Non-Linear Vibration
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Module - 4
Stability and Bifurcation Analysis
of Nonlinear Responses
Lecture - 4
Static and Dynamic Bifurcation

Welcome to today class of non-linear vibration. So, we are studying about stability and bifurcation analysis of non-linear fixed responses. So, previous class we have discussed about the static bifurcations, where we have discussed about the saddle node bifurcation, point pitch bifurcation and trans-critical bifurcation point. Today we will review all the static bifurcation points and along with that we will discuss about this dynamic bifurcation that is, Hopf bifurcation which occur in non-linear systems. Also, we will see different systems where this Hopf bifurcation takes place.

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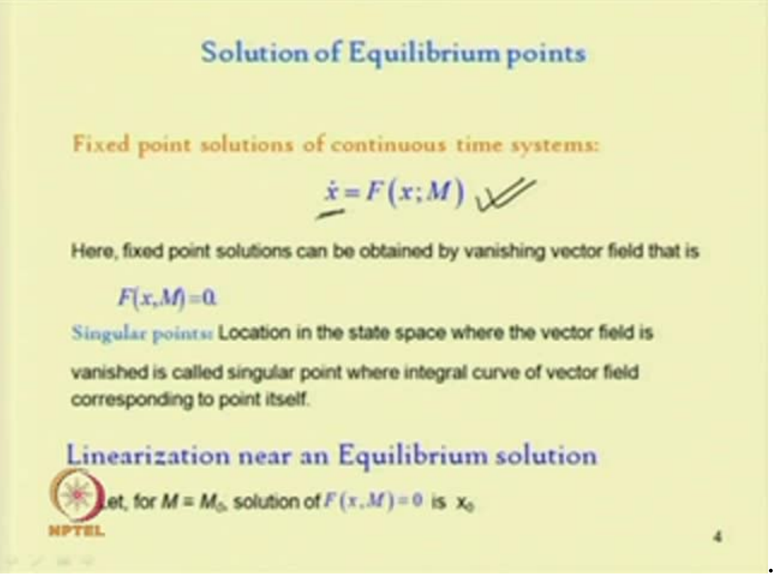

So, to review the static bifurcations or that is bifurcation points so, before that we have discussed about the equilibrium points. So, for a system to find the fixed point response if the system is represented by the governing equation, first order governing equation x dot equal to F x M, where M is the control parameter then by putting the time derivative equal to 0 and solving that equation we obtain the equilibrium equations.

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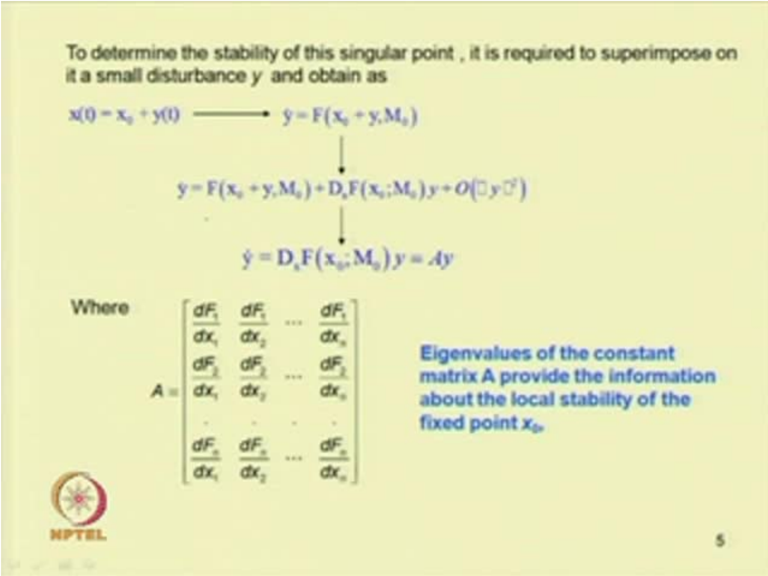

So, after getting these equilibrium equations we should know whether these equilibrium points what we obtained are stable or unstable. To know whether the equilibrium points are stable or unstable so, we perturb that equations or we perturb the governing equation and check whether the equilibrium point is stable. So, in that case so, by perturbing that thing we obtain the Jacobian matrix and by finding the Eigen values of the Jacobian matrix we can find whether the equilibrium point is stable or not. So, in case the negative part of, the negative part of the real part of the Eigen value is negative then, the fixed point response is stable. If the real part of the real part of the Eigen value is positive then, the system is unstable.

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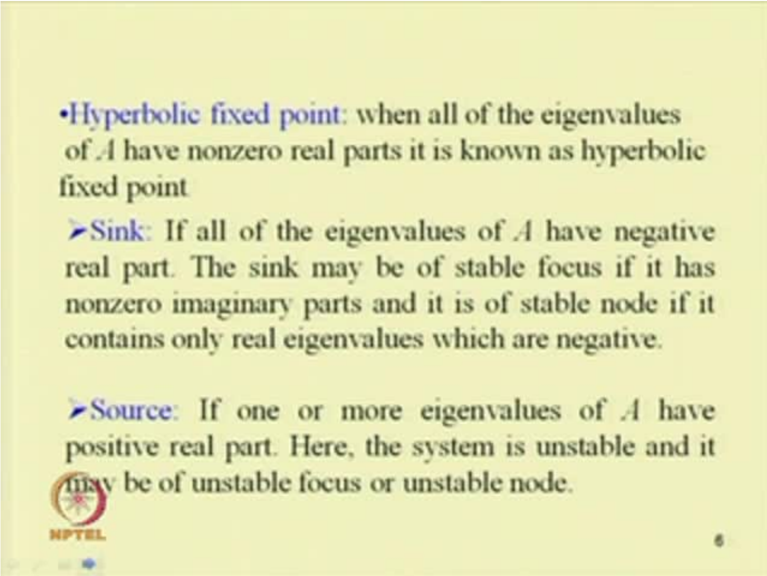

Already we have discussed about this hyperbolic fixed point. So, when all the Eigen values of the Jacobian matrix have non-0 real parts then, it is known as hyperbolic fixed point if some of them are 0 then it is known as non hyperbolic fixed point.

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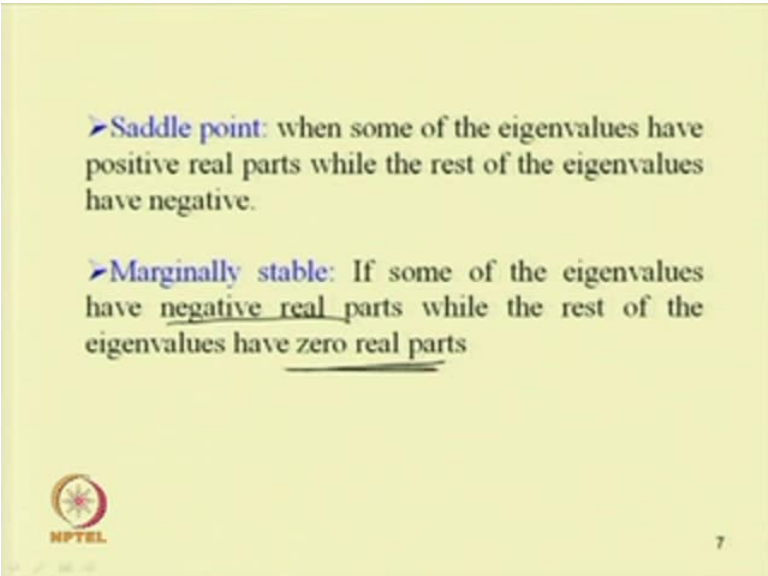

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So, if all the Eigen values of a have negative real parts then it is known as sink. The sink may be stable focus if it has non 0 imaginary parts and it is stable node if it contains only a real Eigen values which are negative. Similarly, source if one or more Eigen values of a have positive real parts here, the system is unstable and it may be unstable focus or unstable nodes. Similarly, saddle point, if some of the Eigen values have positive real parts while rest of the Eigen values have negative real part then, it is a saddle point and if some of the Eigen values have negative real parts while the rest of the Eigen values have the 0 real parts then it is marginally stable. So, the system is marginally stable if some of the real values if some of the Eigen values have negative real parts, some of the Eigen values have negative real parts while rest of the Eigen values have 0 real parts.

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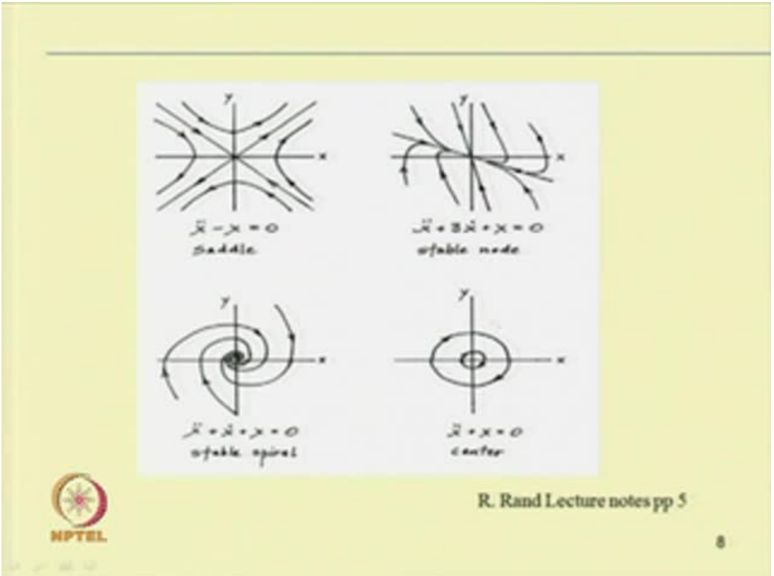

So, we have seen some of the examples which are known as saddle so, in this case this origin is the saddle point. Similarly, this is a stable node and stable spiral or focus so, then, this is the center.

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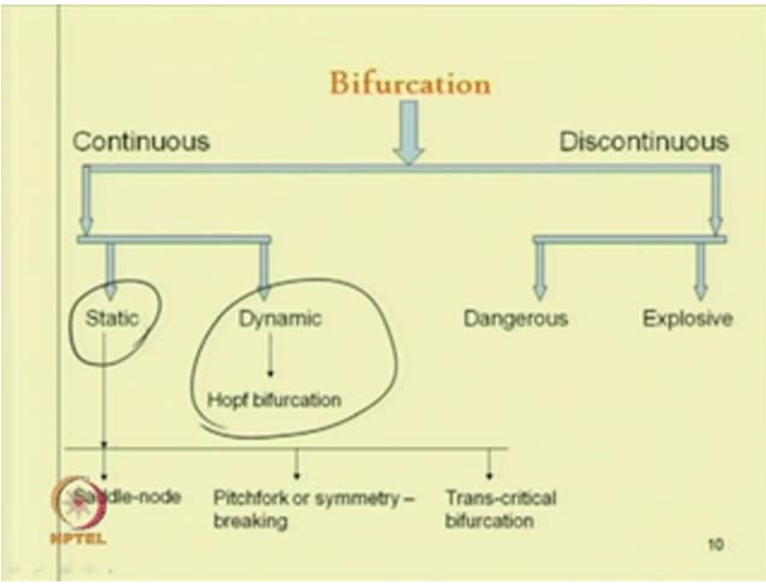

So, x double dot plus x equal to 0. So, here one can find a periodic response and belongs to the center. And already we have discussed about this bifurcation that is static continuous and discontinuous bifurcation static and dynamic. So, in case of discontinuous this may be dangerous and explosive and we have completed these static bifurcation points so, which are divided into saddle node bifurcation point pitch fork or symmetry breaking bifurcation point and trans-critical bifurcation point. So, today we are going to discuss more on this dynamic or Hopf bifurcation point.

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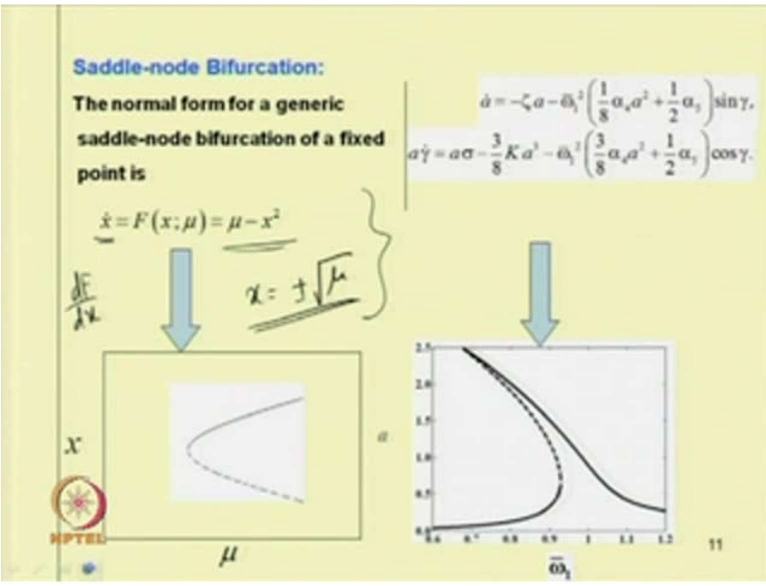

So, before that let us review this saddle node pitch fork and trans-critical which is static bifurcation point. So, already we know the normal form or generic form of this saddle node bifurcation point can be written in this form mu minus alpha x square or so, mu. So, in case of the saddle node bifurcation point this is mu minus x square so, if one plot by putting this x dot equal to 0 so, one can obtain this x equal to plus minus root over mu. So, for mu negative value mu it has no solution and for positive value of mu it has 2 solutions and they are plus minus so, x will be equal to plus minus root over mu and if one find the Eigen value in this case so, Eigen value can be obtained by differentiating this equation. That is so, by differentiating del one can find this d f by d x so, by finding d f by d x one can find the Eigen value so, by d f by d x it becomes minus 2 x so, minus 2 x minus lambda equal to 0 or lambda equal to minus 2 x so, Eigen value becomes minus 2 x. So, by substituting this one can obtain so, this branch is stable and this is unstable branch. Similarly, so, this is for a one dimensional case.

Similarly for a two dimensional case one can find so, these points to be saddle node bifurcation point. So, here also this equation is for a dot and gamma dot so, in these equations first find the Jacobian matrix after finding the Jacobian matrix so, one can plot this frequency response curve, after plotting this frequency response curve and finding the Eigen values of the Jacobian matrix one can see that at these points, at these points the fixed point response one can obtain the fixed point response and one of the Eigen value become 0. So, as one of the Eigen value become 0 so, this is a Static bifurcation point.

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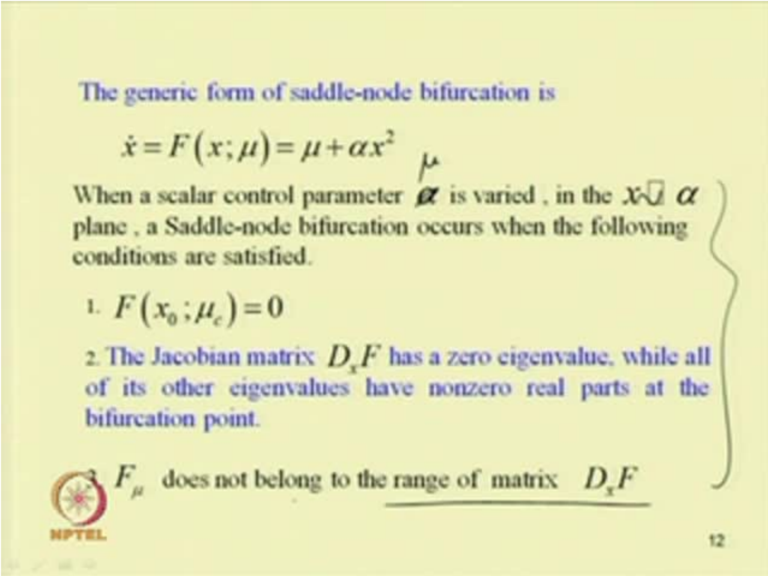

So, in the static bifurcation point to know whether it is saddle node bifurcation point or node. So, these are the conditions so, it has to satisfy. Already we know the generic form of this saddle node is mu plus alpha x square. So, when this control parameter alpha is varied in x versus alpha state plane a saddle node bifurcation will occur if this control parameter that is, x 0 mu becomes 0. Or so, let the control parameter is mu, one can take alpha also as the control parameter. So, if one take mu as the control parameter then, this a x 0 mu control parameter so, it should becomes 0 that means this x dot becomes 0 at that point.

Also, the Jacobian matrix has the 0 Eigen value while all of its other Eigen values have non 0 real parts at the bifurcation point and this F mu does not belong to the range. So, one has to note this point so, F mu does not belongs to the range of matrix D x f. So, in case of saddle node bifurcation point F mu does not belongs to the range of D x f while, in case of other two type of bifurcation points other two type of static bifurcation point it should belongs to the range of D x F.

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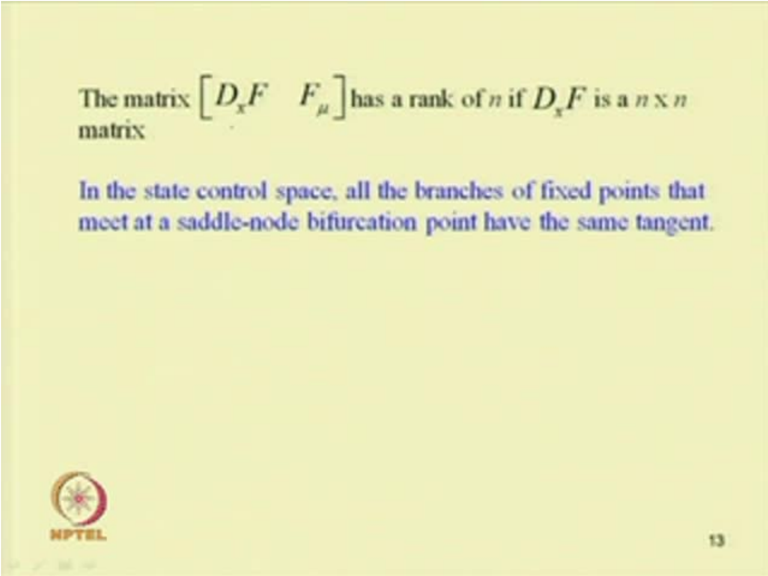

So, in this case by constructing a matrix D x F F mu so, by finding this a rank of that matrix which should be n so, in case of other bifurcation it will be n minus 1 so, in the state control space all the branches of the fixed point that meet at the saddle node bifurcation point have the same tangent.

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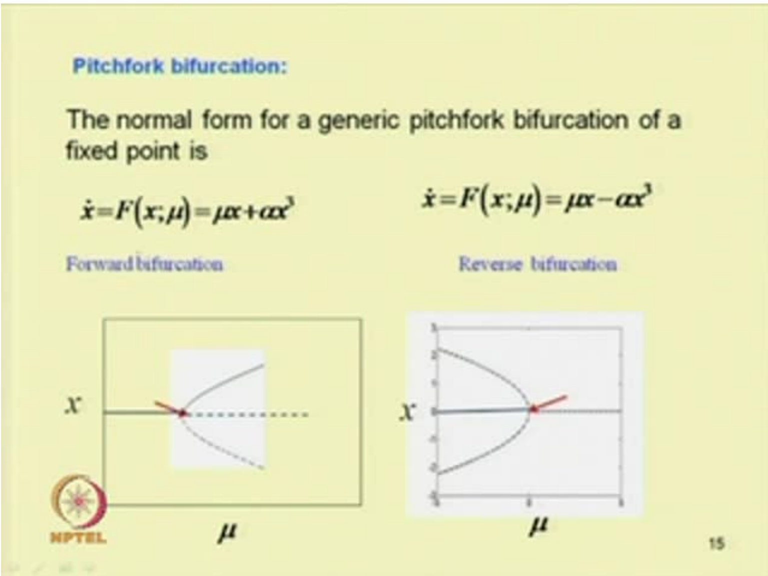

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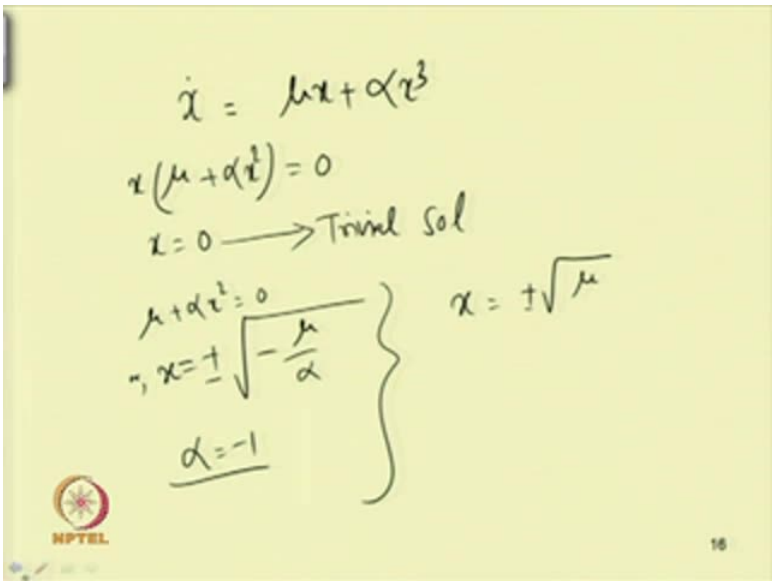

So, now in case of the pitch fork bifurcation point so, the generic form is x dot equal to mu x plus alpha x cube so, in this case by putting this x dot equal to 0. For example, by putting so, let taking this x dot equal to mu x plus alpha x cube now by putting x dot equal to 0 1 obtain this mu by taking this x common so, this equation becomes mu plus alpha x square so, this is equal to 0. So, it has two solutions one solution is x equal to 0 this is the trivial solution and the other one and the other one this is mu plus alpha x square equal to 0 or x becomes minus mu by alpha root over.

So, by taking different value of alpha and mu so, one can have the response in x mu plot or x alpha plot by taking alpha equal to so, let us take alpha equal to minus 1 so, by taking alpha equal to minus 1 then x becomes so, this x one can get this x equal to plus minus root over minus mu by alpha by taking alpha equal to minus 1 so, this becomes x equal to plus minus root mu. So, for negative value of mu the solution becomes the the solution for negative value of mu so, the root over so, one has two solutions so, from the solution one can plot.

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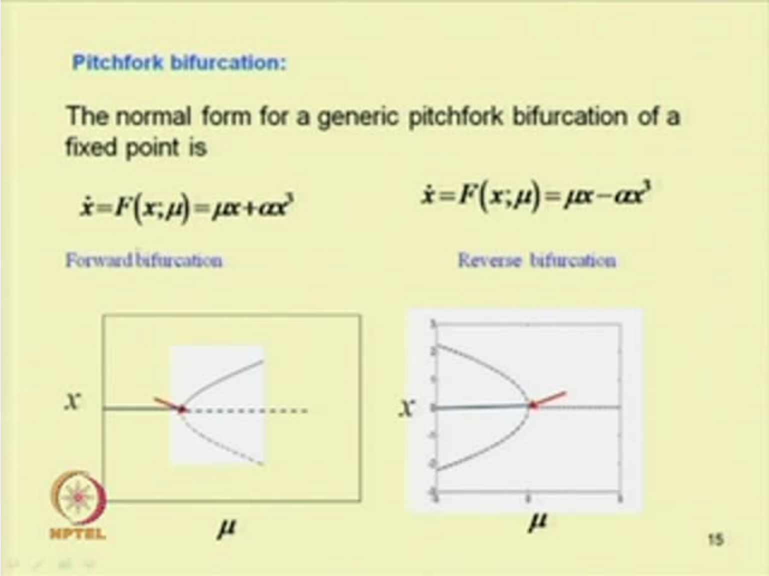

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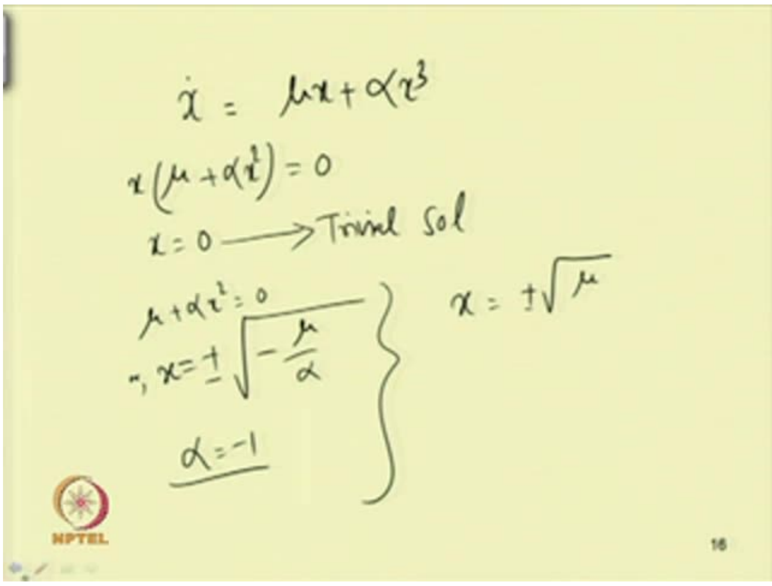

So, let us see this thing. So, by taking alpha equal to minus 1 so, x becomes plus minus root over so, this is minus 1 so, this becomes mu. Now, by plotting this one can obtain the response of the system. So, the response of the systems one can obtain can be like this so, this is the trivial state and for the non trivial state that is up to for negative value of mu so, this becomes imaginary so, this is not the practical solution and for positive value of mu one has two solutions.

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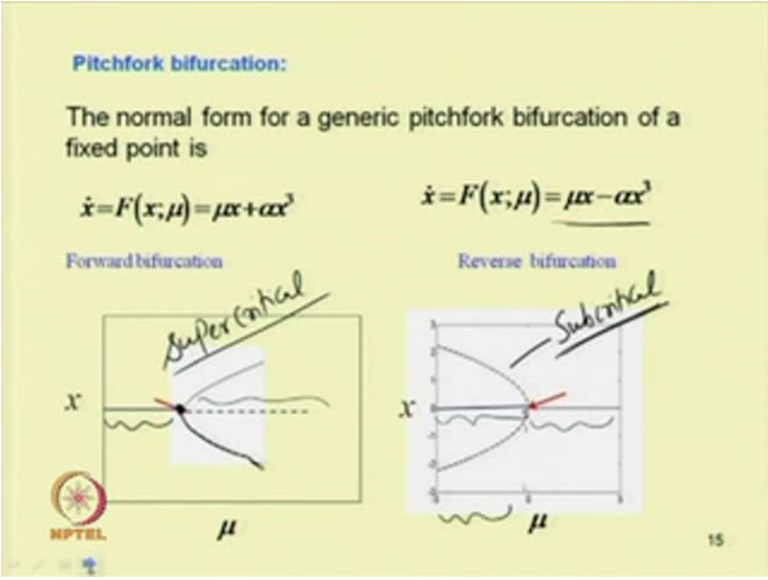

So, for two solutions one can plot plus minus mu plus minus root over mu. So, these point that is x mu corresponding to 0 is a bifurcation point because there is change in the number of solutions and also, there is a change in stability of the system. So, for mu for the trivial state one can obtain that this part is stable and this part is unstable because the Eigen value of the Jacobian matrix so, becomes negative for this part and becomes positive for this one.

And similarly, for these two that is the non trivial state the Eigen value becomes negative. So, as the Eigen value becomes negative this is stable so, these 2 are stable. So, in this case one can observe that by increasing the value of mu so, one obtain the response so, were the response up to 0 that is the trivial state is stable. So, from the stable state one achieve another stable non trivial a non trivial state so, from the stable trivial state one obtain non trivial stable state and a trivial unstable state. So, in this case so, this forward bifurcation so, this is known as super critical pitch fork bifurcation so, this is super critical. So, this bifurcation point is super critical because from the stable branch we are getting for from the stable trivial solution we are getting a stable non trivial solution.

But, in this case if we take this mu x equal to minus alpha x cube and plot this response so, we can observe that this part that is the trivial state is stable but, after this the trivial state is unstable and here the non trivial branch becomes unstable. So, in this case from so, if one increase this mu value so, at mu equal to 0, at mu equal to 0 by further increasing this value of mu so, as the system is unstable so, practically there is the system response will disappear after mu equal to 0 or the system will have a unstable response so, due to the presence of this unstable response the system may suffer a catastrophic failure. So, here the bifurcation is known as sub critical bifurcation. So, the sub critical bifurcation is a dangerous bifurcation or a catastrophic bifurcation. So, the super critical bifurcation so, this is so, in case of super critical from a stable it has gone to a stable but, in case of sub critical after the stable branch as there is no presence of stable branch the system will land to a catastrophic failure, the system response may grow and the system may becomes unstable. So, always it is advisable to avoid the super critical bifurcation or it is advisable to operate the system at a control parameter were the system become stable.

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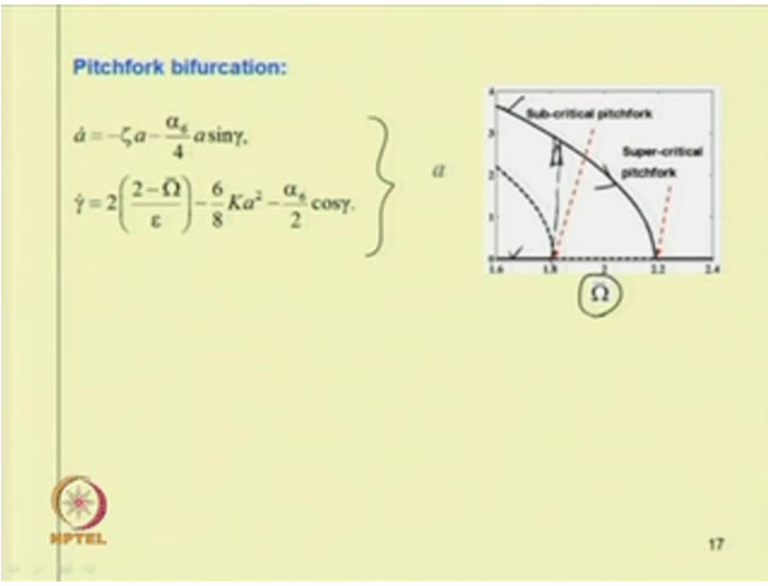

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So, in this case one can operate the system up to a value of mu equal to 0 or value mu from in this negative range and one should not operate the system at a value for mu greater than 0. But, in the previous case so, as we have a super critical pitch fork bifurcation so, from the stable branch we are get going to a stable branch so, there is no problem but, the system will be vibrating with a non trivial value non trivial response so, from the trivial response it is having a non trivial response. So, we observed that in case of a bifurcation point either the stability of those response changes or the number of bifurcation point changes. So, let us already we have so, this is the one dimensional case, one dimensional generic case we have studied similarly, for a two dimensional case one can so, this is the example of a two dimensional case so, in this two dimensional case one can observe this sub critical and super critical pitch fork bifurcation.

Here one can find the Jacobian matrix by perturbing this two equations so, after perturbing these two equation one can obtain the Jacobian matrix and find the Eigen value of the Jacobian matrix. So, if the Eigen values if one of the Eigen value by changing this control parameter in this case the control parameter is omega bar. So, by changing this control parameter so, one can observe that from up to 1.8 the trivial state is stable so, after that the trivial state is unstable and here also, we have one nontrivial state which is unstable and at 2.2 the trivial state becomes again stable and we have also, another branch of nontrivial state which is stable. So, due to the presence of this stable nontrivial and stable trivial branch so, this bifurcation point if the super critical pitch fork bifurcation point and at this point so, the pitch at this point as we have a nontrivial as we have a trivial unstable or on trivial unstable fixed point response then, this bifurcation point is sub critical pitch fork bifurcation. So, by increasing this omega bar here so, from 1.6 to 1.8 the system has a by-stable region because so, in this range so, it has two solution so, this is a by-stable region.

So, in this by-stable region either the response maybe trivial or the nontrivial depending on the depending on the initial conditions so, in this range one has to plot the basin of attraction to know for what initial condition which response of the system will exist. So, one has to plot this basin of attraction so, in case of the basin of attraction one can find whether for or for what initial conditions one will have a trivial state and for what initial condition one will have the nontrivial state. And similarly, from 1.8 to 1.2.2 so, it we have only the nontrivial state so, by increasing the frequency from 1.6 to 1.8 if, initially the response is trivial then, at 1.8 with slight increase in this frequency so, there will be a jump of phenomena and the system response will jump from this trivial state to this nontrivial state and it will follow with further increase it will follow this path so, it will follow this path and it will at 2.2 at omega bar equal to 2.2 again the response becomes trivial.

That means, the nontrivial amplitude will goes on decreasing and at omega bar equal to 2.2 it becomes the trivial response and further increase in omega bar so, the trivial response will exist. Now by sweeping down the frequency so, sweeping down the frequency let from 2.4 so, we have trivial state at 2.2 it becomes nontrivial and it will continues to grow, the response will continues to grow and it will follow this non trivial response. So, in this way we have studied the pitch fork bifurcation where we can have a jump up or sometimes we may have this jump down phenomena also, and we have we know about the sub critical and super critical pitch fork bifurcation.

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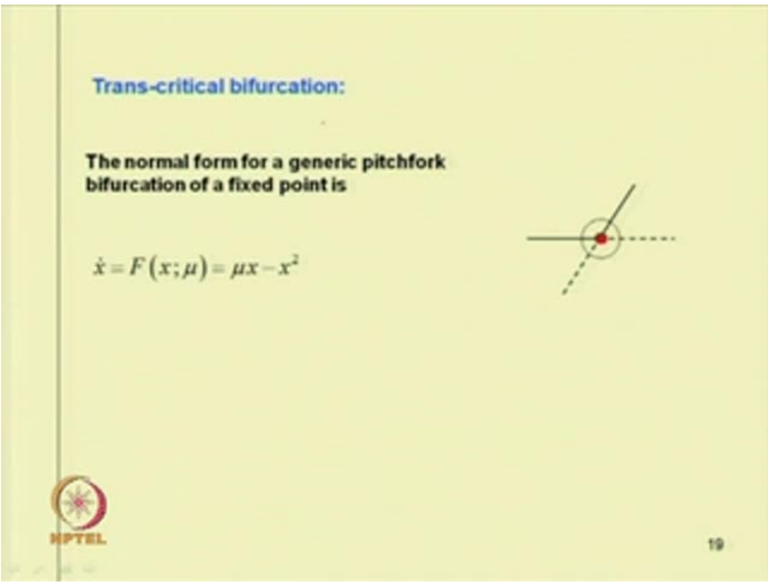

Similarly, we have previously studied this trans critical bifurcation point were a before bifurcation we have a stable and unstable branch and after the bifurcation it becomes the stable branch become unstable and unstable branch become stable that means they have interchanged their stability. Though the number of branches remains same here, the change in the stability occurs in the system.

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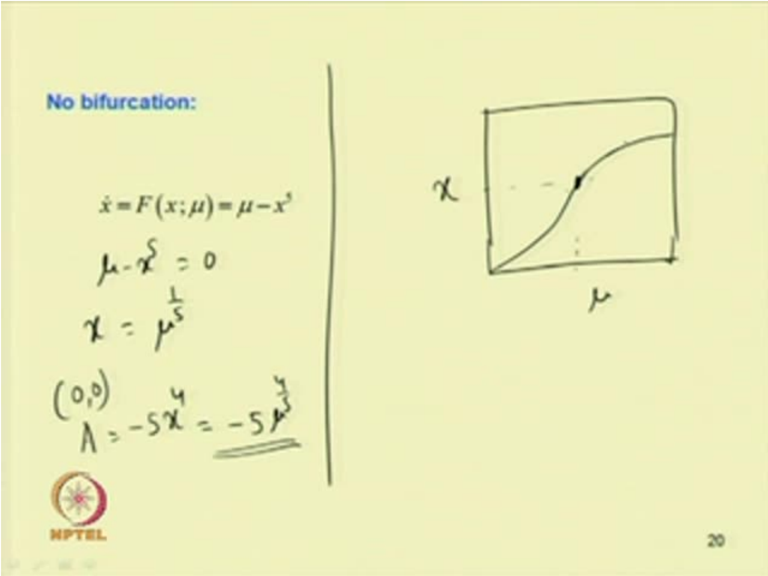

So, due to the change in stability of the system this is known as trans-critical bifurcation point. Let us see one more example so, where x dot equal to mu minus x to the power 5 so, here for equilibrium points we can put this x dot equal to 0. So, as x dot equal to 0 so, we have mu minus x to the power 5 equal to 0 or x becomes so, x to the power 5 become mu so, this x becomes mu to the power 1 by 5. So, x as x equal to mu to the power 1 by 5 for the so, in x mu control space so, if you one plot this x mu control space that in means x versus mu so, at mu equal to 0 that means at 0.00 x equal to 0 and mu equal to 0 or mu equal to 0 and x equal to 0.

So, one can have this filled variable that is mu so, F x mu equal to 0. So, and one can find the Eigen value so, to find this Eigen value our Jacobian matrix a equal to by differentiating this thing one can write this is equal to minus 5 x to the power 4. So, by substituting x equal to mu to the power 1 by 4 so, one can write this a equal to minus 5 mu to the power 4 by 5. So, at x mu equal to 0 this Eigen value also, becomes 0. So, that means this x equal to 0 and mu equal to 0 so, one has a 0 Eigen value.

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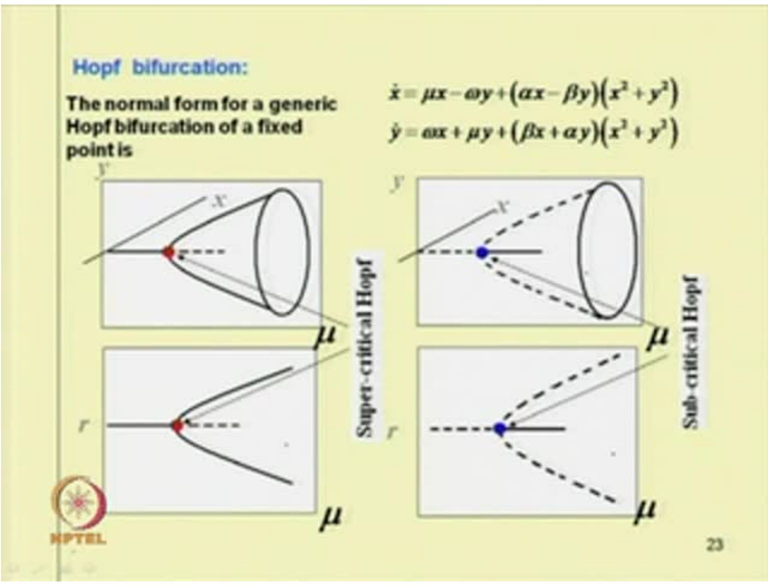

So, it should have been a static bifurcation point. But, if one see this response so, one can observe that neither there is change in stability nor there is change in the number of branches in the system. So, this point is 0 0 point though it is a hyperbolic so, this is a hyperbolic point so, this point is not a bifurcation point as there is no change in stability or there is no change in number of solutions of the system. So, this is all about the static bifurcation point and let us now discuss about the dynamic bifurcation point or the half bifurcation point. So, in case of the half bifurcation point or dynamic bifurcation point so, one can observe that.

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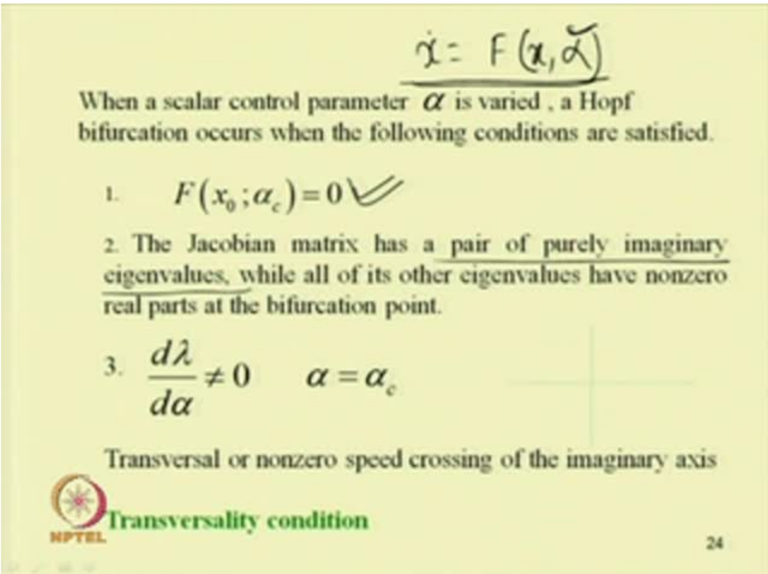

So, these are the conditions so, if we let us have a system x dot equal to same system x dot equal to F x alpha. So, in this case alpha is the control parameter so, for the dynamic bifurcation point first we should have this x 0 alpha c equal to 0. So, that means it has to follow this same as that in case of the static bifurcation but, in this case the difference is that the Jacobian the Jacobian matrix has a pair of purely imaginary Eigen value. So, instead of having 0 Eigen value so, in this case the Jacobian matrix has a pair of purely imaginary Eigen value while all of its other Eigen values have nonzero real parts at the bifurcation part.

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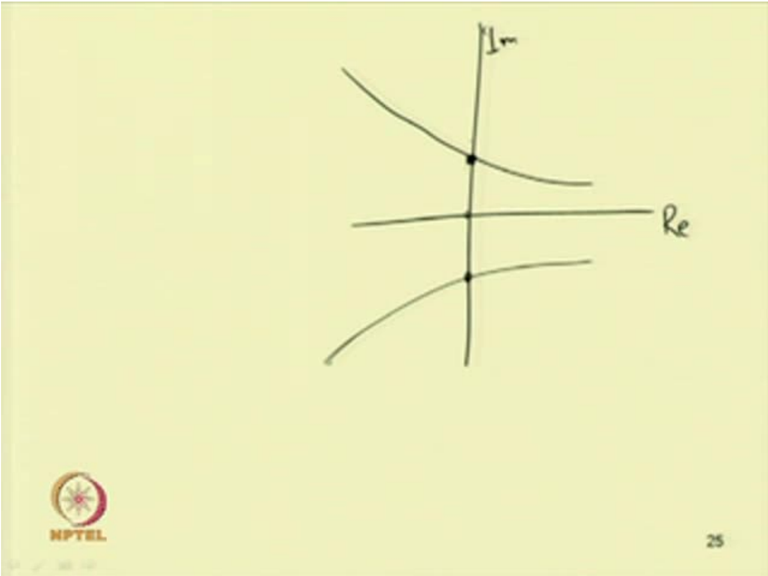

Let alpha c be become the bifurcation point so, if we plot if one plots all the Eigen values real part and imaginary part of the Eigen value so, let this is real part and this is imaginary part of the Eigen value so, at the critical point. So, one can see that a pair of a pair of complex conjugate Eigen value will cross the imaginary axis at this point. So, nonzero so, it will have a nonzero real part nonzero imaginary part. So, the real part becomes 0 the real part of one of the Eigen value becomes 0 and so, one has a pair of imaginary, imaginary roots. So, as so, in case of the Hopf bifurcation due to the presence of this imaginary root so, the system response becomes oscillatory and from so, one can obtain a periodic response at this bifurcation point. So, from the stable or from so, from the so, from this fixed point response one can obtain a periodic response at this bifurcation point. So, at this bifurcation point the periodic response which is generated may be stable or unstable. So, in case of Hopf bifurcation point so, the fixed point response becomes unstable but, it generate a periodic response due to the presence of this imaginary part of the Eigen value.

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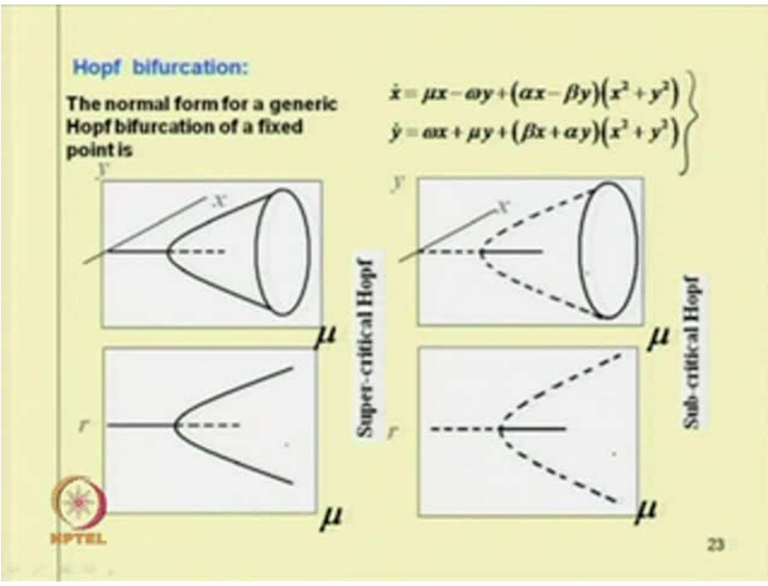

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Let us take the example one example. So, in this case let us take the generic form of the Hopf bifurcation. So, the generic form the normal form or the generic form of bifurcation can be given by this equation that is x dot equal to mu x minus omega y plus alpha x minus beta y into x square plus y square and y dot equal to mu omega x plus mu y plus beta x plus alpha y into x square plus y square. Let us take this example and check so, what will happen to the response of the system.

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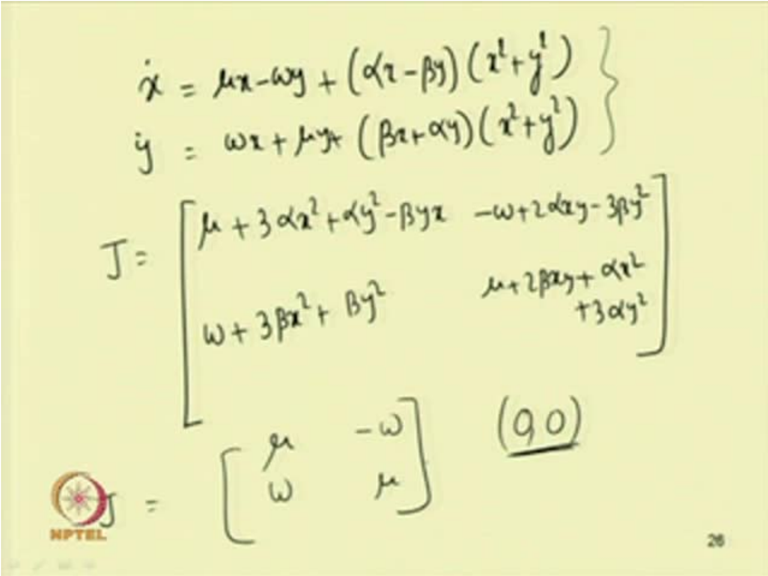

So, in this case x dot equal to mu x minus omega y plus alpha x minus beta y into x square plus y square and y dot equal to omega x plus mu y plus beta x plus alpha y into x square plus y square. So, for the equilibrium positions our x dot and y dot will becomes 0 so, in this case we can find the Jacobian matrix. So, Jacobian matrix can be obtained by differentiating or perturbing these two equations and or by finding the first derivative of these 2 equations by finding the first derivative of this equation.

That means, first differentiating this equation with respect to x one can write this is mu plus 3 alpha x square plus alpha y square minus beta y x and by differentiating this with respect to y so, one can write this becomes minus omega plus 2 alpha x y minus 3 beta y square and this equation becomes omega plus 3 beta x square plus beta y square mu plus 2 beta x y plus alpha x square plus 3 alpha y square. So, by putting this x dot equal to 0 and y dot equal to 0 one can observe that this x and y 0 0 so, this is a equilibrium point. So, the equilibrium point becomes x equal to 0 and y equal to 0 because when x dot equal to 0 by putting this x dot x equal to 0 and y dot equal to 0 so, this satisfy this equation also, this satisfy second equation. So, 0 0 is a equilibrium point so, at this equilibrium this Jacobian matrix will reduce to so, this becomes mu and this becomes minus omega and this becomes omega and this becomes mu. Now, one has to find this Eigen value.

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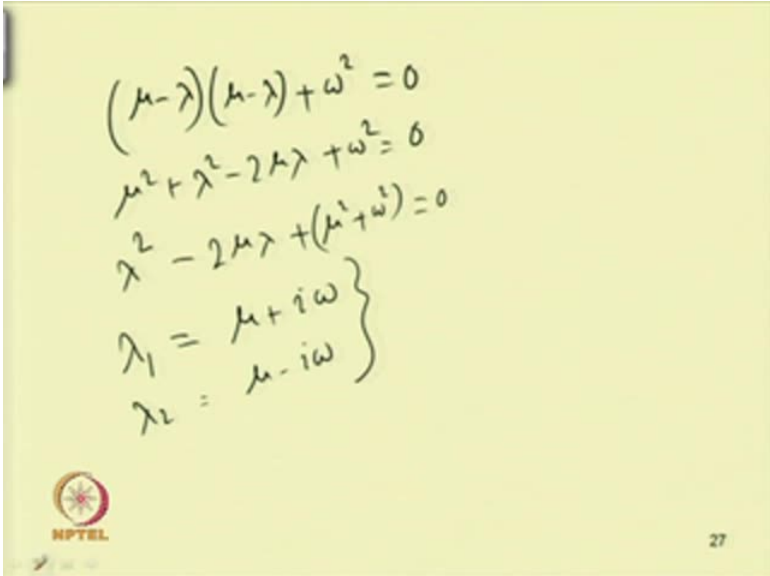

So, for finding this Eigen value one can write this lambda mu minus lambda then omega into mu minus lambda by taking the determinate of this a minus lambda i or j minus lambda i matrix one can find so, one can find this mu minus lambda into mu minus lambda plus omega square equal to 0 or mu square plus lambda square minus 2 mu lambda plus omega square equal to 0. So, this equation to solve these equations or I can write this equation lambda square plus so, lambda square minus 2 mu lambda plus mu square plus omega square so, equal to 0. So, from this I can two value of lambda so, they will be so, lambda 1 will be equal to mu plus i omega and lambda 2 becomes mu minus i omega. So, in this case one can observe by changing this control parameter mu so, for mu equal to 0 this lambda 1 and lambda 2 will have real part equal to 0 but, it will return the imaginary part. So, the system will vibrate with a frequency of omega at that point. So, this lambda 1 becomes mu plus so, from this to visualize these things we can write these two equations.

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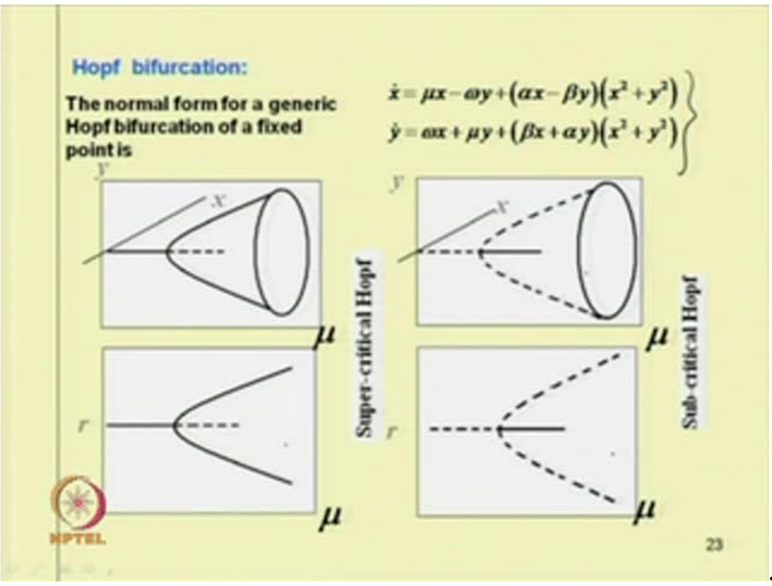

Or by Plotting this equation in x y and mu plan so, at this point at this point so, one can obtain a periodic response and at this point one obtain so, for different value of mu so, one can observe this thing. So, let us see now let us write this equation in polar form.

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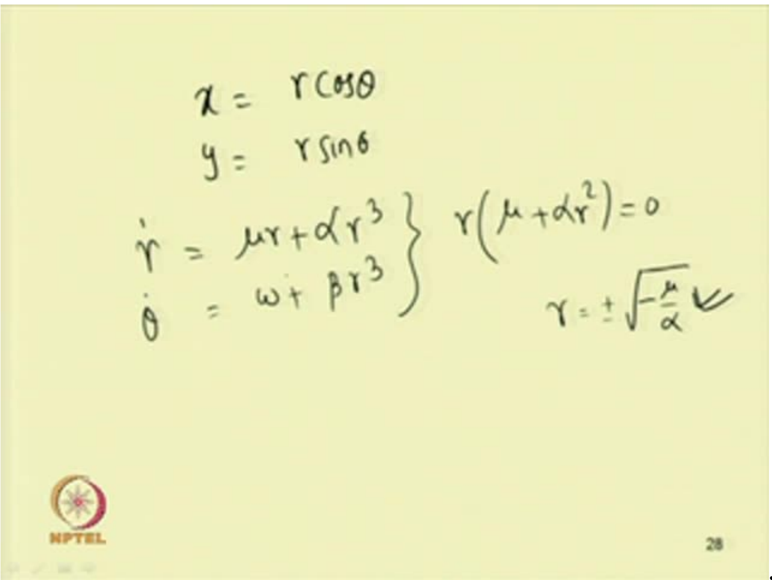

So, instead of writing this in Cartesian form so, we can write in this polar form. So, let us put x equal to r cos theta and y equal to r sin theta. So, by putting x equal to r cos theta and y equal to r sin theta so, this equation x dot and y dot equation will reduce to this form so, r dot will becomes mu r plus alpha r cube and this theta dot equation reduce to omega plus beta r cube. So, in this case for the equilibrium position so, r dot will be equal to 0 and theta dot will be equal to 0. So, by substituting r dot equal to 0 so, we have this equation so, if i will take r common then r into mu plus alpha r square equal to 0 so, in this case r equal to 0 is the trivial state and r so, this is r cube r square so, another one is r square equal to minus mu by alpha or r equal to root over plus minus root over minus mu by alpha. So, this is the nontrivial state and r equal to 0 is the trivial state.

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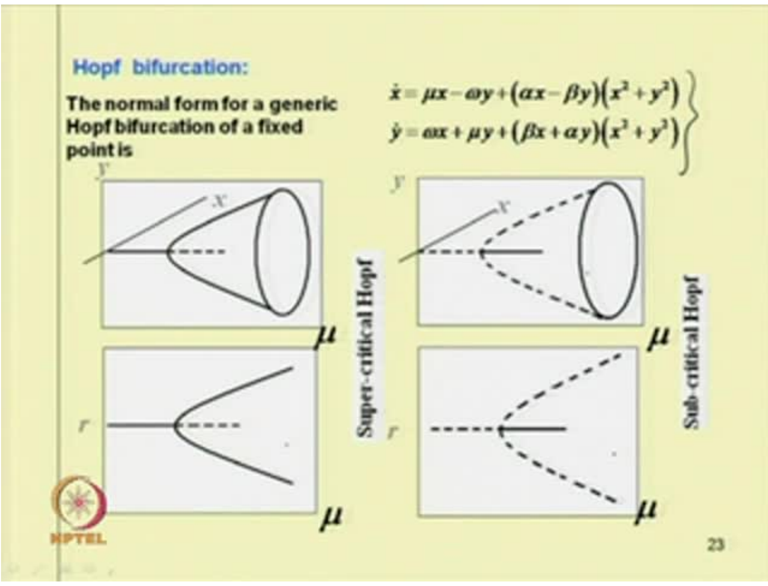

Now, by finding the Eigen value from this so, one can plot so, one can plot this thing in r mu state space so, in r mu state space so, this point that is mu equal to 0 so, this becomes unstable. And at this point one can obtain that so, the system will have a response with frequency omega. Similarly, so, one can see in this case from the stable branch one obtain a nontrivial stable and a trivial unstable branch so, at this point a periodic response as it emanates from this point so, this point so, this is a Hopf bifurcation point. So, in this case this is a super critical Hopf bifurcation point because from the periodic response so, one obtain another periodic response.

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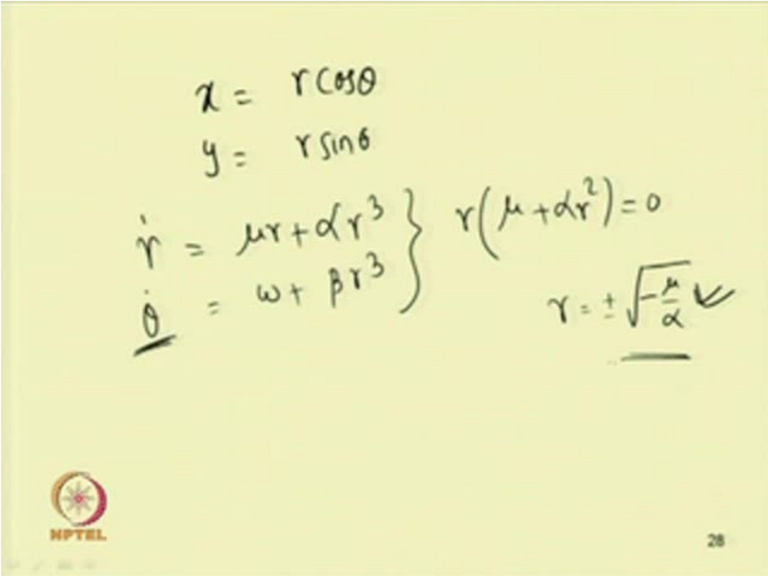

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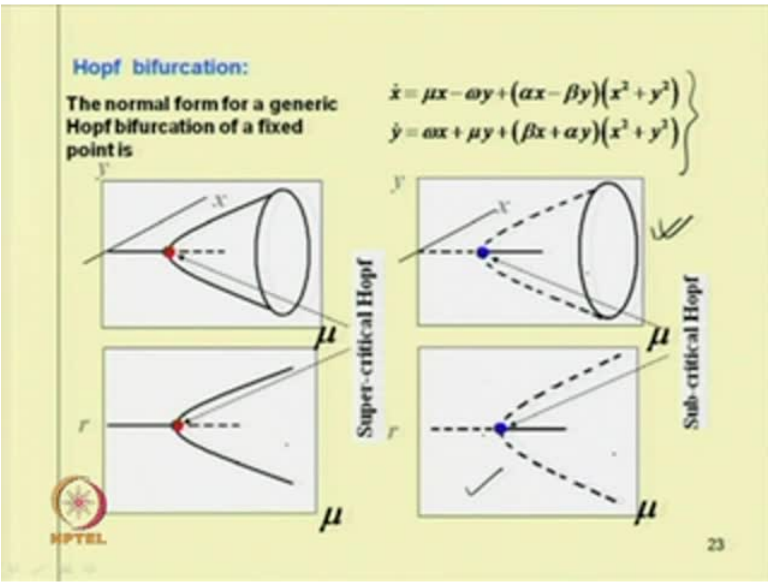

So, this periodic response will have amplitude of this nontrivial response so, this will have amplitude of this and with a frequency of theta dot. So, it will have a frequency of theta dot and with an amplitude of this so, this critical point so, this critical point at this critical point so, from stable as it is going to be stable so, this is known as super critical Hopf bifurcation point and for another value of alpha so, one can obtain this from negative or this is unstable, unstable trivial point.

One obtain a stable periodic solution so, in this case it becomes sub critical pitch fork sub critical Hopf bifurcation point this is also, in r mu plane this is shown and this is in x y mu plane this one. So, in case of the Hopf bifurcation we obtain a stable or unstable periodic response from the fixed point response. So, the fixed point response becomes the trivial fixed point response become unstable and will have a stable periodic response in case of the super critical Hopf bifurcation and in case of the sub critical Hopf bifurcation point so, we will have so, from a stable fixed point response at this bifurcation point we will have a unstable periodic response. So, this is sub critical Hopf bifurcation point. So, in this way so, we know in case of the Hopf bifurcation point the equilibrium point will become 0.

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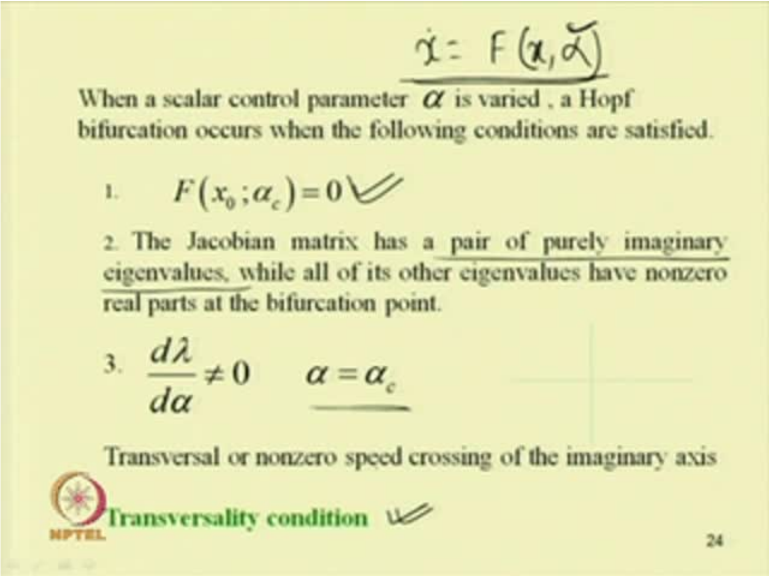

It will have a 0 real part in the Eigen value but, the imaginary part it will have a imaginary part so, that nonzero imaginary part. And also, the other condition that is this transversal condition also, it has to satisfy that means this d lambda by d alpha should not be equal to 0 at alpha equal alpha c. So, the Jacobian matrix as a pair of purely imaginary Eigen values.

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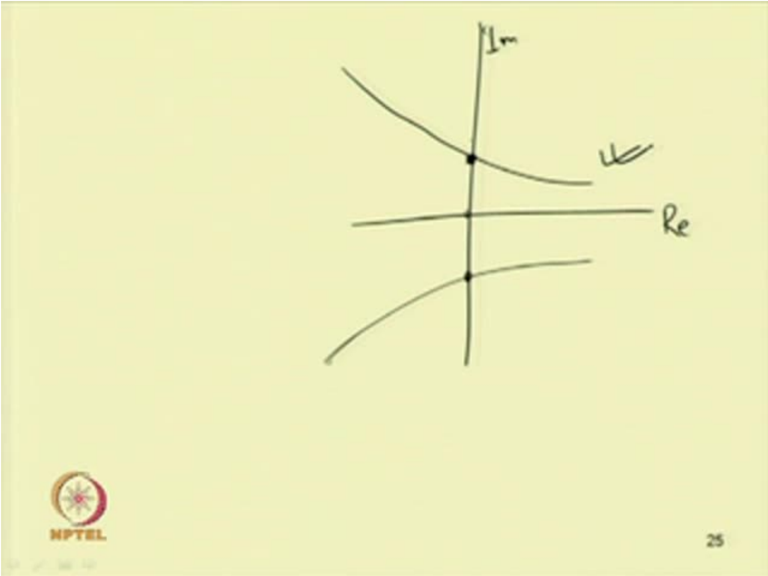

So, in this purely imaginary Eigen value so, one can see this Eigen value this d alpha by d d lambda by d alpha where lambda is the Eigen value is not equal to 0. That means, this transversal or nonzero speed crossing of the imaginary axis that curve we have seen.

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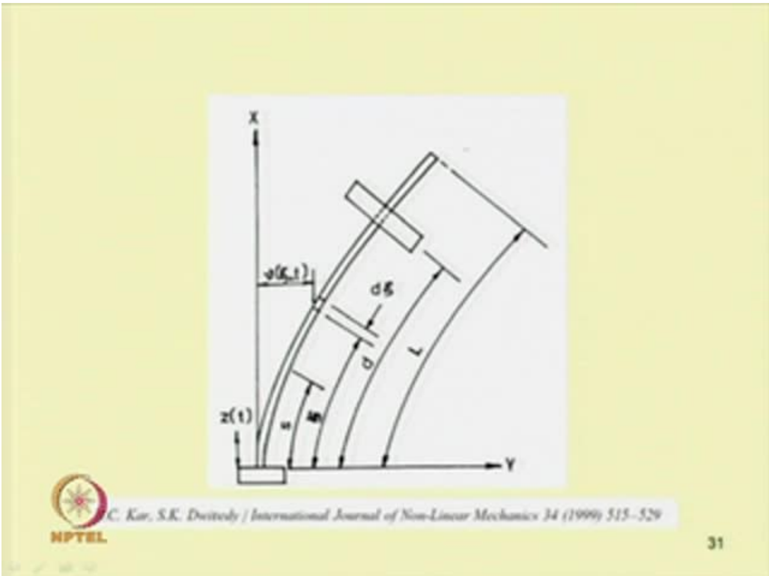

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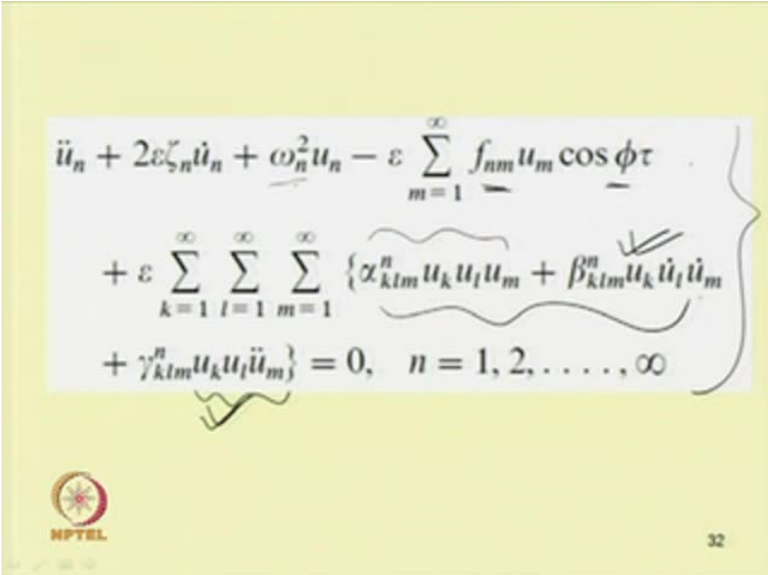

So, in case of the Hopf bifurcation as from the fixed point response we are getting a periodic response that is why this is known as dynamic bifurcation. So, let us take some other examples so, in this case already we have seen. So, this is a base excited system so, in this base excited system so, this equation base excited cantilever beam the equation motion of the system can be written in this form so, here u is the time modulation of the system, zeta is the damping epsilon is the book keeping parameter, omega n is the natural frequency of the system or the model frequency of the system. And here, these are the forcing parameter and phi the phi the frequency non dimensional frequency with which the base is excited.

And so, these are the non-linear terms. So, in this case this u k into u l into u m this k l m will represent the modal interaction of the system so, k l m represent the modes different modes so, this term this first term is known as the geometric non-linear terms and the last 2 terms so, were this u dot and u so, this is u k into u l dot and u m dot and similarly, u k into u l into u m double dot. So, as acceleration terms present in this case product of two velocity term is acceleration also, this is u double dot this is acceleration terms. So, these last two terms are inertia and nonlinearity. So, in this case one can observe a geometric and another two inertia and non-linear terms.

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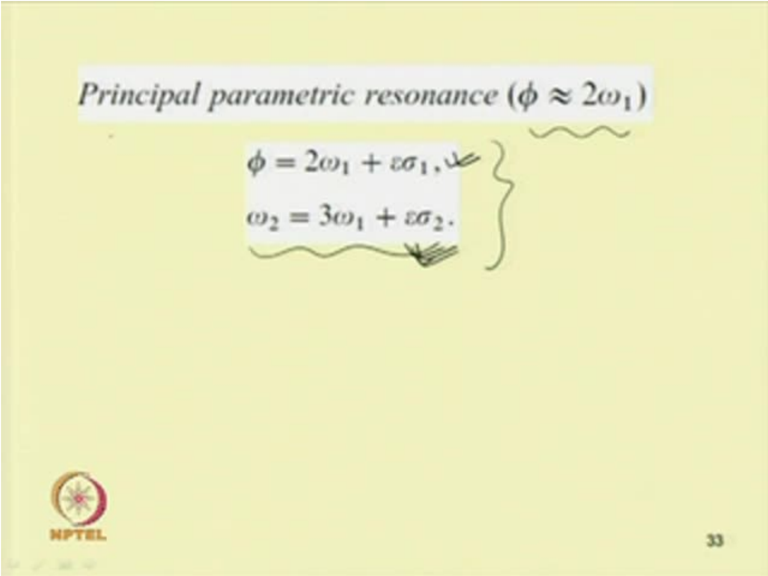

So, while solving this equation so, by applying this method of multiple scale so, here one can study the system for either 2 modes or 3 mode interaction. So, when the system is studied for this two mode interaction that is by taking this external frequency nearly equal to twice the first mode frequency and also, the second mode frequency nearly equal to thrice the first mode frequency. So, the second condition, the second condition is for internal resonance and this first condition gives the condition for the external resonance.

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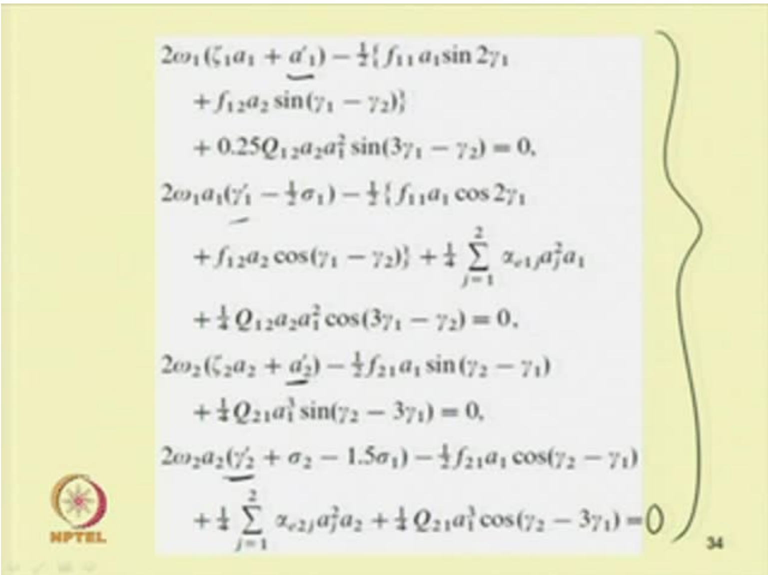

That means, when the external excitation is nearly twice that of the natural frequency first mode natural frequency of the system and when the second mode frequency is nearly thrice the first mode frequency of the system. So, one can study the system for different resonance conditions so, if one study this case for this principle parametric resonance condition so, one can obtain a state of 4 differential equations. So, previously we have considered only single and 2 so, in this case one can obtain this 4 differential equations. So, from this 4 differential equations so, by substituting this time derivative term equal to 0 that means a 1 dash gamma 1 dash a 2 dash gamma 2 dash so, were a 1 and a 2 represent the amplitude of the amplitude and phase for the first mode similarly, a 2 a 2 and gamma 2 represent the amplitude and phase for the second mode. By substituting these time derivative terms to 0 so, one can find the equilibrium points.

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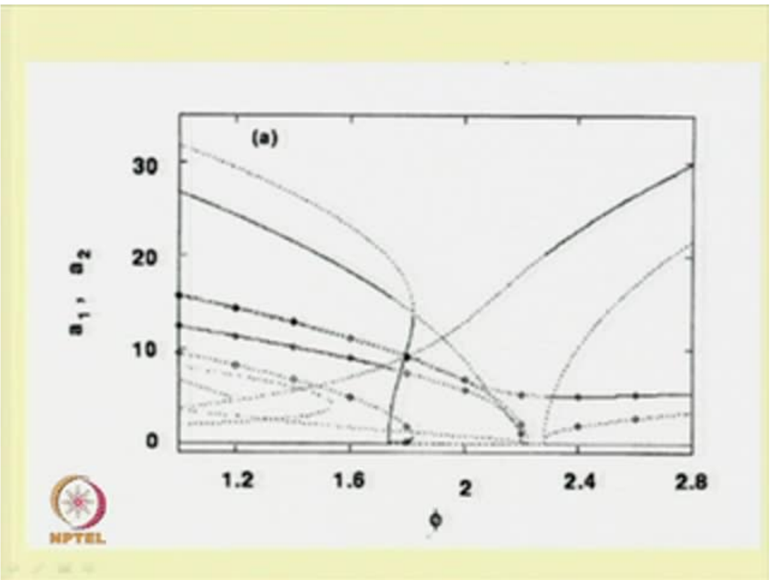

So, in this case one can obtain four algebraic or and transcendental differential transcendental equations. So, to solve these equations one can use this numerical method. So, here by using this Newton’s method one the response has been obtained and in this case by using Newton’s method a 1 and a 2 versus phi has been floated. So, each point on these curves represent a equilibrium position and the solid line represent the stable state and the dotted line present the unstable state of the system. Now by so, one can observe in this case or one can study the stability of the case in similar to the previous way by finding the Eigen value of the Jacobian matrix.

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$$
\begin{aligned}
&2\omega_1(\zeta_1 a_1 + a'_1) - \tfrac{1}{2}\{f_{11} a_1 \sin 2\gamma_1 \\
&\quad + f_{12} a_2 \sin(\gamma_1 - \gamma_2)\} \\
&\quad + 0.25 Q_{12} a_2 a_1^2 \sin(3\gamma_1 - \gamma_2) = 0, \\
&2\omega_1 a_1(\gamma'_1 - \tfrac{1}{2}\sigma_1) - \tfrac{1}{2}\{f_{11} a_1 \cos 2\gamma_1 \\
&\quad + f_{12} a_2 \cos(\gamma_1 - \gamma_2)\} + \tfrac{1}{4}\sum_{j=1}^{2} \alpha_{e1j} a_j^2 a_1 \\
&\quad + \tfrac{1}{4} Q_{12} a_2 a_1^2 \cos(3\gamma_1 - \gamma_2) = 0, \\
&2\omega_2(\zeta_2 a_2 + a'_2) - \tfrac{1}{2} f_{21} a_1 \sin(\gamma_2 - \gamma_1) \\
&\quad + \tfrac{1}{4} Q_{21} a_1^3 \sin(\gamma_2 - 3\gamma_1) = 0, \\
&2\omega_2 a_2(\gamma'_2 + \sigma_2 - 1.5\sigma_1) - \tfrac{1}{2} f_{21} a_1 \cos(\gamma_2 - \gamma_1) \\
&\quad + \tfrac{1}{4}\sum_{j=1}^{2} \alpha_{e2j} a_j^2 a_2 + \tfrac{1}{4} Q_{21} a_1^3 \cos(\gamma_2 - 3\gamma_1) = 0
\end{aligned}
$$

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But, in this case the size of the Jacobian matrix becomes more that is it is 4 is to 4 and by so, one can obtain 4 Eigen values of this system. Now, by checking these Eigen values for each equilibrium point so, one can find without that equilibrium point is stable or unstable.

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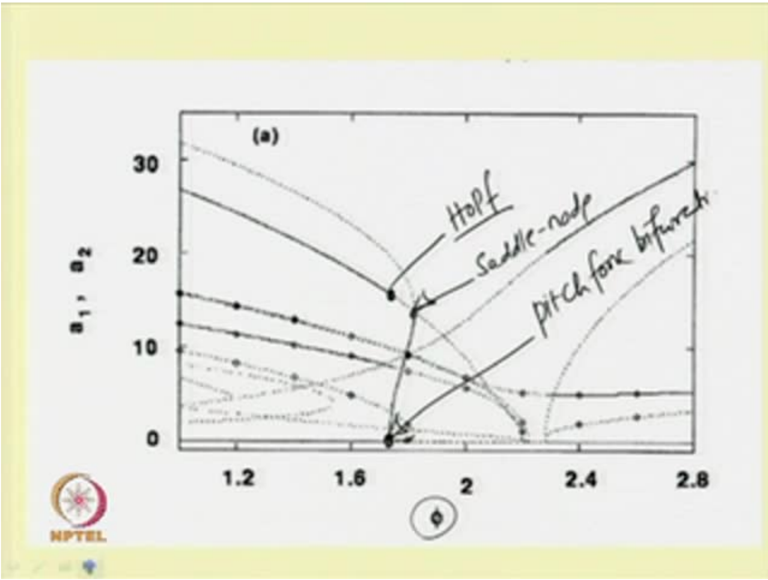

So, in this control space that is a 1 a 2 and phi so, by checking each equilibrium point and studying it is stability one can know whether the system is stable or unstable and by varying this control parameter one can find the bifurcation points. So, in these cases one can see that these bifurcation points so, this is a this is a pitch fork bifurcation point so, this is super critical pitch fork bifurcation point as from a trivial stable trivial response we are getting as trivial stable nontrivial response. Similarly, one can at this point one can see one can observe a saddle node bifurcation point.

And at the at this point one can observe a Hopf bifurcation point so, this point correspond to a Hopf bifurcation point. So, this point corresponds to a saddle node bifurcation point and so, this is a pitch fork bifurcation point. So, at these points if one further investigates so, one can find the generation of a periodic response at this Hopf bifurcation point. So, in this way one can study the stability of higher order systems to find whether the response is stable or not and while plotting this curve so, one can use this Newton's method to solve a state of non-linear equation the solving the set of non-linear equation to get this multi branch response curve is not easy.

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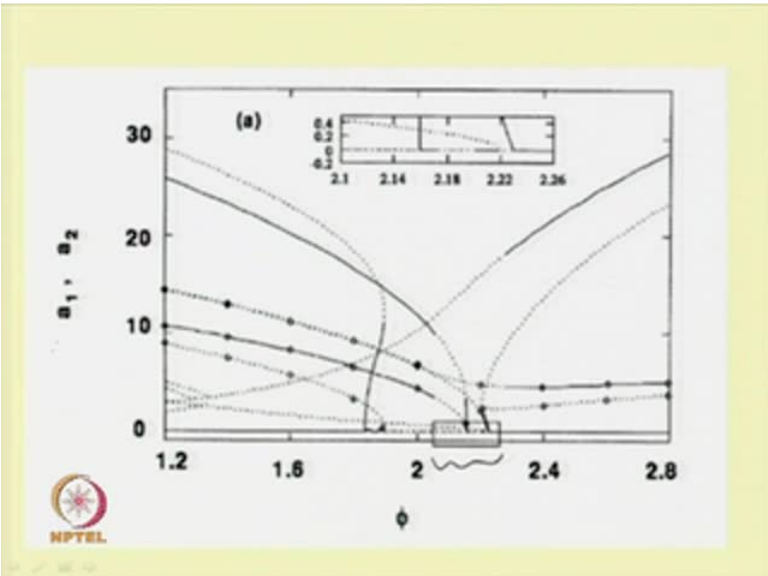

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So, here one can follow different methods one can follow this one can follow the one can follow different methods to find these branches so, one such method is by taking a set of initial conditions in this Newton's method and obtaining different roots. So, here so, in for the same case for a different so, at this point it has been shown in this trivial state there are different so, there is alternate stable and unstable bifurcation points.

So, from this bifurcation points one can observe the existence or observe the origin of the nontrivial states. So, from the trivial bifurcation points one can observe the origin of the nontrivial bifurcation points.

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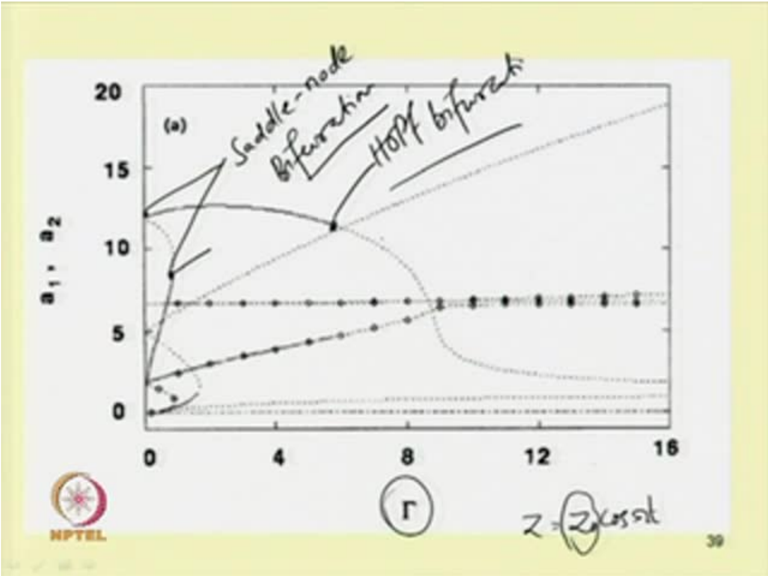

So, in this way one can find the frequency response also one can plot the force response of the system as the system is excited by a force z equal to z 0 cos omega t and in non dimensional form this omega is written in terms of phi so, this z 0 term this is the amplitude so, by changing this control parameter that is the amplitude of the response so, one can plot the force response of the system. So, in this force response of the system one can observe so, this point is a saddle node bifurcation point and this point is a Hopf bifurcation point so, this is a Hopf bifurcation point, and this point is a saddle node so, this point is also a saddle node bifurcation point.

So, at this to find whether it is saddle node or Hopf so, one can plot the variation of the Eigen value with this control parameter and by plotting this variation of this Eigen value with the control parameter for this particular branch one can observe up to what point it is stable and where it becomes unstable.

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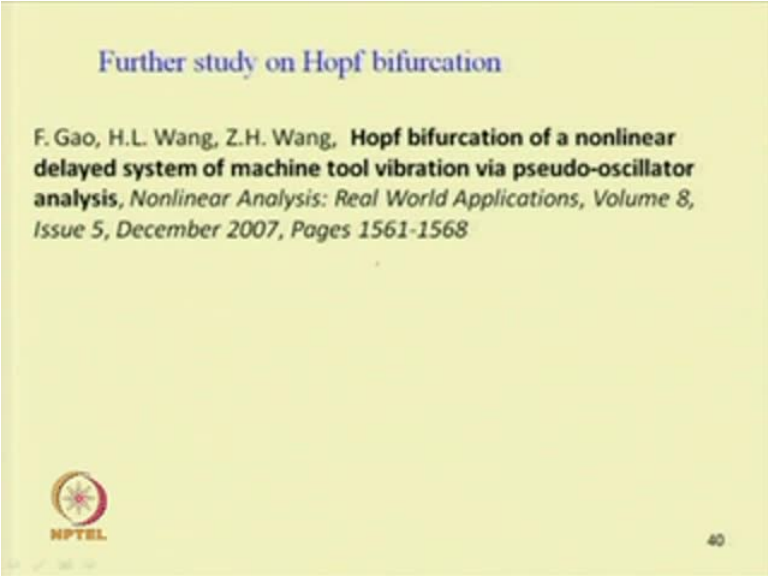

So, for further study of this Hopf bifurcation so, one can study these papers by Gao Wang and Wang Hopf bifurcation of a non-linear delayed system of machine tool vibration via pseudo oscillator analysis.

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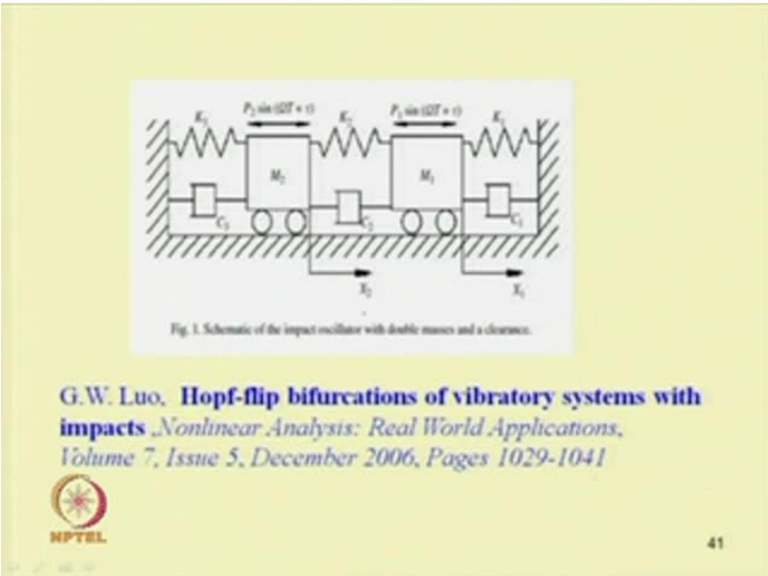

So, this is published in non-linear analysis real world application volume 8 issue 5 December 2007 page 1561 to 68. Also, one can take this system by Luo G W Luo Hopf-flip bifurcation of vibratory system with impacts. So, this is published in non-linear analysis real world application volume 7 issue 5 so, this is December 2006 page 1029 to

1041. So, in this system so, this is 2 mass system so, this is spring and damper system and these 2 mass are subjected to a force the first mass is subjected to a force p 1 sin omega t plus tau and this is the second mass is subjected to a force p 2 sin omega t plus tau.

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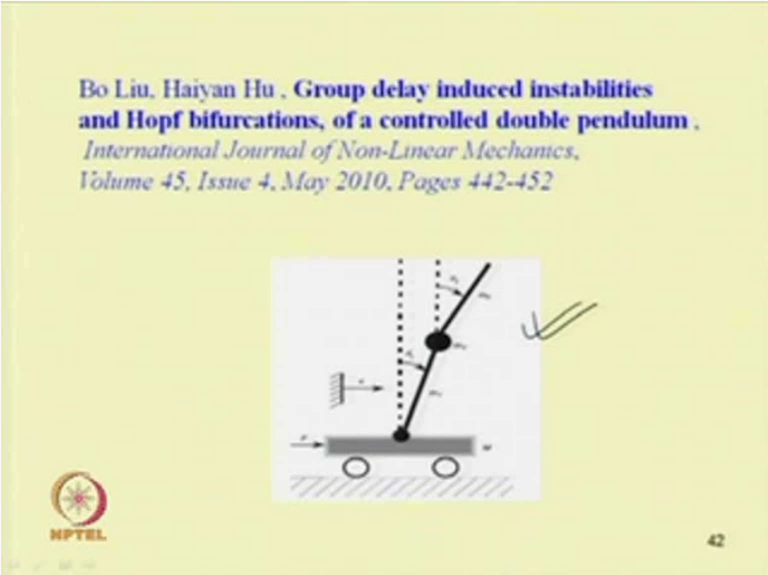

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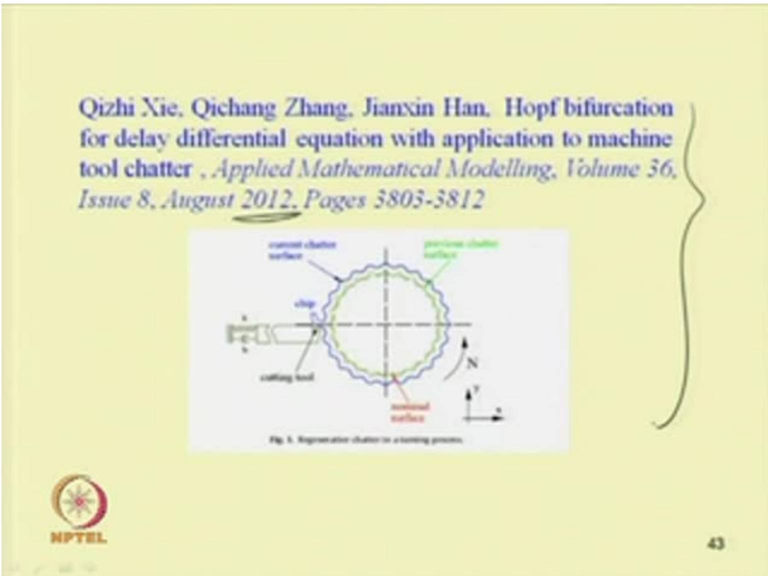

So, the displacement of the first one is x 1 and the displacement of the second one is x 2 and there is so, one can observe so, after sometimes the steady state will one can observe the Hopf or flip bifurcation in the system due to this impact of the system. So, one may also, study this paper by Liu and Ho group delay induced instabilities and Hopf bifurcation of a controlled double pendulum. So, in this case the double pendulum is subjected or this double pendulum is mounted on this card which is subjected to a force. And one can write the equation in terms of theta 1 and theta 2 and by solving this equation one can observe this Hopf or Hopf type of bifurcation in this case. Similarly, one can study the turning process so, which is a regenerative chatter occur in this case so, this paper by Xie Zhang and Hao Hopf bifurcation for delay differential equation with application to machine tool chatter which is published in applied mathematical modeling in 2012. So, one can study further study these papers to know more about this Hopf bifurcation.

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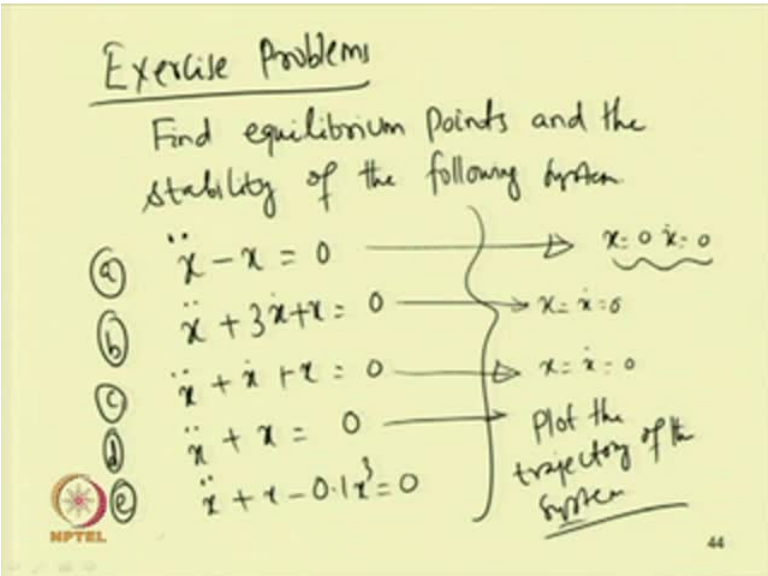

So, let us take some exercise problem. So, in these exercise problems first find the equilibrium points, find equilibrium points and equilibrium points and, and study the stability of the following systems. So, let us take this simple example x double dot minus x equal to 0 so, in case of this case x double dot plus three x dot plus x equal to 0. Then, x double dot plus x dot plus x equal to 0 so, in this case also, x double dot these are the simple equations one can take so, in this case x double dot plus x minus point one x cube equal to 0. So, these systems almost we have studied and one can find as find the equilibrium position.

So, now these are written in terms of second order differential equation has to convert this into a set of first order differential equation. And so, in the first case we can observe that at x equal to 0 and x dot equal 0 you have a saddle node bifurcation and so, for this case also, at x equal to x dot equal to 0 so, you can a stable node so, you can have a stable focus in this case at x equal to x dot equal to 0 and in this case you can have a center.

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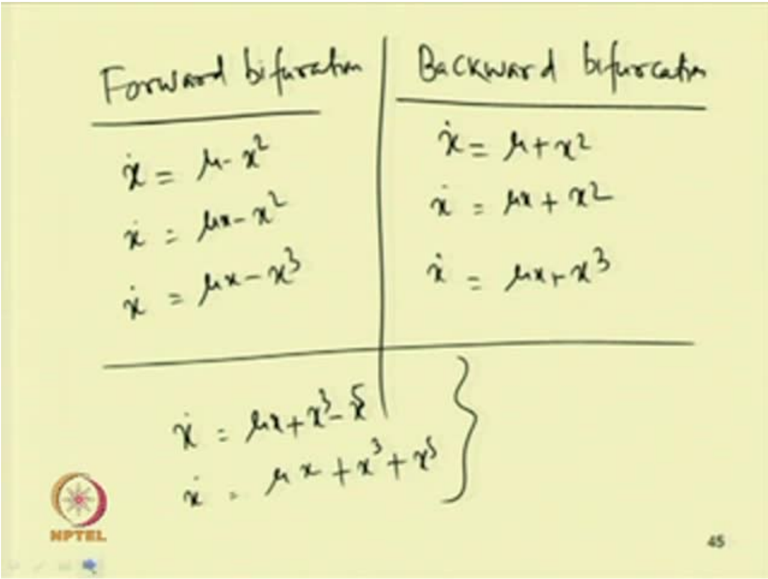

So, you can have a center in this case and in this case you can plot the trajectory in plot the trajectory, find the equilibrium points study the stability and also, you plot the trajectory of the system in this case. So, also, you can plot also, you can study this forward bifurcation and you plot this forward bifurcation and backward bifurcation for different system parameters which can be written in this first order mu minus x square x dot equal to mu x minus x square x dot equal to mu x minus x cube. Similarly, this side in case of backward x dot mu plus x square x dot equal to mu x plus x square and x dot equal to mu x plus x cube also, it is plot this case x dot. So, already we know that these are different generic case for saddle node pitch fork and trans-critical bifurcations. Also you just plot this case that is x dot equal to mu x plus x cube minus x to the power 5 and x dot equal to mu x plus plus x to the power 5.

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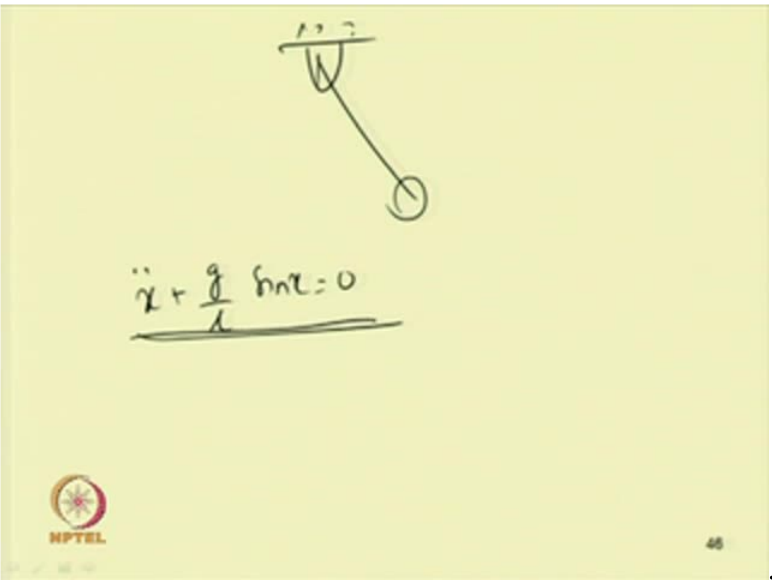

Also, another example you can take that is for the simple pendulum so, the equation is x double dot plus g by l into sin x equal to 0 so, in this case you can expand this sin x term and you can plot the response of the system. So, with this we complete the stability and bifurcation of fixed point response. So, next class we are going to study about the periodic response and its stability.

Thank you.