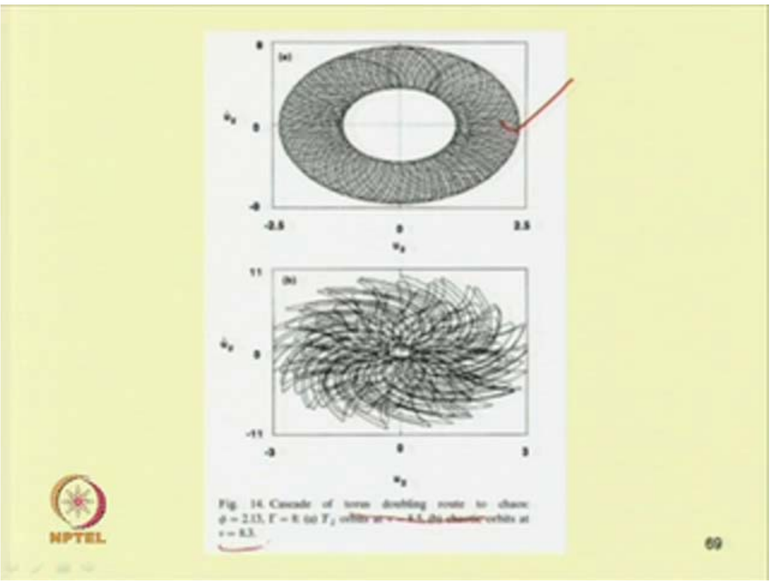

Similarly, one can observe different type of cascade of so initially, we have a torus then this torus doubling route to, so one can obtain a torus doubling route to chaos, in this case by changing this parameter.

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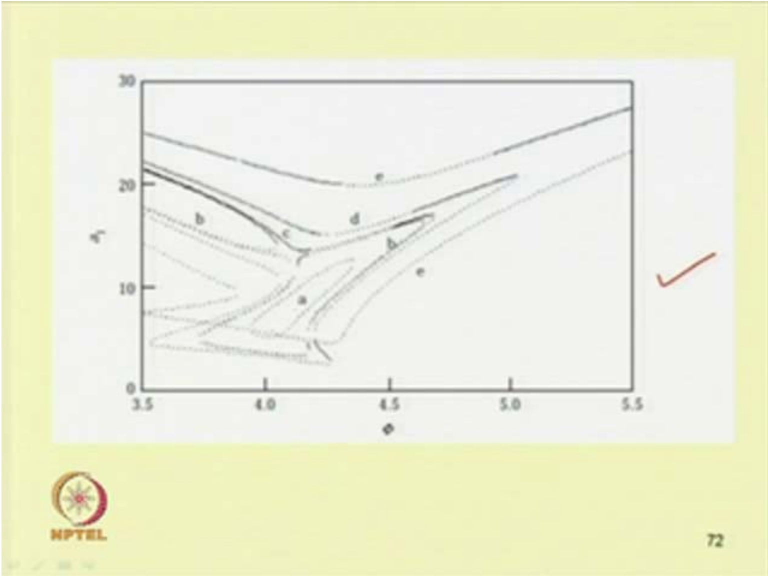

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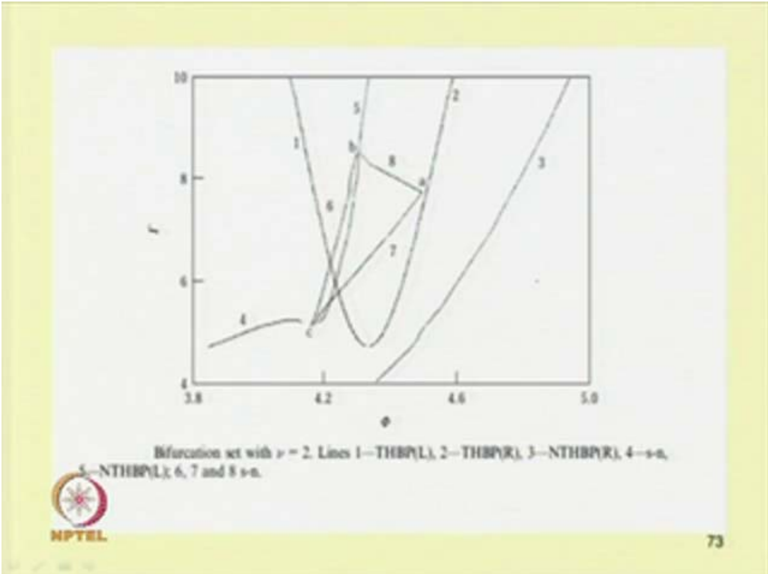

Also for combination resonance, so this is a set of fixed point response; so from this fixed point response one can plot this Bifurcation diagram, so after plotting this Bifurcation diagram one can see, so these are different limit cycles and so this is a set of so, this is a set of periodic response.

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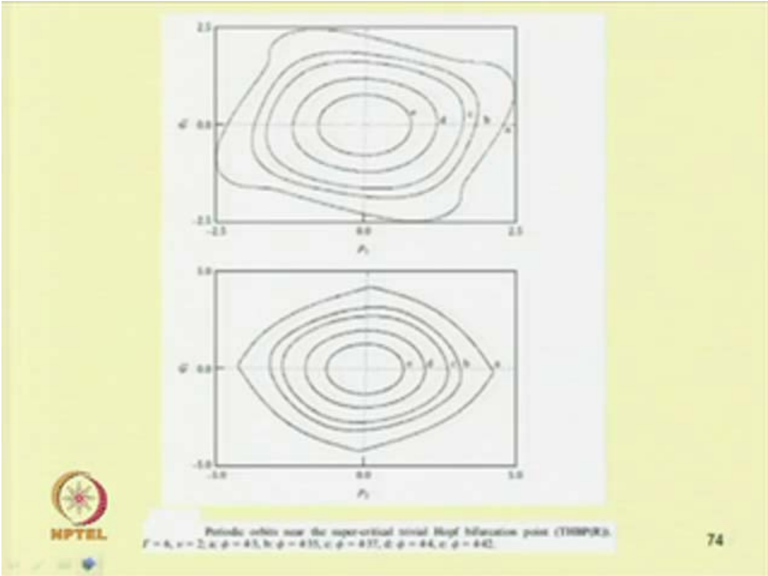

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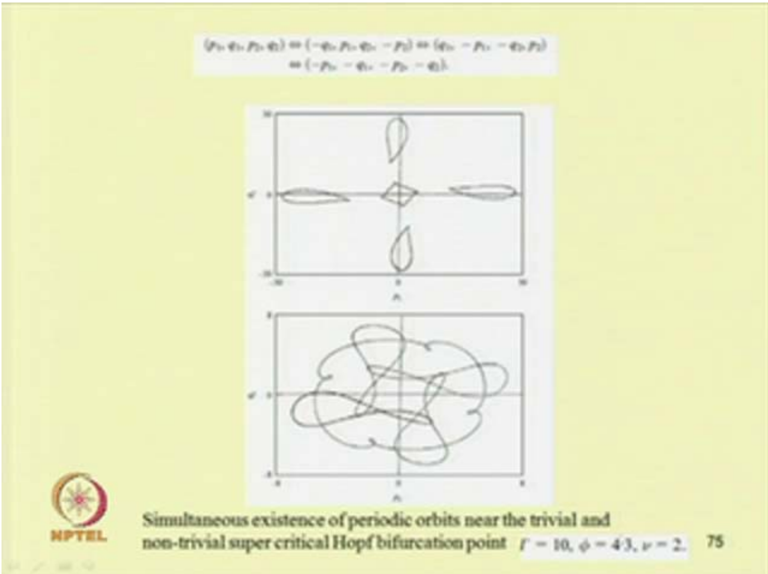

So, symmetric breaking, so one can see the symmetry breaking route to chaos, so initially we have a symmetric periodic response and by changing the system parameter, we can have a chaotic response.

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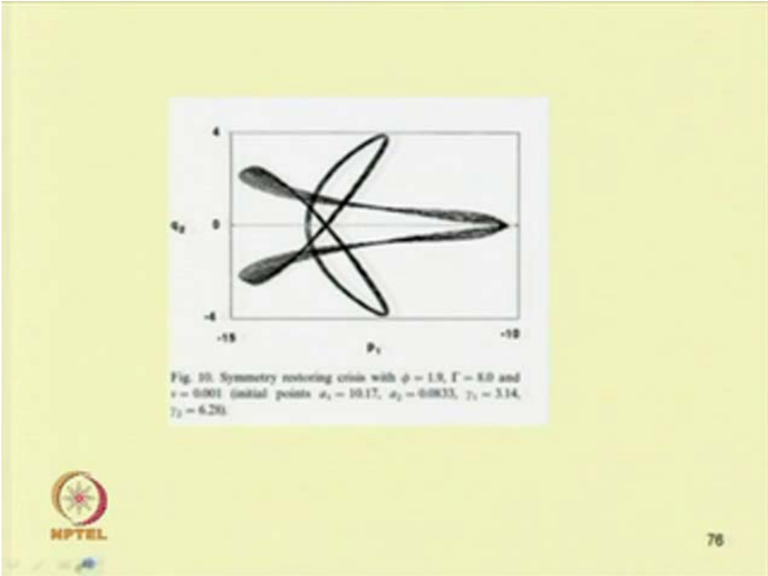

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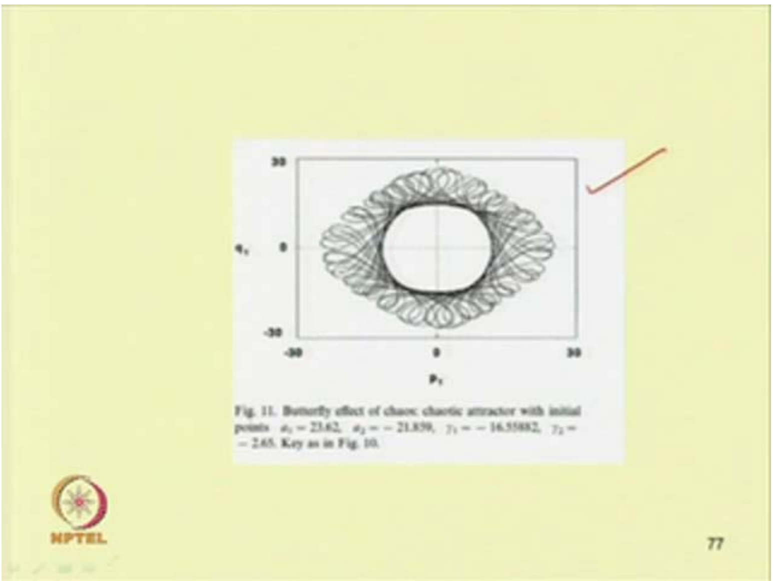

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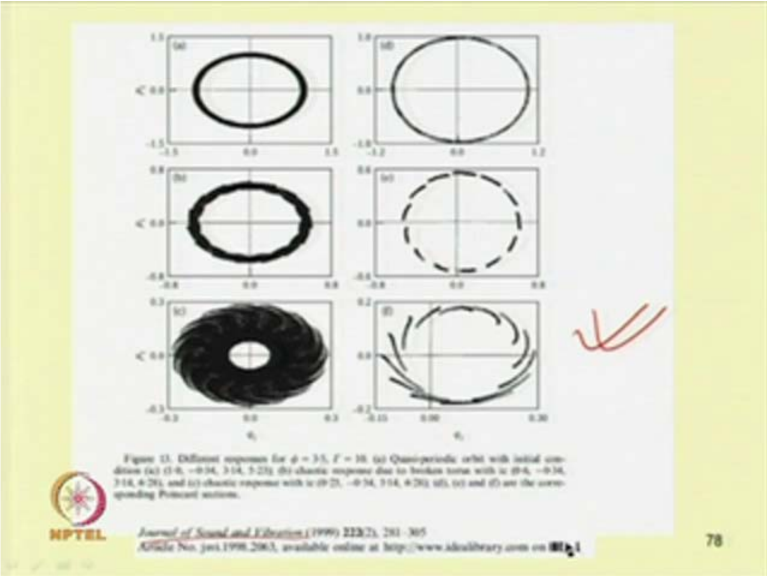

So this is another symmetry restoring crisis, and one can observe a butterfly effect by changing, this initial condition chaotic response one can obtain; so this is one example, where we have seen this initial torus break down route to chaos initially, we have a torus by changing the system parameter, this torus breakdown and finally. So, this torus to this is the Poincare section; in this Poincare section one can see a close loop, these loops slowly it breaks and finally, it becomes chaotic, so this is this published in journal of sound and vibration, and this is a torus break down route to chaos.

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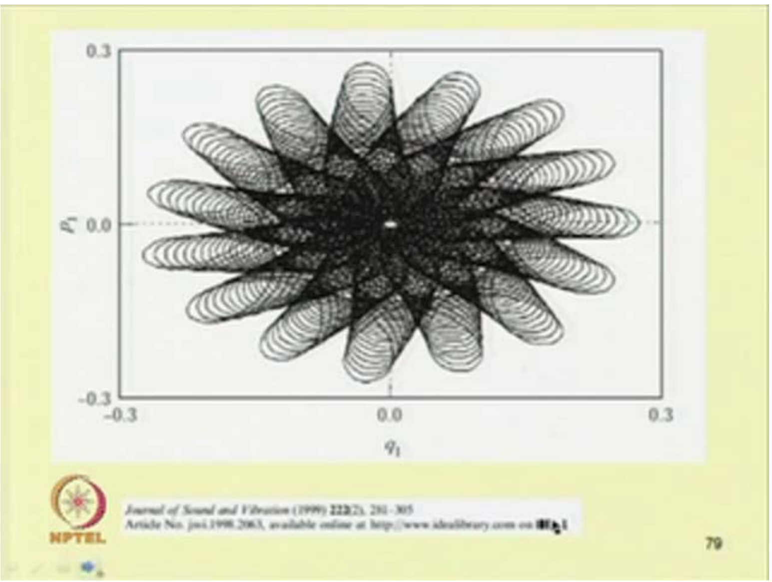

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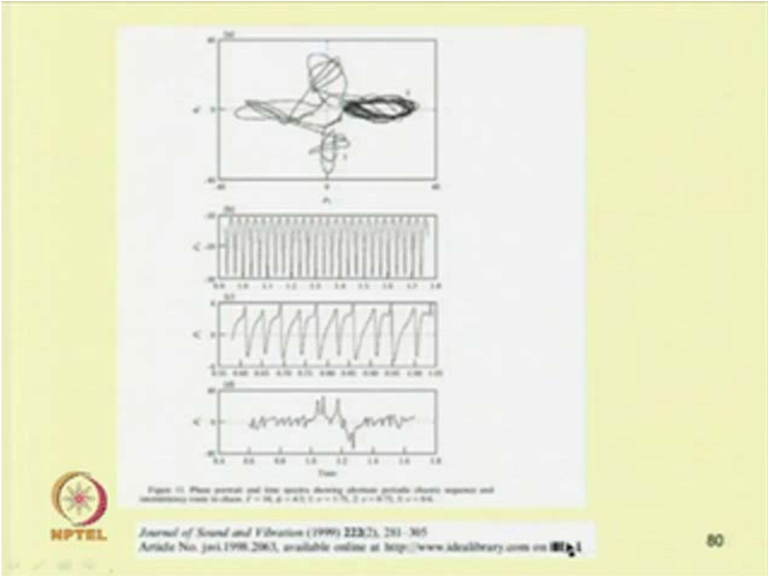

So the, so it is shown in p 1 q 1 portrait phase portrait or state space. So this is also another time response showing alternate, so you can have alternate periodic and chaotic response.

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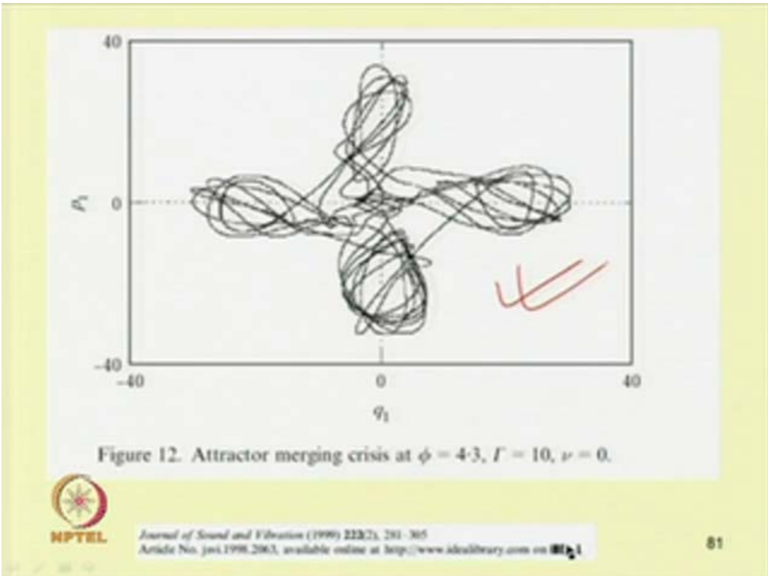

Intermittency, to show the intermittency, this is one example of attractor merging crisis.

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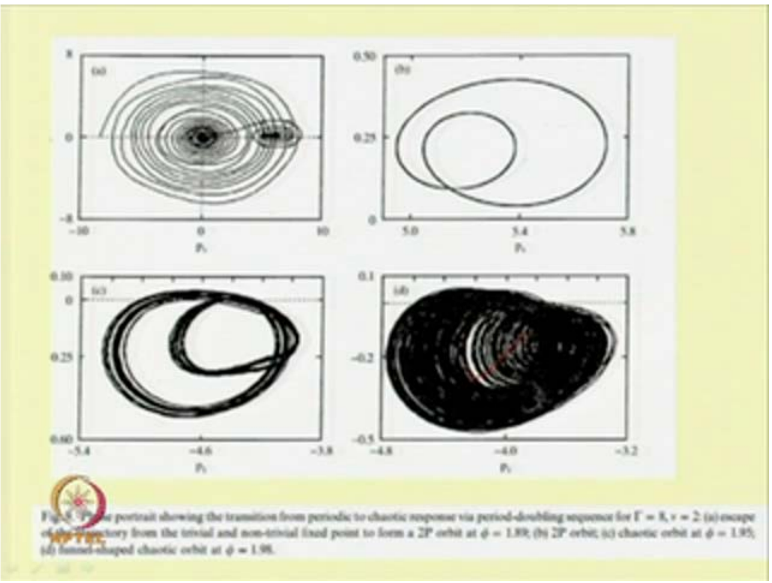

Similarly, in case of three mode interaction also one can observe different type of chaotic response, so here the initial chaotic response come or initial chaotic response come in contact with a unstable fixed point, and it explodes to a bigger attractor, so similar to rosier funnel, here also in this simple example of a cantilever beam, base excited cantilever beam so on.

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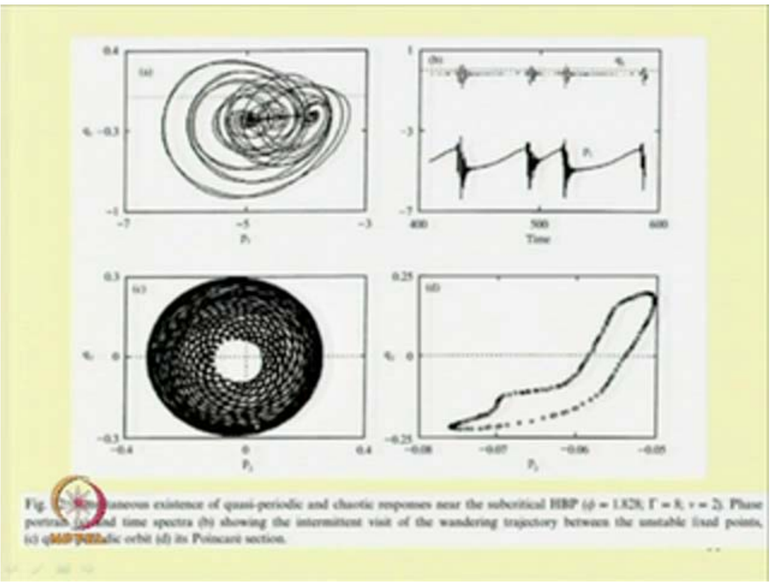

One can obtain this chaotic response, which is similar to that of a funnel. So here also one can obtain different, so it travels or it switch off switch between different unstable periods or different unstable or bits to give a chaotic response, so with this example so we know the types of responses that is fixed point response periodic response quasi periodic response and chaotic response exhibited by a non-linear system, so and also we have studied their stability, stability of the fixed point periodic response and next module, we will study about different vibration response; that is free vibration response force vibration response, and response of parametrically excited system.

Thank you.