Non-Linear Vibration
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Module - 4
Stability and Bifurcation Analysis
of Nonlinear Responses
Lecture - 6
Bifurcation of Periodic Responses Introduction
to Quasi-Periodic and Chaotic Responses

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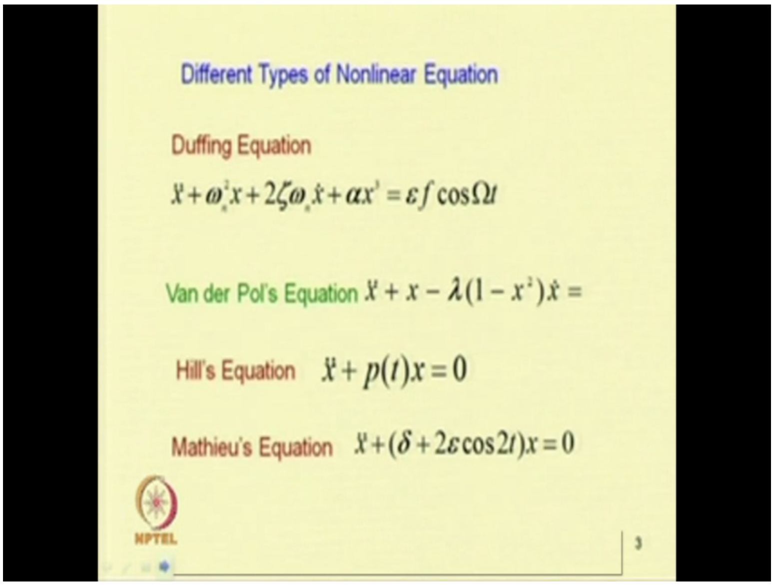

Welcome to today class of non-linear vibration. So, today class, we will discuss about this Bifurcation of Periodic Responses and Introduction to Quasi Periodic and Chaotic Responses. So, before going for this Bifurcation of Periodic Responses, we will discuss about this procured theory in detail and we will solve some problems to find how to find the Floquet multiplier. And already, we have discussed about this Duffing Equation, Van der Pol's Equations, Hill's Equation, Mathieu's Equation where we are getting periodic response in addition to the fixed point responses. So, in those periodic responses whether they are stable or unstable can be determined by using this Floquet theory.

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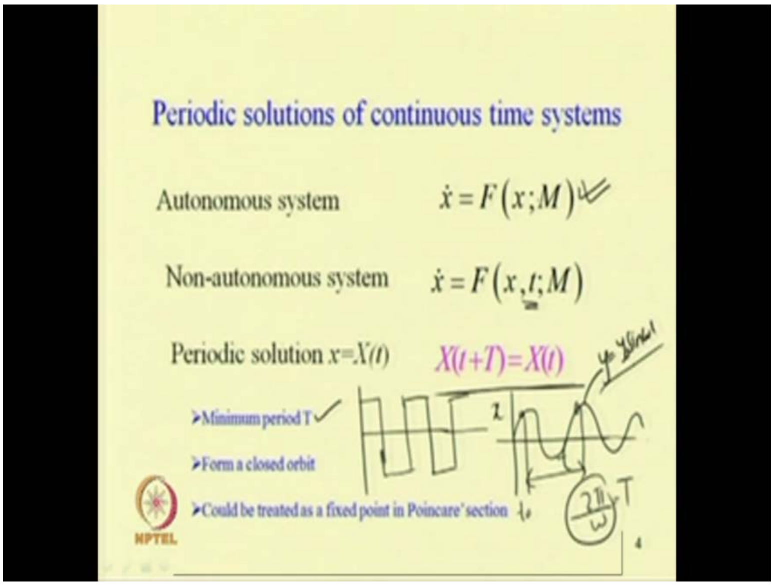

And so periodic solution, what do you mean by periodic solution? So, let us consider this autonomous system x dot is a function of x and M, where M is the control parameter and if the solution x t, that is x with time. So, can be expressed in this form that is it repeats after a time interval of t with a minimum period of t then we can tell the solution to be periodic. For example, so, let us plot this x versus t, so this is x and this t. So, let the solution is periodic. So, this is a periodic solution. So, in this case the solution repeats. So, if the solution, so this is I can write this is y equal to y 0 sin omega t.

So, if the solution repeats with time for example, in this case so it will repeat, so if the frequency is omega, so this time period will be equal to 2 pi by omega. So, the value here and here are same. So, the value; whatever value for x we are getting at this time t equal to t 0, the same value we will get at a time at t 0 plus capital T. Capital T is the time period which is equal to this 2 pi by omega in this case. So, if the response is harmonic. So, this is one type of periodic solution, harmonics will contain both sin and cosine terms, also we can have other different type of periodic solution also. So, where it may be of saw type, rectangular type and many other types. So, for example, in this case it is of rectangular or in control system particularly it is used and this is known as bang bang type of response or bang bang type of excitation.

So, you can give a force like this and we can except a solution or this type of periodic response. Also, we can have saw type also it will have different type of periodic response. So, this periodic response so in this case we can consider the system to be autonomous. So, in case of autonomous, this is not explicit function of time t, but in case of non autonomous system it is an explicit function of time t. So, this field variable can be written or x dot can be written as a function of x, time t and control parameter M, in case of non autonomous system, but in case of autonomous system. So, the time term will not explicitly come into picture. So, in case of the periodic solution already, we have discussed that it will have a minimum period t. So the phase portrait will be a closed orbit. So, if you plot the phase portrait that is x versus x dot, that means velocity displacement versus velocity. So, then it will be a closed circuit.

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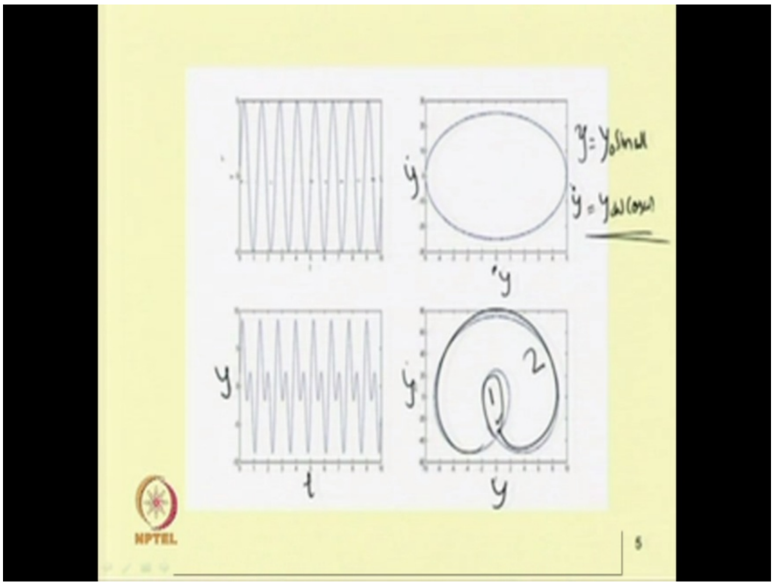

For example, so if y equal to y 0 sin omega t then its velocity will be y dot. So, y dot will be equal to y 0, y 0 omega cos omega t and if you plot this y dot by x y, y dot by y. So, this is y dot by y. So, if 1 plot y dot by y then one can get a close orbit. So, in case of a single periodic system, we have as we have discussed before. So, one can obtain a single closed loop, but in case of 2 frequency, so instead of a single frequency, if you have 2 frequency then this y dot versus y, so y dot versus y will contain 2 loops.

So, this is inside 1 loop, inside we have 1 loop and this is the outer loop. So, we have 2 loops. So, this is loop 1, this correspond to loop 2 and here also from the time response that is your y versus y, y versus t also, one can see the response is periodic, but it contains harmonics. Similarly, in the first case, it contains. So, this is only, it will have a single period. So, in this case it has 2 period, so this is also y versus t. So, one can write a simple mat lab program to study or to find the periodic response of different type of system for example, in case of the Duffing Equation or in case of the Van der pol Equation.

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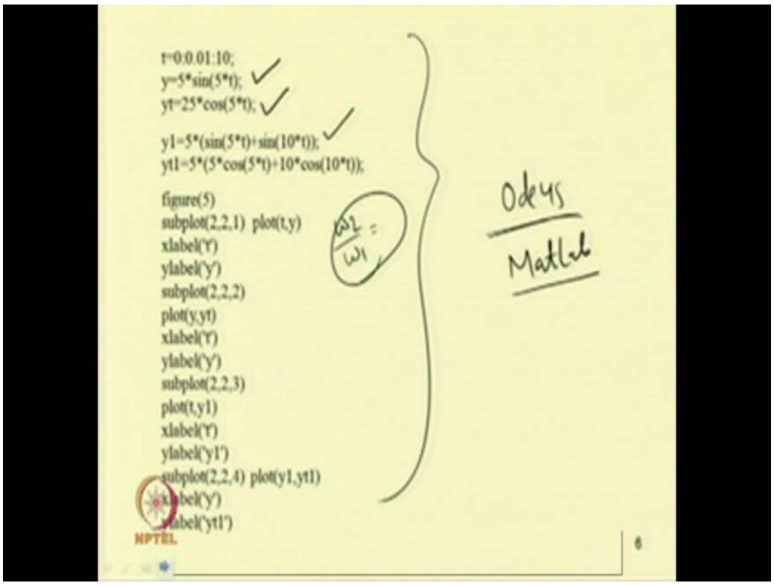

So, one can use this numerical technique ode45 to ode45 of mat lab can be used to find the response in case of the Duffing Oscillator or Van der Pol Oscillation and one can see. So in those cases we can have different time response. So, this time response this time and phase portrait is plotted using this mat lab. So, here this y 1 represent the y 5 sin t plus. So, here this is for y equal to 5 sin t so, y t equal to 25 cos t, 25 cos 5 t. Similarly, for 2 periodic one can have 2 5 into sin 5 t plus sin 10 t. So, 2 frequency have been taken. So, in this case, it can be noted that this omega 2 and omega 1, omega 2 by omega 1. So, this is an integer. So, if omega 2 by omega 1 is an rational number then or if it is integer then one can find a periodic or 2 periodic 3 periodic response of the system, but if this ratio is an irrational number. So, later we will see this will lead to Quasi Periodic Response.

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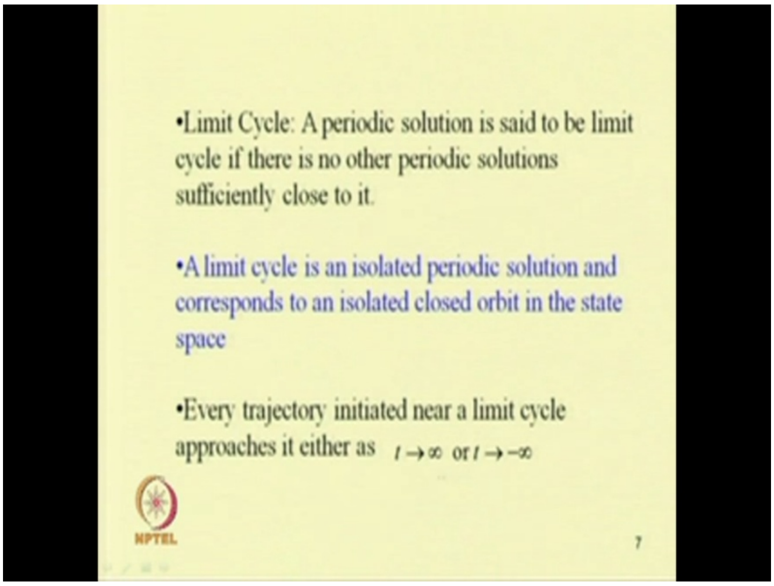

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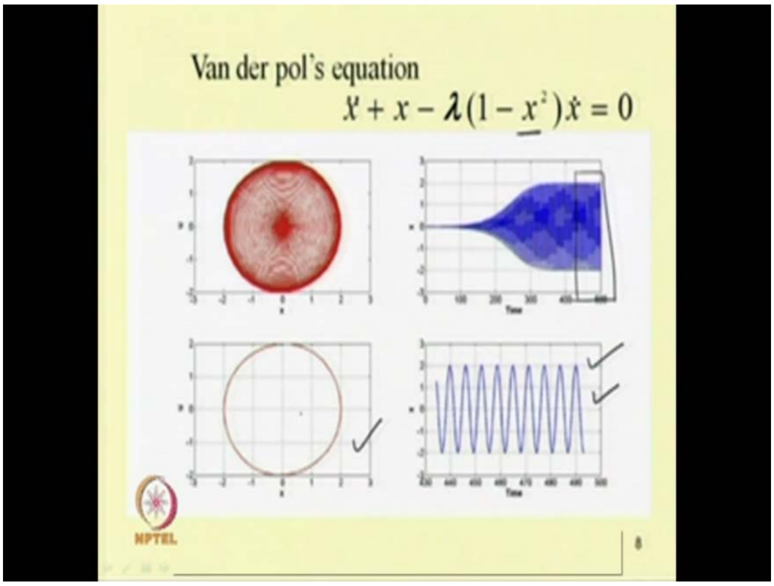

So, one can plot this periodic or Quasi Periodic Response. Already, we have discussed about this limit cycle, a periodic solution is said to be limit cycle, if there is no other periodic solution; sufficiently close to it. So, a limit cycle is an isolated periodic solution and correspond to an isolated closed orbit in the state space, every trajectory initiated near a limit cycle approaches it either as t tends to infinity or t tends to minus infinity. So, this plot is for this Van der Pol Equation. So, here one can see. So, in case of the Van der pol Equation, the equation is given x double dot plus x minus lambda into 1 minus x square into x dot equal to 0.

So, here is the non-linear term x square into x dot and in this case one can find a isolated limit cycles. So starting by taking a starting point here, so one can see with time, so it grows and finally, one obtain the limit cycle. So, this plot in this, this figure and this figure it is plot only from 430 to 500; that means, this portion of the curve, this portion of the plot is plotted here. So, which show the steady state response in case of the Van der Pol Equation. So, if you can one start a one point outside this thing also if one take a initial point outside this u versus x curve. So, then also one can find that it will come back to this limit cycle.

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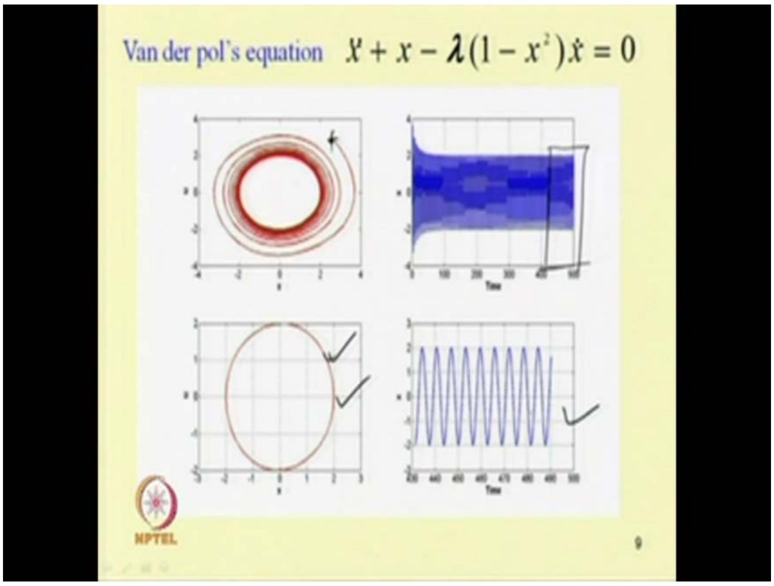

So, this is the starting point. So, if one take this is the starting point. one can see with time. So, it come back to the or comes to the limit cycle. So, in this case also this last portion that is the steady state solution part is plotted in this 2 case. So, this shows the initial condition with the along with the steady state solution, and in this case the steady state part is shown in the response. So, in these examples, we have seen. So, there are several systems were we have periodic solution. So, one has to study; whether, the periodic solution is stable or unstable, also one can get this periodic, periodic solution as in case of the Hopf bifurcation. So, if the solution is periodic.

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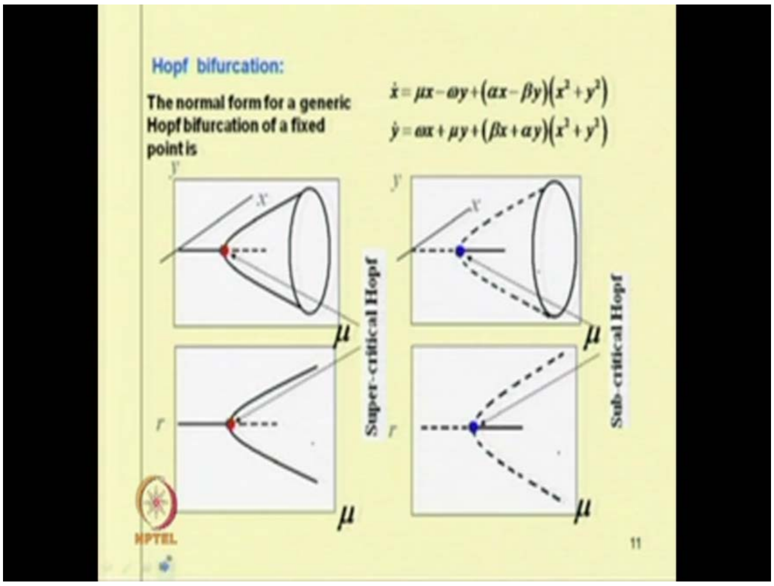

Then how, so in case of the Hopf bifurcation already, we have seen, this one obtain a periodic solution after this Hopf bifurcation. So, this may be either super critical or sub critical as we have discussed before. So, one has to study whether, these periodic responses obtained are stable or unstable, by using this persuade multiplier.

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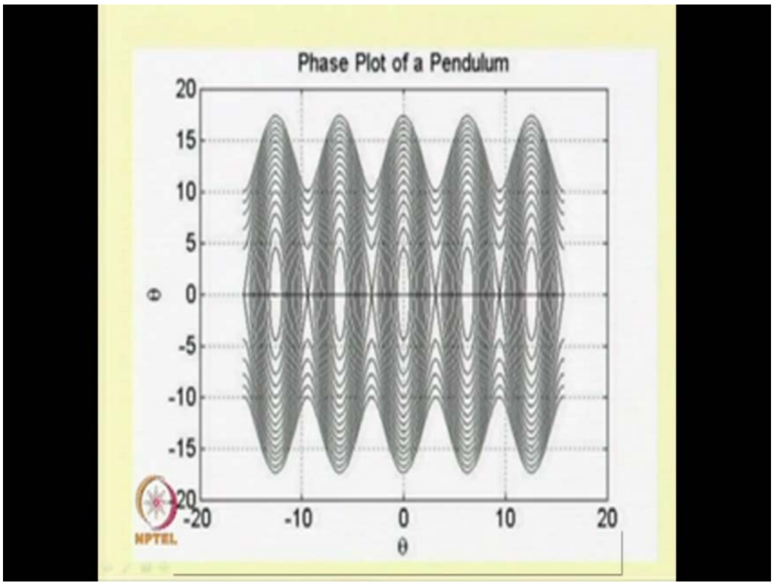

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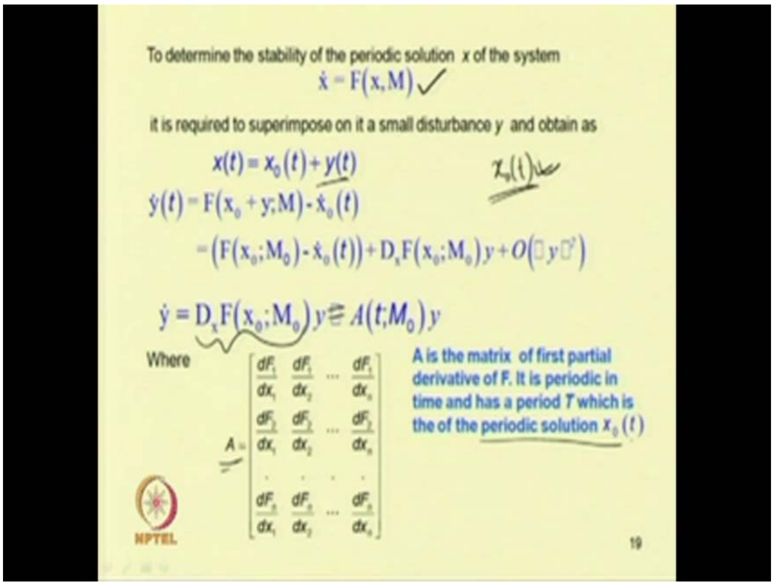

So, you can see so, here also, in case of the pendulum. So, we have periodic solution in between these homo clinic orbits. So, these are the several homo clinic orbit. So, in between these homo clinic orbit, we have infinite sets of periodic orbits, so to study the stability of the system. So, as already told Floquet, Floquet theory can used. So, let us revisit this Floquet theory. So, let we have a system autonomous system x dot equal to F x M so, in this case to study the stability.

So, let the solution steady state solution which is period be x 0 t. So, to study whether, this x 0 t is stable or unstable. Let us add a perturbation to this that means, let us write x t equal to x 0 t x t equal to x 0 t plus y t then by substituting this in this equation. So, one can obtain this y dot t equal to F x 0 y M minus x 0 t. So, which will give so, which will give us y dot equal to d x that is first derivative of F at x 0 and M 0 or so this is equal to, so i dot equal to by taking this d x F 0 M 0 as a, so one can write y dot equal to, so this y dot equal to a into t a t M 0 a is a function of t and M 0. So, a matrix so, a is the matrix of first partial derivative of f. So, one can see that this is periodic. So, this is periodic in time and has a periodic t which is the period of the periodic solution x 0 t.

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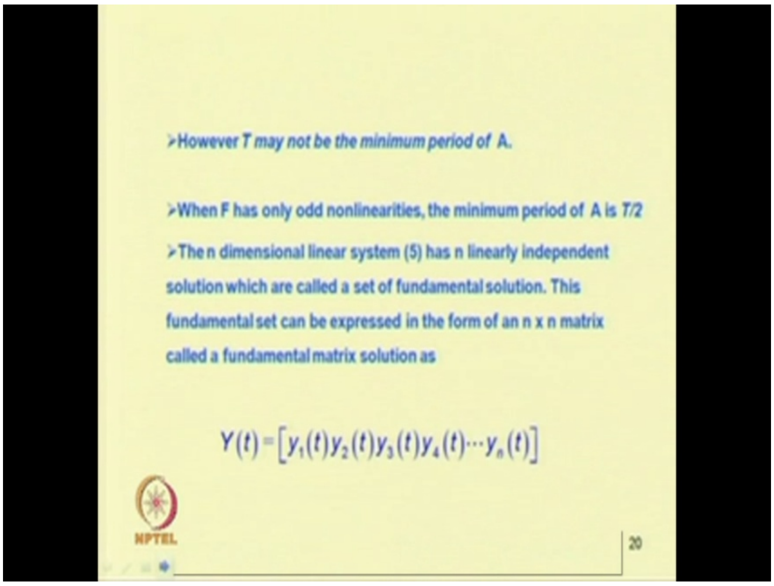

So, however T may not be the minimum period of T. So, when F has only odd non-linearity the minimum period of t is T by 2. When n the n dimensional linear system has so this equation the n dimensional, if you take n number of equations, this n dimensional equation will have n linearly independent solution, which are called as set of fundamental solution. So, this fundamental set can be expressed in the form of n cross n matrix called a Fundamental matrix solution and this Fundamental matrix solution can be written as y 1 t, y 2 t, y 3 t and y n t.

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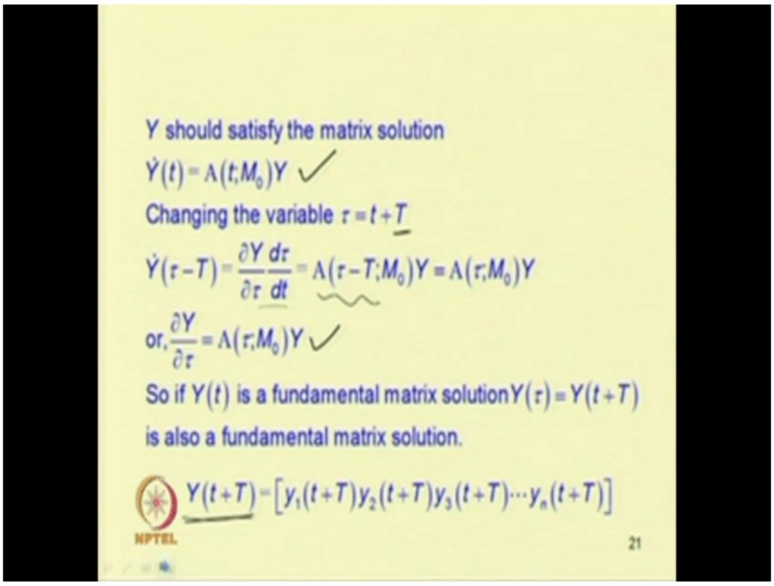

So, this Y t with the fundamental set of solution and in this fundamental set of solution. So, we can write this Y should satisfy the matrix equation Y dot equal to, so it should satisfy this equation. So, it should satisfy this Y dot equal to A T M 0 y. So, if it satisfy then we can write this Y dot T equal to A T M 0 y. So, changing the variable or writing tau equal to t plus T, where T is the period of this solution then one can write this y dot for T we can write tau minus T. So, Y dot tau minus T equal to d y by d tau into d tau by d T by applying this chain rule. So, this will be equal to a tau minus t M 0 y as a is periodic with a period T.

So, one can write that a tau minus, tau minus T should be equal to also A T M 0 Y. So, or one can write this Y dot as T is constant. So, thing one can write this is d Y by d T or d T by d tau, d tau by d T equal to one. So, d Y by d T equal to a tau M 0. So, comparing this equation and this equation, so one can see this Y dot T is same as dot, in case of Y t plus T. So, Y dot T equal to A T M 0 Y, it is same as a d Y by d, tau A, tau M 0 Y. So, from this it can be seen. So, if Y t is a fundamental set of matrix solution then Y t plus T Y small t plus capital t also is a fundamental matrix solution, but as the equation Y t equal to A T M 0 Y has n only n linearly independent solution. So, if Y tau is the solution then Y t plus tau must be linearly dependent on this Y T. So, we can write this Y t plus T that is T, t plus T should be a linearly dependent function of Y t.

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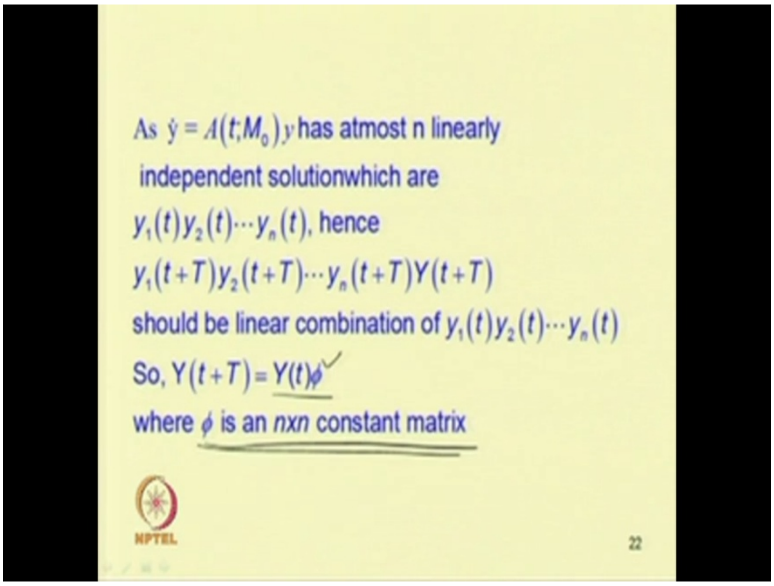

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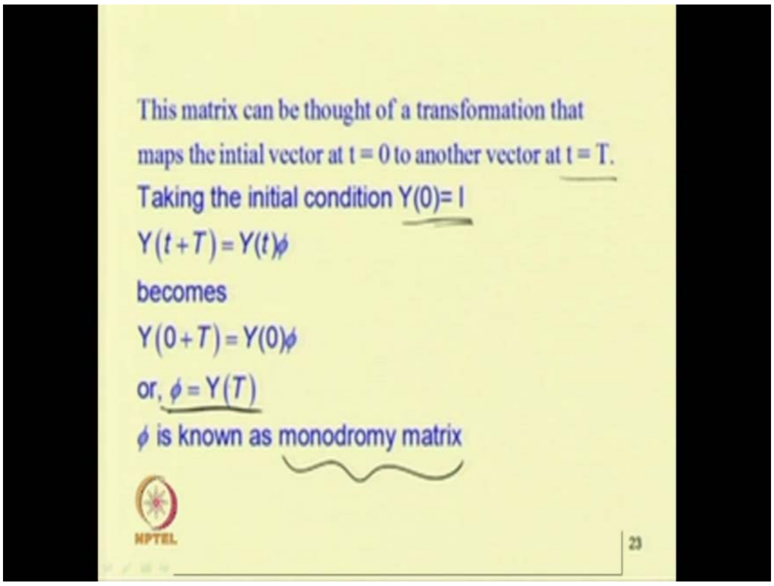

So, one can write then Y t plus T should be equal to Y t into phi. So, this, so phi is a n plus n constant matrix. So, this constant matrix n plus n constant matrix phi is known as the monodromy matrix. So, this matrix now, one can see that this matrix can be obtain. So, if you know the solution Y t then, we can phi, this matrix phi, so how to find this thing. So, this matrix can be thought of as a transformation that maps the initial vector at T equal to 0 to another vector at t equal to T. So, taking initial condition that is Y 0 equal to a unit matrix. So, if you take this Y 0 equal to a unit matrix then by putting this T equal to 0.

So, Y t plus T can be equal to y 0 into phi, but as we are taking Y 0 equal to unit matrix. So, we can write this phi equal to Y t. So, one can obtain this phi, that is the monodromy matrix. By finding the solution; by finding the solution at t equal to T, t equal to that is time equal to capital t. So, later we can see that the Eigen values of this. Monodromy matrix phi are called the Floquet or characteristic multipliers.

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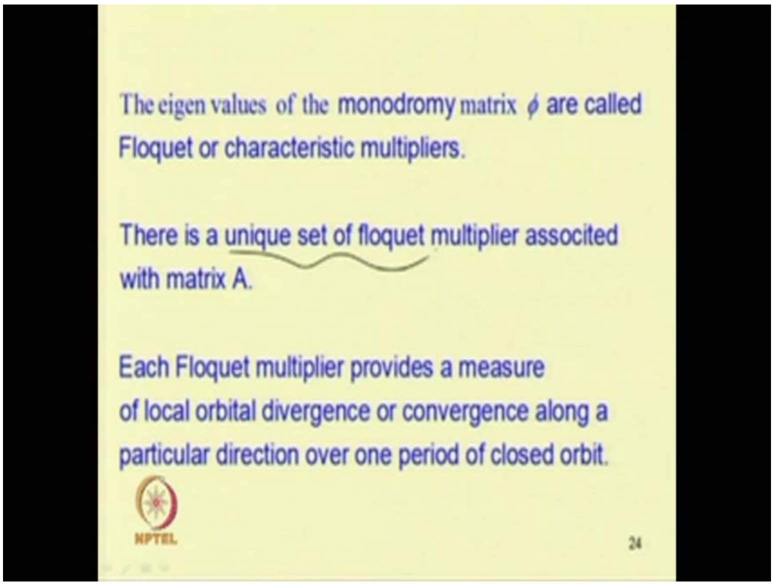

So, there is a unique set of Floquet multiplier associated with this matrix a and each Floquet multiplier proves a measure of local orbital divergence or convergence along a particular direction over one period of closed orbit. So, in this case as one can take this initial condition differently. So, this phi or this monodromy matrix cannot be a unit matrix. So, it, it depends on the initial condition, but if one can see that this Floquet multiplier or a unique set. So, these are the unique set of unique set associated with a with the matrix A. So, in so one can so, while finding this Eigen value phi to find the Floquet multiplier.

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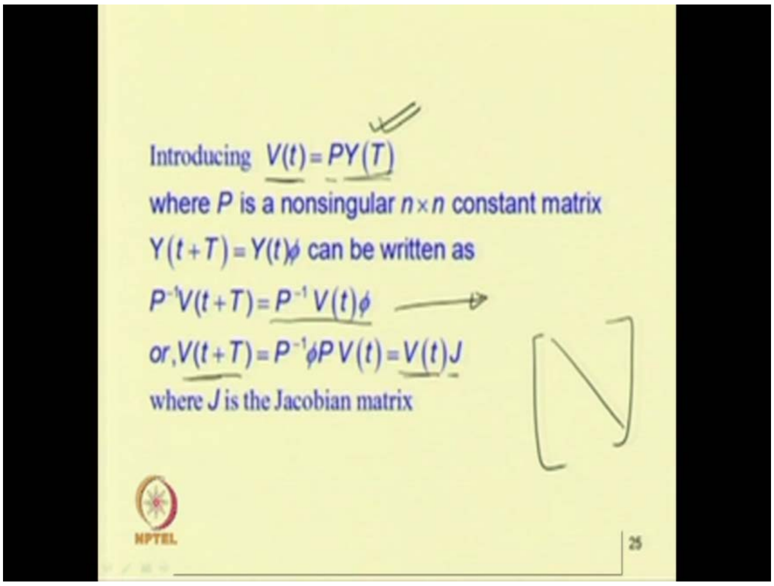

So, one can also use this transformation to make the analysis more simpler by taking. instead of taking. So, let us assume this V t, V t is another variable. So, V t equal to P into Y t. So, already Y t is known to us or we can write this V t equal to PY T square P is a non singular n cross n constant matrix. So, we can write this Y t plus T equal to Y t into phi. So, by substituting this equation, that means; Y t equal to P, P inverse V t. So I can write this P inverse V t plus T equal to P inverse V t into phi or V t plus T can be written as.

So, post multiplying this equation by P. So, one can write this P transpose as P transpose, P equal to I. So, this equation reduces to V t plus T. So, V t plus T will be equal to P transpose phi PV t or this is equal to V t into J. So, where J is a Jacobian matrix or J is a matrix. So, this matrix will be a diagonal matrix. So, this matrix is a diagonal matrix, containing the Eigen values; containing the Eigen values of the matrix phi.

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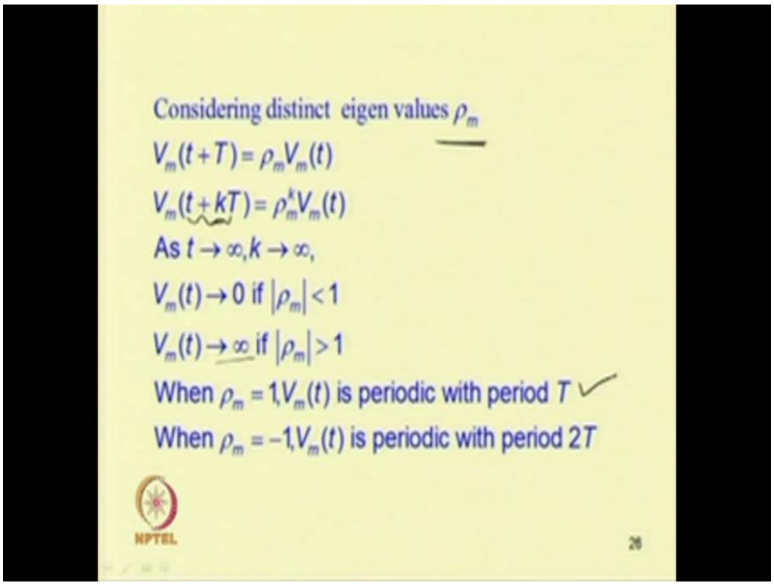

So, considering distinct Eigen values of phi as rho M so, one can write this V m t plus T equal to rho M V M t, so V M t plus. So, if one take V M k T. So, after k, time period, so this will be equal to rho M to the power k V M t. So, as T tends to infinity as T tends to this small t tends to infinity; that means, this inside term also tends to infinity which can happen when k also tends to infinity. So, by putting k tends to infinity in this equation. So, one can see. So, V M t will tends to 0, if rho M so, rho M less than equal to 1. So, if rho M equal less than equal to 1.

So, in that case, it will tends to 0 and it will tends to infinity, if rho M greater than, so if rho M mod rho M greater than 1. So, that means, if rho M to the power so, this minus, if rho M is minus less than, less than 1. So, in that case, we can obtain a solution, which will tends to 0, otherwise the system will grow and we can have, so if rho M greater than 0. So, it will grow and we will have infinite response so, when rho M equal to 1. So, V M is periodic with a period T and rho M equal to minus 1. So, V M is periodic with period 2 T.

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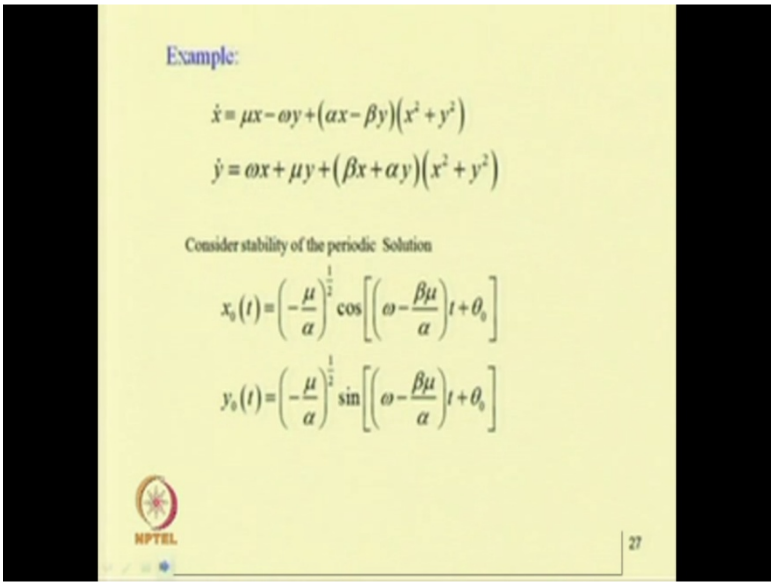

So, for example, so we can take a simple example, let us take the simple example of the simple spring mass damper system. So, that is M x double dot plus k x equal to 0.

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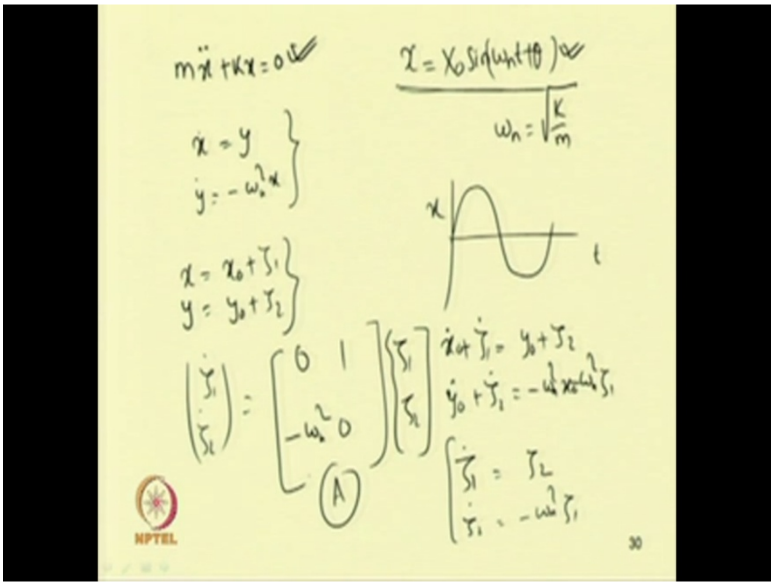

So, all of us are, all of us know the solution is periodic. So, the solution can be written x equal to x 0 sin omega n t plus phi so, omega n t plus, so I can put some other notation so, omega plus theta so, where omega n equal to root over k by M and this theta and x 0 depends on the initial condition. So, this equation also, one can write in terms of first order equation, that is x dot, one can write x dot equal to. So, x dot equal to y and y dot will be equal to minus omega n square x. So, one can write using a set of first order equation or one can write this solution directly were the solution is known to us. So, that is x equal to x 0 sin omega n t plus theta.

So, in this case, so let us find or let us check whether, the solution, what we obtained is stable or not. So, we can take, so the solution, So, if one plot the solution that is x versus t, the solution is a sin omega t. So, we have a sin curve. So, we can take, so let us take. So, for this case, we can substitute x equal to x 0 plus zeta 1 and y equal to. So, y equal to y 0 plus theta 2. So, if we substitute this in this equation and we can find the equation, that is zeta dot zeta 1 dot zeta 2 dot equal to. So, this equation becomes so by substituting in this. So, we can have or let me write this equation.

So, we have this x dot equal to y, y dot equal to minus omega n square x. So, we can have this zeta x 0 dot plus zeta 1 dot equal to y 0 plus zeta 2 and second equation becomes y 0 dot plus zeta 2 dot equal to minus omega n square x 0 minus omega n square zeta 1, but for initial condition that is x equal to x 0 sin omega n t plus theta 0. So, we know. So, in this case, x 0 dot equal to y 0 and y 0 dot equal to minus omega n square x 0. So, they will cancel. So, we have this zeta 1 dot equal to zeta 2 and zeta 2 dot equal to minus omega n square zeta 1 so, our equation a.

So, will be equal to, so 0 1 minus omega n square 0 into. So, this is zeta 1 zeta 2. So, this is this matrix A, now we have to or to study the solution whether, this is stable or unstable. So, we have to find, so taking 2 initial condition. So, let us take 2 initial condition and study. So, first we can take the initial condition that is zeta 1 0. So, let us take the initial condition 1 is 1 0 and other 1 is 0 1. So, by taking these 2 initial condition that is. So, we can obtain after 1 time period. So, the function after 1 time period and then we can see whether, the response is stable or unstable.

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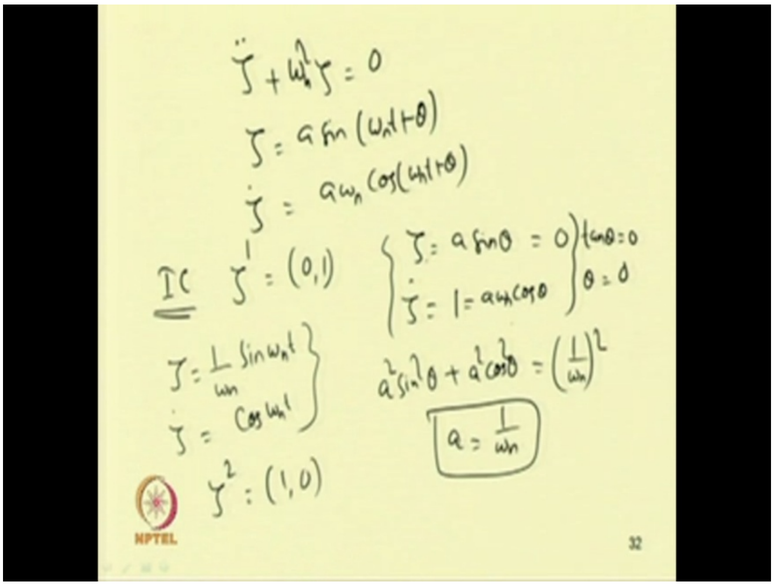

So, here our zeta the solution, this equation also one can write this equation can also be written, if somebody want to write the second order equation. So, this is also zeta double dot equal to omega n square zeta equal to 0. So, this zeta equal to a sin omega n t plus theta and zeta dot equal to a omega n cos omega n t plus theta. So, in this case now, taking initial condition so, let us take this initial condition that is zeta 1 equal to 0 1; that means, we will get this zeta equal to a so, at t equal to 0. So, this becomes at t equal to 0. So, zeta 0 becomes a sin theta and so this will be equal to 0 similarly, zeta dot equal to 1. So, this is 1. So, this is equal to a omega n cos theta so, from this. So, one can see that this tan theta equal to 0. So, tan theta equal to 0 or theta equal to 0. So, by squaring and adding these 2 by squaring and adding these 2 equations. So, one can see this a square or a square sin square theta equal to 0 plus a square omega n square cos square theta or by writing this squaring this thing a square sin square theta plus a square cos square theta.

So, this is equal to 1 by omega M square. So, from this one can obtain this a equal to 1 by omega n. So, the solution what we obtain, so this is zeta equal to 1 by omega n sin omega n t plus theta, as theta is already equal to 0. So, this solution, what we will have. So, zeta equal to 1 by omega n sin omega n t and this zeta dot equal to a omega n a equal to 1 by omega n. So, this becomes cos omega n t so, taking this initial condition. So, we got this zeta and zeta dot and so, after a time period t so, after a time period t. So, what we obtain, so we will obtain this zeta will be same as 0 and zeta dot also will be equal to

1 similarly, taking another initial condition that is zeta 2 equal to 1 0. Similarly, proceeding in the same way we can find the solution that is by taking zeta equal to 1.

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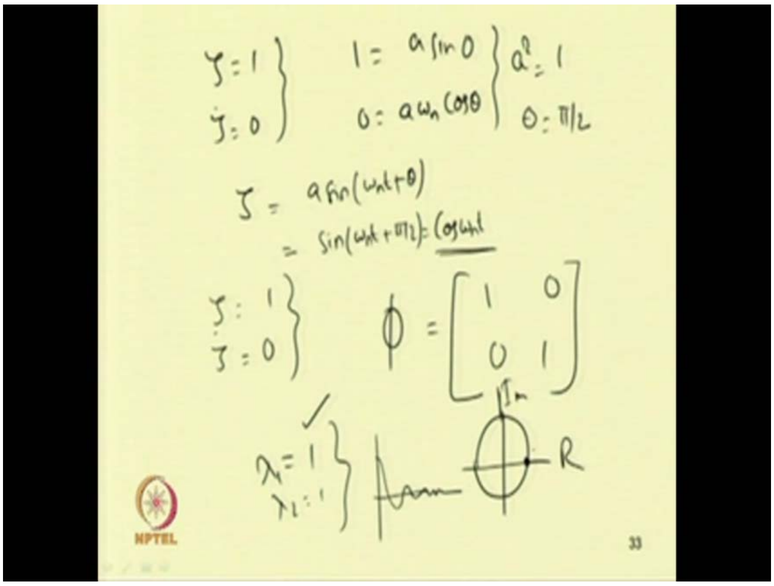

So, by taking zeta equal to 1 and zeta dot equal to 0 so, we can write 1 equal to a sin theta and 0 equal to a omega n cos theta. So, this gives, so squaring and adding. So, we can have this, a square equal to 1 or and this theta equal to pi by 2. So, in this case we got this zeta equal to a sin omega n t plus theta. So, this is equal to, so a equal to 1 so, this become sin omega n t plus pi by 2. So, this is cos omega n t so, the 2 fundamental sets of solutions. So, one become cos omega n t, and the other become 1 by omega n sin omega n t. So, we have 2 fundamental set of solution, we obtain. So, 1 is 1 by omega n sin omega n t and other 1 is cos omega n t. So, after 1 time period t so, we can obtain so, this zeta, in this case, we obtained zeta equal to 1 and zeta dot equal to 0. So, our monodromy matrix that is phi which is obtained after 1 time period.

So, this becomes 1 0 and 0 1. So, this is 1 0 and 0 1. So, from this one can obtain this lambda. So, a minus lambda equal to 0. So, one can see this lambda 1 equal to 1 and lambda 2 equal to 1. So, in case of the periodic system so, at least one of the Eigen value should be Eigen value of the Monodromy matrix so, should be equal to 1 so, in this case, we have, we have seen. So, if we plot the real and imaginary axis. So, this is imaginary axis and this is real axis. So, in this case, we one can see. So, the solution lies on the unit circle itself or the solution lies on the unit circle.

So, this periodic solution is marginally stable; that means, with slight adding in damping to the system, the system response will, will decay. So, already we known that by adding damping the system response will decay and finally, it will become a fixed point response. So, the system is marginally stable similarly, one can study for the Duffing Equation also, in case of the Duffing Equation by taking. So, let us take this Duffing Equation and see, so if we can find.

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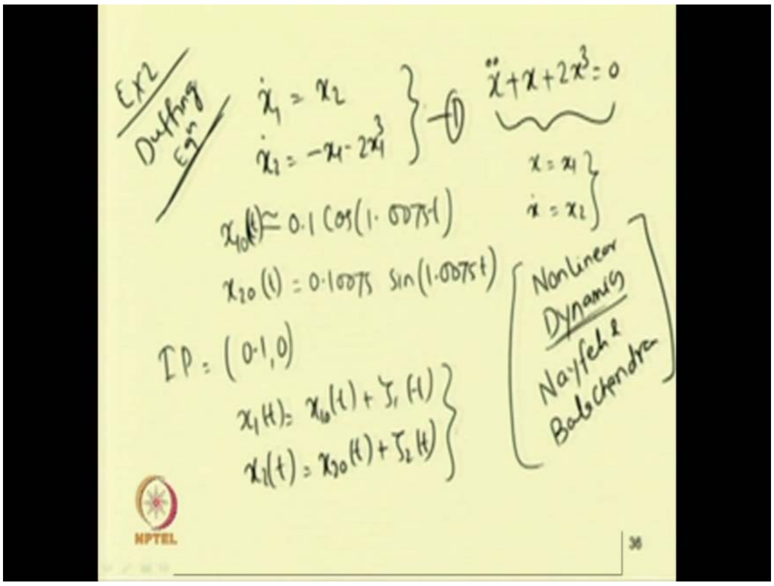

So, Duffing Equation you can write like this that is x 1 dot equal to x 2 and x 2 dot equal to minus x 1 minus 2 x 1 cube. So, in this case by taking this equation; that means, we have taken the equation this way, that is x double dot plus x plus, let us take this thing to x cube equal to 0. So, let us take this equation. So, in this case by writing x equal to x 1 and x dot equal to x 2, we can write this equation in this form that is x dot, x dot, which is equal to x 2. So, x dot equal to x 1 dot so, that is equal to x 2 and x 2 dot that is which is equal to x double dot can be written as minus x minus 2 x cube. So, which can be written as minus 2 x 1 minus 2 x 1 cube.

So, this another example of Duffing Equation. So, let us see how we can find the Monodromy matrix, in this case how to obtain this Monodromy matrix. So, from method of multiple scale already, we know or we known that one of the solution can be or the solution can be written in this form. So, x 1 0 can be written as 0.1 cos 1.0075 t and x 2 0 t. So, this is x 1 0. So, x 1 0 t equal to this and x 2 0 t equal to 0.1 0075 sin 1.0075 t, it may be noted that these examples are and the studies have been taken or adopted from the book by non-linear dynamics, non-linear dynamics by non-linear dynamics by Nayfeh and Balachandra. So, the examples and this chapter have been adopted from the book non-linear dynamics by Nayfeh and Balachandra.

So, in this example this Duffing Equation x double dot plus x plus 2 x cube equal to 0, the solution using method of multiple scale can be written in this way were the initial conditions was taken as point 1 0. So, to study whether, the solution what we have written is stable or not. So, we can put this perturbation like this or we can write this x 1 t equal to x 1 0 t plus zeta 1 t and x 2 t can be written as x 2 0 t plus zeta 2 t, now substituting this equation in this original equation number 1. So, substituting this equation in equation 1, so we can write.

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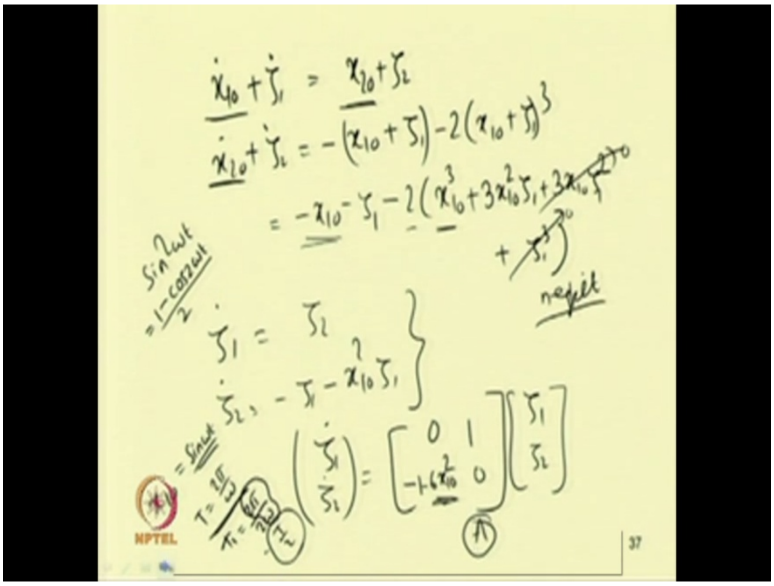

So, this equation can be written x one 0 dot plus zeta 1 dot equal to x 2 0 plus zeta 2 and x 2 0 dot plus zeta 2 dot will be equal to minus x 1 0 plus zeta 1 minus 2 x 1 0 plus zeta 1 cube. So, by expanding this thing, we can write this is equal to x 1 0. So, this is minus. So, this is the thing. So, x 1 0 minus zeta so, minus 2 so, x 1 0 cube so, plus 2 x 1 0 zeta 1 plus. So, this is cubic. So, this is written wrongly.

So, it will be equal to a plus b whole cube so, a cube plus 3 a square b. So, 3 x 1 0 square into zeta 1 plus 3 a b square that is x 1 0 into zeta 1 square plus b cube, that is zeta 1 cube and here as zeta 1 is small perturbation of this zeta 1 0. So, we can neglect the higher order terms that means, we can neglect this zeta 1 square. So, this term and zeta 1 cube. So, these 2 terms we can neglect; we can neglect these 2 terms and so we can write this equation also. We, know that this x 1 0 dot equal to x 2 0 and x 2 0 dot equal to minus x 1 0 minus 2 into x 1 cube. So, we can write this zeta 1 dot equal to zeta 2 and zeta 2 dot equal to minus zeta 1 minus x 1 0 square zeta 1.

So, one can write this way. So, that one can write this zeta 1 dot zeta 2 dot. So, this is equal to 0 1 minus 1 minus 6 x 1 0 square 0 into zeta 1 zeta 2, this is the thing. So, previously, so as zeta 1 0 zeta 1 0 equal to 0.1 cos 1.007 t so, zeta 1 square. So, let me take for example, let me take this or write this, if this x 1 0, let x 1 0 is in the form of sin omega t. So, it will have a period t equal to 2 pi by omega. So, as this contain this cubic nonlinearity. So, what we have told before that this a matrix. So, this is the matrix.

So, a matrix will contain or a matrix which will have a period of t by 2. So, this thing can be seen from this as this contain this x 1 0 square term. So, x 1 0 will be equal to sin square omega t sin, the sin square omega t can be written sin square omega t can be written in this form. So, sin square omega t will be equal to 1, 1 minus cos 2 omega t. So, 1 minus 2 can omega t by 2. So, this is sin square omega t. So, as it contain a term cos 2 omega t cos 2 omega t. So, that means with a frequency twice that of the frequency of the original response. So, the time period will be equal to 2 pi by this 2 omega as the time period t 1 will be equal to 2 pi by 2 omega, but as this 2 pi by omega equal to t. So, this becomes t by 2.

So, what, what is told before that if it contains, if the equation governing equation contain a cubic non-linear term or non-linear of cubic time, then this a matrix will contain a term. So, a matrix will be periodic and it will contain will have a period of t by 2. So, this shows this matrix will have a period of t by 2 now. So, we have to find this Monodromy matrix from this a matrix to find that thing. So, taking this initial condition zeta equal to so, we can take initial condition zeta 0 so, taking different initial condition. So, let us take, so taking different initial condition.

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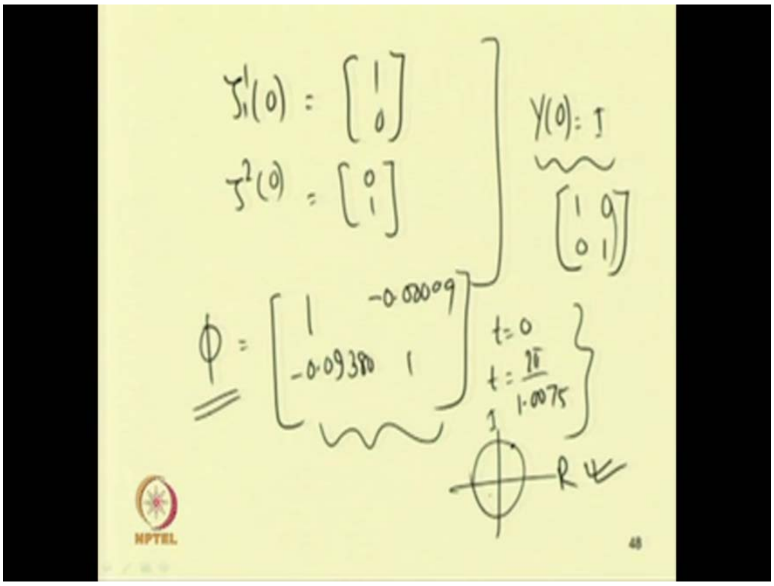

Let us take this initial condition zeta 1 0 equal to 1 0 and zeta 2 0. So, taking initial condition zeta 2 zeta 2 0. So, we will take 2 initial condition. So, we have told before that. So, to find this Monodromy matrix, why should take we should take this initial condition y 0 equal to I. So, in this case by taking this initial condition that is y 0 equal to I; that means, as this is 2 is to 3 2 matrix. So, this will be 1 0 and 0 1. So, the first initial condition or the first y will take 1 0 and second y will take equal to 0 1. So, taking zeta 1 0 equal to 1 0 and zeta 2 0 equal to 0 1. So, we can find this phi matrix will be. So, after one time period so, we have to integrate it so, or we have to find the response after 1 time period so, after one time period, if we will find 1 time period means, in this case 1 time period t equal to 0.

So, we have to solve this equation from t equal to 0 to t equal to 2 pi by 1.0075. So, in this case, we will obtain this phi equal to 1. So, this will be equal to point minus 0.000409, then this is minus 0.09380 and this is 1 now finding the Eigen value of this matrix, this Monodromy matrix. So, one can study the stability of the system. So, for the system to be stable all the Eigen values of this thing should be. So, one of the Eigen value should be on this unit circle and this is the real part and this is imaginary part. So, other root should be inside this unit circle. So, if all the roots are inside this unit circle, then only it will be stable. So, with these examples, we know how to find this Monodromy matrix and also after finding this Monodromy matrix. Now, we will see how to study the Bifurcation of the system similar to the Bifurcation of the system, what we have studied in case of periodic system here also, in case of; in case of fixed point system, in case of periodic system also.

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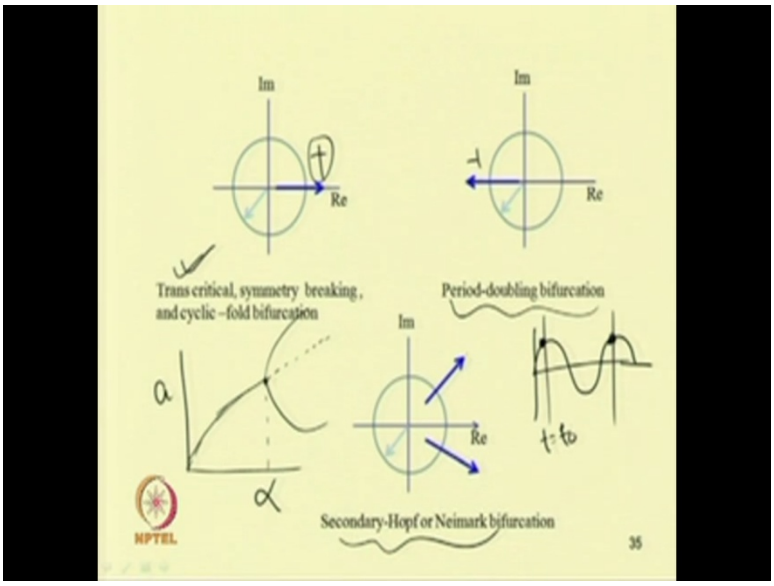

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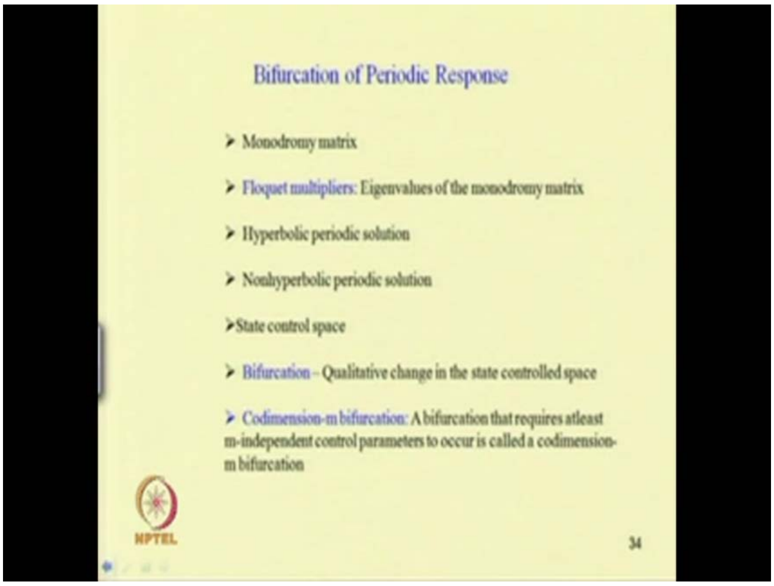

We have different type of Bifurcations similar different type of Bifurcations. So, these Bifurcations can be of Trans Critical Symmetry Breaking and Cyclic Fold Bifurcation or Period Doubling bifurcation or secondary Hopf or Neimark Bifurcation. So, in case of the trans critical symmetry breaking or cyclic fold bifurcation. So, the one of the Eigen value or one of the Floquet multiplier will leave this unit circle through this plus 1.

So, if one of the Floquet multiplier, we have a value more than plus then one can observer or one can find these 3 type of bifurcation, it may be trans critical symmetry breaking or cyclic fold bifurcation. So, if it is leaving the unit circle. So, if it leaving this circle through minus 1 then one will obtain a period doubling bifurcation and if a pair of Eigen values are complex conjugate and they leave this unit circle like this that that is a pair of complex conjugate Eigen values leave this unit circle, then we can have secondary Hopf or Neimark Bifurcation. So one may note that the Poincare Section of this periodic section is that of a fixed point response for example, a periodic solution, let us take this periodic solution sin omega t.

So, if we will take the Poincare Section of this thing; that means, if we sample at a time period t, let we start the sample at t equal to t 0 here and we will find the next point here. So, the next point will also will have the same value. So, in case of the periodic response its poincare section will be a fixed point. So, the stability of the periodic solution also can be studied from by find this Poincare Section of the system. So, as the poincare Section of the periodic system is a fixed point. So, Similar to the study, we made for the fixed point bifurcation and stability one can study the stability of the periodic solution. And if one want to study the stability of the periodic solution by using this Floquet multiplier, one can check whether, it is leaving to plus 1 minus 1 or a complex conjugate leaves the circle.

So, in case of in case of symmetry breaking so, if we plot the poincare section of the system. So, initially, we will have a set of periodic solution and let this contour parameter alpha. So, initially we have a set of periodic solution. So, its poincare section will be a single point. So, these are the Poincare Section of this thing single point. So, after super critical symmetry breaking so, we may have. So, we will have some periodic solution like this. So, here the symmetry is broken. So, at this point at this alpha equal to alpha c, the symmetry of the system is broken. So, that is why, we will have a Symmetry

Breaking Bifurcation here. So, the original periodic solution will continue as an unstable periodic solution. So, in case of, so this is super critical symmetry breaking similarly, we can have a sub critical symmetry breaking. So, in case of sub critical symmetry breaking so, in case of sub critical symmetry breaking we can see. So, here some examples are given.

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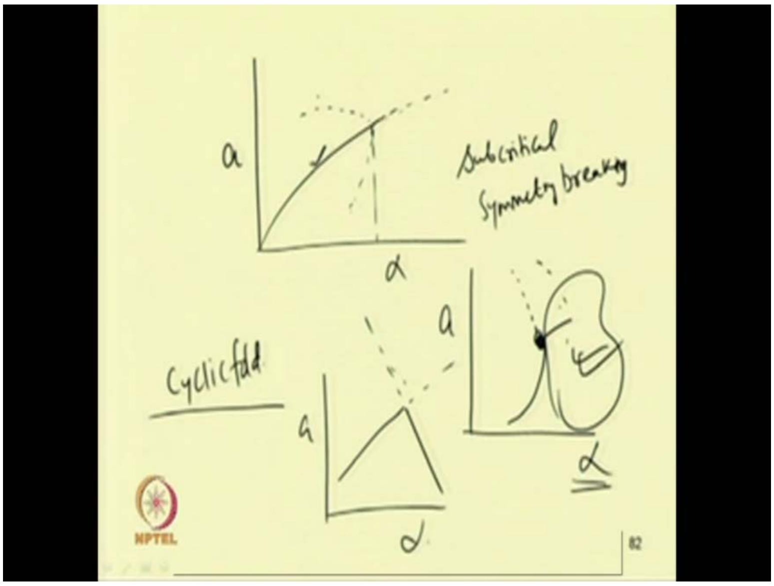

So, in case of sub critical so, we can have a and alpha in this way. So, this is stable and this is unstable. So, you have a stable periodic solution, the poincare sections become a single points and so before bifurcation, we have unstable. So, this is stable periodic solution along with 2 sets of unstable periodic solution and after this Bifurcation, it becomes also unstable single period solution. So, previously 1 has a single period solution, stable period solution with other two solutions which are unstable and after the Bifurcation point. So, one has a unstable periodic solution. So, this is sub critical symmetry breaking, symmetry breaking solution. So, later we will see some examples, where the symmetry breaking type of solutions occur similarly, in case of cyclic fold, this is symmetry breaking. So, we can have this cyclic fold, so in case of cyclic fold.

So, this is similar to the saddle node bifurcation, in case of a fixed point response. So, in this case, so one can see that at this critical point let us take this critical point will have the stable point solution and this is unstable. So, at this point, so this is a saddle node. So, similar to the saddle node in case of the fixed point response. So, in this case also, we will have there is a sudden. So, there is no solution after this alpha equal to alpha c. So, in this range, there is no solution. So, before that we have a set of stable and unstable periodic solution.

So, for a particular value of control parameter alpha so, for alpha less than alpha c solution, we have; we have two solution one is stable periodic and other one is unstable periodic and after the Bifurcation point so, there exist no periodic solution. So, this is cyclic fold bifurcation similarly, one can have this Trans Critical Bifurcation. So, in case of trans critical, we have a stable and unstable. So, this is the poincare Section. So, we are showing the Poincare Section. So, this is a and alpha and in case of the period doubling.

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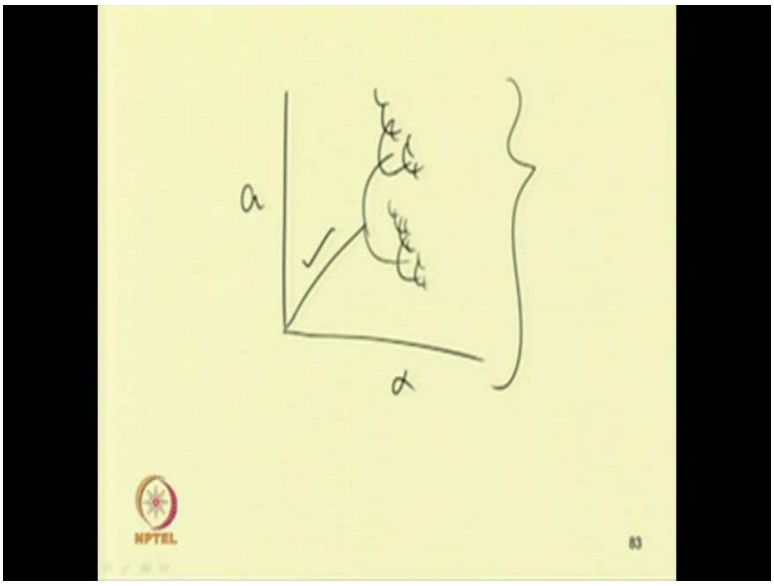

So, one can have a set of period doubling also, here this is the alpha. So, one can find, so from one branch. So, one can have 2 branch then 2 to 4, 4 to 8. So, this way the branching will continue and one can have a set of chaotic response. So, this is a set of chaotic response obtained due to period doubling. So, initially one has single period, then the period goes on increasing and it becomes more and more chaotic. So, in the next class, we will study about the quasi periodic solution and the chaotic Solution. So, here you have seen the example of this cavity solution so, for example, so in this case.

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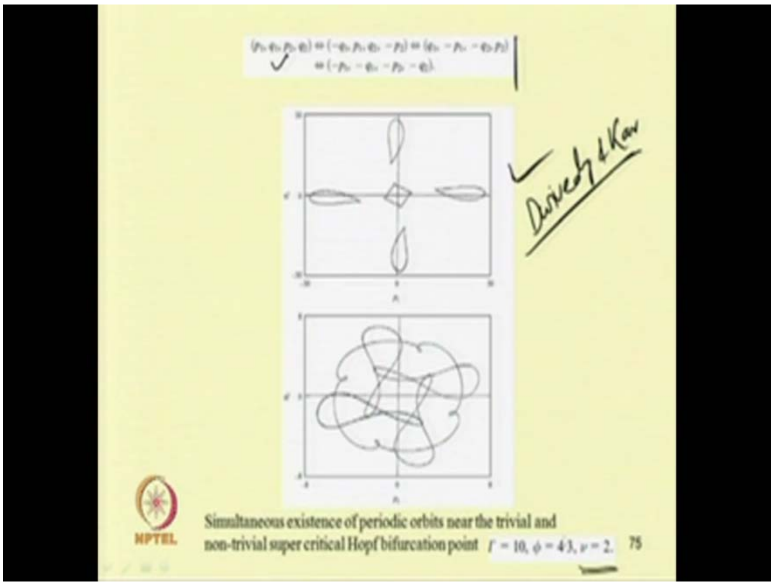

So, initially so this is a case taken from Dwivedy and Kar so, this is periodic response and cavity response. So, in this case 1 has a set of periodic response and one can see there will be the symmetry will break in this case and one can find a symmetry breaking Bifurcation, after the critical parameter. So, that is by changing this critical parameter mu one can; one can see this stable. So, before bifurcation there exist 4 sets of solution, 4 sets of periodic solution and after the bifurcation, it becomes different that symmetry is broken.

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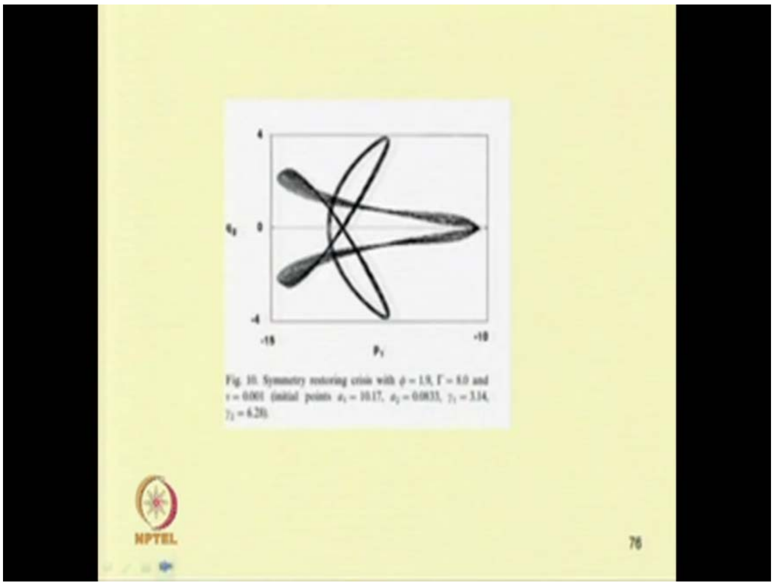

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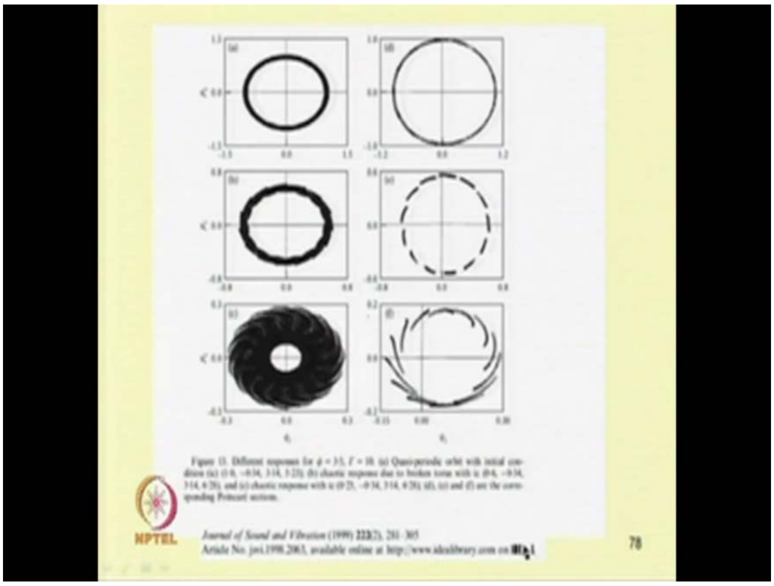

Similarly, one can see some chaotic response also chaotic response in case of this is some system. So, we will study in the next class.

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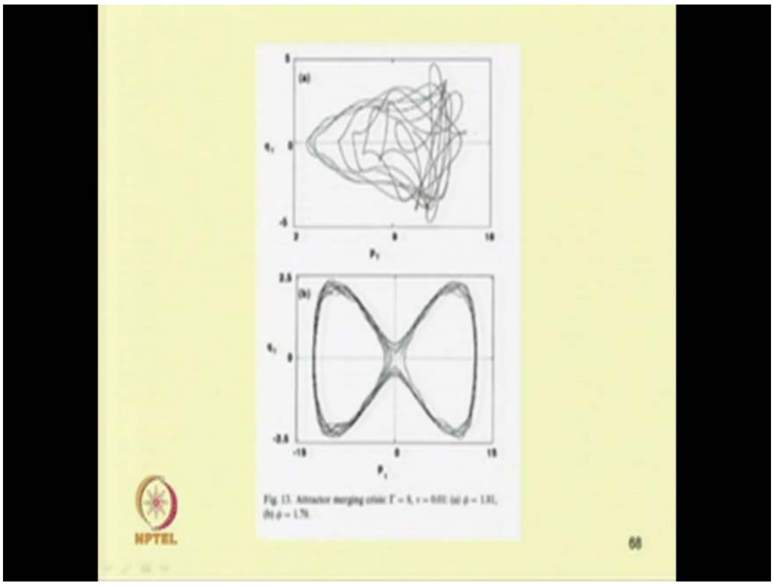

And so, these are some of the Chaotic response also obtained.

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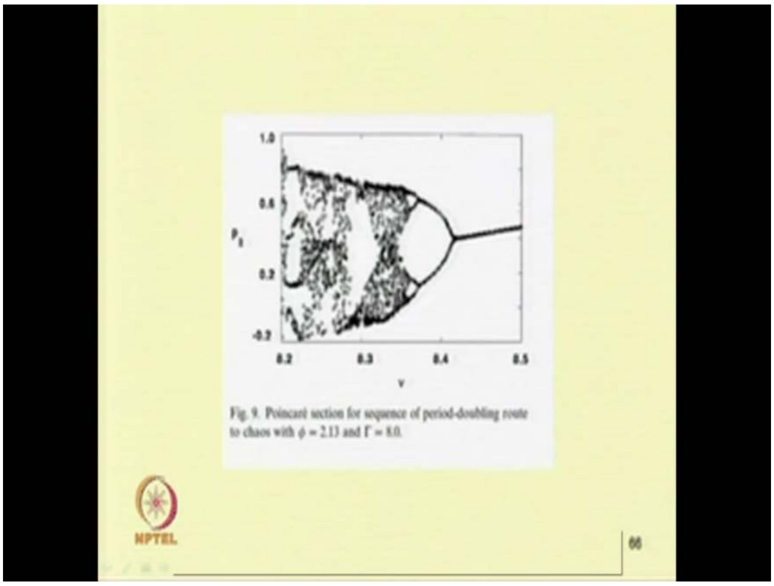

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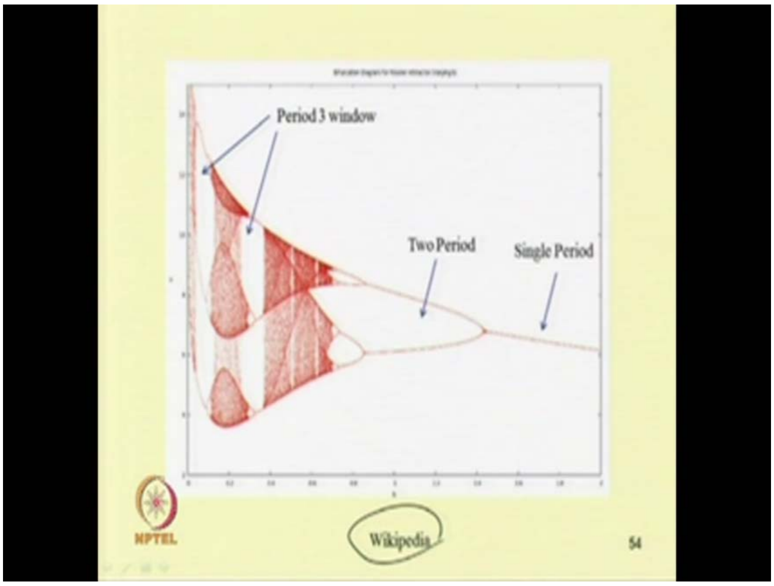

So, this is a period doubling route to chaos. So, initially one has a period, single period now, two periods and this period goes on increasing. So, this is due to this period doubling route to chaos. So, we will study different type of responses and how this periodic and quasi periodic responses, this is also one example of so which is taken from this Wikipedia that is initially a single period then we have 2 period, then we have a window of 3 periods window of so, this is also a window of 1 2 3 and 4 period so, here also 1 2 3 and 4.

So, other portions the response is chaotic. So, you have seen some of the examples of chaotic response also and next class, we will complete this module by studying this quasi periodic and chaotic response along with we will study another example of a base excited cantilever beam. So, in that example all the fixed point periodic quasi periodic and chaotic responses and different routes to chaos will be discussed.

Thank you.